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A POSSIBLE GROUP THEORETICAL REPRESENTATION OF THE GENERALIZED HEISENBERG RELATIONS*

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ABSTRACT

A Generalized set of canonically conjugate variables obeying commutation relations in terms of universal constants is represented in a group theoretical formalism. All physical variables can be uniquely expressed in terms of universal constants in analogy to Wheeler's "wormhole" length and are termed quantal units. The commutation relations of the canonically conjugate variables are termed the Generalized Heisenberg Relations, two of which are the usual $[x, p] \geq \hbar$ and $[E, t] \geq \hbar$. The quantal units are manifest in classical, quantum, and relativistic physics.

I. INTRODUCTION

J. A. Wheeler[1] discussed a quantum of length, termed the Wheeler "wormhole" $\ell = (G\hbar/c^3)^{1/2}$ and interpreted it in a topological description of the space-time manifold. For a more detailed description of Wheeler's geometrizing procedure, see Appendix A. He also discusses a quantum of mass, $m = (c\hbar/G)^{1/2}$, a quantum of energy, $E = (c^5\hbar/G)^{1/2}$, and a quantum of density, $\rho = c^5/G^2\hbar$. Earlier, M. Planck[2] derived the dimensioned quantities of length, time, mass, and temperature in terms of the universal constants $\hbar$ (Planck's constant), G (universal gravitational constant), c (velocity of light). The quantities Planck introduced were:

$$\ell = (G\hbar/c^3)^{1/2} = 1.60 \times 10^{33} \text{ cm}, \tag{1a}$$

$$t = (G\hbar/c^5)^{1/2} = 5.36 \times 10^{-44} \text{ sec}, \tag{1b}$$

$$m = (c\hbar/G)^{1/2} = 2.82 \times 10^{-5} \text{ gm}, \tag{1c}$$

$$T = 1/k(c^5\hbar/G)^{1/2} = 3.60 \times 10^{32} \text{ degrees} \tag{1d}$$

for length, time, mass, and temperature, respectively. The values of the universal constants used in evaluating the Planck quantities are: $c = 2.998 \times 10^{10}$ cm/sec, $\hbar = 1.055 \times 10^{-27}$ erg-sec, $G = 6.673 \times 10^{-8} \frac{\text{cm}^3}{\text{gm-sec}^2}$ and $k = 1.34 \times 10^{-16}$ erg/degree. These values are taken from the recent work of B. N. Taylor, W. H. Parker, and D. N. Langenberg[3] on the theoretical and experimental implications of the universal constants.

Planck discussed the universality of the expressions in (1a), (1b), (1c), (1d), which results from their unique expression in terms of the universal constants. The quantities in (1a), (1b), (1c), (1d) and all physical variables, can be uniquely expressed in terms of universal

constants and, in this form, are termed "quantal units." The quantal units are defined and discussed in Refs. 4 and 5.

Table I contains the set of quantal units and their numerical values relevant to the calculations in this paper.

The quantal units obey paired relations as a set of canonically conjugate variables expressible in terms of non-zero commutations. Two such pair relations are the usual Heisenberg relations, $[x, p] \geq \hbar$ and $[E, t] \geq \hbar$. Recently a set of Generalized Heisenberg Relations were developed in terms of quantal units.[6] We consider the pair relations obeyed by the quantal units as elements of non-Abelian algebras. The non-zero commutation relations are interpreted as representing a limit on observability of obtainable values of physical variables in the sense of the uncertainty interpretation of the usual Heisenberg relations.

Heisenberg developed his uncertainty relations, in terms of E, t and x, p, which represents a limit on the simultaneous observability of canonically conjugate physical variables in relation to the magnitude of $\hbar$. The commutation relations of the non-Abelian elements are expressed in terms of universal constants, combinations of universal constants and physical variables, expressed in quantal unit form. The Generalized Set of Heisenberg Relations in terms of the interdependency of physical variables is interpreted in terms of a scale of observability in physical phenomena which, in part, accounts for the comprehension of various branches of physics.

This is an extension of the concept recently expressed by Taylor, Parker, and Langenberg[3] in their statement that the "universal constants are an important link in the chain of physical theory which binds all

the diverse branches of physics together." The quantal units are uniquely expressed in terms of universal constants and are manifest in classical, quantum, and relativistic physics. See Refs. 5 and 6 for the cosmological aspect of the quantal units.

II. SIX GENERALIZED HEISENBERG RELATIONS

In Ref. 6, six Generalized Heisenberg Relations are developed in terms of pair relations of quantal units and in terms of the ordinary form of quantized variables. Two of these relations are the well-known Heisenberg relations and four are new relations. Two distinct quantization procedures are defined; they are denoted primary and secondary. Primary quantization is that quantization procedure in terms of the quantal units and secondary quantization is that quantization procedure which is the ordinary or standard form of quantization.

We have the usual Heisenberg relations between (ℓ, p) and (t, E) as indicated by the horizontal arrow in Eq. (2a), (2b),

$$\begin{array}{ll} \longrightarrow & \\ [\ell,\ p]_{\hbar} \geq \hbar & \text{(2a)} \\ \downarrow \quad \times & \\ [t,\ E]_{\hbar} \geq \hbar \ . & \text{(2b)} \end{array}$$

(Note ℓ or x is length.) The sub $\hbar$ on the bracket denotes the Generalized Heisenberg Relation in terms of $\hbar$. As discussed by Bohr the canonically conjugate pairs ℓ, p and t, E relate in a quantum mechanical manner to $\hbar$ as a limit on the simultaneous observability of ℓ and p and also t and E. These relations are termed horizontal relations as the relations of ℓ to p and t to E (see arrow in Eq. (2a) and (2b)). Six paired relations are represented in Eq. (2a), (2b) as indicated by the arrows.

The first pair of new relations are obtained as paired relations between the diagonal pairs of quantum variables (indicated by the diagonal arrows). These pairs are E, ℓ and p, t and form the relations,

$$[E, \ell] \geq c\hbar \equiv \ell' \tag{3a}$$

$$[p, t] \geq \hbar/c \equiv \ell \ , \tag{3b}$$

using the quantal unit form of the physical variables given in Table I. We define $\ell' \equiv c\hbar$ and $\ell \equiv \hbar/c$ as is done in Table I. The commutation relations between one space-like component of a four vector with the time-like component of a four vector is equal to ℓ or ℓ'. Now let us look at the third possibility of the relation between pairs in Eq. (2a), (2b). We form the vertical pairs (ℓ, t) and (p, E).

In each case, we are considering one component of length ℓ, is an isotopic space as $\ell_1 = \ell_2 = \ell_3$ and similarly for momentum, $p_1 = p_2 = p_3$.

For the two pair relations (ℓ, t) and (p, E) related by the vertical arrows, we have,

$$[\ell, t] = \frac{(\ell'\ell)}{F}^{1/2} = \frac{\hbar}{F} = \frac{\ell'}{cF} \tag{4a}$$

$$[p, E] = (\ell'\ell)^{1/2} F = \hbar F, = \ell cF \tag{4b}$$

where F is the quantal force, $F = c^4/G$ and $p = cF$ is the quantal power (see Table I).

Also, ℓ and t and also p and E are related in Einstein four vector notation as,

$$\underset{\sim}{x} = (\underline{\ell}, t) \tag{5a}$$

$$\underset{\sim}{p} = (\underline{p}, E) \tag{5b}$$

where t is the fourth component of ℓ and E is the fourth component of p. See Sec. V for the metric interpretation of the relation (ℓ, t) and (p, E) as related to the quantum interpretation of the relation of

$[\ell, t]$ and $[p, E]$ in Eq. (4a) and Eq. (4b). We can also write $[\ell, t]$ as $[x_1, x_4]$ and $[p, E]$ as $[p_1, p_4]$ for $x_4 = ict$ and $p_4 = i\frac{1}{c}E$ for indices running 1 to 4.

The same relationship holds for $[\ell, p] = \hbar$ and $[t, E] = \hbar$ in Eq. (2a), (2b) and $[E, \ell] = \ell'$ and $[p, t] = \ell$ in Eq. (3a), (3b) and $[\ell, t] = \ell'/p$ and $[p, E] = \ell p$ in Eq. (4a), (4b) for both primary and secondary quantization.

The primary and secondary form of quantization of the variables ℓ, p, t and E are given in Eq. (6a), (6b), (6c), (6d) and Eq. (7a), (7b), (7c), (7d).

The primary or quantal unit quantized form from Table I gives,

$$\ell = \left(\frac{\ell'}{F}\right)^{1/2} \qquad p = (\ell F)^{1/2} \qquad (6a),(6b)$$

$$t = (\ell/F)^{1/2} \qquad E = (\ell' F)^{1/2} \qquad (6c),(6d)$$

and the secondary or ordinary quantized form is given as,

$$\ell_o = \frac{\hbar}{mc} \qquad \Delta p = hk = \hbar/\lambda \qquad (7a),(7b)$$

$$t_o = \frac{\hbar}{mc^2} \qquad \Delta E = h\nu = \hbar\omega \qquad (7c),(7d)$$

where the quantity ℓ_o is the Compton wave length for a particle of mass, m and t_o is the corresponding time, $t_o = \ell_o/c$. Eq. (7b), $p = \hbar/\lambda$ is the usual DeBroglie momentum and wave length expression and $E = \hbar\omega$ is the Einstein quantum energy relation.

The primary quantization procedure for the Generalized Heisenberg Relations $[E, \ell]$ and $[p, t]$ gives Eq. (3a) and (3b). The same results are obtained for secondary quantization (see Ref. 6). The primary

quantization procedure for the Generalized Heisenberg Relations [ℓ, t] and [p, E] gives Eq. (4a) and (4b) but the secondary quantized variables in the Generalized Heisenberg Relations give,

$$[\ell, t] = \frac{\ell'}{\hbar\omega^2} \tag{8a}$$

$$[p, E] = \ell(\hbar\omega^2) \tag{8b}$$

Because of the relation of quantal power $p = cF$ and the quantal frequency $\omega = (F/\ell)^{1/2}$, as $p = cF = \hbar\omega^2$ the results in Eq. (8a) and (8b) reduce respectively to Eq. (4a) and (4b). The manner in which primary and secondary quantization are related is discussed in more detail in Ref. 6.

More new Generalized Heisenberg Relations are presented in Sec. IV but before proceeding to these new cases derived by group theoretical methods, let us present some relations presented by G. N. Lewis that also represent specific cases of the Generalized Heisenberg Relations.

III. SOME COMMENTS ON G. N. LEWIS' ULTIMATE RATIONAL UNITS

In his attempt to establish a universal and fundamental set of units, G. N. Lewis developed a set of quantities which represent some specific examples of the Generalized Heisenberg Relations. G. N. Lewis and M. Randall[7] developed a set of units termed Ultimate Rational Units, in which a set of physical quantities are expressed in terms of one other physical quantity or dimension (in this case length).

They write:

$$1 \text{ sec} = 2.999 \times 10^{10} \text{ cm} \tag{9a}$$

$$1 \text{ gm} = 2.499 \times 10^{37} \frac{1}{\text{cm}} \tag{9b}$$

$$1 \text{ erg} = 2.779 \times 10^{16} \frac{1}{\text{cm}} \tag{9c}$$

$$1 \text{ deg} = 3.813 \frac{1}{\text{cm}} \tag{9d}$$

The unit of dimension of centimeter is arbitrary but relative magnitudes are constant. In Sec. IV, it is demonstrated that Eq. (9a), (9b), (9c), (9d) are specific cases of the Generalized Heisenberg Relations. First, let us more closely examine G. N. Lewis' units. These units, like those developed by M. Planck[2], are expressible in terms of universal constants.

In the first case, Eq. (9a), of the relation of the time unit, sec to the unit of length, cm; the multiplicative factor or 2.999×10^{10} is c, the velocity of light (which is not dimensionless).

From Lewis, $t = c\ell$. Actually velocity is given by $c = \ell/t$ so that the time unit should read,

$$1 \text{ sec} = \frac{1}{c} \text{ cm} = \frac{1}{2.999} \times 10^{-10} \text{ cm} \tag{10a}$$

or the reciprocal multiplicative factor of what G. N. Lewis wrote.

In the case of his mass unit to length relation, Eq. (9b) (or the reciprocal of length), we have,

$$1 \text{ gm} = \frac{\hbar}{c} \frac{1}{\text{cm}} = \frac{1}{2.499} \times 10^{-37} \frac{1}{\text{cm}} \tag{10b}$$

and for energy to the reciprocal of length

$$1 \text{ erg} = c\hbar \frac{1}{\text{cm}} = \frac{1}{2.779} \times 10^{-16} \frac{1}{\text{cm}} \tag{10c}$$

and temperature to the reciprocal of length,

$$1 \text{ deg} = \frac{1}{k} c\hbar \frac{1}{\text{cm}} = \frac{1}{3.813} \frac{1}{\text{cm}} \tag{10d}$$

where k is the Boltzmann constant.

If one examines dimensionality of the universal constants and units, one concludes that the actual result of a conversion of sec, gm, erg, and deg into cm or 1/cm is the reciprocal of what G. N. Lewis wrote, but nevertheless the concept is the same. We have expressed the numerical values given in Eq. (9a), (9b), (9c), (9d) from G. N. Lewis in terms of universal constants, in Eq. (10a), (10b), (10c), and (10d).

With this in mind, let us proceed and construct a new correct set of "conversion" or multiplicative factors,

$$t = \frac{1}{c} \ell = 3.33 \times 10^{-11} \text{ cm} \tag{11a}$$

$$m = \frac{\hbar}{c} \frac{1}{\ell} = \ell \frac{1}{\ell} = 4.001 \times 10^{-38} \frac{1}{\text{cm}} \tag{11b}$$

$$E = (c\hbar) \frac{1}{\ell} = \ell' \frac{1}{\ell} = 3.60 \times 10^{-17} \frac{1}{\text{cm}} \tag{11c}$$

$$T = \frac{1}{k} (c\hbar) \frac{1}{\ell} = \frac{\ell'}{k} \frac{1}{\ell} = 2.62 \times 10^{-1} \frac{1}{\text{cm}} \tag{11d}$$

where we defined $\mathcal{L} = \hbar/c$ and $\mathcal{L}' = c\hbar$ in Table I.

It is noted that Eq. (10b) can be written as

$$[m, \ell] = \mathcal{L} \tag{12}$$

compared to Eq. (3b) which can be written as,

$$[p, t] = \mathcal{L} \tag{13}$$

and we see that there can be more than one variable canonical to each variable. Note that it is true that $m\ell = pt$ or $m = p(t/\ell) = p/c$ or $p = mc$ as we would believe. This is true for other physical variables as seen in Table V, which is discussed in Sec. IV.

Also, Eq. (10c) can be written as,

$$[E, \ell] = \mathcal{L}' \tag{14}$$

which corresponds to Eq. (3a). It appears that the G. N. Lewis rational ultimate units are specific cases of the Generalized Heisenberg Relations.

We could write a set of physical variables in terms of cm or 1/cm; we can also do this for sec or 1/sec, etc. for each physical variable.

We can set up units such that each set of physical variables can be written in terms of another variable. These relations comprise part of the set of laws of physics that are expressed by the group representation of the Generalized Heisenberg Relations presented in Sec. IV. In the group theoretical interpretation, certain laws of physics result which we term Primary Laws of physics.

IV. A GROUP THEORETICAL REPRESENTATION OF THE GENERALIZED HEISENBERG RELATIONS

A. Group Symmetry Principles

We shall now look at a possible group representation in which the physical variables, in the form of the quantal units, Table I, form the elements of a group. A set of Generalized Heisenberg Relations are represented in this manner. In Ref. 6, the pair relations of these quantal units are represented as the noncommutative elements of a non-Abelian group, the well-known cases being $[\ell, p] \geq \hbar$ and $[t, E] \geq \hbar$.

Primary Physical Laws come about through group operations on physical variables expressed as quantal units. Some examples of Primary Laws of physics are: $F = ma$, $E = mc^2$, $p = mc$, and $[\ell, p] \geq \hbar$. Primary Laws will appear in our Tables III and IV as we will demonstrate. We define Secondary Laws as those laws of physics which do not result in this manner. Secondary Laws follow indirectly from group operations on physical variables in quantal unit form. Such laws as the radioactive decay law containing differential and exponential operations are examples of Secondary Laws. We shall discuss briefly how the Secondary Laws come about later on in this section. First, let us return to Primary Laws.

The algebra or set of group generators that give rise to Primary Laws do not contain the elements of differentiability or exponentiality and neither do they contain the respective inverse operations of integrability or the logarithmic operation. The group generators of operations that do not involve the above mentioned operations generate the group we denote as R, the representation of R that we choose which

involves the basis vectors and group generators or algebra can be constructed isomorphic to the symmetry group so that a one-to-one mapping of each element can be made to a specific permutation, $\mathcal{P}$, of objects that will specify the symmetry relations between the physical variables. The symmetry relations are describable by the permutation of indices in the tensorial representation.

We can construct the generators of a group we denote as $\mathcal{D}$, which contain the operations of differentiability, exponentiality, integrability, and the logarithmic operation which can generate the Secondary Laws. This group is constructed isomorphic to $\mathcal{R}$ and also has the constraints of the symmetry group, $\mathcal{P}$. We shall deal only with the development of the Primary Laws of physics in the present paper.

The space presently considered denoted, $\mathcal{R}_s$, where the sub s denotes isomorphism to the symmetry group does not contain the special operations of differentiation and integration directly, but one can see that its operations are connected to these operations through the relation of the principle of least action (integral containing element $d\eta$ where η is a physical variable) which results in the Heisenberg Uncertainty Relations. If we have the Poisson bracket for two canonically conjugate variables (μ, η), then we can write the principle of least action as $\int \mu d\eta = \zeta$ which is an integral expression involving the differential element $d\eta$ (see Ref. 6).

Let us turn our attention to the description of the Group $\mathcal{R}_s$ and its operators.

B. Group Multiplication Table for the Generalized Heisenberg Relations

Let us first construct the multiplication table for the subgroup of physical variables expressed as quantal units that are mechanical and thermodynamic (see Table I). In Table II we have constructed this multiplication table which is the table of Generalized Heisenberg Relations. The product of all pair elements is formed and the elements are in the quantal unit form, which of course results in a product and division of universal constants to various exponents and can therefore be written in terms of universal constants or combinations of universal constants. These elements can then in turn be expressed in terms of the physical variable or variables, that the set of universal constants represent in quantal unit form. When this is done, some of the laws of physics result from the pair relations that are formed. Six cases of the Generalized Heisenberg Relations were presented and discussed in Sec. II. The multiplication tables represent a Generalized Set of the Heisenberg Relations.

In Table III we have expressed the elements that result from the multiplication operation, in terms of physical variables or variables which are equal to the resultant quantal form in Table II. Certain laws of physics appear in Table III. Such "simple" laws as cp = E (relating momentum and kinetic energy) and F = ma (Newton's second law of motion) are expressed in Table III, the product (c, p) = E is apparent when one finds the resultant product of row c and column p as being equal to E. Also, the product that results from the multiplication of row p and column c gives E. In an analogous manner, one sees that the product of

row m and column a, or row a and column m, gives F or F = ma. Some of the laws of physics that appear in Table III appear in one or more forms. Another form of F = ma can be determined from the product of row p, column a or row a, column p which gives p = cF or pa = cF. Using the law p = mc, we have mca = cF or F = ma. If we use (m, c) = E/c or $E = mc^2$ and from above cp = E, we then have p = mc which we used to obtain F = ma.

The Einstein relation $E = mc^2$ is obtained from Table III by the resultant product of row m, column c or row c, column m as mc = E/c = > mc^2 = E or $E = mc^2$. This relation also appears in Table IV. So that squared quantities, such as c^2, can appear in the laws of physics that are expressed in Tables III and IV from the formation of the product of pairs of physical variables.

The standard Heisenberg Relations, (E, t) = $\hbar$ and (ℓ, p) = $\hbar$, appear in Table III. These relations appear as Eqs. (2a), (2b) in Sec. II. Also, there are two Heisenberg-like relations [m, ℓ] = $\lambda\!\!\!^{-}$ and [p, t] = $\lambda\!\!\!^{-}$ where $\lambda\!\!\!^{-} \equiv \hbar/c$, which can be derived from G. N. Lewis' units in Eq. (12) and Eq. (13), respectively. We have the relation [p, t] = $\lambda\!\!\!^{-}$ in Eq. (3b). Also, Eq. (3a) [E, ℓ] = $\lambda\!\!\!^{-}{}'$ appears in Table III for the product E or ℓ. Eq. (11b) and Eq. (11c) for [E, ℓ] = $\lambda\!\!\!^{-}$ and [p, t] = $\lambda\!\!\!^{-}$, as implied by G. N. Lewis' Ultimate Rational Units, which also appear in Table III, as does Eq. (3a) [E, ℓ] = $\lambda\!\!\!^{-}{}'$ also deduced from G. N. Lewis' units, Eq. (14). As we see, G. N. Lewis' work is one small aspect of ours and, as we mentioned in Sec. III, a generalized set of Ultimate Rational Units could be constructed by expressing each variable in terms of one

other. (See starred quantities in Table III.) This procedure would result in a table equivalent to our Table III; Eq. (11a), $t = 1/c\ell$ is again derivable from G. N. Lewis' work and also $(c, t) = \ell$ appears in our Table III, and $c \equiv \ell/t$ is the basic definition of velocity. Considering Eq. (11d) we have $T = \frac{1}{k}(c\hbar)\frac{1}{\ell} = \frac{1}{k}\hbar'\left(\frac{1}{\ell}\right)$ from G. N. Lewis units upon substitution of the quantal unit of length, $\ell = \left(\frac{c\hbar}{F}\right)^{1/2}$ into $T = \frac{1}{k}(c\hbar)\frac{1}{\ell}$, we have $T = \frac{1}{k}(c\hbar F)^{1/2}$. Also from Table I we have the quantal unit of entropy as $S = k$ (where k is the Boltzmann constant) so that the product $(T, k) = (T, S) = (c\hbar F)^{1/2}$. This relation appears in Table II. Also, using the quantal unit form of energy from Table I, $E = (c\hbar F)^{1/2}$, we have $T = \frac{E}{k}$ or $(T, k) = E$. It is clear, then, that all the G. N. Lewis relations appear in our tables.

The laws of physics that appear in Table III and/or Table IV are termed Primary Laws of physics and represent a most fundamental aspect of physics. Primary Laws do not contain explicitly the operation of differentiation or integration, but these operations can be implied, such as the form $c = \ell/t$ or $c = \frac{d\ell}{dt}$, or in the uncertainty principle as $\Delta E \Delta t \geq \hbar$. Integration can be implied by the principle of least action $\int p\,dq$ as related to $\Delta p \Delta q \geq \hbar$ in the uncertainty principle. Also Primary Laws do not contain exponential or logarithmic operations directly. As previously mentioned, these operations are contained in Secondary Laws only. Those laws of physics which are not Primary, are termed Secondary.

Let us look at some more examples of the type of Primary Laws that appear in Tables II and III. As we stated previously, in Table II, the elements are expressed in terms of universal constants and quantal force.

From Table I, we substituted physical variables, in the product array, for their equivalent quantal unit expressions, and this is the manner in which Table III was formed. We have seen that such laws as $F = ma$ and $E = mc^2$ appear. Also we have $[\ell, p] > \hbar$ and $[\ell, m] = \hbar$ and also $[E, t] \geq \hbar$ and $[p, t] = \hbar$, and $[\ell, E] = \hbar'$. Also, we have seen that G. N. Lewis' Ultimate Rational Units appear in Tables II and III, and thus they do express some Primary Laws of physics. New Primary Laws of physics become apparent through examination of these tables. Some such laws are $[\ell, m] = \hbar$ and $[p, t] = \hbar$ as canonically conjugate pairs.

From Table IV for the Generalized Reciprocal Heisenberg Relations we have Primary Physics Laws as $c = \ell/t$, $E = mc^2$, $F = ma$, $F = E/\ell$, $F = p/c$, $p = mc$, $[\ell, p] > \hbar$ and $[t, E] \geq \hbar$, which come out of the group representation of quantal units. (In a sense, the rule: "Good" physical laws should be simple is an aspect of this formalism.) Such classical laws as $F = ma$ or relativistic laws such as $E = mc^2$ and quantum laws, such as $(E, t) = \hbar$ are put on an equal footing in this formalism.

Again from the reciprocal relations in Table IV, we have such Generalized Inverse Heisenberg Relations as $[F, \frac{1}{m}] = a$, $[c, \frac{1}{t}] = a$, and $(p, \frac{1}{p}) = a$ where a is the acceleration, p is the power, $p = cF$ and p is the linear momentum. As stated before, we see that each physical variable is canonically conjugate to more than one other physical variable. We have sets such as in Table V.

C. Generalized Reciprocal Heisenberg Relations

Since the group operations of the Group R_s allow an inverse operation, it is possible, from a group theoretical standpoint, to form

reciprocal relations $[\mu, 1/\eta] = \zeta$ analogous to the conventional Heisenberg Relations $[\mu, \eta] = \zeta$, discussed previously. The physical quantities, μ and η, are canonically conjugate to each other and their uncertainty relation is expressed by the Poisson bracket in terms of a universal constant, ζ, or a set of universal constants.

The "principle of least action," in the Sommerfeld notation, can be expressed as $\int \mu d\eta = n\zeta$ where n is a small integer or integer fraction.

Before we interpret the meaning of the uncertainty relations in the reciprocal Heisenberg Relations, let us define what is meant in the usual interpretation by the conventional Heisenberg Relation.

We can write $[\mu, \eta] = \zeta$ as $\Delta\mu\Delta\eta \geq \zeta$ which states that the larger $\Delta\mu$, or the greater the uncertainty in μ, then the more precisely we know η, where $\Delta\mu$ and $\Delta\eta$ represent the uncertainties in μ and η, respectively. The uncertainties or limit of observability is defined in terms of a constant, $\hbar$ (or Planck's constant). In our notation, $\zeta = \hbar$.

The Heisenberg theory involves an observer (probe particle) and observed (target particle) and expresses the essential relation that the observed system and the observer system which affects the probability of obtaining a certain value in the observed system. This is the conventional interpretation of the Heisenberg Relation. Stated briefly for $\Delta\mu\Delta\eta \geq \hbar$, the smaller $\Delta\mu$ is, the larger $\Delta\eta$ will be and vice versa.

Now let us turn our attention to the reciprocal Heisenberg Relations which may be expressed as $[\mu, 1/\eta] = \zeta$ or $\Delta\mu\ 1/\eta \geq \zeta$. For this expression to hold true, there are two possibilities: (1) As $\Delta\mu$ becomes small

so does $\Delta\eta$ or both can become larger (both can go in the same direction of small or larger uncertainties). (2) The uncertainty in $\Delta\mu$ can become larger as that in $\Delta\eta$ becomes smaller (the reverse is not true, that is, there is a limit imposed by the "greater than" sign such that, for a fixed uncertainty in $\Delta\mu$, the uncertainty, $\Delta\eta$, cannot increase without limits).

In the reciprocal relations there is a larger range in value of the uncertainties in the respective value of two conjugate variables, but there are still restrictions on observability.

D. Generalized Heisenberg Relations and Physical Law

In Sec. I, the recent statement of Taylor, Parker, and Langenberg[3] was quoted on the fundamental significance of the universal constants as a possible unifying element to "bind together" the diverse branches of physics. Various of the universal constants appear to be more prominently manifest in particular branches of physics such as classical, quantum, and relativistic physics, but the mechanical quantal units appear to be prominently manifest in each of these branches.

The construction of the multiplication tables for canonically conjugate pairs of quantal units to form the Generalized Heisenberg Relations results in laws of physics in classical, quantum, and special relativistic physics (general relativistic physics contains differentiation and therefore involves operations in the group, $\mathcal{D}$; see Ref. 4 for the general relativistic manifestation of the quantal units). The fact that the Primary Laws of physics come about through group operations on canonical pairs of quantal units demonstrates that relativistic, quantum,

and classical physics do have the same basic origin! This origin, it is here proposed, relates to a geometric interpretation of the space-time manifold. See Ref. 5 for a more detailed discussion.

The Primary Laws of physics, as all laws of physics, represent a constant aspect in the manifold as a constant relation between variable quantities which are represented by mathematical symbols. We term these quantities physical variables. The reason laws can result from a formulation dependent on universal constants is that they represent a fundamental constant aspect in the manifold that can give rise to the constancy of physical law. Since the universal constants give rise to the constancy in all physical laws, therefore, they unify the various branches of physics, for example, we again have $F = ma$ (classical), $E = mc^2$ (special relativistic), and $[\ell, p] \geq \hbar$ (quantum mechanical).

It should be mentioned, also, laws can contain universal constants which represent constancy which can be in the form of a "proportionality constant," for example, G in Newton's law of universal gravitation, but the constancy of law represents a deeper aspect of Nature than the universal constants as proportionality constants or coefficients in algebraic expressions.

In the next section, we discuss a generalized geometrizing procedure and the development of a generalized metric, for the space, in terms of the quantal units.

V. GENERALIZED MINKOWSKI SPACES

In order to geometrize a force field such as the gravitational force field, one must assign a metric to the space. Minkowski[8] introduced a four dimensional geometry for ($\underline{x}$, t). In the case of ($\underline{x}$, t) and ($\underline{p}$, E) an invariant relation holds in terms of a metric that depends on the constancy of a universal constant, c. We have the usual special relativistic expression,

$$s^2 = x^2 + y^2 + z^2 - c^2t^2 \tag{15}$$

where s^2 is the four vector invariant[9] developed as a generalization of the three space Pythagorian theorem. In Eq. (15), we denote $s^2 = s_1{}^2$. The usual Einstein four vector invariant for ($\underline{p}$, E) is given as,

$$s_2{}^2 = p_x{}^2 + p_y{}^2 + p_z{}^2 - \frac{1}{c^2} E^2 \tag{16}$$

The invariant relation in Eq. (15) and Eq. (16) relates to Eq. (5a), (5b) relating to the vertically related (arrows) expressions in Eq. (2a), (2b).

Let us demonstrate the manner in which it might be possible to extend the concept of invariant relations in terms of the velocity of light, c, and other universal constants to a number of physical variables and a multi-dimensional space. In Minkowski's words,[8] when he introduced time as the fourth dimension of three space, "Nobody has ever noticed a place except at a time, or a time except at a place." This also appears to be true of matter and energy and space-time, as matter and energy can not exist without space-time to exist in and, conversely

space-time cannot exist completely devoid of a matter-energy content. Although this latter state is commonly referred to as a "complete" vacuum, it appears that space-time contains, at all times and locations, a matter-energy content that is detected, in one manifestation, as vacuum state polarization.[10] In one sense, this matter-energy content is also referred to as the "Fermi sea." H. P. Stapp[11] discussed some conceptual problems of empty space. Present and early concepts are presented.

Rene' Descorte (1596-1650), the French mathematician and philosopher, conjectured that there exists a geometry of many dimensions, where each dimension is represented as a "physical variable" such as length, time, velocity, force, etc. Based on this concept, we introduce n orthogonal dimensions, where n is the number of physical variables considered for a particular physical problem. For Table VI, $n = 6$ for the physical variables x_1, t, m, E, and p_1. (For vectorial quantities, only one dimension is counted.) This set of variables, considered in the form of the following equations in this section, are x, y, z, t, m, E, and p_x, p_y, p_z; then we have $n = 12$.

In an analogous manner to s_1^2 and s_2^2 relating variables that are related by the vertical arrows in Eq. (2a), (2b), let us develop the invariant expressions for quantities related by the horizontal arrows. From (ℓ, p) we have,

$$s_3^2 = x^2 + y^2 + z^2 - \frac{c^2}{F} \left(p_x^2 + p_y^2 + p_z^2\right) \tag{17}$$

where F is the quantal force. This is a six dimensional space. From (t, E) we have,

$$s_4^{\,2} = t^2 - (c^2F)\ E^2 \tag{18}$$

a two dimensional space of t and E.

From quantities related by the diagonal arrows we have for (ℓ, E)

$$s_5^{\,2} = x^2 + y^2 + z^2 - \frac{1}{F^2}\ E^2 \tag{19}$$

and for (p, t)

$$s_6^{\,2} = p_x^{\,2} + p_y^{\,2} + p_z^{\,2} - (F^2)\ t^2 \tag{20}$$

both of which are four dimensional spaces. New "light cone" spaces are implied by these relationships.

Relations between (x, m), (t, m), (t, p), (m, E), and (m, p) are given as elements of Table VI. Also included are the metrical elements for (y, t), (p, E), (x, E), (x, p), (t, E), and (E, p) in Table VI; these are starred and are from Eq. (16) through Eq. (20) for above and below diagonal elements.

We denote the elements, m_{ik}, in Table VI as elements of a matrix, M, where m_{ik} represent the metrical elements between two physical variables (μ, η) in the form of the invariant $s_n^{\,2}$ as,

$$s_n^{\,2} = \mu_j^{\,2} + m_{ik}\ \eta_j^{\,2} \tag{21}$$

where the index, j, runs from 1 to 3, for three "spacial" dimensions, and i and k run from 1 to n where n is the number of physical variables considered. The index, n, on $s_n^{\,2}$ denotes the relation between a particular pair of variables for which the metric, m_{ik}, is being formed.

For each $s_n^{\,2}$, the metrical element can be represented by the submatrix m_{ik} to M as for example for $s_1^{\,2}$, Eq. (15), we have,

$$s_1^{\,2} = x^2 + y^2 + z^2 - c^2t^2 \tag{22}$$

which can also be written in general relativistic notation as,[12]

$$ds^2 = g_{\mu\nu}\, dx_\mu^{\,2}\, dx_\nu^{\,2} \tag{23}$$

where indices μ and ν run from 1 to 4 and for $\mu_j = \underline{x}$ for $j = 1$ to 3 and $\eta_j = t$, for $j = 1$, $m_{ik} = g_{\mu\nu}$ and for the signature $x_4 = -\,ict$,

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} . \tag{24}$$

For each s_n, analogous m_{ik}'s can be formed for two, four, and six dimensional spaces or for m_{ik} as 2 x 2, 4 x 4, and 6 x 6 dimensional matrices.

To form the upper half non-diagonal elements of M, in Table VI, we use the expression in Eq. (21). Another equivalent form can be written as,

$$s_n^{\,2} = \mu_j^{\,2} + m_{ik}^{-1}\, \eta_j^{\,2} \tag{25}$$

where m_{ik}^{-1} is the inverse matrix of m_{ik}. The lower half non-diagonal elements of M are formed from Eq. (25). The diagonal elements of M are unity, for (μ, η) where $\mu = \eta$, and therefore the trace of M is given by,

$$M^{tr} = n \tag{26}$$

where n is the number of physical variables considered on the dimensionality of M.

The elements of matrix M in Table VI are expressed in terms of universal constants c, $\hbar$, and G, where $t = (G\hbar/c^5)^{1/2}$ and $F = c^4/G$. It

is conjectured that the universal constants represent a geometrized constraint on the space-time manifold. One example of this constraint is presented in Refs. 4 and 5 where the additional constraint of the quantal units give closed cosmological solutions to Einstein's field equations. The form or "shape" of the universe appears to result from the constant constraint of the geometrical interpretation of the universal constants and the constant relation between variables in primary and secondary laws of physics. In Sec IV, some primary laws of physics were presented.

The manner in which the universal constants are manifest in and unify the four force fields of gravitation, weak interactions, electromagnetic interactions, and strong interactions is given in Ref. 13. A unifying formalism of the four force fields uniquely in terms of universal constants is presented.

VI. CONCLUSIONS

A set of Generalized Heisenberg canonical Relations have been developed and, in a group theoretical interpretation in a set of multiplication tables, certain laws of physics result which are termed Primary Laws of physics. The physical variables that form the canonical relations of the Generalized Heisenberg Relations are expressed in quantal unit form or uniquely in terms of universal constants.

The quantal units are interpreted in their pair relations as elements of non-Abelian algebras, i.e., having commutation relations that are non-zero, leading to a possible geometric description of the space-time manifold.

For each canonical pair a metric is assigned and for a generalized set of variables a generalized metric is constructed. This discussion is related to the earlier work of Wheeler's on the topological description of space-time in terms of variables uniquely expressed in terms of universal constants.

The Generalized Heisenberg Relations non-commutation relations in terms of universal constants, combinations of universal constants and quantal units is conjectured to represent a scale of observability in the space-time manifold. This scale of observability may be a unifying description of the various branches of physics or "areas of comprehension."

As in the recent work of Taylor, Parker and Langenberg, the universal constants may bind the diverse branches of physics together as earlier proposed by Dirac, Gamow, and others. The space-time structure

includes the matter-energy content and the content of all physical variables, in this description, and therefore the content of all the branches of physics.

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APPENDIX A

Electromagnetic Heisenberg Relations and Metric Spaces

In order to geometrize the electromagnetic field, one must develop a metric for electromagnetic variables in the manner presented in Sec. V. Geometrization of the electromagnetic field is what J. A. Wheeler attempted to do by introducing the variable $\ell = (G\hbar/c^3)^{1/2}$, termed the Wheeler "wormhole" as representing a discontinuous multi-connected structure in space-time. The geometry of space-time is represented by a complete set of physical variables which can be represented as a set of "dimensions." This is the hypothesis in this present work.

Electromagnetic variables are also considered as representing dimensionality in this geometric interpretation. Therefore, another set of multicomponent vectors such as the four vectors (x, t) and (p, E) exist involving electromagnetic and "mechanical" quantum variables (as denoted in Table I). As we stated before many such relations exist interconnecting the quantal units geometrically in a metrical structure in the space-time manifold. Let us examine the relation,

$$\begin{tikzcd} \overrightarrow{[a,\ \underline{E}]} \arrow[d] \arrow[dr, leftrightarrow] & \\ [m,\ Q] & \end{tikzcd} \tag{1A}$$

where $\underline{a}$ is the acceleration (a three vector), $\underline{E}$ is the electric field, and m is the mass and Q is charge. Note that the "format" of Eq. (1A)

is similar to that of Eq. (2a), (2b), Sec. II.

Again considering the generalized "Heisenberg" relation between quantities related by the vertical arrow we have,

$$[a, m] = F \tag{2A}$$

the quantal force and the other vertical relations is,

$$[E, Q] = F \tag{3A}$$

as new "four vector" relations with quantal variables where the quantal electric field $E = c^4/GQ = F/Q$ and the quantal charge, Q, is considered to be a universal constant so that Q in quantal unit form is Q.

Let us consider the horizontal relations (horizontal arrows) for (a, E) and (m, Q). We have,

$$[a, E] = \left(\frac{c^2 F}{\ell}\right)^{1/2} \left(\frac{F}{Q}\right) = \left(\frac{c^2 F^3}{\ell Q^2}\right)^{1/2} \tag{4A}$$

which can be expressed as

$$(a, E) = \left(\frac{\rho F}{\varepsilon}\right)^{1/2} \tag{5A}$$

for the quantal density, $\rho = c^5/G^2\hbar = F^2/c^2\ell$ and where ε is the permutivity in quantal unit form given as, $\varepsilon = Q^2/c\hbar = Q^2/\ell'$.

$$[m, Q] = \left(\frac{\ell F}{c^2}\right)^{1/2} Q = \left(\frac{\ell F Q^2}{c^2}\right)^{1/2} \tag{6A}$$

which can be expressed as,

$$[m, Q] = (\varepsilon F \ell)^{1/2} \tag{7A}$$

where ε is the quantal unit of permutivity.

Lastly let's consider the diagonal relations,

$$[a, Q] = Q\left(\frac{c^2F}{\ell}\right)^{1/2} = \left(\frac{c^2FQ^2}{\ell}\right)^{1/2} \tag{8A}$$

where one component of $\underline{a}$ is considered, and for $[E, m]$,

$$[E, m] = \frac{F}{Q}\left(\frac{\ell F}{c^2}\right)^{1/2} = \left(\frac{F^3\ell}{Q^2c^2}\right)^{1/2} \tag{9A}$$

where one component of $\underline{E}$ is considered. We can express $[a, Q]$ as

$$[a, Q] = \left(\frac{\varepsilon F}{\ell'}\right)^{1/2} \tag{10A}$$

and $[E, m]$ as

$$[E, m] = \left(\frac{\rho F}{\varepsilon\ell'}\right)^{1/2} \tag{11A}$$

Again the quantal form of ε is used, as $\varepsilon = Q^2/\ell'$.

Let us look at some invariant "light cone" relations corresponding to the electromagnetic Generalized Heisenberg Relations just presented.

We turn now to the invariant relations such as (vertical relations)

$$T_1^{\,2} = a_x^{\,2} + a_y^{\,2} + a_z^{\,2} - \left(\frac{c^4}{\ell^2}\right) m^2 \tag{12A}$$

and

$$T_2^{\,2} = E_x^{\,2} + E_y^{\,2} + E_z^{\,2} - \left(\frac{Q^4}{F^2}\right) Q^2 \tag{13A}$$

For the horizontal relations, we have for $[a, E]$,

$$T_3^{\,2} = a_x^{\,2} + a_y^{\,2} + a_z^{\,2} - \left(\frac{Q^2c^2}{\ell F}\right)\left(E_x^{\,2} + E_y^{\,2} + E_z^{\,2}\right) \tag{14A}$$

and for $[m, Q]$

$$T_4^{\,2} = m^2 + \left(\frac{c\hbar}{GQ^2}\right) Q^2 \tag{15A}$$

For the diagonal relations $[a, Q]$

$$T_5^{\,2} = a_x^{\,2} + a_y^{\,2} + a_z^{\,2} - \left(\frac{c^2 F}{\ell Q^2}\right) Q^2 \tag{16A}$$

and for [E, m]

$$T_6^{\,2} = E_x^{\,2} + E_y^{\,2} + E_z^{\,2} - \left(\frac{c^2 F}{\ell Q^2}\right) m^2 \tag{17A}$$

Symmetries become apparent such as for [a, E] in Eq. (4A) and [E, m] in Eq. (9A). Also [m, Q] in Eq. (6A) and [a, Q] in Eq. (8A).

$$\left(\frac{c^2}{\ell}\right) [a, E] = [\underline{E}, m] \tag{18A}$$

and

$$\left(\frac{\ell}{c^2}\right) [m, Q] = [\underline{a}, Q] \tag{19A}$$

Also, Eqs. (16A) and (17A) are related by the same set of constants,

$$\left(\frac{c^2 F}{\ell Q^2}\right) = \frac{[(a, E)]^2}{F^2} \tag{20A}$$

Wheeler[1] had attempted to geometrize the electromagnetic field and derive electric charge from his discontinuous topological structure describable in terms of the "wormhole" length, ℓ, where ℓ represented the size of the diameter of the mouth of a wormhole structure in the space-time manifold. Electrical charge is then conjectured to be due to the emission of field lines from and entering into the mouths of a wormhole embedded in the space. Many such structures would give rise to virtually produced pairs of oppositely charged particles. This formulation of a geometric description of the electromagnetic field, briefly described above, is as developed in analogy to the geometric interpretation of Einstein's for the gravitational field, but not with the overwhelming success as in general relativity. It is suggested here

that a first step in geometrizing the electromagnetic field is to assign a generalized metric in terms of many or all physical variables relevant to the particular force field to be considered. A method for constructing such a metric has been presented in Sec. V.

FOOTNOTE AND REFERENCES

* Work performed under the auspices of the U. S. Atomic Energy Commission.

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9. Note that the Minkowski notation, $s^2 = x^2 + y^2 + z^2 - c^2t^2$ is used, rather than the Einstein notation of $s^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2$ where indices run from 1 to 4 and $x_4 = -ict$, in order to more graphically demonstrate the relation of one variable time, t, to another variable space, x.

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TABLE CAPTIONS

Table I. Universal Quantal Units: Physical variables are uniquely expressed in terms of universal constants. They are also expressed in terms of ℓ, ℓ', c and the quantal force F where we define $\ell \equiv \hbar/c$ and $\ell' \equiv c\hbar$.

Table II. Generalized Heisenberg Relations: The commutation relations of the physical variables in quantal unit form involve non-zero elements c, $\hbar$, F and k which result from the product of the physical variable. The quantal force, F, is prominently manifest in the Generalized Heisenberg Relations and simplifies the formalism. The array is symmetric about the diagonal so that one-half of Table II contains all the information, including the diagonal elements which are squares of the physical quantities such as ℓ^2, t^2, m^2, etc. (where $m^2 = [m, m]$). Note: The G. N. Lewis Ultimate Rational Units results are starred in Table II and Table IV.

Table III. Generalized Heisenberg Relations and Primary Physics Laws: The elements in Table II, which are expressed in terms of universal constants and the quantal force, are rewritten in terms of their equivalent physical variables; certain physical laws result from this process, for example, $mc = E/c$ or $E = mc^2$, $a = \omega c$, $E = \ell F$, $L = p\ell$ and others. These laws are termed Primary Laws.

Table IV. Reciprocal Operations Generalized Heisenberg Relations and Primary Physics Laws: The elements of the reciprocal Generalized Heisenberg Relations are formed by the canonical product of a variable and the reciprocal of another variable. Again, as in Table III, the resulting universal constants are re-expressed in terms of their quantal unit form. Again Primary Laws of physics result.

Table V. Some Generalized Canonical Relations: Some examples of the Generalized direct and reciprocal Heisenberg Relations are given in Table V, where a variable can be canonically conjugate to more than one other variable and also where direct and inverse canonical relations such as $[\mu, \eta] = \zeta$ or $[\mu, 1/\eta] = \zeta$ result in the same constant, ζ, for different μ and η.

Table VI. Generalized Minkowski Metric: The metrical elements in Table VI are expressed in terms of the velocity of light, c, and the quantal force, F. The metrical element for (E, m) is expressed in terms of c and the quantal time, t. Note that starred elements are from Eq. (16) through Eq. (20) where complementary reciprocal forms are also represented.

Table I. Universal Quantal Units

Mechanical Quantal Units		Quantal Unit in Terms of Force, $\hbar$ and $\hbar'$.[a]	Numerical Value of Quantal Unit[b]	
$\ell = \left(\frac{G\hbar}{c^3}\right)^{1/2}$	length	$\ell = \left(\frac{\hbar'}{F}\right)^{1/2}$	1.60×10^{-33}	cm
$t = \left(\frac{G\hbar}{c^5}\right)^{1/2}$	time	$t = \left(\frac{\hbar}{F}\right)^{1/2}$	5.36×10^{-44}	sec
$m = \left(\frac{c\hbar}{G}\right)^{1/2}$	mass	$m = \left(\frac{\hbar F}{c^2}\right)^{1/2}$	2.82×10^{-5}	gm
$E = \left(\frac{c^5\hbar}{G}\right)^{1/2}$	energy	$E = (\hbar' F)^{1/2}$	1.25×10^{16}	ergs
$p = \left(\frac{c^3\hbar}{G}\right)^{1/2}$	momentum	$p = (\hbar F)^{1/2}$	4.16×10^{10}	$\frac{\text{gm-cm}}{\text{sec}}$
$L = \hbar$	angular momentum	$L = \hbar$	1.06×10^{-27}	erg-sec
$F = c^4/G$	force	$F = F$	1.22×10^{49}	dynes
$c = c$	velocity	$c = c$	3.00×10^{10}	cm/sec
$a = \left(\frac{c^7}{G\hbar}\right)^{1/2}$	acceleration	$a = \left(\frac{c^2 F}{\hbar}\right)$	5.72×10^{53}	cm/sec^2
$p = \frac{c^5}{G}$	power	$p = cF$	3.66×10^{59}	dyne $\frac{\text{cm}}{\text{sec}}$
$P = \frac{c^7}{G^2\hbar}$	pressure	$P = \frac{F^2}{\hbar'}$	4.75×10^{114}	dyne/cm^2
$\rho = \frac{c^5}{G^2\hbar}$	density	$\rho = \frac{F^2}{c^2\hbar}$	6.50×10^{93}	gm/cm
$\omega = \left(\frac{c^5}{G\hbar}\right)^{1/2}$	frequency	$\omega = \left(\frac{F}{\hbar}\right)^{1/2}$	1.91×10^{43}	cycles/sec

Table I Cont.

Thermodynamic Quantal Units		Quantal Unit in Terms of Force, ℓ and ℓ' [a]	Numerical Value of Quantal Unit [b]
$T = \frac{1}{k}\left(\frac{c^5\hbar}{G}\right)^{1/2}$	temperature	$T = \frac{1}{k}(\ell F)^{1/2}$	3.60×10^{32} degrees
$s = k$	entropy	$s = k$	1.34×10^{-16} erg/degree

[a] The quantal units are expressed in terms of the universal quantal force, $F = c^4/G$, ℓ, ℓ', and c. The quantities ℓ and ℓ' are defined as $\ell \equiv \hbar/c$ and $\ell' \equiv c\hbar$.

[b] In the evaluation of the quantal units, the values of $\ell = 3.50 \times 10^{-38}$ gm-cm and $\ell' = 3.15 \times 10^{-17}$ erg-cm have been used.

Table II. Generalized Heisenberg Relations

Physical Variable	ℓ	t	m	E	p	F	L	c	a	p	P	ρ	ω	T	S
ℓ	$\frac{c\hbar}{F}$	$\frac{\hbar}{F}$	$\frac{\hbar}{c}$ ★	$c\hbar$ ★	$\hbar$	$(c\hbar F)^{\frac{1}{2}}$	$\left(\frac{c\hbar^3}{F}\right)^{\frac{1}{2}}$	$\left(\frac{c^3\hbar}{F}\right)^{\frac{1}{2}}$	c^2	$(c^3\hbar F)^{\frac{1}{2}}$	$\left(\frac{F^3}{c\hbar}\right)^{\frac{1}{2}}$	$\left(\frac{F^3}{c^3\hbar}\right)^{\frac{1}{2}}$	c	$\frac{c\hbar}{k}$ ★	$k\left(\frac{c\hbar}{F}\right)^{\frac{1}{2}}$
t	$\frac{\hbar}{F}$	$\frac{\hbar}{cF}$	$\frac{\hbar}{c^2}$	$\hbar$	$\frac{\hbar}{c}$	$\left(\frac{\hbar}{c}\right)^{\frac{1}{2}}$	$\left(\frac{\hbar^3}{cF}\right)^{\frac{1}{2}}$	$\left(\frac{c\hbar}{F}\right)^{\frac{1}{2}}$ ★	c	$(c\hbar F)^{\frac{1}{2}}$	$\left(\frac{F^3}{c^3\hbar}\right)^{\frac{1}{2}}$	$\left(\frac{F^3}{c^5\hbar}\right)^{\frac{1}{2}}$	1	$\frac{\hbar}{k}$	$k\left(\frac{\hbar}{cF}\right)^{\frac{1}{2}}$
m	$\frac{\hbar}{c}$	$\frac{\hbar}{c^2}$	$\frac{\hbar F}{c^3}$	$\frac{\hbar F}{c}$	$\frac{\hbar F}{c^2}$	$\left(\frac{\hbar F}{c^3}\right)^{\frac{1}{2}}$	$\left(\frac{\hbar^3 F}{c^3}\right)^{\frac{1}{2}}$	$\left(\frac{\hbar F}{c}\right)^{\frac{1}{2}}$	F	$\left(\frac{\hbar F^3}{c}\right)^{\frac{1}{2}}$	$\left(\frac{F^5}{c^5\hbar}\right)^{\frac{1}{2}}$	$\left(\frac{F^5}{c^9\hbar}\right)^{\frac{1}{2}}$	$\frac{F}{c}$	$\frac{\hbar F}{kc}$	$k\left(\frac{\hbar F}{c^3}\right)^{\frac{1}{2}}$
E	$c\hbar$	$\hbar$	$\frac{\hbar F}{c}$	$c\hbar F$	$\hbar F$	$(c\hbar F^3)^{\frac{1}{2}}$	$(c\hbar^3 F)^{\frac{1}{2}}$	$(c^3\hbar F)^{\frac{1}{2}}$	$c^2 F$	$(c^3\hbar F^3)^{\frac{1}{2}}$	$\left(\frac{F^5}{c\hbar}\right)^{\frac{1}{2}}$	$\left(\frac{F^5}{c^5\hbar}\right)$	cF	$\frac{c\hbar F}{k}$	$k(c\hbar F)^{\frac{1}{2}}$
p	$\hbar$	$\frac{\hbar}{c}$	$\frac{\hbar F}{c^2}$	$\hbar F$	$\frac{\hbar F}{c}$	$\left(\frac{F^3\hbar}{c}\right)^{\frac{1}{2}}$	$\left(\frac{\hbar^3 F}{c}\right)^{\frac{1}{2}}$	$(c^3\hbar F)^{\frac{1}{2}}$	cF	$(c\hbar F^3)^{\frac{1}{2}}$	$\left(\frac{F^5}{c^3\hbar}\right)^{\frac{1}{2}}$	$\left(\frac{F^5}{c^7\hbar}\right)^{\frac{1}{2}}$	F	$\frac{\hbar F}{k}$	$k\left(\frac{\hbar F}{c}\right)^{\frac{1}{2}}$
F	$(Fc\hbar)^{\frac{1}{2}}\left(\frac{\hbar}{c}\right)^{\frac{1}{2}}$		$\left(\frac{F^3\hbar}{c^3}\right)^{\frac{1}{2}}$	$(c\hbar F^3)^{\frac{1}{2}}$	$\left(\frac{F^3\hbar}{c}\right)^{\frac{1}{2}}$	F^2	$\hbar F$	cF	$\left(\frac{c^3F^3}{\hbar}\right)^{\frac{1}{2}}$	cF^2	$\frac{F^3}{c\hbar}$	$\frac{F^3}{c^3\hbar}$	$\left(\frac{cF^2}{\hbar}\right)^{\frac{1}{2}}$	$\frac{(c\hbar F^3)^{\frac{1}{2}}}{k}$	kF
L	$\left(\frac{c\hbar^3}{F}\right)^{\frac{1}{2}}\left(\frac{\hbar^3}{cF}\right)^{\frac{1}{2}}$		$\left(\frac{\hbar^3 F}{c^3}\right)$	$(c\hbar^3 F)^{\frac{1}{2}}$	$\left(\frac{\hbar^3 F}{c}\right)^{\frac{1}{2}}$	$\hbar F$	$\hbar^2$	$c\hbar$	$(c^3\hbar F)^{\frac{1}{2}}$	$c\hbar F$	$\frac{F^2}{c}$	$\frac{\hbar F^2}{c^3}$	$(c\hbar F)^{\frac{1}{2}}$	$\frac{(c\hbar^3 F)^{\frac{1}{2}}}{k}$	$k\hbar$
c	$\left(\frac{c^3\hbar}{F}\right)^{\frac{1}{2}}\left(\frac{c\hbar}{F}\right)^{\frac{1}{2}}$		$\left(\frac{\hbar F}{c}\right)^{\frac{1}{2}}$	$(c^3\hbar F)^{\frac{1}{2}}$	$(c\hbar F)^{\frac{1}{2}}$	cF	$c\hbar$	c^2	$\left(\frac{c^5F}{\hbar}\right)$	c^2F	$\frac{F^2}{\hbar}$	$\frac{F^2}{c^2\hbar}$	$\left(\frac{c^3F}{\hbar}\right)^{\frac{1}{2}}$	$\frac{c^2}{k}(\hbar F)^{\frac{1}{2}}$	kc

Table II. Cont.

Physical Variable	ℓ	t	m	E	p	F	L	c	a	p	P	ρ	ω	T	S
a	c^2	c	F	c^2F	cF	$\left(\frac{c^3F^3}{\hbar}\right)^{\frac{1}{2}}$	$(c^3\hbar F)^{\frac{1}{2}}$	$\left(\frac{c^5F}{\hbar}\right)^{\frac{1}{2}}$	$\frac{c^3F}{\hbar}$	$\left(\frac{c^5F^3}{\hbar}\right)^{\frac{1}{2}}$	$\left(\frac{cF^5}{\hbar^3}\right)^{\frac{1}{2}}$	$\left(\frac{F^5}{c^3\hbar^3}\right)^{\frac{1}{2}}$	$\left(\frac{c^5F^2}{\hbar^2}\right)^{\frac{1}{2}}$	$\frac{c^2F}{k}$	$k\left(\frac{c^3F}{\hbar}\right)^{\frac{1}{2}}$
p	$(c^3\hbar F)^{\frac{1}{2}}$	$(c\hbar F)^{\frac{1}{2}}$	$\left(\frac{\hbar F^3}{c}\right)^{\frac{1}{2}}$	$(c^3\hbar F^3)^{\frac{1}{2}}$	$(c\hbar F^3)^{\frac{1}{2}}$	cF^2	$c\hbar F$	c^2F	$\left(\frac{c^5F^3}{\hbar}\right)^{\frac{1}{2}}$	c^2F^2	$\frac{F^3}{\hbar}$	$\frac{F^3}{c^2\hbar}$	$\left(\frac{c^3F^3}{\hbar}\right)^{\frac{1}{2}}$	$\frac{(c^2\hbar F^3)^{\frac{1}{2}}}{k}$	kcF
P	$\left(\frac{F^3}{c\hbar}\right)^{\frac{1}{2}}$	$\left(\frac{F^3}{c^3\hbar}\right)^{\frac{1}{2}}$	$\left(\frac{F^5}{c^5\hbar}\right)^{\frac{1}{2}}$	$\left(\frac{F^5}{c\hbar}\right)^{\frac{1}{2}}$	$\left(\frac{F^5}{c^3\hbar}\right)^{\frac{1}{2}}$	$\frac{F^3}{c\hbar}$	$\frac{F^2}{c}$	$\frac{F^2}{\hbar}$	$\left(\frac{cF^5}{\hbar^3}\right)^{\frac{1}{2}}$	$\frac{F^3}{\hbar}$	$\frac{F^4}{c^2\hbar^2}$	$\frac{F^4}{c^4\hbar^2}$	$\left(\frac{F^5}{c^5\hbar^5}\right)^{\frac{1}{2}}$	$\frac{1}{k}\left(\frac{F^5}{c\hbar}\right)^{\frac{1}{2}}$	$\frac{kF^2}{c\hbar}$
ρ	$\left(\frac{F^3}{c^3\hbar}\right)^{\frac{1}{2}}$	$\left(\frac{F^3}{c^5\hbar}\right)^{\frac{1}{2}}$	$\left(\frac{F^5}{c^9\hbar}\right)^{\frac{1}{2}}$	$\left(\frac{F^5}{c^5\hbar}\right)^{\frac{1}{2}}$	$\left(\frac{F^5}{c^7\hbar}\right)^{\frac{1}{2}}$	$\frac{F^3}{c^3\hbar}$	$\frac{\hbar F^2}{c^3}$	$\frac{F^2}{c^2\hbar}$	$\left(\frac{F^5}{c^3\hbar^3}\right)^{\frac{1}{2}}$	$\frac{F^3}{c^2\hbar}$	$\frac{F^4}{c^4\hbar^2}$	$\frac{F^4}{c^6\hbar^2}$	$\left(\frac{F^5}{c^5\hbar^3}\right)^{\frac{1}{2}}$	$\frac{1}{k}\left(\frac{F^5}{c^5\hbar}\right)^{\frac{1}{2}}$	$\frac{kF^2}{c^3\hbar}$
ω	c	1	F/c	cF	F	$\left(\frac{cF^2}{\hbar}\right)^{\frac{1}{2}}$	$(c\hbar F)^{\frac{1}{2}}$	$\left(\frac{c^3F}{\hbar}\right)^{\frac{1}{2}}$	$\frac{(c^5F^2)^{\frac{1}{2}}}{\hbar}$	$\left(\frac{c^3F}{\hbar}\right)^{\frac{1}{2}}$	$\left(\frac{F^5}{c^5\hbar^5}\right)^{\frac{1}{2}}$	$\left(\frac{F^5}{c^5\hbar}\right)^{\frac{1}{2}}$	$\frac{cF}{\hbar}$	$\frac{1}{k}$ cF	$k\left(\frac{cF}{\hbar}\right)^{\frac{1}{2}}$
T	$\frac{c\hbar}{k}$	$\frac{\hbar}{k}$	$\frac{\hbar F}{kc}$	$\frac{c\hbar F}{k}$	$\frac{\hbar F}{k}$	$\frac{(c\hbar F^3)^{\frac{1}{2}}}{k}$	$\frac{(c\hbar^3F)^{\frac{1}{2}}}{k}$	$\frac{c^2}{k}(\hbar F)^{\frac{1}{2}}$	$\frac{c^2F}{k}$	$\frac{(c^2\hbar F^3)^{\frac{1}{2}}}{k}$	$\frac{1}{k}\left(\frac{F^5}{c\hbar}\right)^{\frac{1}{2}}$	$\frac{1}{k}\left(\frac{F^5}{c^5\hbar}\right)^{\frac{1}{2}}$	$\frac{1}{k}$ cF	$\frac{1}{k^2}(c\hbar F)$	$(c\hbar F)^{\frac{1}{2}}$
S	$k\left(\frac{c\hbar}{F}\right)^{\frac{1}{2}}$	$k\left(\frac{\hbar}{cF}\right)^{\frac{1}{2}}$	$k\left(\frac{\hbar F}{c^3}\right)^{\frac{1}{2}}$	$k(c\hbar F)^{\frac{1}{2}}$	$k\left(\frac{\hbar F}{c}\right)^{\frac{1}{2}}$	kF	$k\hbar$	kc	$k\left(\frac{c^3F}{\hbar}\right)$	kcF	$\frac{kF^2}{c\hbar}$	$k\left(\frac{cF}{k}\right)^{\frac{1}{2}}$	$(c\hbar F)^{\frac{1}{2}}$	k^2	

Table III. Generalized Heisenberg Relations and Primary Physics Laws

Physical Variable	ℓ	t	m	E	p	F	L	c	a	$\mathcal{p}$	$\mathcal{P}$
ℓ	ℓ^2	$\hbar/F$	$\hbar/c$	$c\hbar$	$\hbar$	E	$\hbar\ell$	$\hbar/m$	c^2	cE	$\hbar p$
t	$\hbar/F$	t^2	$\hbar/c^2$	$\hbar$	$\hbar/c$	$(\hbar/c)^{\frac{1}{2}}$		ℓ	c	E	
m	$\hbar/c$	$\hbar/c^2$	m^2	p^2	p^2/c	$\frac{p^3}{\hbar}$	$\frac{p^3}{F}$	$\frac{E}{c}$	F	Fp	
E	$c\hbar$	$\hbar$	p^2	E^2	$\hbar F$				c^2F	$\frac{E^3}{\hbar}$	
p	$\hbar$	$\hbar/c$	p^2/c	$\hbar F$	p^2	$\frac{cp^3}{\hbar}$		E	$\mathcal{p}$	FE	
F	E	$\left(\frac{\hbar}{c}\right)^{\frac{1}{2}}$	$\frac{p^3}{\hbar}$			F^2		$\mathcal{p}$		cF^2	$\frac{F^2}{\ell^{\frac{1}{2}}}$
L	$\hbar\ell$						L^2		cE	E^2	
c	$\hbar/m$	ℓ	E/c		E	$\mathcal{p}$		c^2	c^2/t	c^2F	
a	c^2	c	F	c^2F	$\mathcal{p}$		cE	c^2/t	a^2	$c\hbar t^3$	
$\mathcal{p}$	cE	E	Fp		FE	cF^2	E^2	c^2F	$c\hbar t^3$	$\mathcal{p}^2$	
$\mathcal{P}$	$\hbar/\mathcal{p}$					$\frac{F^2}{\ell^{\frac{1}{2}}}$					$\mathcal{p}^2$

Table IV. Reciprocal Operations Generalized Heisenberg Relation and Primary Physics Laws

	ℓ	t	m	E	p	F	L	c	a	p
$1/\ell$	1	$1/c$	F/c^2	F ★	F/c ★	$(Fp)^{\frac{1}{2}}$	cm	$1/t$	$1/t^2$	tF
$1/t$	c	1	F/c	cF ★	F ★	$c(Fp)^{\frac{1}{2}}$	$(cF)^{\frac{1}{2}}$	a ★	c/t^2	aF
$1/m$	c^2/F	c/F	1	c^2 $(E = mc^2)$	c	a ★	$c\ell$	$\left(\frac{c^2}{\hbar}\right)\ell$	$\frac{c^3}{\hbar}$	ca
$1/E$	$1/F$	$\frac{1}{cF}$	$1/c^2$	1	$1/c$	$1/\ell$	c/a	$1/p$	$c/\hbar$	$1/t$
$1/p$	c/F	$1/F$	$1/c$	c	1	$1/t$	ℓ	$1/m$	$c^2/\hbar$	a
$1/F$	$\frac{1}{(Fp)^{\frac{1}{2}}}$	$\frac{1}{c(Fp)^{\frac{1}{2}}}$	$\frac{1}{a}$	ℓ	t	1	$\hbar/F$	c/F	$1/m$	c
$1/L$	$1/cm$	$\frac{1}{(cF)^{\frac{1}{2}}}$	$1/c\ell$	a/c	$1/\ell$	$F/\hbar$	1	$c/\hbar$	$\frac{c}{\hbar^2}E^{\frac{1}{2}}$	$\frac{cF}{\hbar}$
$1/c$	t	$1/a$	$\frac{\hbar}{c^2}\frac{1}{\ell}$	p	m	F/c	$\hbar/c$	1	$1/t$	F
$1/a$	t^2	$\frac{t^2}{c}$	$\frac{\hbar}{c^3}$	$\hbar/c$	$\hbar/c^2$	m	$\frac{\hbar^2}{c}\frac{1}{E^{\frac{1}{2}}}$	$\left(\frac{\hbar}{cF}\right)^{\frac{1}{2}} = t$	1	p
$1/p$	$1/tF$	$\frac{1}{aF}$	$\frac{1}{ca}$	t	$1/a$	$1/c$	$\frac{\hbar}{cF}$	$1/F$	$1/p$	1

Table V. Some Generalized Canonical Relations

$[F, \frac{1}{m}] = a$	$[p, \frac{1}{c}] = F$
$[c, \frac{1}{t}] = a$	$[E, \frac{1}{\ell}] = F$
$[p, p] = a$	$[p, \frac{1}{t}] = F$
$[E, \frac{1}{c}] = p$	$[p, \frac{1}{c}] = m$
$[p, \frac{1}{a}] = p$	$[F, \frac{1}{a}] = m$ ($\Longrightarrow F = ma$)
$[p, \frac{1}{m}] = c$	$[\ell, \frac{1}{c}] = t$
$[p, \frac{1}{F}] = c$	$[E, \frac{1}{p}] = t$
$[E, \frac{1}{p}] = c$	$[E, \frac{1}{a}] = t$
$[\ell, \frac{1}{t}] = c$	

Table VI. Generalized Minkowski Metric

	$1/x^2$	$1/t^2$	$1/m^2$	$1/E^2$	$1/p^2$
x^2	1	c^2 ★	$\frac{c^4}{F^2}$	$\frac{1}{F^2}$ ★	$\frac{c^2}{F}$ ★
t^2	$\frac{1}{c^2}$ ★	1	$\frac{c^2}{F^2}$	c^2F ★	$\frac{1}{F^2}$ ★
m^2	$\frac{F^2}{c^4}$	$\frac{F^2}{c^2}$	1	$\frac{t^2}{c^2}$	$\frac{1}{c^2}$
E^2	F^2 ★	$\frac{1}{c^2F}$ ★	$\frac{c^2}{t^2}$	1	$\frac{1}{c^2}$ ★
p^2	$\frac{F}{c^2}$ ★	F^2 ★	c^2	c^2 ★	1

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