Submitted to Physical
Review

UCRL-19893 c.2 repl.
Preprint

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August 1970

AEC Contract No. W-7405-eng-48

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A SET OF GENERALIZED HEISENBERG RELATIONS AND A NEW FORM OF QUANTIZATION*

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ABSTRACT

A prescription is made for "geometrizing" the space-time manifold as an extension of Wheeler's "wormhole" theory. In the method presented, a set of physical variables, termed quantal units, are uniquely expressed in terms of universal constants. By use of the quantal unit interpretation, a set of Generalized Heisenberg Uncertainty Relations are developed, as a generalized set of canonically conjugate variables obeying commutation relations expressible in terms of universal constants. Two procedures of quantization termed primary and secondary are defined and applications are made to atomic and nuclear systems. The quantal units are interpreted as a representation of the geometrical structure of the space-time manifold and are manifest in nuclear and cosmological physics.

I. INTRODUCTION

J. A. Wheeler[1] attempted to geometrize the space-time manifold in terms of a "quantum of length," termed the Wheeler "wormhole," $\ell = (G\hbar/c^3)^{1/2}$. Wheeler pictures the metric of his space as fluctuations in a multiple-connected, foamlike structure in which the microcurvature has a scale "size" of ℓ characteristic of his topology. The microcurvature sets a lower limit on the meaningful intervals of length and time. He also discusses a quantum of mass, $m = (c\hbar/G)^{1/2}$ and a quantum of density, $\rho = c^5/G^2\hbar$.

Earlier, M. Planck[2] derived dimensioned quantities in terms of the fundamental universal constants $\hbar$ (Planck's constant), G (universal gravitational constant), c (velocity of light). The quantities Planck introduced were:

$$\ell = (G\hbar/c^3)^{1/2} = 1.60 \times 10^{33} \text{ cm}, \tag{1a}$$

$$t = (G\hbar/c^5)^{1/2} = 5.36 \times 10^{-44} \text{ sec}, \tag{1b}$$

$$m = (c\hbar/G)^{1/2} = 2.82 \times 10^{-5} \text{ gm}, \tag{1c}$$

for length, time, and mass, respectively. The values of the universal constants used in evaluating the Planck quantities are: $c = 2.998 \times 10^{10}$ cm/sec, $\hbar = 1.055 \times 10^{-27}$ erg-sec, and $G = 6.673 \times 10^{-8} \frac{\text{cm}^3}{\text{gm-sec}^2}$. These values are taken from the recent work of B. N. Taylor, W. H. Parker, and D. N. Langenberg[3] on the theoretical and experimental implications of the universal constants.

Planck discussed the universality of the expressions in (1a), (1b), (1c), which comes about through their unique expression in terms of the universal constants. The quantities in (1a), (1b), (1c), and all physical variables, can be uniquely expressed in terms of universal constants and,—

in this form, are here termed "quantal units." Table I contains the set of quantal units and their numerical values relevant to the calculations in this paper.

E. R. Harrison[4] discusses some aspects of what he terms "quantum cosmology" in terms of the Planck quantities or quantal units and their limiting values in the space-time manifold.

P. A. M. Dirac[5] and more recently R. A. Alpher and G. Gamow[6] have proposed that the close agreement of ratios of universal constants expressing both atomic and cosmological aspects is not coincidence but is an expression of the theoretical significance of universal constants.

There has been much interest in the recent work of B. A. Taylor, W. H. Parker, and D. A. Langenberg[3] on the theoretical implications of the universal constants. They demonstrate the manner in which the universal constants, on a fundamental theoretical framework, may possibly unify various diverse branches of physics. This is what is proposed and may be possible by use of the quantal units form of physical variables which are manifest in atomic, nuclear, and cosmological physics. It is suggested that the quantal unit form of all physical variables may represent a more complete geometric description of the space-time manifold than that proposed by Wheeler. For a more detailed discussion, see Ref. 7. Presented here are some nuclear aspects of the theory and in Refs. 7 and 8 some of the cosmological aspects of the theory are presented.

We define two distinct quantization procedures, primary and secondary, where primary or maximal quantization is that quantization procedure in terms of the quantal units and secondary is that quantization procedure which is the ordinary or standard form of quantization.

In Sec. II, six Generalized Heisenberg Relations are presented and the manner in which secondary quantization reduces to primary, by use of the relation of the quantal force and the quantal frequency, is shown. In Sec. III, wave functions are constructed for primary quantization. In Sec. IV, we define some nuclear quantities in terms of primary quantization.

II. GENERALIZED HEISENBERG RELATIONS IN PRIMARY AND SECONDARY QUANTIZATION

We shall consider the pair relations obeyed by the quantal units as elements of non-Abelian algebras. That is, the canonically conjugate pairs of physical variables in terms of both primary and secondary quantization form commutation relations that are non-zero but are $\hbar$, c, ℓ, ℓ', F, and ω or combinations of these constants. These constants or combination of constants represent a limit on obtainable information of the physical world.

Heisenberg developed the relations in terms of E, t and x, p which represents a limit on observation of physical phenomena as relates to the magnitude of $\hbar$.

We have the well known relations

$$\begin{array}{l} \longrightarrow \\ \downarrow \; [\ell, p]_{\hbar} \geq \hbar \\ \quad \times \\ \quad [t, E]_{\hbar} \geq \hbar \end{array} \tag{2}$$

(Note ℓ or x is the dimension of length.) The sub $\hbar$ on the bracket denotes the Generalized Heisenberg Relation in terms of $\hbar$. As discussed by Bohr the canonically conjugate pairs ℓ, p and t, E relate in a quantum mechanical manner to $\hbar$ as a limit on the simultaneous observability of ℓ and p and also t and E. These relations we call horizontal relations as the relations of ℓ to p and t to E (see arrow in Eq. (2)). Eq. (2) represents six Generalized Heisenberg Relations.

Looking at the vertical relation between these four variables we see that t is the fourth component of ℓ and E is the fourth component of p, in Einstein four vector notation,

$$\underset{\sim}{x} = (\underline{\ell}, t) \tag{3a}$$

and

$$\underset{\sim}{p} = (\underline{p}, E) \ . \tag{3b}$$

We shall now note a "network" of paired relations as indicated by the arrows in Eq. (2). The first new relations are obtained as paired relations between the diagonal pairs of quantum variables (indicated by the diagonal arrows). These pairs are E, ℓ and p, t and form the relations,

$$[E, \ell] \geq c\hbar \equiv \ell' \tag{4a}$$

$$[p, t] \geq \hbar/c \equiv \ell \tag{4b}$$

where we define $\ell' \equiv ch$ and $\ell \equiv h/c$ as in Table I. The commutation relations between one space-like component of a four vector with the time-like component of a four vector is equal to ℓ or ℓ'. Now let us look at the third possibility of the relation between pairs in Eq. (2). We form the vertical pairs (ℓ, t) and (p, E). As before, we are considering one component of ℓ, is an isotopic space as $\ell_1 = \ell_2 = \ell_3$ and similarly for $p_1 = p_2 = p_3$. We have,

$$[\ell, t] = \frac{(\ell'\ell)^{1/2}}{F} = \hbar/F = \frac{\ell'}{cF} \tag{5a}$$

and

$$[p, E] = (\ell'\ell)^{1/2} F = \hbar F, = \ell cF \tag{5b}$$

using the quantal units given in Table I. Note that $p = cF$ is the quantal power. So we have a generalized set of interrelated canonically conjugate relations for primary quantization. We can also write [ℓ, t] as [x_1, x_4] and [p, E] as [p_1, p_4]. We have presented six cases of the Generalized Heisenberg Relations.

In primary quantization we have* (from Table I),

$$\ell = \left(\frac{\mathcal{L}'}{F}\right)^{1/2} \qquad p = (\mathcal{L}F)^{1/2} \qquad \text{(6a), (6b)}$$

$$t = (\mathcal{L}/F)^{1/2} \qquad E = (\mathcal{L}'F)^{1/2} \qquad \text{(6c), (6d)}$$

In ordinary quantization we have,

$$\ell_o = \frac{\hbar}{mc} \qquad \Delta p = hk = \hbar/\lambda \qquad \text{(7a), (7b)}$$

$$t_o = \frac{\hbar}{mc^2} \qquad \Delta E = h\nu = \hbar\omega \qquad \text{(8a), (8b)}$$

The quantity ℓ_o is the Compton wave length for a particle of mass, m and t_o is the corresponding time, $t_o = \ell_o/c$. A.H. Compton[9] studied the scattering of x-rays off of electrons and the constant, $\hbar/mc$, depending on the target mass, arises in the expression of relating wave length of radiation and scattering angle.

In primary quantization we have for the horizontal arrows in Eq. (2),

$$[\ell, p] = (\mathcal{L}'\mathcal{L})^{1/2} = \hbar \qquad \text{(9a)}$$

$$[t, E] = (\mathcal{L}\mathcal{L}')^{1/2} = \hbar \qquad \text{(9b)}$$

In ordinary quantization we have for the horizontal arrows,

$$[\ell, p] = \ell_o\, p = \frac{\hbar}{mc} \cdot \frac{h}{\lambda} = \frac{\hbar^2}{mc\lambda} = \frac{\hbar}{mc\lambda} \qquad \text{(10a)}$$

from ℓ_o in Eq. (6a) and p in Eq. (6b) and wave numbers, $k = 1/\lambda$.

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* The factor $F/\mathcal{L}'$, Eq. (6a), has cosmological significance in Einstein's field equations. The term $\frac{8\pi G}{c^4}\, T_{ik} = \frac{8\pi}{F} \cdot \frac{F^2}{\mathcal{L}'} = 8\pi(F/\mathcal{L}')$. This is discussed in detail in Refs. 7 and 8.

Multiplying top and bottom of Eq. (10a) by c and using the relation $E = mc^2 = \hbar\omega$, then,

$$[\ell, p] = \frac{\hbar^2 c}{mc^2 \lambda} = \frac{\hbar^2 c}{h\omega\lambda} = \frac{\hbar c}{c} = \hbar \tag{10b}$$

and

$$[t, E] = t_o E = \frac{\hbar}{mc^2} \cdot \hbar\omega = \frac{\hbar \cdot \hbar\omega}{mc^2} \tag{10c}$$

from t_o in Eq. (7c) and E in Eq. (7d) and then using the relation $mc^2 = E = \hbar\omega$, we have,

$$[t, E] = \frac{\hbar \cdot \hbar\omega}{\hbar\omega} = \hbar \tag{10d}$$

For primary quantization, we had from before in Eq. (4a) and (4b) for the diagonal arrow relations in Eq. (2),

$$[E, \ell] = E\ell = (\ell' F)^{1/2} \left(\frac{\ell'}{F}\right)^{1/2} = (\ell'\ell')^{1/2} = \ell' \equiv c\hbar \tag{11a}$$

and

$$[p, t] = pt = (\ell F)^{1/2} \left(\frac{\ell}{F}\right)^{1/2} = (\ell\ell)^{1/2} = \ell \equiv \hbar/c \tag{11b}$$

from the quantities in Eq. (6a), (6b), (6c), and (6d).

For secondary quantization we have,

$$[E, \ell] = E\,\ell_o = \hbar\omega \cdot \frac{\hbar}{mc} = \frac{\hbar^2\omega}{mc} \tag{11c}$$

from Eq. (7a) and Eq. (7b). Multiplying top and bottom by c and using the relation $E = mc^2 = \hbar\omega$, we have,

$$[E, \ell] = \frac{\hbar^2\omega c}{mc^2} = \frac{\hbar^2\omega c}{\hbar\omega} = \hbar c = \ell' \tag{11d}$$

and

$$[p, t] = pt = \frac{\hbar}{\lambda} \cdot \frac{\hbar}{mc^2} = \frac{\hbar^2}{mc^2\lambda} \tag{11e}$$

from Eq. (7b) and E of (7c), and $\bar{k} = 1/\bar{\lambda}$. Using the relation $E = mc^2 = \hbar\omega$, we have,

$$[p, t] = \frac{\hbar^2}{\hbar\omega\bar{\lambda}} = \frac{\hbar}{\omega\bar{\lambda}} = \frac{\hbar}{c} = \bar{\ell} \tag{11f}$$

for $c = \omega\bar{\lambda}$.

So we obtain the same result in the case of the ordinary quantization and so these are indeed the primary and secondary quantization representation of the physical variables, E, ℓ and p, t. These are the same results, Eq. (11d) and Eq. (11f), for [E, ℓ] and [p, t] as for Eq. (4a) and Eq. (4b).

Now for primary quantization for the vertical arrows, Eq. (2), we have,

$$[\ell, t] = \ell t = \left(\frac{\bar{\ell}'}{F}\right)^{1/2} \left(\frac{\bar{\ell}}{F}\right)^{1/2} = \frac{\bar{\ell}'}{cF} = \frac{\hbar}{F} \tag{12a}$$

for $\bar{\ell}'/c = \hbar$ since $\bar{\ell}' \equiv c\hbar$.

$$[p, E] = pE = (\bar{\ell}F)^{1/2} (\bar{\ell}'F)^{1/2} = \bar{\ell}cF = \hbar F \tag{12b}$$

for $\bar{\ell}c = \hbar$ since $\bar{\ell} \equiv \hbar/c$.

Note the analogy of c and F in Eq. (11a) and (12b) and Eq. (11b) and (12b). These results are the same as for [ℓ, t] and [p, E] in Eq. (5a) and Eq. (5b). And for secondary quantization we have for the vertical arrows,

$$[\ell, t] = \ell_o t_o = \frac{\hbar}{mc} \cdot \frac{\hbar}{mc^2} = \frac{\hbar^2 c}{m^2 c^4} = \frac{\hbar^2 c}{(mc^2)^2} \tag{13a}$$

from Eq. (7a) and Eq. (7c) and then multiplying top and bottom by c. Using the relation $E = mc^2 = \hbar\omega$ and $\bar{\ell}' = \hbar c$, we have,

$$[\ell, t] = \frac{\hbar^2 c}{(\hbar\omega)^2} = \frac{\hbar c}{\hbar\omega^2} = \frac{\bar{\ell}}{\hbar\omega^2} = \frac{c}{\omega^2} \tag{13b}$$

and

$$[p, E] = pE = \frac{\hbar}{\lambda} \cdot \hbar\omega = \frac{\hbar^2\omega}{\lambda} \tag{13c}$$

from Eq. (4b) and Eq. (4d). Using the relation $c = \omega\lambda$ so that $\lambda = c/\omega$ then,

$$[p, E] = \frac{\hbar^2\omega^2}{c} = \ell(\hbar\omega^2) = \frac{E^2}{c} \tag{13d}$$

for $\ell \equiv \hbar/c$.

The primary quantization criterion give a product or ratio of cF to ℓ or ℓ' (Eq. (12a) and (12b) and equivalently the ordinary quantization process gives the quantity $\hbar\omega^2$ and ℓ and ℓ' (Eq. (13b) and (13d) which is equivalent to cF, the quantal power. A frequency ω is associated with the quantal force, F, or quantal power $p = cF$.

The relation of the quantal power $p = cF$ and quantal frequency, ω, is a relationship between primary and secondary quantization. Also, we have $\ell_o = \frac{\hbar}{mc}$ as the Compton wave length; for an electron we have $\frac{\hbar}{m_e c} = 3.86 \times 10^{-11}$ cm and for a proton, $\frac{\hbar}{m_p c} = 2.10 \times 10^{-14}$ cm. Let us calculate the Compton wave length for the primary quantal mass $m = \left(\frac{c\hbar}{G}\right)^{1/2}$. We have $\ell_o = \frac{\hbar}{mc} = \frac{\hbar}{c}\left(\frac{G}{ch}\right)^{1/2} = \left(\frac{\hbar G}{c^3}\right)^{1/2}$ which is the quantal unit of length (see Table I).

Primary and secondary quantization are related by the same equations, but primary quantization deals with the geometric structure of the space-time manifold and ordinary quantization deals with the physical particles, electrons, etc., of the space-time manifold.

Then we have the fundamental quantal frequency in the case of primary quantization for $cF = \hbar\omega^2$ or $\hbar/F = c/\omega^2$ so $\omega^2\hbar = cF$ in Eq. (12) and (13) from the primary quantization commutation relations of (ℓ, t) as $[\ell, t] = \ell'/cF$ and the secondary quantization relations $[\ell, t] = \ell'/\hbar\omega^2$ (Eq. (12a) and (13b)). Also these relations can be obtained from the (p, E)

commutations (Eq. (12b) and Eq. (13d)). In relating primary and secondary quantization, we have $p = cF = \hbar\omega^2$ as we have first presented. We can uniquely express the quantal force in terms of the frequency as,

$$F = \frac{\hbar}{c}\ \omega^2 = \frac{\omega^2}{\ell} \tag{14a}$$

and uniquely express the quantal frequency in terms of force as,

$$\omega = \pm\sqrt{\frac{cF}{\hbar}} = \pm\sqrt{\frac{p}{\hbar}} = \pm\sqrt{F\ell} \tag{14b}$$

Let us see if the relation $c = \omega\lambda$ or $c = \omega\ell$ holds. We have,

$$\omega = \left(\frac{cF}{\hbar}\right)^{1/2} = \left(\frac{F}{\ell}\right)^{1/2} = \left(\frac{Fc^2}{\ell'}\right)^{1/2} = \left(\frac{F}{\ell'}\right)^{1/2} c = \frac{c}{\ell} \tag{15}$$

so that $\omega = c/\ell$ or $c = \omega\ell$ for $\ell/\ell' = 1/c^2$.

As we see the quantal force has associated with it the quantal frequency.

We have presented six Generalized Heisenberg Relations in both their primary and secondary quantized forms. To better understand the limits on observability as set by the four new relations, one can study their wave function representation and the wave equations satisfied by these wave functions. In the next section, we discuss some cases of the Generalized Wave Functions. In his recent work, W. Silvert[10] discusses a set of Generalized Uncertainty Relations and the nature of the uncertainty principle for periodic systems.

III. GENERALIZED PRINCIPLE OF LEAST ACTION AND GENERALIZED WAVE FUNCTIONS

A Generalized Principle of Least Action can be formed in terms of $\hbar$, ℓ, ℓ', ℓ/cF and ℓcF. The "least action principle" or the variation of the action integral is given by,

$$S = \int L dt, \tag{16}$$

or the variation of the action variable S, gives the infinitesimal generators that obey canonically conjugate pair relations. We can express this as

$$\delta \int_{t_2}^{t_1} L dt = \delta G_i = G_2 - G_2 \tag{17}$$

where the G_i's are the infinitesimal generators of the group of transformations that act on the dynamical variables. These generators and their commutation relations make up the algebra of the group, of which we require Lorentz invariance when expressed as an equation of motion.

Some of the expressions of the least action principle are (for the horizontal arrow for ℓ, P and t, E (note q is the generalized coordinate of length)).

$$\oint p dq = nh \tag{18a}$$

$$\oint E dt = nh \tag{18b}$$

This is why Planck termed $\hbar$ the action variable in his theory of radiation.[2]

For the cross arrows, for (E, ℓ) and (p, t),

$$\oint E dq = n'\ell' \tag{19a}$$

$$\oint p dt = n''\ell \tag{19b}$$

and for the vertical arrows (ℓ, t) and (p, E) we have,

$$\oint q\,dt = \frac{m'\ell'}{cF} \tag{20a}$$

$$\oint p\,dE = m\ell cF \tag{20b}$$

where n, n', n", m, m' are integers. There are now a set of so-called phase spaces such as (p, q) space. There is also a (E, t) space and (E, q), (p, t) space and the (q, t) and (p, E) space.

The usual expression for a "free particle" plane wave (i.e., a particle not under the influence of a potential field) is,

$$\psi = e^{i\left[\frac{\underline{p}\cdot\underline{x} - Et}{\hbar}\right]} \tag{21}$$

The purpose in mind was to show how the relations develop and occur for E and ℓ and also p and t in terms of ℓ' and ℓ, respectively.

Now we can construct a wave packet from the plane wave expression,

$$\psi = e^{i\left[\frac{xE}{\ell'} - \frac{pt}{\ell}\right]} \tag{22}$$

Also a plane wave expression for the pairs x and t in terms of ℓ'/cF and p and E in terms of ℓcF can be formed. We now have,

$$\psi = e^{i\left[\frac{(xt)\ (cF)}{\ell'} - \frac{pE}{\ell cF}\right]} \tag{23a}$$

This ψ can also be expressed as,

$$\psi = e^{i\left[\frac{(xt)\ (\hbar\omega^2)}{\ell'} - \frac{pE}{\ell h\omega^2}\right]} \tag{23b}$$

The phase space of p and q relates to the constant $\hbar$. The phase space in terms of ℓ, t and E, p relates to ℓ' and ℓ.

The phase space in terms of (ℓ, t) and (p, E) is expressed and relates to $\frac{\ell'}{Fc}$ and ℓFc or $\hbar\omega^2/\ell'$ and $\ell\hbar\omega^2$.

There is another alternative expression, from Eq. (12a) and (12b),

$$\psi = e^{2} \left[(xt) \left(\frac{F}{\hbar} \right) - \frac{(pE)}{(\hbar F)} \right] \tag{23c}$$

In Eq. (23a), (23b), and (23c), x and t and also p and E are considered as variables and F and ω are considered as constants in their quantal unit form. One can construct a plane wave or wave packet for any two pairs of canonically conjugate variables.

In the next section, we shall discuss the primary and secondary quantization procedures as applied to nuclear lengths, energies, and magnetic quantities. We demonstrate the manner in which the two quantization procedures reduce to the same form for the six Generalized Heisenberg Relations discussed in Sec. II. A generalized form of the "quantum lengths" in nuclear physics is also presented.

IV. PRIMARY AND SECONDARY QUANTIZATION AND NUCLEAR QUANTITIES

A. Primary and Secondary Quantization of Atomic Lengths

Let us look at the usual or ordinary quantized form of some of the atomic lengths. This is the form we term secondary quantization. Using the secondary quantized lengths and substituting the quantal mass for the particle mass in these expressions and using $e^2 = \alpha\hbar c$ for the charge where α is the fine structure constant, we form a generalized primary form of these lengths in terms of the quantal length.

The first of the usual "atomic lengths" is the Bohr orbit radius represented in its secondary quantized form as,

$$a_o = \frac{\hbar}{me^2} . \tag{24}$$

A second length is the Compton wave length for particle of mass, m,

$$\lambda_c = \frac{\hbar}{mc} \tag{25}$$

and the third length is the Lorentz electron radius,

$$r_e = \frac{e^2}{m_e c^2} \tag{26}$$

where m_e is the mass of the electron and the fourth length is the Rydberg length defined in atomic spectra as,

$$R_{cm} = \frac{4\pi\hbar^3 c}{me^4} = \frac{4\pi a_o}{\alpha} \tag{27}$$

and $1/R_{cm}$ is termed the Rydberg constant.

In their primary quantized form we have first for the Bohr radius,

$$a_o = \frac{\hbar^2}{me^2} = \frac{\hbar^2}{e^2}\left(\frac{G}{c\hbar}\right)^{1/2} \tag{28}$$

where the quantal mass is substituted for m. Using the expression for the fine structure constant, we substitute $e^2 = \alpha\hbar c$ into the above expression and obtain,

$$a_o = \frac{\hbar^2}{\alpha\hbar c}\left(\frac{G}{c\hbar}\right)^{1/2} = \frac{1}{\alpha}\left(\frac{\hbar G}{c^3}\right)^{1/2} . \tag{29}$$

And from Table I we see that $\ell = \left(\frac{\hbar G}{c^3}\right)^{1/2}$, which is the quantal length. So that

$$a_o = \frac{1}{\alpha}\,\ell \tag{30}$$

where α is the fine structure constant.

The Compton wave length, as we had before in Sec. II, is,

$$\lambda_c = \frac{\hbar}{mc} = \frac{\hbar}{c}\left(\frac{G}{c\hbar}\right)^{1/2} = \left(\frac{\hbar G}{c^3}\right)^{1/2} = \ell \tag{31}$$

which is directly equal to the quantal length, ℓ.

The Lorentz electron radius becomes, using $e^2 = \alpha\hbar c$ and the quantal mass for m_e,

$$r_e = \frac{e^2}{m_e c^2} = \alpha^2 a_o = \alpha\ell , \tag{32}$$

where comparison to Eq. (29) is made to express r_e in terms of a_o.

The primary quantized form of the Rydberg length is,

$$R_{cm} = \frac{4\pi\hbar^3 c}{me^4} = \frac{4\pi a_o}{\alpha} = \frac{4\pi}{\alpha^2} . \tag{33}$$

So in summary, the primary quantized lengths are:

$$R_{cm} = \frac{4\pi}{\alpha^2}\,\ell \tag{34a}$$

$$a_o = 1/\alpha\ \ell \tag{34b}$$

$$\lambda_c = \ell \tag{34c}$$

$$r_e = \alpha\ell \tag{34d}$$

B. The Relation of Length and Energy

The manner in which energy and length are related in both secondary and primary quantization is the same.

In secondary quantization we have $c = \lambda\nu$ where λ is wave length and ν is frequency and $E = h\nu$ so that,

$$E = \frac{\hbar c}{\lambda} = \frac{\mathcal{E}'}{\lambda} . \tag{35}$$

In primary quantization, we have the quantal energy and length as,

$$E = \left(\frac{c^5\hbar}{G}\right)^{1/2} \tag{36a}$$

and

$$\ell = \left(\frac{G\hbar}{c^3}\right)^{1/2} \tag{36b}$$

so that $E = \mathcal{E}'/\ell$ or

$$\ell E = (\ell, E) = \mathcal{E}' \tag{37}$$

This is also our Eq. (11d).

Each of our lengths, Eq. (34a), (34b), (34c), and (34d), have associated energy such as the Rydberg length,

$$R_{cm} = \frac{4\pi\hbar^3 c}{me^4} \tag{38}$$

and the Rydberg energy,

$$R_E = \frac{me^4}{2\hbar^2} \tag{39}$$

The variables of length and energy are related to each other in forming a product of ℓ and E to equal $\mathcal{E}'$ (see Eq. (11d).) Length can be obtained

from energy and from length by dividing length or energy by ℓ'. This is true in both primary and secondary quantized representation.

C. Magnetic Moment, Bohr Magneton and Cyclotron Angular Frequency

The magnetic moment of an electron in a fired orbit (or any rotating charge) producing an induced magnetic field, B, is given by,

$$\mu = g \frac{e}{2m_e c} \tag{40}$$

where m_e is the mass of the electron and g in general is the Landé g factor. For example,

$$\mu = \sqrt{\ell(\ell+1)} \quad \left(\frac{e\hbar}{2m_e c}\right) \tag{41}$$

for an atomic orbital electron. The Bohr magneton is given as,

$$\mu = \frac{e\hbar}{2m_e c} = 9.28 \times 10^{-21} \text{ erg/gauss.} \tag{42}$$

Some references such as Reference 11 define the Bohr magneton (the natural unit of magnetic moment) as,

$$\mu_o = \frac{e\hbar}{2m_e} \tag{43}$$

where $\frac{e}{2m}$ is the gyromagnetic ratio. From Eq. (42) and the substitution of e = Q and quantal mass m,

$$\mu = \frac{e\hbar}{2mc} = \frac{Q\hbar}{2c}\left(\frac{G}{c\hbar}\right)^{1/2} \tag{44}$$

$$\mu = \frac{Q}{2}\left(\frac{\hbar G}{c^3}\right)^{1/2} = \frac{Q}{2}\,\ell \tag{45}$$

where ℓ is the quantal length, and from Eq. (43) we have,

$$\mu = \frac{\mu_o}{c}\,. \tag{46}$$

The electron cyclotron frequency is given by,

$$\omega_c \equiv \omega_e = \frac{eB}{m_e c} = \frac{QB}{m_e c} \tag{47}$$

for $e \equiv Q$, and for quantal magnetic field, $\mathcal{B}$ (see Ref. 7),

$$\mathcal{B} = B = \frac{c^3}{GQ} \tag{48}$$

We substitute the quantal mass for m_e as,

$$m = m_e = \left(\frac{c\hbar}{G}\right)^{1/2} = \left(\frac{\ell F}{c^2}\right)^{1/2} \tag{49}$$

and then substituting Eq. (49) into Eq. (47) and again defining $e \equiv Q$, we have,

$$\omega_c = \frac{QB}{mc} = QB\left(\frac{G}{c\hbar}\right)^{1/2} \tag{50}$$

Now substituting the quantal magnetic field from Eq. (48) into Eq. (50), we obtain,

$$\omega_c = \frac{Qc^3}{GQ}\left(\frac{G}{c\hbar}\right)^{1/2} = \left(\frac{c^5}{G\hbar}\right)^{1/2} = \left(\frac{cF}{\hbar}\right)^{1/2} = \left(\frac{F}{\ell}\right)^{1/2} \tag{51}$$

Using the quantal force, $F = c^4/G$ and the quantal frequency,

$$\omega = \left(\frac{cF}{\hbar}\right)^{1/2} = \left(\frac{F}{\ell}\right)^{1/2} \tag{52}$$

we have,

$$\omega_c = \omega \tag{53}$$

We have investigated some of the relationships between primary and secondary quantization in atomic and nuclear quantities. These relations for length are summarized in Eq. (34a), (34b), (34c), and (34d). Also, the primary and secondary expressions for energy were investigated using the relation $(\ell, E) = \ell'$ (Eq. (11d)). The primary quantal frequency can

also be related to the secondary cyclotron frequency (see Eq. (53)). In subsection D, we introduce a primary form of the fine structure constant incorporating both nuclear and cosmological features and relate it to the secondary or usual form of the fine structure constant, $\alpha = e^2/\hbar c$. Through the quantal mass, the two forms are found to be equivalent.

D. Primary Form of the Fine Structure Constant

The ratio of the Coulomb to the gravitational potential between two particles of charge magnitude e and mass, m, is given by,

$$\frac{e^2}{Gm^2} \quad . \tag{54}$$

It is interesting to note that upon substitution of the quantal mass, $m = (c\hbar/G)^{1/2}$ in Eq. (54), we have,

$$\frac{e^2}{Gm^2} = \frac{e^2}{G(\frac{c\hbar}{G})} = \frac{e^2}{c\hbar} = \frac{e^2}{\ell'} \ , \tag{55}$$

where $e^2/\hbar c = \alpha$, the ordinary fine structure constant and $\ell' \equiv c\hbar$, as before. The form in Eq. (55) represents another equivalent form of the fine structure constant in primary quantization. The charge, $e = 4.80 \times 10^{-10}$ esu is considered to be a fundamental universal constant.

The quantal mass can be identified with the matter-energy content of the space-time manifold.[7] The expression e^2/Gm^2 can be identified macroscopic (gravitational and electromagnetic) phenomena and $\alpha = e^2/\hbar c$ is identified with atomic phenomena. If Eq. (54) is used to calculate the ratio of the constant force between an electron and a proton or two pions, one obtains a value of about 2.3×10^{39} which is approximately equal to inverse of the gravitational coupling constant relative to the

electromagnetic coupling constant, $\alpha \cong 1/137$. The charge e, on a proton or on an electron is an atomic feature and G, the universal gravitational constant is a macroscopic feature.

V. CONCLUSION

By expanding Wheeler's wormhole description of the topology of the space-time manifold, we have developed a set of physical variables termed quantal units. This set of variables represents a quantization procedure which is termed primary quantization as opposed to the standard or ordinary quantization procedure which is termed secondary quantization.

As in Wheeler's interpretation, we propose that the quantal units represent a geometric description of the space-time manifold which is more complete than that proposed by Wheeler for the quantal unit of length termed the Wheeler wormhole.

The quantal units are expressed in terms of the universal constant and as suggested recently by Taylor, Parker, and Langenberg, the "universal constants are an important link in the chain of physical theory which binds all the diverse branches of physics together." Four new Heisenberg like relations were introduced as canonically conjugate pairs of variables with commutation relations which are expressed in terms of universal constants, other than $\hbar$, combinations of universal constants and quantal units. It is suggested that these non-commuting elements may represent a scale of observality which may unify various branches of physics. Both primary and secondary forms of the Generalized Heisenberg Relations were discussed and primary and secondary forms of nuclear quantities were also presented.

ACKNOWLEDGMENTS

The author is grateful to Professor Paul Lieber for his extensive discussions of this and other work and for the privilege of working with him. The author would also like to express appreciation to Professor John O. Rasmussen for assistance on this and earlier work.

FOOTNOTE AND REFERENCES

*Work performed under the auspices of the U.S. Atomic Energy Commission.

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Table I. Universal Quantal Units.

		Quantal unit in terms of force, $\mathscr{L}$ and $\mathscr{L}'$ [a]	Numerical value of quantal unit [b]
$\ell = \left(\frac{G\hbar}{c^3}\right)^{1/2}$	length	$\ell = \left(\frac{\mathscr{L}'}{F}\right)^{1/2}$	1.60×10^{-33} cm
$t = \left(\frac{G\hbar}{c^5}\right)^{1/2}$	time	$t = \left(\frac{\mathscr{L}}{F}\right)^{1/2}$	5.36×10^{-44} sec
$m = \left(\frac{c\hbar}{G}\right)^{1/2}$	mass	$m = \left(\frac{\mathscr{L}F}{c^2}\right)^{1/2}$	2.82×10^{-5} gm
$E = \left(\frac{c^5\hbar}{G}\right)^{1/2}$	energy	$E = (\mathscr{L}'F)^{1/2}$	1.25×10^{16} ergs
$p = \left(\frac{c^3\hbar}{G}\right)^{1/2}$	momentum	$p = (\mathscr{L}F)^{1/2}$	4.16×10^{10} $\frac{\text{gm-cm}}{\text{sec}}$
$L = \hbar$	angular momentum	$L = \hbar$	1.06×10^{-27} erg-sec
$F = c^4/G$	force	$F = F$	1.22×10^{49} dynes
$c = c$	velocity	$c = c$	3.00×10^{10} cm/sec
$a = \left(\frac{c^7}{G\hbar}\right)^{1/2}$	acceleration	$a = \left(\frac{c^2F}{\mathscr{L}}\right)$	5.72×10^{53} cm/sec^2
$p = \frac{c^5}{G}$	power	$p = cF$	3.66×10^{59} dyne $\frac{\text{cm}}{\text{sec}}$
$P = \frac{c^7}{G^2\hbar}$	pressure	$P = \frac{F^2}{\mathscr{L}'}$	4.75×10^{114} dyne/cm^2
$\rho = \frac{c^5}{G^2\hbar}$	density	$\rho = \frac{F^2}{c^2\mathscr{L}}$	6.50×10^{93} gm/cm
$\omega = \left(\frac{c^5}{c\hbar}\right)^{1/2}$	frequency	$\omega = \left(\frac{F}{\mathscr{L}}\right)^{1/2}$	1.91×10^{43} cycles/sec

[a] The quantal units are expressed in terms of the universal quantal force, $F = c^4/G$, $\mathscr{L}$, $\mathscr{L}'$, and c. The quantities, $\mathscr{L}$ and $\mathscr{L}'$ are defined as $\mathscr{L} = \hbar/c$ and $\mathscr{L}' = c\hbar$.

[b] In the evaluation of the quantal units, the values of $\mathscr{L} = 3.50 \times 10^{-38}$ gm-cm and $\mathscr{L}' = 3.15 \times 10^{-17}$ erg-cm have been used.

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