Submitted to Physics of Fluids

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June 24, 1970

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REFORMULATION OF QUASILINEAR THEORY*

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June 24, 1970

ABSTRACT

A new formulation of quasilinear theory is presented, which allows for only resonant diffusion, caused by both growing and damped waves. Nonresonant terms do not appear in the diffusion equation, but contribute to wave momentum and energy, and ensure conservation of total momentum and energy. Analyticity is not required, since the complex plane is not used.

* This work was supported by the U. S. Atomic Energy Commission and by NASA.

I. INTRODUCTION

Users of quasilinear theory[1] have often expressed uneasiness about its applicability to damped waves,[2] because the standard expression of the diffusivity is then negative. Attempts[3,4] to establish a quasilinear theory uniformly valid for growing and decaying waves have been based on the Klimontovich[5] picture, rather than on a Vlasov analysis, and have further required analytic continuation into the complex wave-number and velocity planes. As pointed out by Weitzner,[6] analytic continuation may impose unrealistic continuity requirements on physical distributions.

The present paper is devoted to a fresh look at quasilinear theory, with a treatment of growing and decaying waves on an equal footing. The complex plane is completely avoided, even for frequency, and Landau's analysis of the linear Vlasov problem is replaced by a different technique. The nonlinear Vlasov equation is manipulated to a form exhibiting only resonant diffusion, without the need for the limit $\gamma \to 0+$. The term previously denoted non-resonant, adiabatic, or "fake" diffusion[1] no longer appears explicitly; it contributes to wave momentum and energy, and ensures the conservation of total momentum and energy.

No attempt is made here to maintain mathematical rigor. The ordering of terms is made heuristically, without the explicit introduction of multiple-time techniques.

To keep things as simple as possible, only the uniform one-dimensional Coulomb plasma is treated here. The generalization to more dimensions,[7] and to a nonuniform[8] Maxwell[9] system is straightforward.

II. LINEAR ANALYSIS

When the Vlasov equation

$$\frac{\partial f(x,v;\ t)}{\partial t} + v\frac{\partial f}{\partial x} + \frac{e}{m}E(x,t)\frac{\partial f}{\partial v} = 0 \tag{1}$$

is linearized about a uniform state $f_0(v;\ t)$ which is <u>slowly</u> varying in time, the first-order solution, obtained by integration along characteristics, is

$$f_1(x,v;\ t) = -(e/m)\ f_0'(v;\ t)\int_0^\infty d\tau\ E(x - v\tau,\ t - \tau). \tag{2}$$

Here the prime denotes velocity-differentiation. The time-integration is on the fast time-scale; accordingly $f_0'(v;\ t - \tau)$ has been replaced by $f_0'(v;\ t)$ and taken outside the integral.

We look for solutions of the form:

$$\begin{aligned} E(x,t) &= E_k(t)\exp(ikx - i\omega_k t), \\ f_1(x,v;\ t) &= f_k(v;\ t)\exp(ikx - i\omega_k t), \end{aligned} \tag{3}$$

where k and ω_k are both real, and the amplitudes have a slow time dependence representing wave growth or decay. But we note that the variation of $E_k(t)$, which is of order γ_k, is not as slow as that of $f_0(v;\ t)$, which is of order E_k^2. (Our ordering thus breaks down when E_k^2 and γ_k are of comparable size, in appropriate units.)

Insertion of (3) into (2) yields

$$f_k(v;\ t) = -(e/m)\ f_0'(v;\ t)\int_0^\infty d\tau\ E_k(t - \tau)\exp i(\omega_k - kv)\tau. \tag{4}$$

The Poisson equation then reads

$$ikE_k(t) = -(4\pi e^2/m)\int dv\, f_0'(v;\, t)\int_0^\infty d\tau\, E_k(t-\tau)\exp i(\omega_k - kv)\tau, \tag{5}$$

where a sum over species, if needed, is implicit.

Appealing to the slow variation of $E_k(t)$, we expand

$$E_k(t-\tau) = E_k(t) - \tau\,\dot{E}_k(t) + \cdots \tag{6}$$

and keep only the two terms shown. We then use

$$\begin{aligned}\int_0^\infty d\tau \exp i(\omega_k - kv)\tau &= \pi\,\delta(\omega_k - kv) + i\,P(\omega_k - kv)^{-1} \\ &\equiv \pi\,\delta_+(\omega_k - kv)\end{aligned} \tag{7}$$

and

$$\int_0^\infty d\tau\,\tau \exp i(\omega_k - kv)\tau = -i\,\partial[\pi\delta_+(\omega_k - kv)]/\partial\omega_k, \tag{8}$$

to obtain

$$\begin{aligned}ikE_k(t) = -(4\pi e^2/m)[&E_k(t)\int dv\, f_0'(v;\, t)\,\pi\,\delta_+(\omega_k - kv) \\ &+ i\,\dot{E}_k(t)(\partial/\partial\omega_k)\int dv\, f_0'(v;\, t)\,\pi\,\delta_+(\omega_k - kv)].\end{aligned} \tag{9}$$

We now multiply (9) by $E_k^*(t)$, and multiply the complex conjugate of (9) by $E_k(t)$. Adding and subtracting the resulting equations, we find equations for the growth rate and for the eigenfrequency ω_k. The subtraction yields

$$1 = -(4\pi e^2/mk)\int dv\, f_0'(v;\, t)\,P(\omega_k - kv)^{-1} \tag{10}$$

(plus terms of relative order $\dot{E}_k/E_k$), which is the standard dispersion equation for the real eigenfrequency $\omega_k(t)$. The addition yields

$$\frac{d\ \ell n\ |E_k(t)|^2}{dt} = \frac{2\int dv\ f_0'(v;\ t)\ \pi\delta(\omega_k - kv)}{(\partial/\partial\omega_k)\int dv\ f_0'(v;\ t)\ P(\omega_k - kv)^{-1}} \tag{11}$$

(plus terms of relative order $\dot{E}_k/E_k$), which is the standard equation for the growth (or decay) rate $2\ \gamma_k(t)$. The slow time-dependence of $\omega_k(t)$, implied by (10), is of order E^2, and therefore is not included in (11), which is zero-order in E^2.

III. QUASILINEAR ANALYSIS

In expanding the nonlinear Vlasov equation (1) to second order in the wave amplitudes, we shall distinguish between $f_2(v;\, t)$, the second-order perturbation in f (averaged over x and over oscillation period), whose <u>size</u> is proportional to E^2, and $f_0(v;\, t)$, the part of f independent of E, whose <u>evolution</u> is proportional to E^2.

Inserting (2) into the space-time average of (1), we thus obtain

$$\frac{\partial f_0}{\partial t} + \frac{\partial f_2}{\partial t} = \left(\frac{e}{m}\right)^2 \frac{\partial}{\partial v}\left[I(v,t)\,\frac{\partial f_0(v;\, t)}{\partial v}\right], \tag{12}$$

where

$$I(v,t) \equiv \int_0^\infty d\tau\, \langle E(x,t)\, E(x - v\tau,\, t - \tau)\rangle \tag{13}$$

(the angular bracket denotes spatial and fast-time averaging), and where $\partial f_0(v;t)/\partial t$ represents the slow evolution of f_0 due to diffusion, while $\partial f_2(v;\, t)/\partial t$ represents the slow variation of f_2 due to wave growth and decay. On the right side of (12), we have dropped $\partial f_2/\partial v$, but have retained $\partial f_0/\partial v$; we justify this below.

In expression (13), we generalize (3) to include all k, and several branches ℓ:

$$E(x,t) = \sum_{k,\ell} E_k^{\ell}(t)\, \exp(ikx - i\omega_k^{\ell}\, t), \tag{14}$$

with $E_{-k}^{\ell}(t) = E_k^{\ell *}(t)$, and $\omega_{-k}^{\ell} = -\omega_k^{\ell}$; we then obtain

$$I(v,t) = \sum_{k,\ell} \int_0^\infty d\tau\, E_k^{\ell *}(t)\, E_k^{\ell}(t-\tau) \exp i(\omega_k^{\ell} - kv)\tau. \tag{15}$$

Again we use Eqs. (6), (7), and (8), and find

$$I(v,t) = \sum_{k>0,\ell} \{|E_k^{\ell}(t)|^2\, 2\pi\, \delta(\omega_k^{\ell} - kv) - (d|E_k^{\ell}|^2/dt)\, \partial P(\omega_k^{\ell} - kv)^{-1}/\partial\omega_k^{\ell}\}. \tag{16}$$

We insert (16) in (12), associating the first term of (16) with $\partial f_0/\partial t$, and the second term of (16) with $\partial f_2/\partial t$. We then obtain for f_0 the resonant diffusion equation:

$$\frac{\partial f_0(v;t)}{\partial t} = \frac{\partial}{\partial v}\left[D(v,t)\,\frac{\partial f_0}{\partial v}\right], \tag{17}$$

with

$$D(v,t) \equiv (e/m)^2 \sum_{k>0,\ell} |E_k^{\ell}(t)|^2\, 2\pi\, \delta(\omega_k^{\ell} - kv), \tag{18}$$

while the equation for f_2 can be explicitly integrated to yield

$$f_2(v;\, t) = -\left(\frac{e}{m}\right)^2 \frac{\partial}{\partial v}\left[f_0'(v;\, t) \sum_{k>0,\ell} |E_k^{\ell}(t)|^2\, \partial P(\omega_k^{\ell} - kv)^{-1}/\partial\omega_k^{\ell}\right]. \tag{19}$$

The diffusion equation (17) includes the effects of <u>all</u> waves in the diffusivity (18), whether growing or damped. The second-order perturbation f_2 appears singular, but is actually quite smooth for a continuous spectrum of waves. Replacing $\sum_k$ by $V\int dk/2\pi$, and defining the spectral density

$$\mathcal{E}^{\ell}(k;\, t) \equiv \lim_{V\to\infty} V|E_k{}^{\ell}(t)|^2, \tag{20}$$

we have

$$f_2(v;\, t) = \left(\frac{e}{m}\right)^2 \frac{\partial}{\partial v}\left\{ f_0'(v;\, t) \right.$$

$$\left. \times \int_0^\infty (dk/2\pi) \sum_{\ell} P(\omega_k{}^{\ell} - kv)^{-1} \frac{\partial}{\partial k}\left[\mathcal{E}^{\ell}(k;\, t)\left(\frac{\partial\omega_k{}^{\ell}}{\partial k}\right)^{-1}\right]\right\} . \tag{21}$$

Estimating the ratio of f_2' to f_0', we find it to be of order $(\omega_k\, \tau_D)^{-1}$ (where τ_D is the diffusion time for f_0), and therefore very small. Hence we are justified in keeping only f_0' on the right side of (12) and (17).

The moments of f_2 yield the particle contribution to the wave energy and momentum. The second moment, which is the wave kinetic energy density, is

$$\begin{aligned} K_2(t) &\equiv \int dv\, \frac{1}{2}\, m\, v^2\, f_2(v;\, t) \\ &= \left(\frac{e}{m}\right)^2 \sum_{k>0,\ell} |E_k{}^{\ell}(t)|^2 \frac{\partial}{\partial\omega_k{}^{\ell}} \int dv\, m\, v\, f_0'(v;\, t)\, P(\omega_k{}^{\ell} - kv)^{-1} \\ &= (e^2/m) \sum_{k>0,\ell} |E_k{}^{\ell}(t)|^2 \frac{\partial}{\partial\omega_k{}^{\ell}}\left[(\omega_k{}^{\ell}/k)\int dv\, f_0'(v;t) P(\omega_k{}^{\ell} - kv)^{-1}\right]. \end{aligned} \tag{22}$$

When we add this to the wave electric energy density

$$\begin{aligned} W_E(t) &= \langle |E(x,t)|^2/8\pi \rangle \\ &= (4\pi)^{-1} \sum_{k>0,\ell} |E_k{}^{\ell}(t)|^2, \end{aligned} \tag{23}$$

we obtain the total wave energy density:

$$W(t) \equiv W_E(t) + K_2(t)$$

$$= (4\pi)^{-1} \sum_{k>0,\ell} |E_k{}^{\ell}(t)|^2 \frac{\partial}{\partial\omega_k{}^{\ell}} [\omega_k{}^{\ell}\ \epsilon(k,\omega_k{}^{\ell};\ t)], \tag{24a}$$

where

$$\epsilon(k,\omega_k{}^{\ell};\ t) \equiv 1 + (4\pi e^2/mk) \int dv\ f_0'(v;\ t)\ P(\omega_k{}^{\ell} - kv)^{-1} \tag{25}$$

is the <u>real</u> part of the dielectric function. We note that Eq. (10) is equivalent to $\epsilon(k,\omega_k{}^{\ell};\ t) = 0$, whence (24a) may be written

$$W(t) = (4\pi)^{-1} \sum_{k>0,\ell} |E_k{}^{\ell}(t)|^2\ \omega_k{}^{\ell}\ \partial\epsilon/\partial\omega_k{}^{\ell}. \tag{24b}$$

The first moment of f_2 is the wave momentum density:

$$P_2(t) \equiv \int dv\ m\ v\ f_2(v;\ t)$$

$$= (e^2/m) \sum_{k>0,\ell} |E_k{}^{\ell}(t)|^2 \frac{\partial}{\partial\omega_k{}^{\ell}} \int dv\ f_0'(v;\ t)\ P(\omega_k{}^{\ell} - kv)^{-1}$$

$$= (4\pi)^{-1} \sum_{k>0,\ell} |E_k{}^{\ell}(t)|^2\ k\ \partial\epsilon(k,\omega_k{}^{\ell})/\partial\omega_k{}^{\ell}. \tag{26}$$

Comparing (26) with (24b), we see that wave momentum and wave energy satisfy the standard relation

$$\frac{(P_2)_k{}^{\ell}}{W_k{}^{\ell}} = \frac{k}{\omega_k{}^{\ell}}\ . \tag{27}$$

The standard conservation laws follow from these equations, <u>if</u> we ignore the time-dependence of ω_k^{ℓ} and ϵ. From (24b) and (11), we have

$$dW/dt = (e^2/m) \sum_{k>0,\ell} |E_k^{\ell}|^2 (\omega_k^{\ell}/k) \int dv\, f_0'(v;\, t)\, 2\pi\, \delta(\omega_k^{\ell} - kv). \tag{28}$$

The zero order kinetic energy density

$$K_0(t) \equiv \int dv\, \frac{1}{2}\, m\, v^2\, f_0(v;\, t) \tag{29}$$

evolves according to (17) and (18):

$$\begin{aligned} dK_0/dt &= -\int dv\, m\, v\, D(v,t)\, f_0' \\ &= -(e^2/m) \sum_{k>0,\ell} |E_k^{\ell}|^2 (\omega_k^{\ell}/k) \int dv\, f_0'(v;\, t)\, 2\pi\, \delta(\omega_k^{\ell} - kv). \end{aligned} \tag{30}$$

Comparing (28) and (30), we see that the total energy density

$$U(t) \equiv W(t) + K_0(t) \tag{31}$$

is indeed conserved.

The total momentum density

$$P(t) \equiv P_2(t) + P_0(t) \tag{32}$$

is proved constant in the same way, using

$$P_0(t) \equiv \int dv\, m\, v\, f_0(v;\, t) \tag{33}$$

and Eq. (26), with Eqs. (17), (18), and (11) for their evolution.

ACKNOWLEDGMENT

I have benefited from discussing these ideas with Blake Patterson and David Montgomery.

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