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RESONANT INTERACTIONS BETWEEN PARTICLES AND NORMAL MODES IN A CYLINDRICAL PLASMA*

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ABSTRACT

A plasma configuration with cylindrical symmetry is studied, containing axial and azimuthal magnetic field and radial electric field, with arbitrary radial variation. The particle motion is parameterized by three exact action-invariants: radial action, canonical angular momentum, and canonical axial momentum; in the limit of small gyroradius they are equivalent to magnetic moment, radial guiding-center position, and parallel velocity. The perturbed Vlasov-Maxwell equations lead to a set of normal modes, which can interact resonantly with the particles. The quantum rate equations for this interaction, together with the laws for conservation of energy, angular momentum, and axial momentum, lead (in the classical limit) to a Fokker-Planck equation in action-space for the particles, and to an equation of evolution for mode energy. These coupled kinetic equations satisfy an H-theorem, which implies a monotonic approach to a canonical distribution: a rigid-rotor distribution for particles, and a generalized Rayleigh-Jeans distribution for the modes. This asymptotic state may however be unconfined. The quantum transition probability is deduced from a classical calculation of emissivity. Explicit expressions are obtained for the mode growth

rate and for the particle diffusion tensor. Finally, the Vlasov conductivity kernel is deduced from the growth rates, by the use of the Kramers-Kronig relations.

I. INTRODUCTION

The study of resonant interactions between plasma waves and particles has been well developed for the case of a uniform system, and for local interactions in a slightly nonuniform system.[1-5] It is now of interest to extend these ideas to essentially nonuniform systems, where the particles may traverse an appreciable part of the system, and where the waves are normal modes (not necessarily describable by the WKB method).

Since the difference between a uniform and a nonuniform equilibrium configuration is the lack of translational invariance in the latter, it seems advisable to study first the case where this loss of invariance occurs only for one of the dimensions, while the other two dimensions retain the invariance. Such systems[6] are the plasma slab (variation in x, symmetry in y,z) and the plasma cylinder (variation in r, symmetry in φ,z). The latter system has been chosen for the study in this paper, since the slab can be obtained from the cylinder by appropriate changes.

We consider thus an equilibrium configuration with vector potential $A_\varphi(r)$, $A_z(r)$, to describe the magnetic field $B_z^0(r) = r^{-1}\, d(rA_\varphi)/dr$, $B_\varphi^0(r) = -dA_z/dr$. In addition, there may be a scalar potential $\phi(r)$ to describe the equilibrium electric field $E_r^0(r) = -d\phi/dr$. There is no limitation on the magnitude of the electric field, nor on the shear of the magnetic field. The equilibrium particle distributions must be self-consistent with these fields, providing the appropriate charge and current densities. (The plasma need not be quasi-neutral, and may be single-component.)

The particle motion may be taken as nonrelativistic or relativistic. Since φ and z are ignorable coordinates in the Hamiltonian, their conjugate momenta, p_φ and p_z, are invariants of the unperturbed particle orbits. The radial motion is periodic, and is characterized by its action J_r. (We exclude the possibility of radially untrapped particles, corresponding to an unconfined system.) In this description there is no requirement for small gyroradius; the radial motion may be over a small region, or over the whole system.

It is helpful to keep in mind a special limiting case, to provide intuitive guidance, namely the case of uniform B_z, with $B_\varphi^{\ 0} = E_r^{\ 0} = 0$, and small gyroradius. In that case the action J_r is proportional to the magnetic moment, the canonical angular momentum p_φ is proportional to the radial position (squared) of the guiding center, and the axial momentum p_z is proportional to the axial particle velocity $v_\parallel$ [see Eq. (5).]. If instead $B_z^{\ 0} = 0$, while $B_\varphi^{\ 0} \neq 0$, then p_φ represents azimuthal velocity $v_\parallel$, while p_z represents radial guiding-center position. In the general case, the invariants p_φ and p_z represent combinations of radial position and $v_\parallel$.

Again because φ and z are ignorable, the normal modes of the Maxwell equations have the form

$$\underset{\sim}{E}^a(r)\, \exp(im\varphi + ikz - i\omega_a t)\ , \tag{1}$$

where m is an integer and k is real. In this paper we restrict our study to systems whose instabilities arise only from first-order resonant interactions between the normal modes and the particles. In that case, we may take ω_a real, since its small imaginary part, representing resonant growth or decay, will appear as a slowly varying mode amplitude.

The condition for resonant interaction between a normal mode and a particle is found to be

$$\omega_a - m\dot{\bar{\phi}} - k\dot{Z} = \ell\,\omega_r\,, \tag{2}$$

where $\dot{\bar{\phi}}$ is the azimuthal drift velocity (the average of $\dot{\phi}$ over a radial bounce), $\dot{Z}$ is the axial drift velocity, ω_r is the radial bounce frequency, and ℓ is an integer. (For small gyroradius, ω_r is the particle's gyrofrequency.) As a result of this interaction, there are changes in the particle's action variables J_r, p_φ, p_z. These changes are represented by a (relatively) slow evolution of the particle distribution $f(J_r, p_\varphi, p_z; t)$, which satisfies a Fokker-Planck equation (71) in the action space. The diffusion tensor (73) and the dynamic friction (72) are both simply expressed in terms of a mode-particle coupling coefficient $\alpha_a(J_r, p_\varphi, p_z)$. This coefficient is calculated (105) from the radial variation of the mode and the unperturbed particle orbit.

The diffusion tensor also involves the mode energies. These evolve according to a linear equation (65), which includes the spontaneous emission rate by discrete particles, and the linear Vlasov growth (or damping) rate. Both these quantities are expressed in terms of the coupling coefficient.

The coupled equations (70) and (63), for particles and mode energies, satisfy an H-theorem, Eq. (82), representing the monotonic increase of entropy, and approach to a stable thermal equilibrium, with the particle distribution (for species s):

$$f_s(J_r, p_\varphi, p_z) \propto \exp(-\beta)[H^s(J_r, p_\varphi, p_z) - \omega_\varphi\, p_\varphi] \tag{3}$$

and with the mode energy

$$W_a = \beta^{-1}[1 - m(\omega_\varphi/\omega_a)]^{-1} . \tag{4}$$

The parameters β and ω_φ are determined by the conservation of energy and angular momentum. (We assume, in this introduction, that the axial momentum parameter vanishes.)

In a single-species plasma, Eq. (3) represents a confined system, if ω_φ lies within a certain range.[7] Note that the generalized Rayleigh-Jeans law (4) allows for negative-energy modes $(m\,\omega_\varphi > \omega_a)$. However, for an electron-ion plasma, ω_φ must be the same for both species (in order to satisfy the H-theorem), and this indicates that at least one of the species is unconfined, as will be shown in Sec. V. Hence the approach to stable equilibrium is to be interpreted in the two-species case as a perpetual radial diffusion towards uniformity.

The groundwork for the present study was laid in a previous paper,[8] to be referred to as KN. That paper proceeded, by purely classical means, to derive the coupled kinetic equations for an inhomogeneous one-dimensional plasma. In these equations each term was found to involve the same mode-particle coupling coefficient. It was then shown that the classical limit of the quantum rate equations led to equations of the same form, with much less labor. The quantum equations involved a transition probability for the emission or absorption of a normal mode quantum by a particle. By comparison of the quantum and classical

equations, the quantum transition probability was identified with the classical coupling coefficient.

From the experience of that paper, it was felt justifiable to use the quantum approach[9] in the present work. In this way, the emission rate, the linear growth rate, the dynamic friction, and the diffusion tensor are all expressed in terms of the coupling coefficient, which in turn is calculated classically by studying the emission rate, only one of these four quantities. The emission rate is found from the test particle theorem[10] applied to the Maxwell equations, whose Green function is expressed in terms of the normal modes.[11]

The Vlasov conductivity tensor can be determined indirectly, by applying the Kramers-Kronig relation to its hermitian part, which is deduced from the mode growth rate expression. The conductivity is expressed in terms of the microcurrent correlation tensor, as a generalization of the Kubo relation,[12] and is also expressed explicitly in terms of the particle orbits.

In the final section, a critique is presented on the methods of this paper, on the limits of validity, and on the possibility of generalizations in various directions.

II. THE PARTICLE INVARIANTS

In a static configuration with cylindrical symmetry, the canonical momenta for a relativistic particle are

$$\begin{aligned} p_r &= m_0 \gamma \dot{r} , \\ p_\varphi &= m_0 \gamma r^2 \dot{\varphi} + (e/c) r A_\varphi(r) , \\ p_z &= m_0 \gamma \dot{z} + (e/c) A_z(r) , \end{aligned} \tag{5}$$

where m_0 is the rest mass and $\gamma = (1 - v^2/c^2)^{-1/2}$. In the non-relativistic case, set $\gamma = 1$. The relativistic Hamiltonian is

$$H(p_r, p_\varphi, p_z, r) = \left\{ m_0^2 c^4 + c^2 p_r^2 + [cp_z - eA_z(r)]^2 + [(cp_\varphi/r) - eA_\varphi(r)]^2 \right\}^{1/2} + e\phi(r) , \tag{6a}$$

while the nonrelativistic Hamiltonian is

$$H(p_r, p_\varphi, p_z, r) = (2m_0)^{-1} \left\{ p_r^2 + [p_z - (e/c)A_z(r)]^2 + [(p_\varphi/r) - (e/c) A_\varphi(r)]^2 \right\} + e\phi(r). \tag{6b}$$

In either case, φ and z are ignorable, so p_φ and p_z are invariants of the motion.

For given (p_φ, p_z), consider the phase-plane (r, p_r). Since H is even in p_r, the curves of constant H are nested closed curves, symmetric under $(p_r \to -p_r)$, and represent the periodic radial motion. The area enclosed by a curve is its action

$$2\pi\, J_r(H, p_\varphi, p_z) \equiv \oint p_r(r, H, p_\varphi, p_z)\,dr\ , \tag{7}$$

where p_r is the double-valued function obtained by solving Eq. (6) for $p_r^{\,2}$. The limits of the radial motion are $r_1(H, p_\varphi, p_z)$ and $r_2(H, p_\varphi, p_z)$, obtained by setting $p_r = 0$ in Eq. (6) and solving for r. Although $\dot{r}$ is doubled-valued in r, the other velocity components $\dot{\varphi}$ and $\dot{z}$ are single-valued, as seen from Eq. (5).

It is convenient to use J_r as a new canonical momentum, in place of p_r, since it is an invariant of the unperturbed motion. The generating function for the canonical transformation is

$$G(r, \varphi, z;\ J_r, P_\varphi, P_z) \equiv \int_{r_1}^{r} dr'\ p_r(r', J_r, P_\varphi, P_z) + \varphi\, P_\varphi + z\, P_z,$$

where $p_r(r, J_r, p_\varphi, p_z)$ is obtained by solving (7) for $H(J_r, p_\varphi, p_z)$ to eliminate H from $p_r(r, H, p_\varphi, p_z)$. We note first that the old momenta

$$p_r = \partial G/\partial r = p_r(r, J_r, P_\varphi, P_z)$$

$$p_\varphi = \partial G/\partial \varphi = P_\varphi$$

$$p_z = \partial G/\partial z = P_z$$

are the same as the new momenta, for the φ, z variables.

The new coordinates are, however, quite different. Conjugate to J_r is the radial angle variable

$$w = \partial G/\partial J_r = \int_{r_1}^{r} dr'\ \partial p_r(r', J_r, P_\varphi, P_z)/\partial J_r\ . \tag{8}$$

Since w is ignorable in $H(J_r, P_\varphi, P_z)$, we have

$$\dot{J}_r = -\partial H/\partial w = 0 \,, \qquad \text{and} \qquad \dot{w} = \partial H/\partial J_r \equiv \omega_r(J_r, P_\varphi, P_z) \;; \tag{9}$$

the angular bounce frequency ω_r is also an invariant of the motion. In place of (8), it is then simpler to use

$$w(r, \underset{\sim}{J}) = \int \dot{w} \; dt = \omega_r(\underset{\sim}{J}) \int_{r_1}^{r} dr'/\dot{r}\,(r', \underset{\sim}{J}) \,, \tag{10}$$

where $\underset{\sim}{J}$ denotes the set (J_r, P_φ, P_z). We note that w runs from zero to π as r runs from r_1 to r_2; in a complete radial cycle, w changes by 2π.

The new azimuth conjugate to P_φ is

$$\Phi = \partial G/\partial P_\varphi = \varphi + \int_{r_1}^{r} dr' \; \partial p_r(r', \underset{\sim}{J})/\partial P_\varphi \,. \tag{11}$$

Its time-derivative,

$$\dot{\Phi} = \partial H(\underset{\sim}{J})/\partial P_\varphi \,, \tag{12}$$

is also an invariant of the motion, and is the average of $\dot{\varphi}(p_\varphi, r)$ [see (5)] over a radial bounce period. It thus represents the guiding-center drift, in conventional language. In place of (11), it is simpler to use

$$\Phi = \varphi - \int_{r_1}^{r} dr' \; \dot{r}^{-1}(r', \underset{\sim}{J})[\dot{\varphi}(p_\varphi, r) - \dot{\Phi}(J)] \,. \tag{13}$$

Analogously, the new coordinate conjugate to P_z is

$$Z = \partial G/\partial P_z = z + \int_{r_1}^{r} dr' \; \partial p_r(r',\underset{\sim}{J})/\partial P_z \; . \tag{14}$$

Its time-derivative,

$$\dot{Z} = \partial H(\underset{\sim}{J})/\partial P_z \; , \tag{15}$$

is also invariant, and represents $\dot{z}(p_z,r)$ averaged over a radial period; it is the axial drift velocity.

In equilibrium, the phase-space densities must be functions only of the invariants. Suppressing the species label, they have the form $f(\underset{\sim}{J}) = f(J_r, P_\varphi, P_z)$. Noting the invariance of the phase-space element $d\Gamma \equiv dr\, d\varphi\, dz\, dp_r\, dp_\varphi\, dp_z = dw\, d\mathcal{J}\, dZ\, dJ_r\, dP_\varphi\, dP_z$, we see that the number of particles per unit axial length is $(2\pi)^2 \int d^3J\; f(\underset{\sim}{J})$.

The charge density is (with species summation implied)

$$\begin{aligned} \rho(r) &\equiv e \int (d\Gamma/d^3r)\; f(\underset{\sim}{J}) \\ &= (e/r) \int d^3J\; (dw/dr)\; f(\underset{\sim}{J}) \\ &= (e/r) \int d^3J\; [\omega_r(\underset{\sim}{J})/|\dot{r}(r,\underset{\sim}{J})|]\; f(\underset{\sim}{J}) \; . \end{aligned} \tag{16}$$

The scalar potential must satisfy the quation

$$r^{-1}\, d(r\;\; d\phi/dr)/dr = -\, 4\pi\, \rho(r) \; ; \tag{17}$$

this is highly nonlinear in $\phi(r)$, since $\rho(r)$ is an implicit nonlinear functional of ϕ, as seen from Eq. (6). We shall not be concerned with explicit solutions of (17), nor of the analogous equation for the vector potential $\underset{\sim}{A}$.

III. NORMAL MODES

The linearized Vlasov equation leads to a conductivity relation[13] between the perturbed electric field $\underset{\sim}{E}(\underset{\sim}{r},t)$ and the perturbed current density $\underset{\sim}{j}(\underset{\sim}{r},t)$:

$$\underset{\sim}{j}(\underset{\sim}{r},t) = \int_0^\infty d\tau \int d^3r' \; \underset{\approx}{g}(\underset{\sim}{r},\underset{\sim}{r}';\tau)\cdot\underset{\sim}{E}(\underset{\sim}{r}',t-\tau), \tag{18a}$$

or in terms of Fourier transforms:

$$\underset{\sim}{j}(\underset{\sim}{r},\omega) = \int d^3r' \; \underset{\approx}{g}(\underset{\sim}{r},\underset{\sim}{r}';\omega)\cdot\underset{\sim}{E}(\underset{\sim}{r}',\omega) \; , \tag{18b}$$

where

$$\underset{\approx}{g}(\underset{\sim}{r},\underset{\sim}{r}';\omega) \equiv \int_0^\infty d\tau \; e^{i\omega\tau} \; \underset{\approx}{g}(\underset{\sim}{r},\underset{\sim}{r}';\tau) \; . \tag{19}$$

In (19), the imaginary part of ω must be larger than the growth rates of all instabilities. For the remainder of the ω-plane, $\underset{\approx}{g}$ is to be analytically continued from above.

The hermitian part of $\underset{\approx}{g}$ is responsible for dissipation:

$$\sigma'_{\mu\nu}(\underset{\sim}{r},\underset{\sim}{r}';\omega) \equiv \frac{1}{2}\left[\sigma_{\mu\nu}(\underset{\sim}{r},\underset{\sim}{r}';\omega) + \sigma^*_{\nu\mu}(\underset{\sim}{r}',\underset{\sim}{r};\omega)\right] \tag{20}$$

(for ω restricted to the real axis). The antihermitian part $\underset{\approx}{g}''$, given by $\underset{\approx}{g} \equiv \underset{\approx}{g}' + i\,\underset{\approx}{g}''$, is responsible for the reactive part of the response. We shall assume that $\underset{\approx}{g}'$ is, in some sense, small compared to $\underset{\approx}{g}''$, and that the eigenfrequencies of the normal modes are nearly real.

When (18b) is inserted in the linearized Maxwell equations:

$$\nabla \times \underset{\sim}{E}(\underset{\sim}{r},\omega) - (i\omega/c)\,\underset{\sim}{B}(\underset{\sim}{r},\omega) = 0 \tag{21a}$$

$$\nabla \times \underset{\sim}{B}(\underset{\sim}{r},\omega) + (i\omega/c)\,\underset{\sim}{E}(\underset{\sim}{r},\omega) = (4\pi/c)\,\underset{\sim}{j}(\underset{\sim}{r},\omega) \; , \tag{21b}$$

the result is

$$\nabla \times [\nabla \times \underset{\sim}{E}(\underset{\sim}{r},\omega)] = (\omega^2/c^2) \int d^3r' \, \underset{\approx}{\epsilon}(\underset{\sim}{r},\underset{\sim}{r}';\, \omega)\cdot\underset{\sim}{E}(\underset{\sim}{r}',\omega) \, , \tag{22}$$

where the dielectric kernel is defined as

$$\underset{\approx}{\epsilon}(\underset{\sim}{r},\underset{\sim}{r}';\, \omega) = \delta(\underset{\sim}{r} - \underset{\sim}{r}') \, \underset{\approx}{I} + (4\pi i/\omega) \, \underset{\approx}{g}(\underset{\sim}{r},\underset{\sim}{r}';\, \omega) \, . \tag{23}$$

It is convenient to write the field equation (22) in the concise form

$$\underset{\approx}{K}(\omega)\cdot\underset{\sim}{E}(\underset{\sim}{r},\omega) + \underset{\sim}{E}(\underset{\sim}{r},\omega) = 0 \, , \tag{24}$$

where the operator $\underset{\approx}{K}(\omega)$ is defined by

$$\underset{\approx}{K}(\omega)\cdot\underset{\sim}{E}(\underset{\sim}{r}) \equiv -(c^2/\omega^2)\nabla \times [\nabla \times \underset{\sim}{E}(\underset{\sim}{r})] + (4\pi i/\omega) \int d^3r' \, \underset{\approx}{g}(\underset{\sim}{r},\underset{\sim}{r}';\, \omega)\cdot\underset{\sim}{E}(\underset{\sim}{r}') . \tag{25}$$

For all complex ω, we generalize Eq. (24) to the eigenvalue equation

$$\underset{\approx}{K}(\omega)\cdot\underset{\sim}{E}^n(\underset{\sim}{r},\omega) + \Lambda_n(\omega) \, \underset{\sim}{E}^n(\underset{\sim}{r},\omega) = 0 \, . \tag{26}$$

The complex eigenfrequencies Ω_a for Eq. (24) are then the roots of the equation

$$\Lambda_n(\omega) = 1 \, . \tag{27}$$

We assume that all the eigenfrequencies of interest are nearly real:

$$\Omega_a = \omega_a + i\,\gamma_a \, , \tag{28}$$

with γ_a small. Expanding (27) to first order in γ_a, we obtain the equations

$$\mathrm{Re}\ \Lambda_n(\omega_a) = 1 , \tag{29a}$$

$$\gamma_a = -\mathrm{Im}\ \Lambda_n(\omega_a)/(d\mathrm{Re}\ \Lambda_n/d\omega)\big|_{\omega_a} , \tag{29b}$$

for the determination of ω_a and γ_a. These equations require the study of (26) only on the real axis, where we express $\underset{\approx}{K}(\omega)$ in terms of hermitian and anti-hermitian parts:

$$\underset{\approx}{K}(\omega) = \underset{\approx}{K}'(\omega) + i\,\underset{\approx}{K}''(\omega) , \tag{30}$$

and assume that $\underset{\approx}{K}''$ (proportional to $\underset{\approx}{\sigma}'$) is small.

We therefore at first neglect $\underset{\approx}{K}''$, and consider the <u>hermitian</u> equation

$$\underset{\approx}{K}'(\omega)\cdot\underset{\sim}{E}^{n(0)}(\underset{\sim}{r},\omega) + \Lambda_n^{(0)}(\omega)\ \underset{\sim}{E}^{n(0)}(\underset{\sim}{r},\omega) = 0 , \tag{31}$$

as the zero-order approximation to Eq. (26). The eigenvalues $\Lambda_n^{(0)}(\omega)$ of this equation are thus real, and are given by

$$\Lambda_n^{(0)}(\omega) = -\frac{\int d^3r\ \underset{\sim}{E}^{n(0)*}(\underset{\sim}{r},\omega)\cdot\underset{\approx}{K}'(\omega)\cdot\underset{\sim}{E}^{n(0)}(\underset{\sim}{r},\omega)}{\int d^3r\ |\underset{\sim}{E}^{n(0)}(\underset{\sim}{r},\omega)|^2} . \tag{32}$$

Treating $i\underset{\approx}{K}''(\omega)$ as a first-order perturbation, we find the perturbation in Λ_n by standard means:

$$\Lambda_n^{(1)}(\omega) = -i\ \frac{\int d^3r\ \underset{\sim}{E}^{n(0)*}(\underset{\sim}{r},\omega)\cdot\underset{\approx}{K}''(\omega)\cdot\underset{\sim}{E}^{n(0)}(\underset{\sim}{r},\omega)}{\int d^3r\ |\underset{\sim}{E}^{n(0)}(\underset{\sim}{r},\omega)|^2} . \tag{33}$$

Since $\underset{\approx}{K}''(\omega)$ is itself a hermitian operator, $\Lambda_n^{(1)}(\omega)$ is purely imaginary, and we may thus identify it with $\mathrm{Im}\,\Lambda_n(\omega)$ in (29b), to lowest order. Likewise, we identify $\Lambda_n^{(0)}(\omega)$ with $\mathrm{Re}\,\Lambda_n(\omega)$ in (29a). We must now require that the roots of

$$\Lambda_n^{(0)}(\omega_a) = 1 \tag{34}$$

are indeed all real. If complex roots of (34) are found, they occur in complex conjugate pairs, and represent either nonresonant instabilities (which are not covered by our treatment) or extraneous roots (beyond the range of validity of the perturbation expansion).

For (29b), we differentiate (32) with respect to ω, use (31), and find

$$\gamma_a = -\frac{\int d^3r\, \underset{\sim}{E}^{a*}(\underset{\sim}{r})\cdot\underset{\approx}{K}''(\omega_a)\cdot\underset{\sim}{E}^{a}(\underset{\sim}{r})}{\int d^3r\, \underset{\sim}{E}^{a*}(\underset{\sim}{r})\cdot[\partial\underset{\approx}{K}'(\omega)/\partial\omega]_{\omega_a}\cdot\underset{\sim}{E}^{a}(\underset{\sim}{r})}\,, \tag{35}$$

where

$$\underset{\sim}{E}^{a}(\underset{\sim}{r}) \equiv \underset{\sim}{E}^{n(0)}(\underset{\sim}{r},\omega_a) \tag{36}$$

are the zero-order eigenfunctions, i.e., the solutions of

$$\underset{\approx}{K}'(\omega_a)\cdot\underset{\sim}{E}^{a}(\underset{\sim}{r}) + \underset{\sim}{E}^{a}(\underset{\sim}{r}) = 0\,. \tag{37}$$

The reality of $\underset{\sim}{j}(\underset{\sim}{r},t)$ and $\underset{\sim}{E}(\underset{\sim}{r},t)$ in (18a) implies that

$$\underset{\approx}{g}(\underset{\sim}{r},\underset{\sim}{r}';\,-\omega) = \underset{\approx}{g}^{*}(\underset{\sim}{r},\underset{\sim}{r}';\,\omega)\,. \tag{38}$$

It follows from the previous development that the eigenfrequencies Ω_a occur in pairs $+\omega_a + i\gamma_a$, $-\omega_a + i\gamma_a$, so that we may limit our attention to positive ω_a. The corresponding eigenfunctions are $\underset{\sim}{E}^{a}(\underset{\sim}{r})$, $\underset{\sim}{E}^{a*}(\underset{\sim}{r})$.

The time-development of the energy in a normal mode includes not only the Vlasov growth rate γ_a, but also the effect of spontaneous emission, to be studied below. We therefore introduce a slowly varying complex amplitude $A_a(t)$ for each mode, and express the field as

$$\underset{\sim}{E}(\underset{\sim}{r},t) = \sum_a \left[A_a(t)\, \underset{\sim}{E}^a(\underset{\sim}{r})\, e^{-i\omega_a t} + \text{complex conjugate}\right], \tag{39}$$

where the eigenfunctions are normalized below. The electric energy, averaged over several periods of the oscillations, is

$$\begin{aligned} W_E(t) &= T^{-1}\int dt \int d^3r\, [\underset{\sim}{E}(\underset{\sim}{r},t)]^2/8\pi \\ &= \sum_a |A_a(t)|^2 \int d^3r\, |\underset{\sim}{E}_a(\underset{\sim}{r})|^2/4\pi\ . \end{aligned} \tag{40}$$

The <u>total</u> wave energy is found by the method of Landau and Lifshitz:[14]

$$\begin{aligned} W(t) = \sum_a |A_a(t)|^2\, (4\pi)^{-1} \Bigg\{ &\int d^3r\, |\underset{\sim}{B}^a(\underset{\sim}{r})|^2 \\ &+ \int d^3r \int d^3r'\, \underset{\sim}{E}^{a*}(\underset{\sim}{r}')\cdot\partial[\omega\, \underset{\approx}{\epsilon}'(\underset{\sim}{r},\underset{\sim}{r}';\omega)]/\partial\omega\Big|_{\omega_a}\cdot\underset{\sim}{E}^a(\underset{\sim}{r})\Bigg\}, \end{aligned} \tag{41}$$

where $\underset{\sim}{B}^a(\underset{\sim}{r}) = (c/i\omega)\nabla \times \underset{\sim}{E}^a(\underset{\sim}{r})$, by (21a). In terms of the operator $\underset{\approx}{K}$, this is simply

$$W(t) = \sum_a |A_a(t)|^2\, (4\pi)^{-1} \int d^3r\, \underset{\sim}{E}^{a*}(\underset{\sim}{r})\cdot\omega_a\, \partial\underset{\approx}{K}'(\omega)/\partial\omega\Big|_{\omega_a}\cdot\underset{\sim}{E}^a(\underset{\sim}{r})\ . \tag{42}$$

The previous development has been independent of geometry. In the plasma cylinder, the eigenfunctions have the form

$$\underset{\sim}{E}^a(\underset{\sim}{r}) = \underset{\sim}{E}^a(r)\exp(im\varphi + ikz) , \tag{43}$$

where the symbol a includes m, k, and the radial mode number. The operator $\underset{\approx}{K}$ becomes

$$\underset{\approx}{K}(\omega)\cdot\underset{\sim}{E}^a(\underset{\sim}{r}) = \exp(im\varphi + ikz)\,\underset{\approx}{K}^{mk}(\omega)\cdot\underset{\sim}{E}^a(r) ; \tag{44}$$

the new operator $\underset{\approx}{K}^{mk}(\omega)$ acts only on the radial dependence of the normal mode. The wave energy (42) is then

$$W(t) = L \sum_a |A_a(t)|^2 \frac{1}{2}\int r\,dr\, \underset{\sim}{E}^{a*}(r)\cdot\omega_a\, \partial\underset{\approx}{K}'^{mk}(\omega)/\partial\omega\big|_{\omega_a}\cdot\underset{\sim}{E}^a(r) , \tag{45}$$

where a factor 2π arises from the azimuthal integration, and L is the axial length of the cylinder, with the limit $L \to \infty$ implicit.

It is now convenient to choose the normalization of $\underset{\sim}{E}^a(r)$ so as to simplify this form. Since the integral is real, we normalize according to

$$\frac{1}{2}\int r\,dr\, \underset{\sim}{E}^{a*}(r)\cdot\omega_a\, \partial\underset{\approx}{K}'^{mk}(\omega)/\partial\omega\big|_{\omega_a}\cdot\underset{\sim}{E}^a(r) = \sigma_a , \tag{46}$$

where $\sigma_a = \pm 1$ is the sign of the integral. The wave energy than reads

$$W(t) = L \sum_a \sigma_a\, |A_a(t)|^2 , \tag{47}$$

and we may interpret

$$W_a(t) \equiv L\,\sigma_a\, |A_a(t)|^2 \tag{48}$$

as the energy of a normal mode. The sign of σ_a represents the sign of the mode energy.

The modes also have angular momentum and axial momentum; these include not only the field contributions, but also the canonical contributions of nonresonant perturbed particles. Their relation to mode energy is found from Lagrangian field theory[15] to be

$$P_\varphi{}^a/W_a = m/\omega_a ,$$
$$P_z{}^a/W_a = k/\omega_a . \tag{49}$$

IV. QUANTUM RATE EQUATIONS

In the quantum picture, the energy of a normal mode is quantized:

$$W_a(t) = N_a(t)\hbar\Omega_a , \tag{50}$$

where

$$\Omega_a = \sigma_a \omega_a . \tag{51}$$

(Recall that $\omega_a > 0$.) The particle invariants are also quantized:

$$J_r = \mathcal{J}_r \hbar , \tag{52a}$$

$$P_\varphi = \mathcal{P}_\varphi \hbar , \tag{52b}$$

$$P_z = \mathcal{P}_z \hbar . \tag{52c}$$

The quantum numbers N_a, $\mathcal{J}_r$, $\mathcal{P}_\varphi$, $\mathcal{P}_z$ must all be integers. When a particle with action $\underset{\sim}{J}'' \equiv (J_r'', P_\varphi'', P_z'')$ makes a transition to state $\underset{\sim}{J}'$ by emitting a single normal mode quantum, the conservation laws for energy, angular momentum, and axial momentum require that

$$H(\underset{\sim}{J}'') = H(\underset{\sim}{J}') + \hbar\, \sigma_a\, \omega_a , \tag{53a}$$

$$P_\varphi'' = P_\varphi' + \hbar\, \sigma_a\, m , \tag{53b}$$

$$P_z'' = P_z' + \hbar\, \sigma_a\, k . \tag{53c}$$

The quantization of radial action (52a) implies further that

$$J_r'' = J_r' + \hbar\, \sigma_a\, \ell , \tag{53d}$$

where ℓ can be any integer.

In the classical limit, the relative change in action

$$\Delta \underset{\sim}{J} \equiv \underset{\sim}{J}'' - \underset{\sim}{J}'$$

is small, and we may Taylor-expand ΔH:

$$\Delta H \equiv H(\underset{\sim}{J}'') - H(\underset{\sim}{J}')$$

$$= (\Delta J_r)(\partial H/\partial J_r) + (\Delta P_\varphi)(\partial H/\partial P_\varphi) + (\Delta P_z)(\partial H/\partial P_z)$$

$$= \omega_r \Delta J_r + \dot{\Phi} \Delta P_\varphi + \dot{Z} \Delta P_z \ . \tag{54}$$

Using Eqs. (53) for ΔH, ΔJ_r, ΔP_φ, ΔP_z, we obtain the resonance condition:

$$\omega_a = \ell\, \omega_r(\underset{\sim}{J}) + m\, \dot{\Phi}(\underset{\sim}{J}) + k\, \dot{Z}(\underset{\sim}{J}) \ , \tag{55}$$

for the interaction of a particle with a normal mode.

In terms of the Doppler-shifted frequency seen by the guiding center:

$$\omega_a'(\underset{\sim}{J}) \equiv \omega_a - m\, \dot{\Phi}(\underset{\sim}{J}) - k\, \dot{Z}(\underset{\sim}{J}) \ , \tag{56}$$

the resonance condition reads

$$\omega_a'(\underset{\sim}{J}) = \ell\, \omega_r(\underset{\sim}{J}) \ , \tag{57}$$

i.e., the Doppler-shifted frequency must be an integral multiple of the radial bounce frequency. This result is a generalization of the condition found in the one-dimensional model (KN). In Section VI, the resonance condition is derived by a purely classical calculation.

For a given mode (m, k, ω_a), the resonance condition (55) defines a discrete set of nonintersecting surfaces in $\underset{\sim}{J}$-space. Each surface corresponds to a particular value of ℓ. All the particles on these surfaces interact resonantly with the same mode; we call this set of surfaces S_a.

When a resonant particle emits (or absorbs) a mode quantum, it moves in the direction $\pm(\ell, m, k)$ in action-space, by Eq. (53). It is then no longer in resonance with ω_a, having moved off S_a. However, the normal modes themselves are a continuum, since k is a continuous parameter. Thus every point in action-space lies on a resonant surface of some mode $S_{a'}$. The particle then moves from one surface S_a to another $S_{a'}$, always being in resonance with some mode. Since the sign of its velocity in action-space is random, this is a Brownian motion, describable by a Fokker-Planck equation. We now proceed to derive this equation, and the corresponding equation for the mode energy.

Let $\rho(\underset{\sim}{J}'' \leftrightarrow \underset{\sim}{J}', a)$ denote the probability density (in action-space) for mode emission or absorption, per unit time. The rate equation for $N_a(t)$ is

$$dN_a/dt = [(2\pi)^2 L]^2 \int d^3J' \int d^3J'' \, \rho(\underset{\sim}{J}'' \leftrightarrow \underset{\sim}{J}', a)[f(\underset{\sim}{J}'')(N_a+1) - f(\underset{\sim}{J}')N_a]$$
$$\times \delta(\Delta H - \hbar\Omega_a)\, \delta(\Delta P_\varphi - m\hbar\sigma_a)\, \delta(\Delta P_z - \hbar k\sigma_a) , \tag{58}$$

while the equation of evolution for $f(\underset{\sim}{J};t)$ is

$$\partial f(\underset{\sim}{J};t)/\partial t = (2\pi)^2 L \int d^3J' \int d^3J'' \sum_a \rho(\underset{\sim}{J}'' \leftrightarrow \underset{\sim}{J}', a)$$
$$\times [f(\underset{\sim}{J}'')(N_a + 1) - f(\underset{\sim}{J}')N_a]$$
$$\times \delta(\Delta H - \hbar\Omega_a)\, \delta(\Delta P_\varphi - m\hbar\sigma_a)\, \delta(\Delta P_z - \hbar k\sigma_a)$$
$$\times \{\delta(\underset{\sim}{J}' - \underset{\sim}{J}) - \delta(\underset{\sim}{J}'' - \underset{\sim}{J})\} . \tag{59}$$

The factors $(2\pi)^2 L$ come from integration over (w, φ, z).

In the square bracket, the first term represents induced and spontaneous emission, while the second represents absorption. Expanding the square bracket in $\Delta \underset{\sim}{J}$, we find [with the help of (53)]

$$[\cdots] = f(\underset{\sim}{J}'') + N_a \, \Delta \underset{\sim}{J} \cdot \partial f/\partial \underset{\sim}{J}$$

$$= f(\underset{\sim}{J}'') + N_a \{\Delta J_r (\partial f/\partial J_r) + \Delta P_\varphi (\partial f/\partial P_\varphi) + \Delta P_z (\partial f/\partial P_z)\}$$

$$= f(\underset{\sim}{J}'') + (W_a/\omega_a)\, \mathcal{D} f \,, \tag{60}$$

where

$$\mathcal{D} \equiv \ell\,(\partial/\partial J_r) + m(\partial/\partial P_\varphi) + k(\partial/\partial P_z) \,. \tag{61}$$

On performing the integration over $\underset{\sim}{J}'$, and dropping the primes on $\underset{\sim}{J}''$, we find for the mode equation (58)

$$dN_a/dt = (2\pi)^4 L^2 \int d^3J \; \rho(\underset{\sim}{J} \leftrightarrow \underset{\sim}{J}', a) \; \omega_r^{-1}(\underset{\sim}{J})[f(\underset{\sim}{J}) + (W_a/\omega_a)\mathcal{D} f] \,. \tag{62}$$

In using (50) to eliminate N_a, let us ignore the possible slow variation in ω_a due to adiabatic changes in the configuration. We then obtain the wave kinetic equation

$$dW_a/dt = \sigma_a (2\pi)^2 L \int d^3J \; \alpha_a(\underset{\sim}{J})[f(\underset{\sim}{J}) + (W_a/\omega_a)\, \mathcal{D} f] \,, \tag{63}$$

where

$$\alpha_a(\underset{\sim}{J}) \equiv (2\pi)^2 L \hbar \, \omega_a \, \omega_r^{-1}(\underset{\sim}{J}) \; \rho(\underset{\sim}{J} \leftrightarrow \underset{\sim}{J}', a) \tag{64}$$

is the classical mode-particle coupling coefficient, to be evaluated in Section VI. The resonance condition (55) is contained in $\alpha_a(\underset{\sim}{J})$, as we shall see there.

In conventional notation, (63) may be written as

$$dW_a/dt = \dot{W}_a + 2\gamma_a W_a , \tag{65}$$

where the rate of spontaneous emission by discrete particles,

$$\dot{W}_a \equiv \sigma_a (2\pi)^2 L \int d^3J\, \alpha_a(\underset{\sim}{J})\, f(\underset{\sim}{J}) , \tag{66}$$

is proportional to f ; while the linear Vlasov growth rate,

$$\gamma_a \equiv \frac{1}{2} (2\pi)^2 L\, \Omega_a^{-1} \int d^3J\, \alpha_a(\underset{\sim}{J})\, \not{\partial} f(\underset{\sim}{J}) , \tag{67}$$

is proportional to derivatives of f .

If the particle evolution can be ignored and if the mode considered is stable $(\gamma_a < 0)$, Eq. (65) can be solved for the quasi-steady state $(dW_a/dt = 0)$ wave energy W_a^{QS}:

$$\begin{aligned} W_a^{QS} &= \dot{W}_a / 2|\gamma_a| \\ &= \frac{\omega_a \int d^3J\, \alpha_a(\underset{\sim}{J})\, f(\underset{\sim}{J})}{- \int d^3J\, \alpha_a(\underset{\sim}{J})\, \not{\partial} f(\underset{\sim}{J})} . \end{aligned} \tag{68}$$

This expression generalizes the result of Kent and Taylor,[5] who found the steady-state energy using WKB methods for the normal modes and small gyroradius for the particles.

In the particle equation (59), we change variables to $\underset{\sim}{J}^0 \equiv \frac{1}{2}(\underset{\sim}{J}' + \underset{\sim}{J}'')$ and $\Delta \underset{\sim}{J}$, and expand the curly bracket in $\Delta \underset{\sim}{J}$:

$$\{\delta(\underset{\sim}{J}' - \underset{\sim}{J}) - \delta(\underset{\sim}{J}'' - \underset{\sim}{J})\} = \Delta \underset{\sim}{J} \cdot \partial\delta(\underset{\sim}{J}^0 - \underset{\sim}{J})/\partial \underset{\sim}{J} .$$

We commute $\partial/\partial \underset{\sim}{J}$ to the left, obtaining

$$\frac{\partial f(\underset{\sim}{J};t)}{\partial t} = (2\pi)^2 \, L \int d^3J^0 \int d^3\Delta J \sum_a \rho(\underset{\sim}{J}^0, \Delta \underset{\sim}{J}, a) \, \delta(\underset{\sim}{J}^0 - \underset{\sim}{J})$$

$$\times \, [\cdots] \times \delta\delta\delta \, , \qquad (69)$$

the three final δ factors being those of (59). We integrate over $\Delta \underset{\sim}{J}$ and $\underset{\sim}{J}^0$, to obtain the classical particle kinetic equation:

$$\partial f(\underset{\sim}{J};t)/\partial t = \sum_a \Omega_a^{-1} \, \mathfrak{d} \, \alpha_a(\underset{\sim}{J})[f(\underset{\sim}{J}) + (W_a/\omega_a) \, \mathfrak{d} \, f(\underset{\sim}{J})] \, . \qquad (70)$$

This kinetic equation is manifestly of Fokker-Planck form:

$$\partial f(\underset{\sim}{J};t)/\partial t = -\left[\partial(\dot{J}_r \, f)/\partial J_r\right] - \left[\partial(\dot{P}_\varphi \, f)/\partial P_\varphi\right] - \left[\partial(\dot{P}_z \, f)/\partial P_z\right]$$

$$+ (\partial/\partial \underset{\sim}{J}) \cdot [\underset{\approx}{D}(\underset{\sim}{J}) \cdot (\partial f/\partial \underset{\sim}{J})] \, . \qquad (71)$$

The "dynamic friction" coefficients are given by

$$(\dot{J}_r, \dot{P}_\varphi, \dot{P}_z) = - \sum_a \Omega_a^{-1} \, \alpha_a(\underset{\sim}{J})(\ell, m, k), \qquad (72)$$

and represent the radiation reaction from spontaneous emission. [The ratios of the three coefficients are understandable from Eq. (53).]

The quasilinear diffusion tensor in action-space is

$$\underset{\approx}{D}(\underset{\sim}{J}) = \sum_a \omega_a^{-2} \, |W_a| \, \alpha_a(\underset{\sim}{J}) \begin{pmatrix} \ell^2 & \ell m & \ell k \\ \ell m & m^2 & mk \\ \ell k & mk & k^2 \end{pmatrix} . \qquad (73)$$

We note that it is positive definite [since $\alpha_a(\underset{\sim}{J})$ is non-negative, by definition (64)]; i.e., for any real vector $\underset{\sim}{A} \equiv A_1 \hat{J}_r + A_2 \hat{P}_\varphi + A_3 \hat{P}_z$,

$$\underset{\sim}{A} \cdot \underset{\approx}{D}(\underset{\sim}{J}) \cdot \underset{\sim}{A} = \sum_a \omega_a^{-2} |W_a| \, \alpha_a(\underset{\sim}{J})(A_1 \ell + A_2 m + A_3 k)^2 \geqslant 0 . \tag{74}$$

Recalling the significance of $\underset{\sim}{J}$, we see that the kinetic equation describes radial diffusion and diffusion in $v_\perp$ and in $v_\parallel$, in the limit where a guiding-center description applies.

V. CONSERVATION LAWS AND H-THEOREM

The coupled kinetic equations for normal modes (63) and particles (70) satisfy conservation laws for energy, angular momentum, and axial momentum, and in addition yield an H-theorem for entropy production.

The conservation laws were of course built into the derivation, and so serve only as an algebraic check. For energy, we have

$$U(t) = (2\pi)^2 L \int d^3J\, H(\underset{\sim}{J})\, f(\underset{\sim}{J};t) + \sum_a W_a(t) . \tag{75}$$

In its derivative,

$$dU/dt = (2\pi)^2 L \int d^3J\, H(\underset{\sim}{J})\, (\partial f/\partial t) + \sum_a (dW_a/dt) , \tag{76}$$

the first term includes the energy change of only the resonant particles, while the second term includes the energy change of the nonresonant particles. (The effect of adiabatic variation of the "static" potentials is consistently ignored here.) Substituting (70) and (63), we have

$$dU/dt = (2\pi)^2 L \sum_a \sigma_a \int d^3J \{H(\underset{\sim}{J})\omega_a^{-1} \mathcal{D} + 1\}\, \alpha_a(\underset{\sim}{J})[f(\underset{\sim}{J}) + (W_a/\omega_a)\mathcal{D} f] . \tag{77}$$

For the operator in the curly brackets, we integrate by parts, using

$$\mathcal{D} H = \ell\omega_r + m\dot{\phi} + k\dot{Z} = \omega_a , \tag{78}$$

whereupon $dU/dt = 0$, as required. The proof for conservation of total angular momentum

$$(2\pi)^2 L \int d^3J\, P_\varphi\, f(\underset{\sim}{J};t) + \sum_a (m/\omega_a)W_a(t) \tag{79}$$

and of axial momentum

$$(2\pi)^2 \, L \int d^3J \; P_z \; f(\underset{\sim}{J};t) + \sum_a (k/\omega_a) W_a(t) \tag{80}$$

is analogous.

The entropy for a system of weakly interacting particles and modes is

$$S(t) \equiv -(2\pi)^2 \, L \int d^3J \; f(\underset{\sim}{J};t) \; \ell n \; f(\underset{\sim}{J};t) + \sum_a \ell n \; |W_a(t)| \; . \tag{81}$$

Its time derivative is, by (70) and (63),

$$dS/dt = (2\pi)^2 \, L \sum_a \sigma_a \int d^3J \; \{-\omega_a^{-1}(\ell n \; f) \not{\partial} + W_a^{-1}\} \; \alpha_a(\underset{\sim}{J})$$

$$\times \; [f + (W_a/\omega_a) \not{\partial} \; f] \; .$$

Integrating by parts, we obtain

$$dS/dt = (2\pi)^2 \, L \sum_a \int d^3J \; \alpha_a(\underset{\sim}{J}) \{f(\underset{\sim}{J}) |W_a|\}^{-1} \; [f(\underset{\sim}{J}) + (W_a/\omega_a) \not{\partial} \; f]^2 \; . \tag{82}$$

This is manifestly non-negative, and implies a monotonic increase of entropy.

When the wave energy is so large that the spontaneous emission terms are negligible on time-scales of interest, the kinetic equations (65) and (71) reduce to quasilinear equations:

$$dW_a/dt = 2 \, \gamma_a \, W_a \; , \tag{83a}$$

$$\partial f(\underset{\sim}{J};t)/\partial t = (\partial/\partial \underset{\sim}{J}) \cdot [\underset{\approx}{D}(\underset{\sim}{J}) \cdot (\partial f/\partial \underset{\sim}{J})] \; . \tag{83b}$$

The conservation laws are still satisfied by these equations, but the H-theorem now pertains to only the resonant-particle entropy:

$$S_R(t) \equiv -(2\pi)^2 L \int d^3J\, f(\underset{\sim}{J};t)\, \ell n\, f(\underset{\sim}{J};t) . \tag{84}$$

From (83b) and (74), we find

$$dS_R/dt = (2\pi)^2 L \int d^3J\, f^{-1}(\partial f/\partial \underset{\sim}{J})\cdot \underset{\approx}{D}(\underset{\sim}{J})\cdot(\partial f/\partial \underset{\sim}{J}) .$$

$$= (2\pi)^2 L \int d^3J\, f^{-1} \sum_a \omega_a^{-2}\, |W_a|\, \alpha_a(\underset{\sim}{J})(\partial f)^2 , \tag{85}$$

which again is manifestly non-negative.

The question now arises, whether an asymptotic stationary state exists. Such a state must correspond to a maximum of the entropy. Let us first maximize the entropy of the particles of one species only. By standard methods, we find for the maximizing distribution the canonical form

$$f^s_{can}(\underset{\sim}{J}) \sim \exp\{ -\beta^s[H^s(\underset{\sim}{J}) - \omega_\varphi{}^s P_\varphi - V_z{}^s P_z]\} , \tag{86}$$

where the parameters β^s, $\omega_\varphi{}^s$, $V_z{}^s$ are to be adjusted to account for the appropriate energy, angular momentum, and axial momentum of this species.

This form is a special case of the rigid rotor distribution

$$f_{RR}{}^s(\underset{\sim}{J}) = F^s(H^s - \omega_\varphi{}^s P_\varphi, P_z) , \tag{87}$$

which has been studied intensively by Davidson and Krall[7] for the case of a single-species plasma. We note[16] the interesting property of (87),

that the radial dependence of the current density is indeed that of a rigid rotor:

$$\underset{\sim}{j}^s(r) = \rho^s(r)\, \omega_\varphi{}^s\, r\, \hat{\varphi} \,. \tag{88}$$

Returning to the canonical form (86), we see that the derivative $\mathcal{D} f$ is

$$\mathcal{D} f^s_{can} = -\beta^s(\omega_a - m\, \omega_\varphi{}^s - k\, V_z{}^s)\, f^s_{can} \,, \tag{89}$$

by (61) and (55). The Vlasov growth rate (67) is then

$$\begin{aligned} \gamma_a &= -\frac{1}{2}(2\pi)^2\, L(\omega_a - m\, \omega_\varphi{}^s - k\, V_z{}^s)\, \Omega_a{}^{-1}\, \beta^s \int d^3J\, \alpha_a(\underset{\sim}{J}) f^s(\underset{\sim}{J}) \\ &= -\frac{1}{2}\beta^s[1 - (m\, \omega_\varphi{}^s + k\, V_z{}^s)/\omega_a]\, \dot{W}_a \,, \end{aligned} \tag{90}$$

by (66). It thus bears a simple relation to the emissivity by discrete particles. The sign of γ_a cannot be determined until one finds ω_a and σ_a (to which $\dot{W}_a$ is proportional).

In the single-species case, we may drop the species index from (86), and proceed to the maximization of the total entropy (81). We then find the generalized Rayleigh-Jeans distribution

$$W_a = W_a{}^{RJ} \equiv \beta^{-1}[1 - (m\, \omega_\varphi + k\, V_z)/\omega_a]^{-1} \,, \tag{91}$$

together with (86). This is then indeed a stationary state of the kinetic equations (63) and (70), since

$$f_{can} + (W_a{}^{RJ}/\omega_a)\, \mathcal{D} f_{can} = 0 \,. \tag{92}$$

(However, it is not stationary for the quasilinear equations (83); by (65), all waves must be damped.)

To complete the proof that an asymptotic state exists, it is necessary to demonstrate that the sign of (91) is consistent with the sign of σ_a, and that (86) is a confined state. According to Davidson and Krall,[7] electrostatic modes are stable for (86); it follows from (90) that the sign of (91) is indeed consistent. (However, no such proof for electromagnetic modes has yet been offered.) Also they have shown that the plasma is confined, only if ω_φ lies within certain limits, for the case $B_\varphi^0 = 0$.

In the two-species case, let us consider the canonical distribution (86) for a quasineutral equilibrium with B_z^0 constant, $B_\varphi^0 = 0$, $E_r^0 = 0$, and small gyroradius. From (86) and (5), we see that the density has a Gaussian shape:

$$n^s_{can}(r) \sim \exp\left(\frac{1}{2}\beta\, e^s\, \omega_\varphi^s\, B_z^0\, c^{-1}\, r^2\right) . \tag{93}$$

This is confined and quasineutral only if

$$\omega_\varphi^s = -\, c/\beta\; e^s\, B_z^0\, R_0^2 , \tag{94}$$

for each species, where R_0 is the common radial plasma size; thus ω_φ^s must have <u>opposite</u> sign for the two species.

However, the maximization of entropy (81) leads to the requirement of the <u>same</u> ω_φ for each species. Further, one again finds (91) for the normal modes; clearly, this requires a common ω_φ if modes can resonate simultaneously with particles of both species. In this limiting case at least, an asymptotic state cannot be a confined state for a two-species plasma. It would be of great interest to determine the generality of this result.

We conclude that resonant interactions drive the system toward a canonical distribution, which may be unconfined.

VI. EMISSIVITY AND COUPLING COEFFICIENT

To obtain an explicit expression for the coupling coefficient $\alpha_a(\underset{\sim}{J})$, which appears in (63), (66), (67), (70), (72), (73), and (82), we shall calculate the emissivity (66) classically, and identify $\alpha_a(\underset{\sim}{J})$. Since the emissivity is the emission rate by discrete particles, we consider the linear plasma response to the current density of a single discrete particle, and then sum over particles.

Treating the particle as an external current source $\underset{\sim}{j}^e(\underset{\sim}{r},t)$, we replace (21b) by

$$\nabla \times \underset{\sim}{B} + (i\omega/c)\,\underset{\sim}{E} = (4\pi/c)[\underset{\sim}{j} + \underset{\sim}{j}^e] , \tag{95}$$

whence (24) becomes

$$\underset{\approx}{K}(\omega)\cdot\underset{\sim}{E}(\underset{\sim}{r},\omega) + \underset{\sim}{E}(\underset{\sim}{r},\omega) = -(4\pi i/\omega)\,\underset{\sim}{j}^e(\underset{\sim}{r},\omega) . \tag{96}$$

The solution of (96) can be written in terms of Green's tensor:

$$\underset{\sim}{E}(\underset{\sim}{r},\omega) = -(4\pi i/\omega)\int d^3r'\,\underset{\approx}{G}(\underset{\sim}{r},\underset{\sim}{r}';\,\omega)\cdot\underset{\sim}{j}^e(\underset{\sim}{r}',\omega) , \tag{97}$$

where $\underset{\approx}{G}$ satisfies

$$\underset{\approx}{K}(\omega)\cdot\underset{\approx}{G}(\underset{\sim}{r},\underset{\sim}{r}';\,\omega) + \underset{\approx}{G}(\underset{\sim}{r},\underset{\sim}{r}';\,\omega) = \underset{\approx}{I}\,\delta(\underset{\sim}{r} - \underset{\sim}{r}') . \tag{98}$$

To zero-order in $\underset{\approx}{K}''$ [see (30)], the solution of (98) is found by standard techniques:

$$\underset{\approx}{G}(\underset{\sim}{r},\underset{\sim}{r}';\,\omega) = \sum_n \left[1 - \Lambda_n^{(0)}(\omega)\right]^{-1} \frac{\underset{\sim}{E}^{n(0)}(\underset{\sim}{r},\omega)\,\underset{\sim}{E}^{n(0)*}(\underset{\sim}{r}',\omega)}{\int d^3r''\,|\underset{\sim}{E}^{n(0)}(\underset{\sim}{r}'',\omega)|^2} , \tag{99}$$

in terms of the eigenvalues and eigenmodes of Eq. (31).

The total transfer of energy to the plasma from the test particle is

$$\Delta W = -\int_{-\infty}^{+\infty} dt \int d^3r \; \underset{\sim}{j}^{e}(\underset{\sim}{r},t) \cdot \underset{\sim}{E}(\underset{\sim}{r},t)$$

$$= -\int \frac{d\omega}{2\pi} \int d^3r \; \underset{\sim}{j}^{e*}(\underset{\sim}{r},\omega) \cdot \underset{\sim}{E}(\underset{\sim}{r},\omega)$$

$$= 2i \int (d\omega/\omega) \int d^3r \int d^3r' \; \underset{\sim}{j}^{e*}(\underset{\sim}{r},\omega)\cdot\underset{\sim}{\underset{\sim}{G}}(\underset{\sim}{r},\underset{\sim}{r}';\omega)\cdot\underset{\sim}{j}^{e}(\underset{\sim}{r}',\omega),$$

where the ω-integral is in the upper half plane, above all singularities. In the perturbation limit $\underset{\approx}{K}'' \to 0$, we use (99):

$$\Delta W = 2i \int (d\omega/\omega) \sum_n [1 - \Lambda_n^{(0)}(\omega)]^{-1} \frac{\left| \int d^3r \; \underset{\sim}{j}^{e*}(\underset{\sim}{r},\omega)\cdot\underset{\sim}{E}^{n(0)}(\underset{\sim}{r},\omega)\right|^2}{\int d^3r' \; |\underset{\sim}{E}^{n(0)}(\underset{\sim}{r}',\omega)|^2},$$

whence the singularities of the integrand are on the real axis, at the eigenfrequencies [see (34)].

Depressing the contour of integration to the real axis, the semicircles about these poles yield

$$\Delta W = -4\pi \sum_a{}' \; \omega_a^{-1} \, [d\Lambda_n^{(0)}/d\omega]_{\omega_a}^{-1} \frac{\left| \int d^3r \; \underset{\sim}{j}^{e*}(\underset{\sim}{r},\omega_a)\cdot\underset{\sim}{E}^{a}(\underset{\sim}{r})\right|^2}{\int d^3r' \, |\underset{\sim}{E}^{a}(\underset{\sim}{r}')|^2}.$$

(A factor of 2 comes from the poles at $-\omega_a$. There is no pole at $\omega = 0$, since $\underset{\sim}{j}^{e}(\underset{\sim}{r},\omega)$ vanishes there. The principal value part of the integral vanishes, since the integrand is odd in ω.) Evaluating $d\Lambda_n^{(0)}/d\omega$ from (32):

$$-[d\Lambda_n^{(0)}/d\omega]_{\omega_a} \int d^3r' \; |\underset{\sim}{E}^a(\underset{\sim}{r}')|^2 = \int d^3r \; \underset{\sim}{E}^a(r)\cdot(\partial\underset{\approx}{K}'/\partial\omega)\cdot\underset{\sim}{E}^a(\underset{\sim}{r}) \; ,$$

we obtain

$$\Delta W = 4\pi \sum_a \omega_a^{-1} \left| \int d^3r \; \underset{\sim}{j}^e(\underset{\sim}{r},\omega_a)\cdot\underset{\sim}{E}^{a*}(\underset{\sim}{r}) \right|^2$$

$$\times \; [\int d^3r \; \underset{\sim}{E}^a(\underset{\sim}{r})\cdot(\partial\underset{\approx}{K}'/\partial\omega)\cdot\underset{\sim}{E}^a(\underset{\sim}{r})]^{-1} . \qquad (100)$$

We now use (43) and the normalization (46) in (100), obtaining

$$\Delta W = L^{-1} \sum_a \sigma_a \left| \int r \, dr \; \underset{\sim}{j}^e(r,m,k;\, \omega_a)\cdot\underset{\sim}{E}^{a*}(r) \right|^2 , \qquad (101)$$

where

$$\underset{\sim}{j}(r,m,k;\, t) \equiv \int d\varphi \; e^{-im\varphi} \int dz \; e^{-ikz} \; \underset{\sim}{j}(\underset{\sim}{r},t) \; . \qquad (102)$$

For the external current density, we now use the incoherent sum of the contributions of all the particles:

$$\Delta W = L^{-1} \sum_a \sigma_a \int d\Gamma \; f(\underset{\sim}{J}) \left| \int r \, dr \; \underset{\sim}{j}(r,m,k;\, \omega_a;\, \Gamma)\cdot\underset{\sim}{E}^{a*}(r) \right|^2 , \qquad (103)$$

where $\underset{\sim}{j}(\underset{\sim}{r},t;\, \Gamma)$ is the current density of a single particle, whose phase-point is at Γ at some reference time $t = 0$, while $\underset{\sim}{j}(r,m,k;\, \omega;\, \Gamma)$ is its φ-z-t Fourier transform. We shall see later that the integrand is independent of the angle variables. Hence (103) becomes

$$\dot{W} = (2\pi)^2 \, T^{-1} \sum_a \sigma_a \int d^3J \; f(J) \left| \int r \, dr \; \underset{\sim}{j}(r,m,k;\, \omega_a;\, \Gamma)\cdot\underset{\sim}{E}^{a*}(r) \right|^2 , \qquad (104)$$

where T is the (formally infinite) time used in the Fourier transforms. Comparing (104) with (66), we identify the coupling coefficient:

$$\alpha_a(\underset{\sim}{J}) = (LT)^{-1} \left| \int r\, dr\, \underset{\sim}{j}(r,m,k;\, \omega_a;\, \Gamma)\cdot \underset{\sim}{E}^{a*}(r) \right|^2 . \qquad (105)$$

To evaluate $\alpha_a(\underset{\sim}{J})$, we transform

$$\underset{\sim}{j}(\underset{\sim}{r},t;\, \Gamma) \equiv e\, \underset{\sim}{v}(t;\Gamma)\ \ \delta[\underset{\sim}{r} - \underset{\sim}{R}(t;\Gamma)] , \qquad (106)$$

and find (after a bit of algebra)

$$\underset{\sim}{j}(r,m,k;\, \omega;\, \Gamma) = e\, r^{-1}\, |\dot{r}(r;\underset{\sim}{J})|^{-1} \sum_{\pm} \underset{\sim}{v}^{\pm}(r;\underset{\sim}{J}) \exp[\pm\, i\, \omega_a'(\underset{\sim}{J})\, \tau(r;\underset{\sim}{J})]$$

$$\times \sum_{\ell=-\infty}^{+\infty} \delta[\ell - \omega_a'(\underset{\sim}{J})/\omega_r(\underset{\sim}{J})] \times \exp[-i\ell w - im\Phi - ikZ] . \qquad (107)$$

Here $\underset{\sim}{v}^{\pm}$ is the double-valued velocity at r for a particle with actions $\underset{\sim}{J}$; $\tau(r;\underset{\sim}{J})$ is the transit time for the particle to reach r from its inner turning point; and (w, Φ, Z) are the angle variables at $t = 0$.

We substitute (107) into (105), express $\underset{\sim}{v}^{\pm}$ in terms of momenta via (3), and find

$$\alpha_a(\underset{\sim}{J}) = 2\pi^{-1} L^{-1} \omega_r(\underset{\sim}{J}) \Bigg| \int dr \Bigg[i\, E_r^{a*}(r) \sin \omega_a'(\underset{\sim}{J})\, \tau(r;\underset{\sim}{J})$$
$$+ |p_r(r,J)|^{-1} \Big\{ E_\varphi^{a*}(r)[(p_\varphi/r) - (e/c)A_\varphi(r)]$$
$$+ E_z^{a*}(r)[p_z - (e/c)A_z(r)] \Big\} \cos \omega_a'(\underset{\sim}{J})\tau(r;\underset{\sim}{J}) \Bigg] \Bigg|^2 \delta[\ell - \omega_a'(\underset{\sim}{J})/\omega_r(\underset{\sim}{J})] . \qquad (108)$$

The methods of KN can be used to simplify this expression in certain limiting cases. In general, numerical methods are needed for the explicit evaluation of (108).

VII. CONDUCTIVITY

We have obtained two expressions, (35) and (67), for the growth rate γ_a of a normal mode. Equating these expressions, we have [with the normalization (46)],

$$\int r\, dr\, \underset{\sim}{E}^{a*}(r)\cdot\underset{\approx}{K}^{''mk}(\omega_a)\cdot\underset{\sim}{E}^{a}(r) = -(2\pi)^2\, L\, \omega_a^{-2} \int d^3J\, \alpha_a(\underset{\sim}{J})\, \eth f(\underset{\sim}{J}) \; .$$

On the left side of this equation, we use [see (25)]

$$\underset{\approx}{K}^{''mk}(\omega)\cdot\underset{\sim}{E}(r) \equiv (4\pi/\omega)\int r'\, dr'\, \underset{\approx}{\sigma}^{'mk}(r,r';\, \omega)\cdot\underset{\sim}{E}(r') \; ;$$

on the right side, we use (105) for $\alpha_a(\underset{\sim}{J})$. We then deduce that

$$\underset{\approx}{\sigma}^{'mk}(r,r';\, \omega) = -\pi\, \omega^{-1}\, T^{-1} \int d^3J\, \eth f(\underset{\sim}{J})\, \underset{\sim}{j}(r,m,k;\, \omega;\, \Gamma)\, \underset{\sim}{j}^{*}(r',m,k;\, \omega;\, \Gamma); \tag{109}$$

here we have used (55) to eliminate ℓ from $\eth f$ (61):

$$\eth f(J) \equiv \omega(\partial f/\partial H) + m(\partial f/\partial P_\varphi) + k(\partial f/\partial P_z) \; , \tag{110}$$

with $(H,\, P_\varphi,\, P_z)$ as the new variables of differentiation.

The Kramers-Kronig relation now enables us to determine $\underset{\approx}{\sigma}''$ from $\underset{\approx}{\sigma}'$, and thus the total $\underset{\approx}{\sigma}$. After some algebra, we find

$$\underset{\approx}{\sigma}^{mk}(r,r';\, \omega) = -(2\pi/\omega) \int_0^\infty d\tau\; e^{i\omega\tau} \int d^3J\, \eth f(\underset{\sim}{J}) \times \underset{\sim}{j}(r,m,k;\, t;\, \underset{\sim}{J})\, \underset{\sim}{j}^{*}(r',m,k;\, t-\tau;\, \underset{\sim}{J}) \; . \tag{111}$$

(Alternatively, we may more easily guess the form (111), and verify the reduction to (109) by the Wiener-Khinchin theorem.)

This result is a generalization to nonequilibrium $f(\underset{\sim}{J})$ of the Kubo expression[12] for $\underset{\approx}{g}$. When f is a canonical distribution (86), it reduces to

$$\underset{\approx}{g}^{mk}(r,r';\,\omega) = 2\pi\,\beta \int_0^\infty d\tau\, e^{i\omega\tau}\,[1 - (m\omega_\varphi + kV_z)/\omega\,]$$

$$\times \int d^3J\, f(\underset{\sim}{J})\, \underset{\sim}{j}(r,m,k;\, t;\, \underset{\sim}{J})\, \underset{\sim}{j}^*(r',m,k;\, t-\tau;\, \underset{\sim}{J})\,. \tag{112}$$

In the special case $\omega_\varphi = 0$, $V_z = 0$, we can find the response kernel in the standard Kubo form:

$$\underset{\approx}{g}^{mk}(r,r';\,\tau) = 2\pi\,\beta \int d^3J\, f(\underset{\sim}{J})\, \underset{\sim}{j}(r,m,k;\, t;\, \underset{\sim}{J})\, \underset{\sim}{j}^*(r',m,k;\, t-\tau;\, \underset{\sim}{J})\,. \tag{113}$$

The explicit evaluation of (111) entails considerable straightforward algebra, involving (107) and the identity

$$\sum_{\ell=-\infty}^{+\infty} (y - \ell)^{-1}\, e^{i\ell\varphi} \equiv \pi(\csc \pi y)\, \exp iy[\varphi(\mathrm{mod}\ 2\pi) - \pi]\,. \tag{114}$$

The result is

$$\underset{\approx}{g}^{mk}(r,r';\,\omega) = 16\pi^2\, e^2\, i(rr'\omega)^{-1} \int d^3J\, \omega_r(\underset{\sim}{J})\, \partial f(\underset{\sim}{J})\, \csc[\omega'(\underset{\sim}{J})\, \tau(\underset{\sim}{J})]$$

$$\times\, [\hat{r}\, \sin \omega'(\underset{\sim}{J})\, \tau_>(\underset{\sim}{J}) + i\, \hat{s}\, \cos \omega'\tau_>][\hat{r}\, \sin \omega'\tau_< + i\, \hat{s}'\, \cos \omega'\tau_<]\,, \tag{115}$$

where

$$\hat{s} \equiv |\dot{r}(r,\underset{\sim}{J})|^{-1} [r\,\dot{\varphi}(r,\underset{\sim}{J})\hat{\varphi} + \dot{z}(r,\underset{\sim}{J})\hat{z}] , \tag{116}$$

$\hat{s}'$ is the same with r replaced by r', $\tau(\underset{\sim}{J})$ is the particle transit time (one-way) between radial turning points, $\tau_{>}(\underset{\sim}{J})$ is the transit time from the greater of (r,r') to the outer turning point, and $\tau_{<}(\underset{\sim}{J})$ is that from the lesser to the inner turning point.

In contrast to the usual approach, we have obtained the Vlasov conductivity as a by-product of the calculation, instead of solving the linearized Vlasov equation directly.

VIII. CONCLUSIONS

This study has concentrated on the lowest-order resonant interactions between particles and normal modes, in a system with two degrees of symmetry. In further work, we plan to extend the theory to include higher-order effects, including departures from exact resonance, resulting in finite-width functions[17] to replace the delta-functions of the present theory. We also plan to study systems with less symmetry, in which adiabatic invariants replace the exact invariants of the present theory.

The primary flaw in this study is the limitation to resonant instabilities. Inasmuch as nonresonant instabilities are of considerable importance, they should be included; however, a satisfactory theory for diffusion due to them does not yet exist even for the uniform case.

The methods used here are a judicious selection of quantum and classical ideas, with little regard for the demands of rigor. It would be intellectually more satisfying, perhaps, but much more fatiguing, to develop a purely classical theory, along the lines of KN. What is needed to justify the quantum approach is a classical theory which develops analogously to the quantum theory. (Alternatively, a rigorous quantum theory needs to be formulated.) Progress along these lines is in evidence for a fluid system,[18] and needs to be extended to the kinetic regime.

The basic validity criterion for our theory concerns the existence of two time scales: a fast one for the mode eigenfrequencies and the particle radial motion, and a slow one for the evolutionary development described by the kinetic equations. The existence of such a clean separation certainly depends upon the details of the configuration under

consideration. In such a case, the Markov assumption may be invoked to decouple the slow irreversible evolution from the fast reversible dynamics. Some estimates of the several time scales may be found in KN.

FOOTNOTES AND REFERENCES

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