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SPINORS FROM VECTORS*

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March 18, 1970

ABSTRACT

Working in general at fixed time, we construct spinor quantities out of vectors. The methods used are not directly applicable to dynamical fields.

I. INTRODUCTION

There has long been a strong feeling in physics that theories should involve as few unmeasurable quantities as possible. In recent years this principle has been applied to the question of whether it might be possible to write theories of spinor fields (which cannot be measured directly) wholly in terms of vector and tensor currents (whose matrix elements are measurable) which are either constructed from the fermion fields, or whose properties (commutation relations, etc.) are specified by some theory of their own.[1,2] The problem then is whether the underlying fermion field can be reconstructed from the bosonic operators.

In this paper we wish to present some work we have done on the construction of spinor fields from vector fields (in isospin, 3-dimensional Euclidean, and Lorentz spaces) throughout space but at a fixed time, and mention the difficulties involved in extending our approach to the case of dynamically evolving fields. Our work provides then a simple kinematical precursor to the dynamical reconstruction procedures considered already in the literature.[3-7]

II. SPINORS OUT OF VECTORS IN A 3-DIMENSIONAL EUCLIDEAN SPACE

In this and the next section we construct several spinor fields from vector fields. These constructions are all kinematical (or geometrical); they have no dynamical content as we are considering the construction of the fields point by point in space, considering no relations that might exist among the fields at different points in time.

The first case that we shall consider is the construction of spinors from 3-vectors. The vectors in this case can be considered to be in either isospin space or ordinary 3-dimensional Euclidean space, with the resulting spinors in the appropriate space.

The procedure used here is just the usual Cayley-Klein formalism[8] which we regularly use to move from representations of $O(3)$ to those of $SU(2)$. Since this procedure amounts to determining the parameters specifying the rotation of a coordinate system we must use two linearly independent vectors (since a single vector does not determine a coordinate system), which we consider as having rotated from a reference set of vectors. These we take to be one vector along the x-axis and another in the xy-plane at the appropriate angle (the lengths of the vectors and the angle between them are invariant under rotation, so this reference system is determined uniquely). We then calculate the Cayley-Klein parameters for this rotation and the two columns of the 2×2 Cayley-Klein matrix provide us with two 2-spinors (which are not independent--each contains all the information specified by the rotation).

We take our two vectors to be $\vec{v}_1 = (v_{1x}, v_{1y}, v_{1z})$ and $\vec{v}_2 = (v_{2x}, v_{2y}, v_{2z})$. Our reference set of vectors is then $\vec{w}_1 = \left[(v_1{}^2)^{\frac{1}{2}}, 0, 0\right]$ and $\vec{w}_2 = (a,b,0)$ where $a^2 + b^2 = v_2{}^2$ and $a(v_1{}^2)^{\frac{1}{2}} = \vec{v}_1 \cdot \vec{v}_2$ $\left[\text{this imples } a = \dfrac{\vec{v}_1 \cdot \vec{v}_2}{(v_1{}^2)^{\frac{1}{2}}} \text{ and } b^2 = v_2{}^2 - \dfrac{(\vec{v}_1 \cdot \vec{v}_2)^2}{v_1{}^2}\right]$.

Then we must solve for the Cayley-Klein parameters:

$$\underset{\sim}{v}_1 = \begin{pmatrix} v_{1z} & v_{1-} \\ v_{1+} & -v_{1z} \end{pmatrix} = Q^\dagger \begin{pmatrix} w_{1z} & w_{1-} \\ w_{1+} & -w_{1z} \end{pmatrix} Q \tag{1}$$

$$\underset{\sim}{v}_2 = \begin{pmatrix} v_{2z} & v_{2-} \\ v_{2+} & -v_{2z} \end{pmatrix} = Q^\dagger \begin{pmatrix} w_{2z} & w_{2-} \\ w_{z+} & -w_{2z} \end{pmatrix} Q \tag{2}$$

$$Q = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad \delta = \alpha^*, \quad \beta = -\gamma^*, \quad |\alpha|^2 + |\gamma|^2 = 1 . \tag{3}$$

We find:

$$\alpha^2 = -\frac{1}{2ib}\left(v_{2-} - \frac{v_{1-}(a + ib)}{(v_1{}^2)^{\frac{1}{2}}}\right) \tag{4}$$

$$\gamma^2 = \frac{1}{2ib}\left(v_{2+} - \frac{v_{1+}(a + ib)}{(v_1{}^2)^{\frac{1}{2}}}\right) . \tag{5}$$

Then β and δ may be found from Eq. (3). We have thus constructed our two 2-spinors, namely

$$\chi_1 = \begin{pmatrix} \alpha \\ \gamma \end{pmatrix} , \qquad \chi_2 = \begin{pmatrix} \beta \\ \delta \end{pmatrix} . \tag{6}$$

We are guaranteed that these objects transform as spinors by the nature of the Cayley-Klein formalism. In particular the double-valuedness characteristic of spinors is apparent from the functional forms for α and γ (they are square roots of the vector directions involved).

To illustrate how these spinors contain at least some of the information given by the original vectors we look at the special case where the vectors are orthogonal. Then we find that

$$\mathrm{Re}\,\vec{V} = \mathrm{Re}(\chi_1^\dagger \vec{\sigma} \chi_2) = \frac{\vec{v}_1}{(v_1^{\,2})^{\frac{1}{2}}} , \qquad \vec{v}_1 \cdot \vec{v}_2 = 0 \tag{7}$$

$$\mathrm{Im}\,\vec{V} = \mathrm{Im}(\chi_1^\dagger \vec{\sigma} \chi_2) = \frac{-\vec{v}_2}{(v_2^{\,2})^{\frac{1}{2}}} , \qquad \vec{v}_1 \cdot \vec{v}_2 = 0 . \tag{8}$$

We see that it is impossible to recover the lengths of the vectors as we are in effect determining the angles of a rotation, which does not depend on the legnths of the vectors involved.

This procedure can be used to construct fields by repeating the construction at each point. One simply writes our vectors and spinors

as functions of position $\vec{x}$. Thus, for example, starting with $\vec{v}_1(\vec{x})$, $\vec{v}_2(\vec{x})$, we construct exactly as above:

$$\chi_1(\vec{x}) = \begin{pmatrix} \alpha(\vec{x}) \\ \gamma(\vec{x}) \end{pmatrix} \tag{9}$$

$$\alpha^2(\vec{x}) = -\frac{1}{2ib(\vec{x})}\left(v_{2-}(\vec{x}) - \frac{v_{1-}(\vec{x})\left(a(\vec{x}) + ib(\vec{x})\right)}{[(v_1(\vec{x}))^2]^{\frac{1}{2}}}\right) \tag{10}$$

and similarly for $\gamma^2(\vec{x})$, $\chi_2(\vec{x})$, etc.

However, the independence of length of the result is a serious drawback if we are considering applying our method to a realistic case of quantum fields. That is to say, these spinors can only be used to construct unit vectors.

To see how we might get around this difficulty we consider forming vectors from quartics in the spinors. In particular we consider two isovectors concocted by Bardakci and Halpern[9,10] in connection with their work on the Sugawara model. These vectors are:

$$\phi^\alpha = i(u^\dagger\tau^\alpha v - v^\dagger\tau^\alpha u) = -2\,\mathrm{Im}(u^\dagger\tau^\alpha v) \tag{11}$$

$$S^\alpha = [(\dot{u}^T\tau^\alpha v + \dot{v}^T\tau^\alpha u) + \mathrm{h.c.}][\dot{u}^T v + \mathrm{h.c.}] \quad . \tag{12}$$

Here $\dot{u}_i = \epsilon_{ij}u_j$ where ϵ_{ij} is the 2-dimensional antisymmetric symbol and h.c. stands for hermitian conjugate. It is evident that ϕ^α and S^α

do indeed transform as isovectors if u and v transform as isospinors; also ϕ^α and S^α are real and independent (there is no relation among the invariants ϕ^2, S^2, and $\vec{\phi}\cdot\vec{S}$). We may then solve for u and v to obtain spinors depending on all six degrees of freedom of the vectors. The solution in complete generality is rather formidable; we shall perform the inversion in one particular coordinate system to show the important features. We may then rotate to a general coordinate system to obtain the complete solution.

We define new spinors θ_1, θ_2 by:

$$u = \frac{1}{2}(\theta_1 - \theta_2) \tag{13}$$

$$v = \frac{1}{2}(\theta_1 + \theta_2) \tag{14}$$

and go to the frame in which[11]

$$\theta_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} . \tag{15}$$

We shall then solve for

$$\theta_2 = \begin{pmatrix} a \\ b \end{pmatrix} , \qquad |a|^2 + |b|^2 = 1 . \tag{16}$$

We find that

$$a = (1 - \phi)^{\frac{1}{2}} \pm i(\phi^2 - |b|^2)^{\frac{1}{2}} \tag{17}$$

$$b = \left(-\frac{2}{\phi^2}\right)\left\{\vec{\phi}\cdot\vec{S} \pm (1-\phi^2)^{\frac{1}{2}}\left[(\vec{\phi}\cdot\vec{S})^2 - \phi^2(S^2-4)\right]^{\frac{1}{2}}\right\}$$

$$\pm\ i\left(\frac{1}{2}\right)\left[(\phi\cdot S)^2 - \phi^2(S^2-4)\right]^{\frac{1}{2}} \quad . \tag{18}$$

This demonstrates clearly the dependence of the spinors on the vector invariants in contrast to the first case we considered. However, the result is somewhat awkward, to say the least, and its physical meaning unclear. Moreover, we still have not completely solved our problem because though not unit vectors, the vectors are still bounded, since the spinors are.

The boundedness problem is easily removed by introducing unbounded vectors defined in terms of the original vectors by:

$$\phi'^{\alpha} = \frac{\phi^{\alpha}}{\phi^2} \tag{19}$$

$$S'^{\alpha} = \frac{S^{\alpha}}{S^2} \tag{20}$$

or equivalently:

$$\phi^{\alpha} = \frac{\phi'^{\alpha}}{\phi'^2} \tag{21}$$

$$S^{\alpha} = \frac{S'^{\alpha}}{S'^2} \quad . \tag{22}$$

Thus we have a scheme involving vectors of arbitrary length, and spinors, with their mutual functional dependence evident at least in principle.

III. DIRAC SPINORS OUT OF 4-VECTORS

We now leave our discussion of 2-spinors to show how a similar construction can be performed in Lorentz space (and by analogy in other higher dimensional spaces). The problem in this case is to start with three linearly independent (with respect to the Lorentz metric) 4-vectors (three vectors being needed to determine a coordinate system in 4-space) and use these to determine Dirac spinors by a method analogous to the Cayley-Klein formalism appropriate to Lorentz space. The required technique is in fact described in the first chapter of Streater and Wightman.[12] The relevant equations are (notation slightly altered) Eqs. (23)-(28):

$$\underset{\sim}{x}' \equiv \begin{pmatrix} x_0' + x_3' & x_1' - ix_2' \\ x_1' + ix_2' & x_0' - x_3' \end{pmatrix} \equiv \begin{pmatrix} X_+' & x_-' \\ x_+' & X_-' \end{pmatrix}$$

$$= A \underset{\sim}{x} A^\dagger$$

$$= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix} \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix} \tag{23}$$

$$= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} X_+ & x_- \\ x_+ & X_- \end{pmatrix} \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}$$

$$\det A = ad - bc = 1 \tag{24}$$

$$S'(A) = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & \zeta^{-1}\bar{A}\zeta \end{pmatrix} \tag{25}$$

$$\zeta = i\tau^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{26}$$

$$\bar{A} \equiv \begin{pmatrix} a^* & b^* \\ c^* & d^* \end{pmatrix} \qquad \text{(the complex conjugate of } A\text{)} \tag{27}$$

$$S(A) = V\, S'(A)\, V^{-1} = \frac{1}{2}\begin{pmatrix} I & I \\ I & -I \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\begin{pmatrix} I & I \\ I & -I \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} A + B & A - B \\ A - B & A + B \end{pmatrix} . \tag{28}$$

Equations (23) and (24) determine the matrix A characterizing the Lorentz transformation taking a reference coordinate system into the coordinate system determined by our three 4-vectors. Equations (25), (26), and (27) determine the matrix S' whose columns are spinors, and Eq. (28) transforms to the more familiar representation of the Dirac algebra (given in Bjorken and Drell[13]). The columns of $S(A)$ then form the usual four Dirac spinors that we are accustomed to dealing with.

Calling our three 4-vectors η, ρ, and σ, requiring them to be perpendicular, of unit spacelike interval, and taking them to be transformations of $\hat{x}$, $\hat{y}$, and $\hat{z}$ respectively, we obtain the equations:

$$\eta_0 = \frac{1}{2}(a^*b + c^*d + ab^* + cd^*)$$

$$\eta_1 = \frac{1}{2}(bc^* + a^*d + ad^* + b^*c)$$

$$\eta_2 = \frac{1}{2i}(-bc^* + a^*d - ad^* + b^*c) \tag{29}$$

$$\eta_3 = \frac{1}{2}(a^*b - c^*d + ab^* - cd^*)$$

$$\rho_0 = \frac{i}{2}(a^*b + c^*d - ab^* - cd^*)$$

$$\rho_1 = \frac{i}{2}(bc^* + a^*d - ad^* - b^*c)$$

$$\rho_2 = \frac{1}{2}(-bc^* + a^*d + ad^* - b^*c) \tag{30}$$

$$\rho_3 = \frac{i}{2}(a^*b - c^*d - ab^* + cd^*)$$

$$\sigma_0 = \frac{1}{2}(aa^* + cc^* - bb^* - dd^*)$$

$$\sigma_1 = \frac{1}{2}(ac^* + a^*c - bd^* - b^*d)$$

$$\sigma_2 = \frac{1}{2i}(-ac^* + a^*c + bd^* - b^*d) \tag{31}$$

$$\sigma_3 = \frac{1}{2}(aa^* - cc^* - bb^* + dd^*) \quad .$$

These twelve equations (six of which are independent), together with Eq. (24), determine the matrix A (to an overall sign). Their actual solution is prohibitively difficult and we have not attempted it.

Similar equations could also be written down for more general choices of the three 4-vectors, but here as in our first example, the result is independent of the invariants.

The spinors determined by our 4-vectors could also be obtained by finding the Lorentz transformation in the SO(3,1) representation [as opposed to the SL(2,C) representation], decomposing into a product of boost and rotation matrices, and subjecting the matrix S determined by the boost (as given in Bjorken and Drell) to a similarity transformation with the rotation matrix so determined. This is the analogue of doing our first case by determining the Euler angles of the rotation and writing the Cayley-Klein parameters in terms of them. This method of finding the spinors is unfortunately even messier than the one we have presented.

Now that we know how to construct spinors from vectors by the above methods we may extend the results to fields by performing the construction point by point, as mentioned earlier, or perhaps specifying some dependence of the spinors on position (such as a plane wave).

IV. SUMMARY

We have seen how to form various classical spinor fields from classical vector fields, working at fixed time. When we try to extend our solution to dynamical fields (classical or quantum), we encounter the problem of satisfying the constraints imposed by the commutation relations (or Poisson brackets), and the equations of motion on the fields involved. In fact, these constraints are quite restrictive and to the best of our knowledge there have been only three cases in which spinors have been constructed from vectors in the context of a dynamical theory; these are the cases considered by Coleman, Gross, and Jackiw,[3] by Prugovečki,[4] and more formally by Halpern.[5]

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FOOTNOTES AND REFERENCES

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