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I. Introduction

Graphical methods for the approximate solution of the equations of motion of a charged particle in a magnetic field are useful in many applications where an analytic solution presents formidable difficulties. This paper presents several methods for obtaining plane trajectories thru a given magnetic field by means of finite step procedures. These methods represent a compromise between desired accuracy and simplicity of application in practice. They will be described in Section II. An evaluation of the stepwise and cumulative errors incurred in their use will be presented in Sections III and IV.

The equation of motion for a charged particle moving in a magnetic field is

$$\frac{d}{dt}\, m\dot{\vec{r}} = \frac{e}{c}\left[\dot{\vec{r}} \times H(\vec{r})\right] , \qquad m = \frac{m_o}{\sqrt{1 - v^2/c^2}} \tag{1}$$

where m_o is the rest mass and e the charge of the particle. $\vec{r}$ is the position vector, $\dot{\vec{r}}$ the velocity of the charge, $v = |\dot{\vec{r}}|$, and $H(\vec{r})$ is the magnetic field strength at the particle's position.

If the motion is restricted to a plane, the vector equation (1) may be expressed in rectangular coordinates

$$m\ddot{x} = e/c \; \dot{y} \, H_z \, (x, y) \tag{2a}$$

$$m\ddot{y} = - e/c \; \dot{x} \, H_z \, (x, y) \tag{2b}$$

or more conveniently in intrinsic coordinates

$$\frac{1}{\rho(\vec{r})} = \frac{e}{mvc} \; H(\vec{r}) \tag{3a}$$

$$v = |\dot{\vec{r}}| = \text{const.} \tag{3b}$$

where $\rho(\vec{r})$ is the magnitude of the radius of curvature of the trajectory at $\vec{r}$ and $H(\vec{r})$ is the component of the magnetic field strength at $\vec{r}$ normal to the plane of motion. Since the acceleration is always perpendicular to the direction of motion, no work is done on the particle, and hence no kinetic energy is imparted to it. Therefore mv is a constant of the motion and $\frac{e}{mvc}$ is simply a proportionality factor relating the magnetic field strength to the curvature of the path.

Fixing a system of polar coordinates in the plane, the path of a particle is given analytically by the solution

$$r = f\left[\theta, r_o, \left(\frac{dr}{d\theta}\right)_{\theta_o}\right]$$ of the differential equation for the radius of curvature, ρ, of a plane curve.

$$\frac{d^2 r}{d\theta^2} = r + \frac{2}{r}\left(\frac{dr}{d\theta}\right)^2 - \frac{1}{\rho(r, \theta) r}\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}^{3/2} \tag{4}$$

where $\rho(r, \theta)$ is related to the magnetic field by equation (3a).

Provided $\rho(r, \theta)$ is not too small compared to the desired path length, nor too rapidly varying, one may hope to obtain a satisfactory approximate path

using finite arcs of circles and appropriate local values of curvature, together with stepwise corrections for position and direction of the endpoints of these arcs.

In the methods to be described, we shall assume that a magnetic field plot is available, having lines of constant field strength which are separated by distances small compared to the radii of curvature involved in the problem. For our purpose, we shall think of the lines as calibrated directly in units of length, i.e., corresponding to the radius of curvature $\frac{mvc}{eH}$.

II. Plotting Methods

A. Direct Method

Since this method involves no corrections for either position or direction, it is the simplest to apply. However, it will be shown that the error accumulated in its use makes it too inexact for most applications.

(a) Description of Procedure (see Fig. 1).

The initial conditions for plotting a trajectory are an initial position P_1 and velocity for the charged particle. The latter includes an initial direction of motion, represented by a unit vector $\vec{n}_1$, and the magnitude of the particle's momentum, $m\,|\vec{v}| = mv$, which as previously noted is a constant of the motion. A knowledge of mv then allows the field to be directly calibrated in terms of the radius of curvature $\rho(\vec{r})$ at each point.

The plotting procedure is as follows:

(1) At P_1 , the normal to $\vec{n}_1$ is constructed.

(2) Along this normal a distance $\rho_1 \equiv \rho(P_1)$ is laid off from P_1, to locate a point C_1.

(3) With C_1 as center and ρ_1 as radius, an arc is swung thru an angle ξ_1 , intersecting the next field line at the point P_2. The tangent to this arc at P_2 gives the new direction vector, $\vec{n}_2$.

The procedure is then repeated with the new quantities P_2, $\vec{n}_2$, ρ_2, C_2, and ξ_2.

B. Method with Compensation for Directional Error.

The initial position, P_1, and direction $\vec{n}_1$, together with mv must be specified as in Method A.

(a) Description of Prodecure (see Fig. 2).

(1) At P_1 the normal to $\vec{n}_1$ is constructed.

(2) Along this normal a distance $\rho_1 \equiv \rho(P_1)$ is laid off from P_1 to locate a point C_1.

(3) With C_1 as center and ρ_1 as radius, an arc is swung thru an angle ξ_1, intersecting the next field line at the point P_2.

(4) Angle ξ_1 is bisected and the bisector is extended thru C_1.

(5) With P_2 as center and $\rho_2 \equiv \rho(P_2)$ as radius an arc is swung, cutting this bisector at C_2.

(6) The corrected direction $\vec{n}_2$ is perpendicular to $\overline{P_2C_2}$ at P_2, and the procedure continues with steps (1), and (2) already accomplished.

C. Method with Compensation for Directional and Positional Errors.

The preliminary specification of P_1, $\vec{n}_1$, and mv is the same as in the preceding methods.

(a) Description of Procedure (see Fig. 3).

(1) At P_1 the normal to $\vec{n}_1$ is constructed.

(2) Along this normal a distance $\rho_1 \equiv \rho(P_1)$ is laid off from P_1 to locate a point C_1.

II. C. (3) With C_1 as center and ρ_1 as radius, an arc is swung thru an angle ξ_1, intersecting the next field line at the point S_1.

(4) Angle ξ_1 is bisected and the bisector is extended thru C_1. Let T_1 designate the point of intersection of the bisector and arc $\overset{\frown}{P_1 S_1}$.

(5) Along this bisector a distance $\rho_2 \equiv \rho(S_1)$ is laid off from T_1, to locate a point C_2.

(6) With C_2 as center and $\rho(S_1)$ as radius an arc is swung from T_1, intersecting the next field line at P_2, which will be near S_1. (Note $\rho(S_1) \equiv \rho(P_2) \equiv \rho_2$). This same arc is then continued to the next field line to reaccomplish step (3) and the procedure is repeated, beginning with step (4).

III. Evaluation of the Stepwise Error.

It will now be shown that the methods considered above are particular cases of a general method involving two parameters, α and β. The stepwise error will then be evaluated for this method.

With initial point P_1 and center of curvature C_1 given, ξ_1 is defined as the angle subtended at C_1 by the arc of radius ρ_1 from P_1 to the next field line. The angles $P_1 C_1 T_1 \equiv \alpha \xi_1$ and $P_1 C_1 V_1 \equiv \beta \xi_1$ are then determined as in Figure 4, where α and β are two chosen numbers. The new center of curvature C_2 is constructed to lie on line $T_1 C_1$, a distance ρ_2 from the Point V_1. By proper choice of α and β, this method is seen to be identical with Methods A, B, or C, as shown in Table 1.

To determine the error incurred by the nth application of the above procedure, the approximate paths will be compared at points P_n, P_{n+1} with the exact path at the corresponding points Q_n, Q_{n+1}. The latter are defined to be the intersections of the exact path with the lines $\overline{C_n P_n}$ and $\overline{C_{n+1} P_{n+1}}$, respectively.

For this purpose the polar coordinates are chosen with origin at C_n and the $\theta = 0$ axis along $\overline{C_n P_n}$. (cf. Fig. 5).

The exact path is given by a solution to equation (4), which is denoted by

$$R = R(\theta) \quad . \tag{5}$$

This particular solution is determined by two initial conditions which may be expressed as

$$R(\theta = 0) = R_n \qquad \text{where } R_n = \overline{C_n Q_n} \tag{6}$$

$$\left(\frac{1}{R}\frac{dR}{d\theta}\right)_{\theta=0} = \tan \psi_n \tag{7}$$

Similarly the approximate path is denoted by

$$r = r(\theta) \tag{8}$$

subject to the initial conditions

$$r(0) = \rho_n \qquad \text{where } \rho_n = \overline{C_n P_n} \tag{9}$$

$$\left(\frac{dr}{d\theta}\right)_{\theta=0} = 0 \qquad \text{, by choice of coordinates.} \tag{10}$$

A quantity $\Delta_{n+1} = \overline{P_{n+1} Q_{n+1}}$ is now defined, which is a measure of the error incurred after n steps. Likewise Ψ_{n+1} is a measure of the error

in direction after n steps.

The problem now is to evaluate Δ_{n+1} and Ψ_{n+1} in terms of Δ_n, Ψ_n and ξ_n, the angle characterizing the n'th step. This will determine the positional and directional errors incurred by the n'th step.

To accomplish this purpose, the coordinates of the endpoint P_{n+1} of the approximate path are first obtained.

Referring to Fig. 4 the angle $P_n C_n P_{n+1} \equiv \zeta_n \equiv h_n \xi_n$, and the distance $\overline{C_n C_{n+1}}$ is denoted by x. (For convenience subscripts 1 and 2 are used in place of n, n+1 and omitted entirely where they are unneccessary for purposes of distinction.)

Applying the law of cosines to triangles $\left\{C_2\ C_1\ V_1\right\}$ and $\left\{C_2\ C_1\ P_2\right\}$, the following equations result

$$\rho_2^2 = \rho_1^2 + x^2 + 2\rho_1 x \cos(\alpha - \beta)\xi \tag{11}$$

$$\rho_2^2 = r^2 + x^2 + 2rx \cos(h - \alpha)\xi \tag{12}$$

Since S_1 and P_2 lie on a line of constant ρ, the following relation is satisfied also

$$d\rho = \rho_r\, dr + \rho_\theta\, d\theta \simeq \rho_r(\rho_1 - r) + \rho_\theta(1 - h)\xi = 0 \tag{13}$$

where $\rho_r \equiv \frac{\partial \rho}{\partial r}$, $\rho_\theta \equiv \frac{\partial \rho}{\partial \theta}$.

Equations (11), (12), and (13) are now solved simultaneously for x, h, and r.

Setting $\rho_2 - \rho_1 = \lambda$ in (11) ,

$$2\rho_1 x \cos(\alpha - \beta)\xi = \lambda(2\rho_1 + \lambda) - x^2 \tag{14}$$

whence, to terms of order ξ^3, the solution is

$$x = \lambda + O(\xi^3) \tag{15}$$

Setting $r = \rho_1 + w$ in equation (12), substituting for ρ_2 from equation (11), and rearranging

$$2w\rho_1 + w^2 + 2(\rho_1 + w)\, x \cos(h - \alpha)\xi - 2\rho_1 x \cos(\alpha - \beta)\xi = 0 . \tag{16}$$

Assuming $w \sim O(\xi^3)$, and neglecting terms $\sim O(\xi^5)$

$$w = \tfrac{1}{2}\lambda\left(1 - \frac{\lambda}{\rho_1}\right)(h - \beta)(h + \beta - 2\alpha)\xi^2 + O(\xi^5) \tag{17}$$

so the assumption that $w \sim O(\xi^3)$ is justified, since

$$\begin{aligned} \lambda = \rho_2 - \rho_1 &= \rho(S_1) - \rho(P_1) \\ &= \rho_1 + \rho_{1\theta}\xi + \tfrac{1}{2}\rho_{1\theta\theta}\xi^2 - \rho_1 + O(\xi^3) \\ &= \rho_{1\theta}\xi + \tfrac{1}{2}\rho_{1\theta\theta}\xi^2 + O(\xi^3) \end{aligned} \tag{18}$$

$$w = \tfrac{1}{2}(h - \beta)(h + \beta - 2\alpha)\left\{\rho_{1\theta}\xi^3 + \left(\tfrac{1}{2}\rho_{1\theta\theta} - \frac{\rho_{1\theta}^2}{\rho_1}\right)\xi^4\right\} + O(\xi^5) . \tag{19}$$

Since $w \equiv r - \rho_1 = O(\xi^3)$, it follows from Eq. (13) that

$$h = 1 + O(\xi^2) \tag{20}$$

and the terms of $O(\xi^2)$ contribute to w only in terms $\gtrsim O(\xi^5)$. So finally, the values of r and θ at P_2 are

$$r = \rho_1 + \tfrac{1}{2}(1-\beta)(1+\beta-2\alpha)\left[\rho_{1\theta}\,\xi^3 + \left(\frac{\rho_{1\theta\theta}}{2} - \frac{\rho_{1\theta}^{\,2}}{\rho_1}\right)\xi^4\right] + O(\xi^5) \tag{21}$$

$$\theta = \zeta_1 = \xi_1 + O(\xi^3) \quad . \tag{22}$$

ϕ_1 may be evaluated easily (see figure 4) using the law of sines applied to triangle $C_2\,C_1\,P_2$

$$\frac{\sin(h-\alpha)\xi}{\rho_2} = \frac{\sin\phi}{x} \tag{23}$$

or

$$\sin\phi = \frac{\lambda}{(\rho_1+\lambda)}\sin(1-\alpha)\xi + O(\xi^4)$$

so

$$\phi = \frac{\lambda}{\rho_1}\left(1-\frac{\lambda}{\rho_1}\right)(1-\alpha)\xi + O(\xi^4)$$

$$\phi = \frac{1}{\rho_1}(1-\alpha)\left[\rho_{1\theta}\,\xi^2 + \left(\frac{\rho_{1\theta\theta}}{2} - \frac{\rho_{1\theta}^{\,2}}{\rho_1}\right)\xi^3\right] + O(\xi^4) \quad . \tag{24}$$

Having evaluated r, θ, and ϕ, the quantities related to the approximate path, the next step is to obtain the coordinates of Q_2, the endpoint of the exact path.

So now an expression for $R(\theta)$, the radius vector of the exact path is obtained in terms of quantities evaluated at $\theta = 0$. To do this a Taylor's expansion in powers of θ is made.

$$R(\theta) = \sum_{n=0}^{\infty} R^{(n)}(0) \frac{\theta^n}{n!} \tag{25}$$

where $R^{(n)}(0) \equiv \left. \frac{d^n R(\theta)}{d\theta^n} \right|_{\theta=0}$.

To satisfy the initial conditions (6), (7)

$$R^{(0)}(0) = \rho_1 + \Delta_1 \quad , \quad R^{(1)}(0) = (\rho_1 + \Delta_1) \tan \psi_1 \equiv \varepsilon_1 \quad . \tag{26}$$

The remaining coefficients $R^{(n)}(0)$ in (25) can be evaluated in terms of the magnetic field and its derivatives measured at P_1 by use of Eq. (4)

$$R^{(2)} = R + 2\frac{R^{(1)^2}}{R} - \frac{1}{\rho R}\left(R^2 + R^{(1)^2}\right)^{3/2}$$

whence, to first order in $(\Delta + \varepsilon)$

$$R^{(2)}(0) = R(0) - \frac{R(0)^2}{\rho(0)} = \rho_1 + \Delta_1 - \frac{(\rho_1 + \Delta_1)^2}{\rho_1 + \Delta_1 \rho_{1r}}$$

$$= -\Delta_1(1 - \rho_{1r}) + o(\Delta + \varepsilon)^2 \quad . \tag{27}$$

Similarly, by differentiating (4), expressions for $R^{(3)}(0)$, and $R^{(4)}(0)$ are obtained

$$R^{(3)}(0) = \rho_{1\theta} + o(\Delta + \varepsilon) \tag{28}$$

$$R^{(4)}(0) = \rho_{1\theta\theta} - 2\frac{\rho_{1\theta}^2}{\rho_1} + o(\Delta + \varepsilon) \quad . \tag{29}$$

In the above expressions the following definitions have been used.

$$\rho_{1r} \equiv \left.\frac{\partial \rho}{\partial r}\right)_{P_1} \qquad \rho_{1\theta} \equiv \left.\frac{\partial \rho}{\partial \theta}\right)_{P_1} \tag{30}$$

and

$$\rho_{1\theta\theta} \equiv \left.\frac{\partial^2 \rho}{\partial \theta^2}\right)_{P_1} .$$

Substituting (26) - (29) into series (25), there results

$$R(\xi_1) = \rho_1 + \Delta_1 + \varepsilon_1 \xi_1 - \tfrac{1}{2}\Delta_1(1 - \rho_{1r})\xi_1^2 + \tfrac{1}{6}\rho_{1\theta}\xi_1^3 + \tfrac{1}{24}\left(\rho_{1\theta\theta} - \frac{2\rho_{1\theta}^2}{\rho_1}\right)\xi_1^4 + \delta R(\xi_1) . \tag{31}$$

This remainder term is of the order

$$\delta R(\xi_1) \sim O\left[(\Delta + \varepsilon)\xi^3 + \xi^5 + (\Delta^2 + \varepsilon^2)\xi^2\right] . \tag{32}$$

Differentiating (31) to obtain $\frac{dR}{d\theta}$

$$\left.\frac{dR}{d\theta}\right)_{\theta = \xi_1} = \varepsilon_1 - \Delta_1(1 - \rho_{1r})\xi_1 + \tfrac{1}{2}\rho_{1\theta}\xi_1^2 + \tfrac{1}{6}\left(\rho_{1\theta\theta} - \frac{2}{\rho_1}\rho_{1\theta}^2\right)\xi_1^3 + \delta R'(\xi_1) \tag{33}$$

where

$$\delta R' \sim O\left[(\Delta + \varepsilon)\xi^2 + \xi^4 + (\Delta^2 + \varepsilon^2)\xi\right] . \tag{34}$$

Now actually the polar angle $Q_1 C_1 Q_2$ of the endpoint Q_2 is not ξ_1 but some almost equal angle η_1. From figure 5, it is clear that

$$\frac{\sin(\theta - \eta)}{\Delta_1} = \frac{\sin(\theta - \eta + \phi)}{r}$$

which may be written

$$\theta - \eta = \frac{\Delta_1}{r} \sin \phi + O(\xi^4) \tag{35}$$

since $\phi \sim O(\xi^2)$ from Eq. (24).

Making use of Eq. (22)

$$\eta = \xi_1 + O(\xi^3) \quad . \tag{36}$$

Furthermore, since $\theta - \eta = O(\xi^3)$, $\cos(\theta - \eta) = 1 + O(\xi^6)$

and

$$\Delta_2 = (R_2^2 + r_2^2 - 2 R_2 r_2 \cos(\theta - \eta))^{\frac{1}{2}}$$

$$= ((R_2 - r_2)^2 + 2 R_2 r_2 (1 - \cos(\theta - \eta))^{\frac{1}{2}}$$

$$= R_2 - r_2 + O(\xi^6) \quad .$$

Referring to Eqs. (21) and (31), the final expression for Δ_2 emerges

$$\Delta_2 = \Delta_1 + \varepsilon_1 \xi_1 - \tfrac{1}{2} \Delta_1 (1 - \rho_{1r}) \xi_1^2 \tag{37}$$

$$+ \frac{1}{6} \rho_{1\theta} \xi_1^3 \left[1 - 3(1 - \beta)(1 + \beta - 2\alpha)\right]$$

$$+ \frac{1}{24} \left(\rho_{1\theta\theta} - 2 \frac{\rho_{1\theta}^2}{\rho_1}\right) \xi_1^4 \left[1 - 6(1 - \beta)(1 + \beta - 2\alpha)\right]$$

$$+ O\left[(\Delta + \varepsilon) \xi^3 + \xi^5 + (\Delta^2 + \varepsilon^2) \xi^2\right].$$

To obtain the expression for the change in the directional error incurred by this step, it may be noted from the construction (cf. Fig. 5) that

$$\psi_2 = \tan^{-1}\left(\frac{1}{R(\eta_1)} R'(\eta_1)\right) - (\phi_1 - \theta_1 + \eta_1) \tag{38}$$

or using definition (26) for the next boundary condition.

$$\varepsilon_2 = (\rho_2 + \Delta_2)\tan \psi_2 \tag{39}$$

which must be evaluated neglecting terms $\sim 0\left[\xi^4 + (\Delta + \varepsilon)\xi^2 + \Delta^2\right]$.

Since $\theta - \eta \sim 0(\Delta \xi^2)$ from (35), (24), it can be neglected. Further since $\psi_2 \sim 0(\xi^2 + \varepsilon + \Delta\xi)$, Δ_2 may be neglected and $\tan \psi_2 \cong \psi_2$

$$\varepsilon_2 \cong \rho_2 \psi_2 \cong (\rho_1 + \rho_{1\theta}\xi_1)\psi_2$$

and from (31), (33) and (24), this may be written

$$\varepsilon_2 \cong (\rho_1 + \rho_{1\theta}\xi_1)\frac{1}{\rho_1}\Big\{\Big[\varepsilon_1 - \Delta_1(1 - \rho_{1r})\xi_1 + \tfrac{1}{2}\rho_{1\theta}\xi_1^2 + \tfrac{1}{6}\left(\rho_{1\theta\theta} - \frac{2}{\rho_1}\rho_{1\theta}^2\right)\xi_1^3\Big] - (1 - \alpha) \times \Big[\rho_{1\theta}\xi_1^2 + \left(\frac{\rho_{1\theta\theta}}{2} - \frac{\rho_{1\theta}^2}{\rho_1}\right)\xi_1^3\Big]\Big\}$$

or

$$\varepsilon_2 = \varepsilon_1 - \Delta_1 (1 - \rho_{1r})\xi_1 + \varepsilon_1 \frac{\rho_{1\theta}}{\rho_1}\xi_1 + \tfrac{1}{2}\rho_{1\theta}\xi_1^2\left[1 - 2(1-\alpha)\right]$$

$$+ \tfrac{1}{6}\left(\rho_{1\theta\theta} - \frac{2}{\rho_1}\rho_{1\theta}^2\right)\xi_1^3\left[1 - 3(1-\alpha)\right] + \tfrac{1}{2}\rho_{1r}'\rho_{1\theta}\xi_1^3\left[1 - 2(1-\alpha)\right]$$

$$+ o\left[(\Delta + \varepsilon)\xi^2 + \xi^4 + (\Delta^2 + \varepsilon^2)\xi\right]. \tag{40}$$

It is clear that Eqs. (37) and (40) represent general expressions for Δ_{n+1} and ε_{n+1} in terms of Δ_n, ε_n and field quantities measured at P_n. From these expressions one may infer the optimum values of the parameters α and β and get a measure of the directional and positional errors incurred in a single application of the procedure. Further, by solving these simultaneous difference equations, expressions for the errors accumulated after n steps may be obtained.

To first evaluate the stepwise errors, it will be assumed that Δ_1 and ε_1 are both zero, i.e., that we are starting with a precise knowledge of the initial position and direction of motion of the charged particle.

Then the equations (37) and (40) reduce to

$$\Delta_2 = \tfrac{1}{6}\rho_{1\theta}\xi_1^3\left[1 - 3(1-\beta)(1+\beta-2\alpha)\right] + o(\xi^4) \tag{41}$$

$$\varepsilon_2 = \tfrac{1}{2}\rho_{1\theta}\xi_1^2\left[1 - 2(1-\alpha)\right] + o(\xi^3) \quad . \tag{42}$$

It is seen that the positional error Δ starts building up as a third order effect whereas the directional error ε starts in as a second order effect. Furthermore from (37) it is seen that even if the coefficient

of ξ^3 in (41) were adjusted to be zero, so that Δ starts in as a fourth order effect, the presence of the $\varepsilon_1 \xi_1$ term will almost immediately predominate as a third order effect, which will grow more rapidly than that due to the ξ^3 term. Therefore, it is most important to adjust α to remove the second order term in (42) .

$$1 - 2(1 - \alpha) = 0 \qquad \text{giving} \qquad \alpha = 1/2 \ . \tag{43}$$

With this choice the coefficient of ξ^3 in (41) is then minimized.

$$1 - 3\beta(1 - \beta) = \text{minimum at} \quad \beta = 1/2 \ .$$

Unfortunately this minimum is not zero, since the roots of the equation

$$1 - 3\beta(1 - \beta) = 0 \tag{44}$$

are both complex.

It is also seen that $\beta(1 - \beta)$ is symmetric about $\beta = 1/2$, so that methods in which $\beta = 1/2 \pm \alpha$ will give the same amount of cancellation and hence are essentially equivalent.

The findings of this last investigation into the initially incurred stepwise error are given in Table 2 for the three methods previously described.

IV. Evaluation of the Cumulative Error

The purpose of this section is to investigate the systematic error accumulated after n-steps. The results obtained will give some information as to the applicability of the methods to a given problem.

Those interested primarily in the general limits of precision of the plotting methods will find the results summarized in Tables 3 to 4 and in the final paragraphs of this section.

The evaluation of the cumulative error involves the solution of the set of simultaneous equations (37) and (40), which may be written in the form

$$\left.\begin{aligned} \Delta_{n+1} &= (1 - a_n)\Delta_n + b_n \varepsilon_n + f_n \\ \varepsilon_{n+1} &= (1 + c_n)\varepsilon_n - d_n \Delta_n + g_n \end{aligned}\right\} \tag{45}$$

where

$$a_n = \tfrac{1}{2}(1 - \rho_{nr})\xi_n^2 + O(\xi^3) \tag{46}$$

$$b_n = \xi_n + O(\xi^3) \tag{47}$$

$$c_n = \frac{\rho_{n\theta}}{\rho_n}\xi_n + O(\xi^2) \tag{48}$$

$$d_n = (1 - \rho_{nr})\xi_n + O(\xi^2) \tag{49}$$

$$\begin{aligned} f_n = {} & \frac{1}{6}\left[1 - 3(1 - \beta)(1 + \beta - 2\alpha)\right]\rho_{n\theta}\xi_n^3 \\ & + \frac{1}{24}\left[1 - 6(1 - \beta)(1 + \beta - 2\alpha)\right]\left(\rho_{n\theta\theta} - \frac{2}{\rho_n}\rho_{n\theta}^2\right)\xi_n^4 + O(\xi^5) \end{aligned} \tag{50}$$

$$\begin{aligned} g_n = {} & \tfrac{1}{2}\left[1 - 2(1 - \alpha)\right]\rho_{n\theta}\xi_n^2 + \left\{\frac{1}{6}\left[1 - 3(1 - \alpha)\right]\left(\rho_{n\theta\theta} - \frac{2}{\rho_n}\rho_{n\theta}^2\right)\right. \\ & \left. + \tfrac{1}{2}\left[1 - 2(1 - \alpha)\right]\rho_{nr}\rho_{n\theta}\right\}\xi_n^3 + O(\xi^4) . \end{aligned} \tag{51}$$

To solve (45), $\mathcal{E}_i$ is first eliminated giving

$$b_n \Delta_{n+2} = \left[b_n(1 - a_{n+1}) + b_{n+1}(1 + c_n)\right]\Delta_{n+1} - b_{n+1}\left[(1 + c_n)(1 - a_n) + b_n d_n\right]\Delta_n + \left[b_n f_{n+1} - b_{n+1}(1 + c_n) f_n + b_n b_{n+1} g_n\right]$$

$$b_n = \xi + O(\xi^3) ; \quad a_n = O(\xi^2) ; \quad c = O(\xi) ; \quad d = O(\xi) ; \tag{52}$$

$$f = O(\xi^3) ; \quad g = O(\xi^2) .$$

choosing all ξ_n's equal to a fixed ξ, and combining unknown terms,

$$(\xi + \xi\alpha_n)\Delta_{n+2} = \xi\left[2 + c_n + \beta_n\right]\Delta_{n+1} - \xi\left[1 + c_n + \gamma_n\right]\Delta_n + \xi\left[f_{n+1} - (1 + c_n) f_n + \xi g_n + \delta_n\right] \tag{53}$$

where $\alpha_n = O(\xi^2)$, $\beta_n = O(\xi^2)$, $\gamma_n = O(\xi^2)$, $\delta_n = O(\xi^5)$ are unevaluated terms of order indicated. Setting $h_n \equiv \xi g_n$ and rearranging

$$\Delta_{n+2} = (2 + c_n + \beta'_n)\Delta_{n+1} - (1 + c_n + \gamma'_n)\Delta_n + \left[f_{n+1} - (1 + c_n) f_n + h_n + \delta'_n\right] \tag{54}$$

$$\beta'_n = O(\xi^2) ; \quad \gamma'_n = O(\xi^2) ; \quad \delta'_n = O(\xi^5)$$

which may be expressed as

$$\Delta_n = \Delta_{n-1} + (1 + c'_{n-2})\left[\Delta_{n-1} - \Delta_{n-2} - f_{n-2}\right] + f_{n-1} + \ell_{n-2} + \lambda_{n-2}\Delta_{n-2} \tag{55}$$

where $c_n' \equiv c_n + \beta_n'$; $\ell_n \equiv h_n + \delta_n' + \beta_n' f_n$; $\lambda_n \equiv \beta_n' - \gamma_n' = O(\xi^2)$.

In equation (55) λ_n is an unknown quantity of order ξ^2 and will hereafter be assumed to be a constant parameter λ .

The equation (55) will first be solved neglecting λ and then the $\lambda\Delta$ term will be treated as a perturbation, with λ as the expansion parameter.

$$\Delta_n = \Delta_n^{(0)} + \lambda\Delta_n^{(1)} + \lambda^2\Delta_n^{(2)} + \dots + \lambda^k\Delta_n^{(k)} \quad . \tag{56}$$

Upon substituting this expansion in (55) and equating equal powers of λ

$$\Delta_n^{(0)} = \Delta_{n-1}^{(0)} + (1 + c_{n-2}')\left[\Delta_{n-1}^{(0)} - \Delta_{n-2}^{(0)} - f_{n-2}\right] + f_{n-1} + \ell_{n-2} \tag{57$^{(0)}$}$$

$$\Delta_n^{(1)} = \Delta_{n-1}^{(1)} + (1 + c_{n-2}')\left[\Delta_{n-1}^{(1)} - \Delta_{n-2}^{(1)}\right] + \Delta_{n-2}^{(0)} \tag{57$^{(1)}$}$$

$$\Delta_n^{(k)} = \Delta_{n-1}^{(k)} \quad (1 + c_{n-2}')\left[\Delta_{n-1}^{(k)} - \Delta_{n-2}^{(k)}\right] + \Delta_{n-2}^{(k-1)} \quad . \tag{57$^{(k)}$}$$

The equation (57$^{(0)}$) must be solved first. Separating the f-dependence explicitly,

$$\Delta_n^{(0)} = \sum_{i=1}^{n-1} f_i + \lambda_n^{(0)} \tag{58}$$

$$\lambda_n^{(0)} = \lambda_{n-1}^{(0)} + (1 + c'_{n-2})\left[\lambda_{n-1}^{(0)} - \lambda_{n-2}^{(0)}\right] + \ell_{n-2} \quad . \tag{59}$$

Substituting

$$\lambda_n^{(0)} = \sum_{i=1}^{n-2} A_{ni}^{(0)} \ell_i \quad \text{and} \quad B_{ni}^{(0)} = A_{ni}^{(0)} - A_{n-1i}^{(0)}$$

and equating coefficients of the ℓ_k

$$B_{ni}^{(0)} = (1 + c'_{n-2}) B_{n-1,i}^{(0)} + \delta_{n-2,i} \tag{60}$$

Calling $S_{n-2} \equiv 1 + c'_{n-2}$, the solution of (60) is

$$B_{ni}^{(0)} = \begin{cases} 0 & i > n-2 \\ 1 & i = n-2 \\ \prod_{i+1}^{n-2} S_k & i < n-2 \end{cases} \tag{61}$$

The values of $B_{ni}^{(0)}$ for $i \geq n - 2$ follow from the boundary conditions $\Delta_o^{(0)} = 0$ and $\Delta_1^{(0)} = 0$.

$$\text{Since} \quad A_{ni}^{(0)} = A_{n-1,i}^{(0)} + B_{ni}^{(0)} \tag{62}$$

$$A_{ni}^{(0)} = \sum_{k=i+2}^{n} B_{ki}^{(0)} \tag{63}$$

whence

$$\Delta_n^{(0)} = \sum_{i=1}^{n-1} f_i + \sum_{i=1}^{n-2} \ell_i \left[1 + \sum_{k=i+3}^{n} \prod_{j=i+1}^{k-2} S_j\right] \quad . \tag{64}$$

or

$$\Delta_n^{(0)} = \sum_{i=1}^{n-1} f_i + \sum_{i=1}^{n-2} \ell_i \left[1 + \sum_{k=i+1}^{n-2} \prod_{j=i+1}^{k} s_j \right] \tag{65}$$

$$= \sum_{i=1}^{n-1} f_i + \sum_{i=1}^{n-2} \ell_i \left[1 + s_{i+1} + s_{i+1} s_{i+2} + \cdots + s_{i+1} \cdots s_{n-2} \right]$$

Having $\Delta_n^{(0)}$, $\Delta_n^{(1)}$ is next obtained. First, introducing the notation

$$0_{i+1}^{n-2}(s) \equiv s_{i+1} + s_{i+1} s_{i+2} + \cdots + s_{i+1} s_{i+2} \cdots s_{n-2} \tag{66}$$

from (57') follows immediately, comparing with ($57^{(0)}$) and (65)

$$\Delta_n^{(1)} = \sum_{i=1}^{n-2} \left[1 + 0_{i+1}^{n-2}(s) \right] \Delta_i^{(0)} \quad . \tag{67}$$

So formally the solution can be expressed

$$\Delta_n = \sum_{\alpha=0}^{\infty} \lambda^{\alpha} \Delta_n^{(\alpha)} = \Delta_n^{(0)} + \sum_{\alpha=1}^{\infty} \lambda^{\alpha} \Delta_n^{(\alpha)} \tag{68}$$

where

$$\Delta_n^{(0)} = \sum_{i=1}^{n-1} f_i + \sum_{i=1}^{n-2} \ell_i \left[1 + 0_{i+1}^{n-2}(s) \right] \tag{69^{o}}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$\Delta_n^{(k+1)} = \sum_{i=1}^{n-2} \left[1 + 0_{i+1}^{n-2}(s) \right] \Delta_i^{(k)} \quad ; \quad k \geq 0 \quad . \tag{69^{k}}$$

As a special case, where a simpler expression for Δ_n may be obtained, consider f, ℓ, S as constants; then

$$\Delta_n^{(0)} = (n-1)f + \frac{\ell}{S-1}\left[\frac{S}{S-1}(S^{n-2}-1) - (n-2)\right] \tag{70}$$

$$\Delta_n^{(1)} = \sum_{i=1}^{n-2}\left(\frac{S^{n-1-i}-1}{S-1}\right)\left\{(i-1)f + \frac{\ell}{S-1}\left[\frac{S}{S-1}(S^{i-2}-1) - (i-2)\right]\right\}$$

$$= \frac{1}{S-1}S^{n-1}\left\{\sum_{i=1}^{n-2}\frac{i}{S^i}\left[f - \frac{\ell}{S-1}\right] + \sum_{i=1}^{n-2}\frac{1}{S^i}\left[-f + \frac{\ell}{S-1}\left(-\frac{S}{S-1}+2\right)\right]\right.$$

$$\left. + (n-2)\frac{\ell}{S(S-1)^2}\right\} - \frac{1}{S-1}\left\{\sum_{i=1}^{n-2} i\left[f - \frac{\ell}{S-1}\right] + (n-2)\left[-f + \frac{\ell}{S-1}\left(-\frac{S}{S-1}+2\right)\right]\right.$$

$$\left. + \sum_{i=1}^{n-2} S^i\left[\frac{\ell}{S(S-1)^2}\right]\right\}.$$

Performing the sums

$$\Delta_n^{(1)} = \frac{S^{n-1}}{S-1}\left\{\frac{1}{S^{n-2}(S-1)^2}\left[n-2-(n-1)S+S^{n-1}\right]\left[f-\frac{\ell}{S-1}\right]\right.$$

$$+\frac{1}{S^{n-1}(S-1)}\left[S^{n-1}-S\right]\left[-f+\frac{\ell(S-2)}{(S-1)^2}\right]$$

$$\left.+(n-2)\frac{\ell}{S(S-1)^2}\right\} - \frac{1}{S-1}\left\{\frac{(n-1)(n-2)}{2}\left[f-\frac{\ell}{S-1}\right]\right.$$

$$\left.+(n-2)\left[-f+\frac{\ell(S-2)}{(S-1)^2}\right]+\frac{S^{n-1}-S}{S-1}\left[\frac{\ell}{S(S-1)^2}\right]\right\}$$

and collecting terms

$$\Delta_n^{(1)} = \frac{1}{(S-1)^3}\left[S^{n-1}-\frac{(n-1)(n-2)}{2}S^2+(n-1)(n-3)S-\frac{(n-3)(n-2)}{2}\right]f$$

$$+\frac{1}{(S-1)^4}\left[(n-4)S^{n-1}-(n-1)S^{n-2}+S^2\frac{(n-1)(n-2)}{2}-(n-1)(n-4)S\right.$$

$$\left.+\frac{(n-3)(n-4)}{2}\right]\ell$$

resubstituting $S = 1 + c$ and regrouping terms

$$\Delta_n^{(1)} = \frac{1}{c^3}\left[(1+c)^{n-1}-1-(n-1)c-\frac{(n-1)(n-2)c^2}{2!}\right]f$$

$$+\frac{1}{c^4}\left[\left[(n-4)c-3\right](1+c)^{n-2}+3+2(n-1)c+\frac{(n-1)(n-2)}{2}c^2\right]\ell \quad . \tag{71}$$

In evaluating the summation above the following relations have been used.

$$\sum_{k=\ell}^{n} z^k = \frac{z^{n+1} - z^{\ell}}{z - 1} \tag{72}$$

$$\sum_{k=1}^{n} kz^k = z \frac{d}{dz} \sum_{k=1}^{n} z^k = \frac{1}{(z-1)^2} \left[nz^{n+2} - (n+1)z^{n+1} + z \right] . \tag{73}$$

Equations (70) and (71) provide an upper limit for the cumulative error if f, ℓ, and S are set equal to the upper bounds of f_n, ℓ_n, and S_n respectively for a given problem.

These limits have been evaluated for the three plotting methods A, B, and C in Tables 3 and 4. The results are tabulated for three different step sizes and total path angles. The two tables correspond to two types of magnetic field.

It should be emphasized that these results represent an upper limit on the cumulative error and do not take into account the effects of variations in f_n, c_n, and ℓ_n from step to step.

The effects of variations in the parameters may be illustrated by the following comparison.

Let $f/\rho = 4 \times 10^{-5}$; $\ell/\rho = 1 \times 10^{-5}$; $\lambda = 10^{-2}$; $\xi = .1$ which corresponds to method C with $\left| \frac{1}{H} \frac{\partial H}{\partial \theta} \right| \leq 1$ $\left| \frac{1}{H} \frac{\partial^2 H}{\partial \theta^2} \right| \leq 1$. Comparing Case I: $\{ c = .1 \}$ with Case II $\{ c = .1 \cos (n\pi/20) \}$ at $n\xi = 30$, one finds $\frac{\Delta^{I}}{\rho} \simeq .03$ and $\frac{\Delta^{II}}{\rho} \simeq .01$.

Thus it is seen that a variation in the parameters will tend to diminish the error quite appreciably.

Tables 3 and 4 show:

1. Method A is too inexact for most applications.

2. Methods B and C give approximately the same degree of accuracy, especially for values of θ corresponding to a large fraction of a full revolution in path.

In the above treatment, no account has been taken of random errors incurred in the application of the plotting procedure. But from Eq. (65), it is seen that a random error in position will give no contribution to $\Delta_n^{(0)}$, nor will a directional error on the average. So if the approximate path is found in practice to be reproducible, the random errors may be safely disregarded.

At this point, the authors wish to express their appreciation to Dr. David L. Judd for many helpful discussions throughout the course of this work. This work was performed under the auspices of the Atomic Energy Commission.

TABLE 1

Method	α	β
A	1	1
B	1/2	1
C	1/2	1/2

TABLE 2

Method	a	b
A	1/6	1/2
B	1/6	0
C	1/24	0

$$\Delta_2 = a\,\rho_{1\theta}\,\xi_1^3 + O(\xi^4)\ ;$$

$$\varepsilon_2 = b\,\rho_{1\theta}\,\xi_1^2 + O(\xi^3)\ .$$

TABLE 3

Cumulative error for a magnetic field such that $\left| \frac{1}{H} \frac{\partial H}{\partial \theta} \right| \leq 1$; $\left| \frac{1}{H} \frac{\partial^2 H}{\partial \theta^2} \right| \leq 1$, where θ is the local polar angle defined in Section III.

Method	$\xi \equiv$ step-angle (radians)	$\left[\Delta^{(0)}/\rho\right]_{\theta=n\xi}$			$\left[\lambda\Delta^{(1)}/\rho\right]_{\theta=n\xi}$			$\left[\frac{\Delta^{(0)}+\lambda^{(1)}}{\rho}\right]_{\theta=n\xi}$		
		θ=1.2	2.1	3.0	θ=1.2	2.1	3.0	θ=1.2	2.1	3.0
A										
	.3	.1	>.1	>.1				.1		
	.1	.05	.2	>.2	.004	.04		.05	.3	
	.03	.02	.07	.2	-.0004	-.003		.02	.07	.2
					(We have assumed $\lambda \equiv \xi^2$.)					
B										
	.3	.02	.06	.1	.002	.02		.02	.08	
	.1	.003	.007	.02	.0005	.004	$>.5\Delta^{0}/\rho$	.003	.01	.03
C										
	.3	.009	.03	.08	.0004	.008	.05	.01	.04	.1
	.1	.001	.004	.01	.0002	.002	.01	.001	.006	.02+

TABLE 4

Cumulative error for a magnetic field such that $\left| \frac{1}{H} \frac{\partial H}{\partial \theta} \right| \leq 3$;

$\left| \frac{1}{H} \frac{\partial^2 H}{\partial \theta^2} \right| \leq 10$, where θ is the local polar angle defined in Section III.

Method	step-angle ξ (radians)	$\left[\Delta^{(0)}/\rho\right]_{\theta=n\xi}$			$\left[\lambda\Delta^{(1)}/\rho\right]_{\theta=n\xi}$			$\left[\frac{\Delta^{(0)}+\lambda^{(1)}}{\rho}\right]_{\theta=n\xi}$		
		θ=1.2	2.1	3.0	θ=1.2	2.1	3.0	θ=1.2	2.1	3.0
A	.3	.5								
	.1	.3								
	.03	.1								
B										
	.3	.1								
	.1	.02	.2		.003			.02		
	.03	.003	.03	.5	.0005	.007		.003	.04	
C										
	.3	.08			.006			.09		
	.1	.02	.2		.0007			.02	.2	
	.03	.002	.03	.5	.0001	.003		.002	.04	

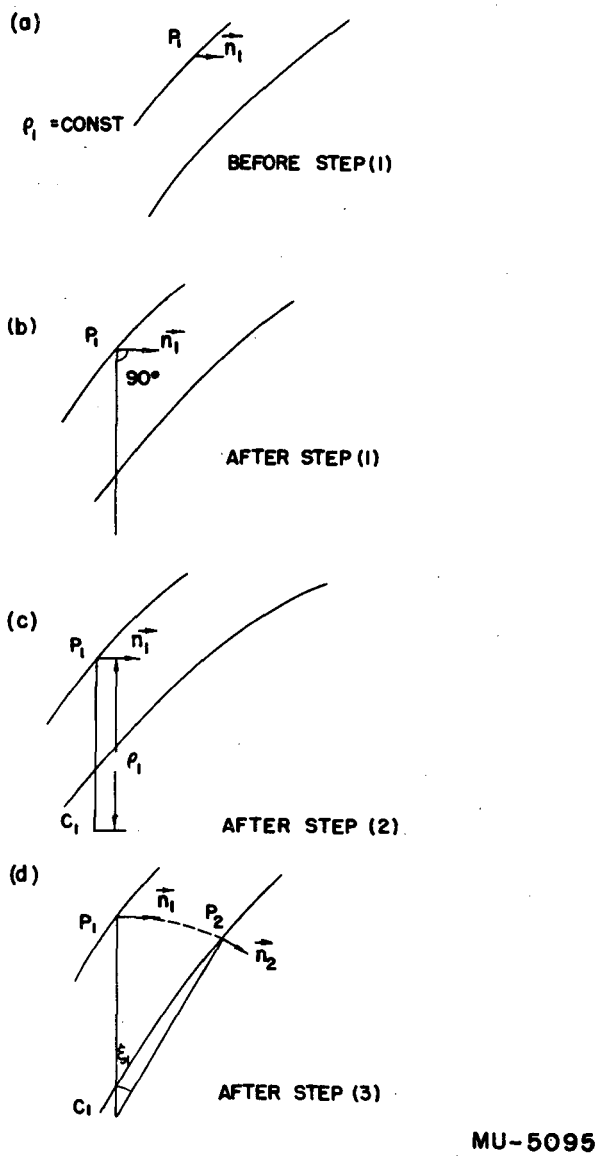
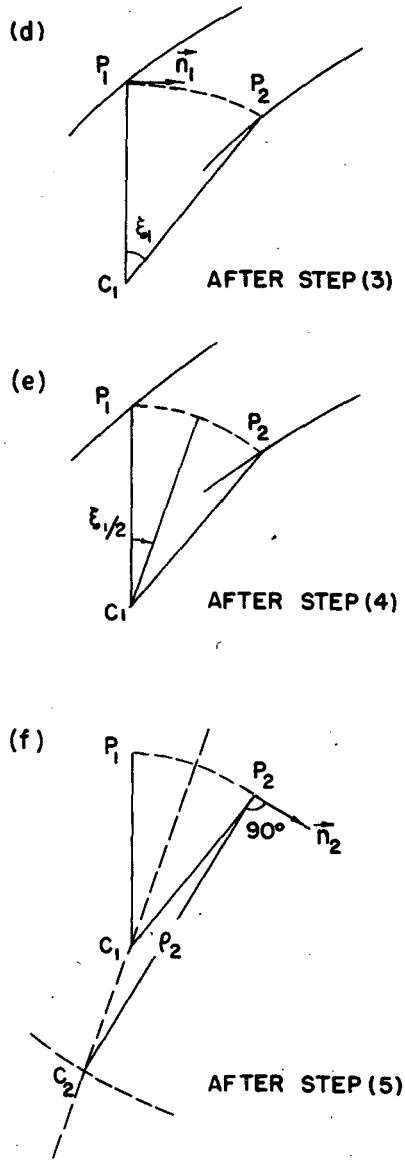
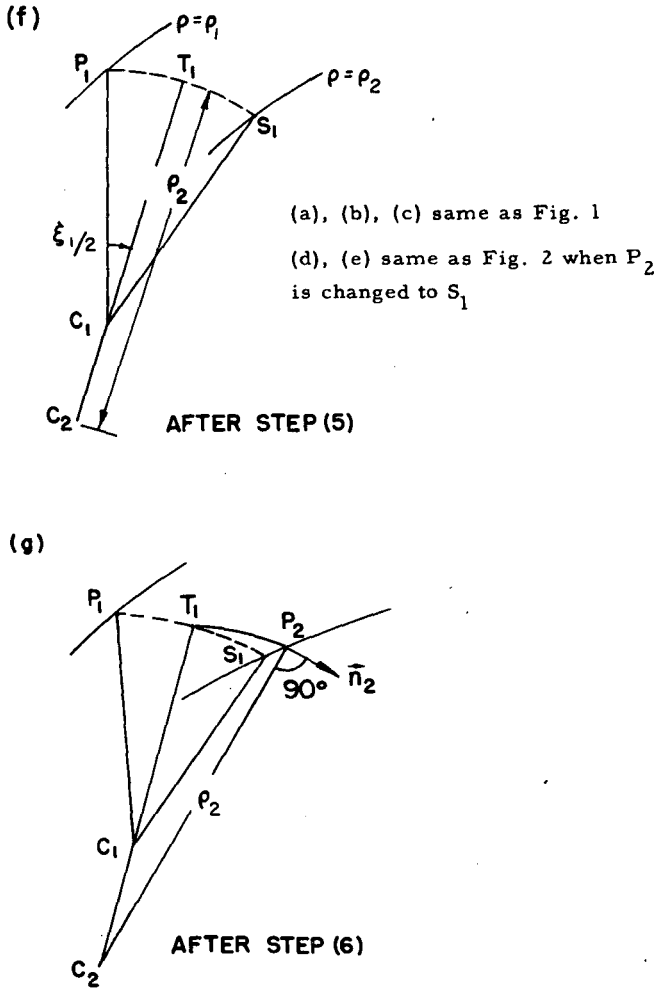

(a)
P_1
$\vec{n}_1$
ρ_1 =CONST
BEFORE STEP (1)
(b)
P_1
$\vec{n}_1$
90°
AFTER STEP (1)
(c)
P_1
$\vec{n}_1$
ρ_1
C_1
AFTER STEP (2)
(d)
P_1
$\vec{n}_1$
P_2
$\vec{n}_2$
ξ_1
C_1
AFTER STEP (3)
MU-5095

Fig. 1



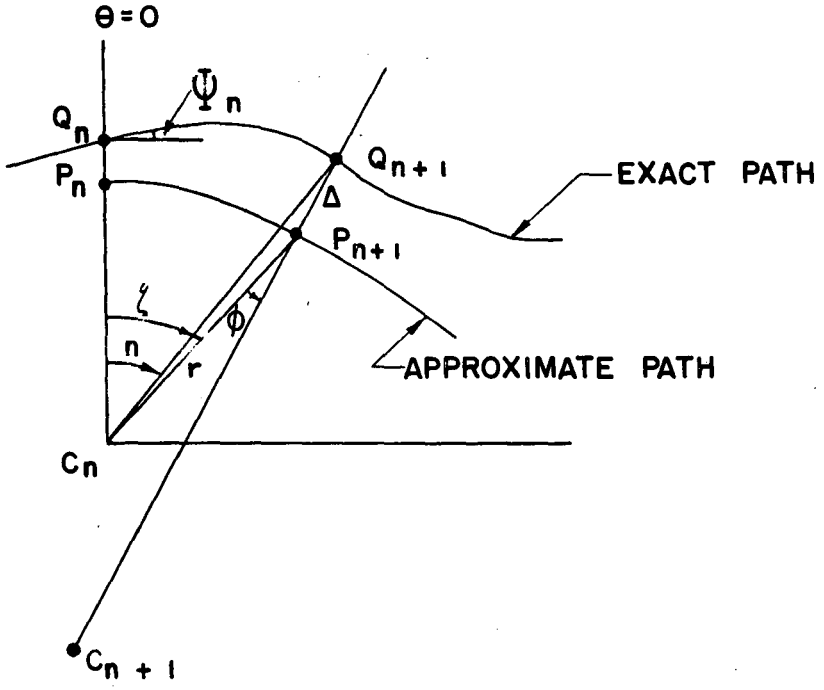
(a), (b), (c) same as Fig. 1

Fig. 2


(f)
ρ=ρ1
P1
T1
ρ=ρ2
S1
ρ2
ξ1/2
C1
C2
AFTER STEP (5)
(a), (b), (c) same as Fig. 1
(d), (e) same as Fig. 2 when P2 is changed to S1
(g)
P1
T1
P2
S1
90°
n2
ρ2
C1
C2
AFTER STEP (6)

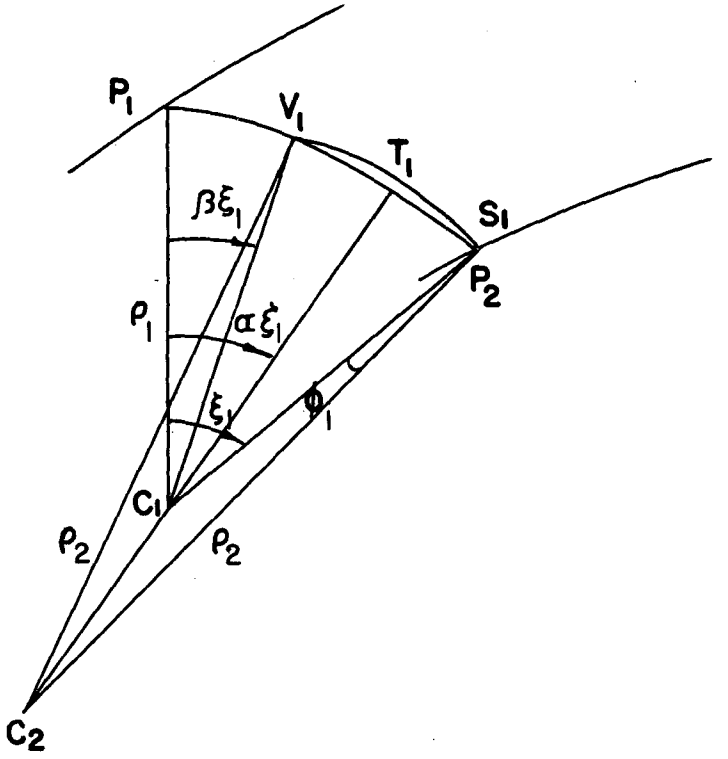
MU-5097

Fig. 3



MU-5098

Fig. 4



MU-5099

Fig. 5