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UNITARIZATION OF THE DUAL RESONANCE AMPLITUDE

I. PLANAR N-LOOP AMPLITUDE*

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August 13, 1970

ABSTRACT

We present the planar multiloop formula in the dual resonance model. Both the multiple factorization formulation and the Sciuto three-reggeon formulation are given in this paper.

I. INTRODUCTION

After the planar single-loop amplitude[1] in the dual resonance model had been found, the nonplanar single-loop amplitude[2] was soon calculated. To complete the unitarization program of the dual amplitude, we require the calculation of the multiloop diagrams. In this series of papers, we will present the results for the planar, nonplanar, and overlapping multiloop amplitudes. In this paper we calculate the planar-loop amplitudes[3] both from the multiply factorized tree[4] and from the Sciuto three-reggeon vertex.[5]

The final form of the N-loop planar amplitude will be expressed in a projectively invariant manner via Koba-Nielsen variables.[6] The first derivation has the advantage of being projectively invariant from the start; the second calculation, however, exhibits projective invariance only when all traces are performed.

Remarkably, the N-loop planar amplitude exhibits much of the same structure as the original single-loop amplitude. Indeed, arguing from projective invariance and the dual diagrams of KSV, Mandelstam was able to predict the essential features of the N-loop amplitude.

The original ideas of KSV assume a particularly intuitive and pleasing form when reexpressed in the language of projective transformations. In their paper, an M-point N-loop amplitude is represented by an M-sided polygon enclosing N internal points; different triangulations of the dual diagram yield the various terms in the integrand of the Veneziano amplitude. In the single-loop amplitude, there are an infinite number of triangulations which circle about the internal point.

Therefore, it is not surprising to find in the integrand such factors as

$$\prod_{r=1}^{\infty} (y_i - w^r y_j)^{-k_i k_j}, \tag{1.1}$$

where k represents the external momenta and y the various Koba-Nielsen variables of integration. Though w is a c-number variable, it can be seen to be a special case of an arbitrary projective transformation. If, for example, an arbitrary projective transformation is performed upon the y's, then the true character of w as a projective operator is easily seen:

$$\prod_{i,j} (y_i - w^r y_j)^{-k_i \cdot k_j} = \prod_{i,j} (R^{-1}Ry_i - R^{-1}Rw^r R^{-1}Ry_j)^{-k_i \cdot k_j}$$

$$= \prod_{i,j} [R^{-1}\bar{y}_i - R^{-1}(RwR^{-1})^r \bar{y}_j]^{-k_i \cdot k_j} = \prod_{i,j} (\bar{y}_i - Q^r \bar{y}_j)^{-k_i \cdot k_j}, \tag{1.2}$$

where

$$\sum_i k_i = 0, \quad \sum_j k_j = 0, \quad \bar{y}_i = Ry_i, \tag{1.3}$$

$$Q = RwR^{-1}, \tag{1.4}$$

and

$$R(z) = \frac{Az + B}{Cz + D}, \qquad AD - BC \neq 0 . \tag{1.5}$$

If we wish to write our amplitude in projectively invariant form, we must therefore replace w by a more general projective transformation Q. One might expect, therefore, that since there is one projective transformation representing the single loop, there might be N projective transformations for N loops. Different combinations of these N projective operators, raised to various powers, would then represent the different paths a line can make through the N points of the dual diagram. We are led, therefore, to consider the integrand

$$\prod_{\{\bar{R}\}} \prod_{i,j} (y_i - \bar{R}y_j)^{-k_i \cdot k_j} , \tag{1.6}$$

where $\bar{R}$ represents the totality of products one can construct out of N-projective operators R_i and their powers.

Likewise, the singularity structure for the N-loop amplitude can be considered a natural generalization of the single-loop case. Since the singularity of the single loop is expressed entirely in terms of w, one would like to express w as an invariant associated with the projective transformation Q. This may be done by simply defining w as the ratio of the eigenvalues of Q. Furthermore, all operators Q' related to Q by a similarity transformation have eigenvalues whose ratio is equal to w or its inverse.

At this point, a small digression on projective transformations will prove beneficial, especially when the singularity structure is analyzed in detail. Many of the definitions used in this paper are taken from Ford's Automorphic Functions.[7]

The parametrization of a projective operator R in terms of A, B, C, and D is simple but clumsy; a much more convenient parametrization can be found if one expresses R in terms of its two "invariant points" x_1 and x_2 and its "multiplier" X_R. With these variables, we can express R quite elegantly:

$$R(z) = z', \tag{1.7}$$

$$\frac{z' - x_1}{z' - x_2} = X_R^{-1} \frac{z - x_1}{z - x_2}, \tag{1.8}$$

$$R(x_1) = x_1, \qquad R(x_2) = x_2. \tag{1.9}$$

Quite simply, we see that w is the multiplier of Q. There are several identities that will prove useful for our discussion:

$$X_{RQ} = X_{QR}, \tag{1.10}$$

$$X_R^{\,n} = X_{(R^n)}, \tag{1.11}$$

$$R(z) = \frac{z(x_2 - x_1 X_R) - x_1 x_2 (1 - X_R)}{z(1 - X_R) + (x_2 X_R - x_1)}, \tag{1.12}$$

$$\begin{aligned} 0 < X_R < 1 \;:\; & R^{\infty}(z_1) = x_2, \quad z_1 \neq x_1, \\ & R^{-\infty}(z_2) = x_1, \quad z_2 \neq x_2. \end{aligned} \tag{1.13}$$

The multiplier is most easily found by taking the trace of R (all matrices shall have unit determinant):

$$\mathrm{Tr}\begin{pmatrix} X_R^{\frac{1}{2}} & 0 \\ 0 & X_R^{-\frac{1}{2}} \end{pmatrix} = \mathrm{Tr}\ R = \frac{A + D}{(AD - BC)^{\frac{1}{2}}} \equiv \Phi_R . \tag{1.14}$$

Then

$$X_R = \frac{\Phi_R^{\ 2} - 2 \pm \Phi_R(\Phi_R^{\ 2} - 4)^{\frac{1}{2}}}{2} . \tag{1.15}$$

Projective transformations are readily categorized by the value of the multiplier:

$$\begin{aligned} &R \text{ is hyperbolic if } X_R \text{ is real and } \neq 1 \quad (X_R > 0), \\ &R \text{ is parabolic if } X_R \text{ is real and } = 1, \\ &R \text{ is elliptic if } |X_R| = 1 \text{ and } X_R \neq 1, \\ &R \text{ is loxodromic if } X_R \text{ is complex and none of the above.} \end{aligned} \tag{1.16}$$

Using these identities, we can show that the singularity of the single loop resembles the closed paths one can make around the internal point of the dual diagram:

$$\prod_{n=1}^{\infty} (1 - X_R^{\ n})^{-4} = \prod_{n=1}^{\infty} [1 - X_{(R^n)}]^{-4}. \tag{1.17}$$

Starting with this intuitive approach, Mandelstam has conjectured that the singularity of the N-loop amplitude is simply

$$\prod_{\{\bar{R}\}} (1 - X_{\bar{R}})^{-4}, \tag{1.18}$$

where $\bar{R}$ now represents the totality of <u>closed</u> paths one can make around N or fewer internal points of the dual diagram. Notice that

this function contains the product of single loop divergences as a subset.

With this projective approach to higher loop amplitudes, one can also easily choose the most convenient set of variables. Since there are $M - 3 + 3N$ variables of integration in an M-point N-loop amplitude, one expects to have one variable for each of the M-scalar external lines. We choose the multiplier and the two invariant points as the variables for each loop for a total of $3N$ variables. Finally, three of these variables are fixed.

By using the methods outlined in a previous paper,[8] linear dependencies will be subtracted out and found to give a correction factor of

$$\prod_{\alpha=\{\mathcal{L}\}} (1 - X_{R_\alpha}) \tag{1.19}$$

(where α ranges over all the N-loop indices $\{\mathcal{L}\}$) when the calculation is performed with Sciuto vertices. The projected propagator[9] is used in the calculation with the multiply factorized tree, with identical results. We now present the N-loop amplitude,

$$\int \prod_{\alpha=\{\mathcal{L}\}} d^4k_\alpha \int_{U_1} \prod_{\beta=\{\mathcal{L}\}} dX_\beta \int_{U_2} \prod_{i=0}^{S+1} dw_i (dw_a dw_b dw_c)^{-1}$$

$$\times \quad X_\beta^{-\alpha(k_\beta)-1} (1 - X_\beta) \prod_{\{Q\}} (1 - X_Q)^{-4} \prod_{n=0}^{\infty} \prod_{\substack{i,j=0 \\ (i\neq\alpha,\alpha+1;\, j\neq\beta,\beta+1) \\ (i\neq j \text{ if } n=0)}}^{S+1} [w_i - (R^{\pm})^{(n)}_{\alpha\beta} w_j)^{-\frac{1}{2}k_i k_j}$$

Equation (1.20) continued

Equation (1.20) continued.

$$\frac{(w_a - w_b)(w_b - w_c)(w_c - w_a)}{\prod_{\substack{i=0 \\ i\neq\{\mathcal{L},\mathcal{L}+1,\mathcal{L}-1\}}}^{S+1}(w_i - w_{i+1})^{1-\alpha_0} \prod_{\gamma=\{\mathcal{L}\}}(w_\gamma - w_{\gamma+1})^2 \left\{\frac{(w_{\gamma+1} - w_{\gamma+2})[w_{\gamma-1} - R_\gamma(w_{\gamma+2})]}{[w_{\gamma+1} - R_\gamma(w_{\gamma+2})]}\right\}^{1-\alpha_0}}, \tag{1.20}$$

where $\alpha_0 = -\frac{1}{2}m^2$, $w_{S+2} = w_0$, (w_a, w_b, w_c) = fixed points,

$\{\mathcal{L}\}$ = set of all loop indices = $\{\alpha, \beta, \gamma, \cdots, \sigma\}$,

w_α = invariant point of $R_\alpha = R_\alpha^{\infty}(z_1), z_1 \neq w_{\alpha+1}$,

$w_{\alpha+1}$ = invariant point of $R_\alpha = R_\alpha^{-\infty}(z_2), z_2 \neq w_\alpha$,

$\{Q\}$ = set of all closed paths,

$(R^{\pm})^{(n)}_{\alpha\beta} = (R_\alpha^{\pm} \cdots R_\beta^{\pm})$ = set of all open paths (where ... means all possible combinations of $n - 2$ R's such that no R ever appears next to R^{-1}).

This amplitude is formed by joining the factorized legs in Fig. 4 in pairs, such that α and $\alpha + 1$ are joined.

Remarkably, the cyclic ordering of the Koba-Nielsen variables mimics the ordering of Fig. 4, if we associate the invariant points w_α and $w_{\alpha+1}$ with the αth and $\alpha + 1$st factorized legs.

If we ignore linear dependences, then we find the following cyclic ordering (see Fig. 6a):

$$U_2 = (w_\alpha \le w_{\alpha-1} \le w_{\alpha-2} \le \cdots \le w_0 \le w_1 \le w_{S+1} \le \cdots$$

$$\le w_{\sigma+2} \le w_{\sigma+1} \le w_\sigma \cdots \le w_{\alpha+2} \le w_{\alpha+1} \le w_\alpha),$$

where $U_1 = (0,1)$ for each loop,

$$(w_\alpha \equiv x_\alpha^{(2)},\ w_{\alpha+1} \equiv x_\alpha^{(1)}). \tag{1.21}$$

The external line variables are interspersed among the invariant points, except that no lines occur between α and $\alpha + 1$ (otherwise, we would have a nonplanar diagram).

We recover the single loop diagram if we let

$$\{\mathcal{L}\} = \{\alpha\}, \quad R_\alpha = X_\alpha, \quad w_{\alpha+2} = w_a = 1,$$

$$w_\alpha = w_b = 0, \quad w_{\alpha+1} = w_c = \infty,$$

$$U_1 = (0,1),\ U_2 = (0 = w_\alpha \le w_{\alpha-1} \le w_{\alpha-2} \le \cdots$$

$$\le w_1 \le w_0 \le w_{S+1} \le \cdots \le w_{\alpha+3} \le w_{\alpha+2} = 1).$$

<u>When linear dependences are added</u>, we find rigid constraints on U_2 which allow periodicities, as demanded by duality. In fact, the conditions are rigid enough to determine U_1 entirely in terms of U_2 (though all multipliers still range from zero to one, they are no longer independent). In studying periodicities, it is more convenient to move all external line variables away from the invariant points, so that all invariant points lie together. This rearranging of external lines is always possible, because the $\alpha + 2$nd line (which lies to the left of $\alpha + 1$ and α) can flip to the right of $\alpha + 1$ and α by duality. [Mathematically, this corresponds to the following: $w_{\alpha+2} \le w_{\alpha+1} \le w_\alpha \rightarrow w_{\alpha+1} \le w_\alpha \le R_\alpha(w_{\alpha+2})$.] This flipping of external line variables past loops is simply a consequence of "rubber band" duality.

Now that all external line variables are outside the region occupied by the invariant points, we find that a simple reordering yields

$$U_2 = (w_{S+1} \leq w_S \leq \cdots \leq w_1 \leq w_0 \leq w_{\sigma+1}$$
$$\leq w_\sigma \leq \cdots \leq w_{\beta+1} \leq w_\beta \leq w_{\alpha+1} \leq w_\alpha \leq w_{S+1}).$$

(See Fig. 6b); $x_\alpha^{(2)} \equiv w_\alpha$, $x_\alpha^{(1)} \equiv w_{\alpha+1}$.) Since the action of R_α flips lines across the $\underline{\alpha}$th loop's invariant points, the action of $R_\alpha R_\beta \cdots R_\sigma$ is to flip lines across <u>all</u> loops, until we regain the original ordering (i.e., we have rotated the external line variables 360°). It has been shown that the invariant points $x^{(1)}$ and $x^{(2)}$ of the product $(R_\alpha R_\beta \cdots R_\sigma)^{-1}$ lie outside the region occupied by the invariant points of the individual R's, and hence $x^{(1)}$ and $x^{(2)}$ divide the external line variables from the invariant points:

$$U_2 = [x^{(1)} \leq w_{S+1} \leq w_S \leq \cdots \leq w_1 \leq w_0 \leq x^{(2)} \leq w_{\sigma+1} \leq w_\sigma$$
$$\leq \cdots \leq w_{\beta+1} \leq w_\beta \leq w_{\alpha+1} \leq w_\alpha \leq x^{(1)}].$$

It can be shown that the region occupied by the original external line variables $w_i (i = 0$ to $S + 1)$ and the regions occupied by $(R_\alpha R_\beta \cdots R_\sigma)^n (w_i)$ are disjoint $(n \neq 0)$. As $n \to \infty$ $(-\infty)$ we merely approach the point $x^{(1)}[x^{(2)}]$. Therefore, we can subtract the periodicities due to these disjoint regions by integrating over only one of them, i.e., integrate one of the variables from y_0 to $R_\alpha R_\beta \cdots R_\sigma (y_0)$, where y_0 lies between $x^{(1)}$ and $x^{(2)}$. In summary, we find

$$U_2 = [w_\alpha \le x^{(1)} \le R_\alpha R_\beta \cdots R_\sigma(w_0) \le w_{S+1} \le w_S \le \cdots \le w_0 \le x^{(2)}$$

$$\le w_{\sigma+1} \le w_\sigma \le \cdots \le w_{\beta+1} \le w_\beta \le w_{\alpha+1} \le w_\alpha]$$

and

$$[R_\alpha R_\beta \cdots R_\sigma(y_0) \le w_0 \le y_0 \le x^{(2)}].$$

[Notice that the factor in the brace in (1.20) changes to reflect the different topology as we move external lines away from loops.] Given the above cyclic ordering, we have also shown that applications of R_α to the various lines does not produce a variable which lies between the invariant points $w_{\beta+1}$ and w_β of a different loop (i.e., planar graphs do not degenerate to nonplanar ones). If linear dependences were not taken into account, we would have had the simple cyclic ordering in (1.21).

In a later paper, we will show that multiloop amplitudes with a different quark topology differ only in the arrangement of invariant points (i.e., the nonplanar graph has external line variables between the invariant points of the <u>same</u> loop).

II. MULTIPLE FACTORIZATION FORMULATION OF THE PLANAR MULTILOOP AMPLITUDE

In order to get the loop amplitude from the multiply factorized tree,[4] we need (a) prescriptions for joining two excited legs and (b) the principal-axes method. We will discuss these two subjects in detail by considering the planar single-loop diagram. The prescriptions and the method used in the single-loop case are applicable without modifications to the multiloop case.

A. The Sewing Prescriptions for Two Excited Legs

We adopt the notations of Kikkawa and Sato.[10] They define

$$|k_i) = \left(\frac{k_i}{1^{\frac{1}{2}}}, \frac{k_i}{2^{\frac{1}{2}}}, \frac{k_i}{3^{\frac{1}{2}}}, \cdots\right) ,$$

$$|a^N) = (a_1^N, a_2^N, a_3^N, \cdots),$$

$$(M_+)_{nm} = (M_+^T)_{mn} = \left(\frac{n}{m}\right)^{\frac{1}{2}} (-)^n \binom{-m}{n} ,$$

$$(M_-)_{nm} = (M_-^T)_{mn} = \left(\frac{m}{n}\right)^{\frac{1}{2}} (-)^m \binom{n}{m},$$

$$(M_0)_{nm} = \delta_{mn},$$

$$(a|x = x|a) = \sum_{n=1}^{\infty} x^n a_n,$$

$$(a|x|b) = \sum_{n=1}^{\infty} a_n x^n b_n,$$

$$(a|xM_\pm yM_\pm z|b) = \sum_{n,m,k=1}^{\infty} a_n x^n (M_\pm)_{nm} y^m (M_\pm)_{mk} z^k b_k. \tag{2.1}$$

In the Appendix A, we give useful identities used in this paper. Let us start from the twice-factorized tree obtained in Ref. 4 (Fig. 1),

$$\langle \lambda_{a^\alpha} | G^{(2)}_{(Y)}(a^{\alpha+}, a^{\alpha+1}) | \lambda_{a^{\alpha+1}} \rangle = \int \prod dy_i \{Y_{S+2}\}$$

$$\times \langle \lambda_{a^\alpha} | \exp \left\{ (a^{\alpha+} | p_\alpha(\alpha+2) | a^{\alpha+1}) + \sum_{\substack{i=0 \\ (i \neq \alpha, \alpha+1)}}^{S+1} (a^{\alpha+} | p_\alpha(i) | k_i) \right.$$

$$\left. + \sum_{\substack{i=0 \\ (i \neq \alpha, \alpha+1)}}^{S+1} (a^{\alpha+1} | p_{\alpha+1}(i) | k_i) \right\} | \lambda_{a^{\alpha+1}} \rangle, \tag{2.2}$$

where

$$p_\alpha(i) = p(\alpha, \alpha+1, \alpha-1, i) \equiv \frac{(y_\alpha - y_{\alpha-1})(y_{\alpha+1} - y_i)}{(y_{\alpha+1} - y_{\alpha-1})(y_\alpha - y_i)}, \tag{2.3a}$$

$$p_{\alpha+1}(i) = p(\alpha + 1, \alpha, \alpha + 2, i) = \frac{p_\alpha(\alpha + 2)}{p_\alpha(i)},$$

and the symbol $\{Y_{S+2}\}$ is the ordinary Koba-Nielsen integrand for the tree containing $S + 2$ scalar legs, i.e.,

$$\{Y_{S+2}\} = (y_a - y_b)(y_b - y_c)(y_c - y_a) \prod_{\substack{i,j=0 \\ (i \neq j)}}^{S+1} (y_i - y_j)^{-\frac{1}{2} k_i \cdot k_j}$$

$$\times \prod_{i=0}^{S+1} (y_i - y_{i+1})^{-\alpha_0 - \frac{1}{2}(k_i^2 + k_{i+1}^2) - 1} \prod_{i=0}^{S+1} (y_i - y_{i+2})^{\alpha_0 + \frac{1}{2} k_{i+1}^2}, \tag{2.4}$$

where y_a, y_b, y_c are any three fixed points of the set $\{y_0, y_1, \cdots y_{S+1}\}$. The ordering of the y_i's is

$$y_i \geq y_j, \quad \text{if} \quad i < j. \tag{2.5}$$

We have to join the $a^{\alpha+}$ leg with the $a^{\alpha+1}$ leg to get the planar single-loop amplitude.[1] We observe that the coherent states $|\lambda_{a^\alpha}\rangle$ and $|\lambda_{a^{\alpha+1}}\rangle$ are defined in exactly the same way.[4] Thus, in joining the $a^{\alpha+}$ leg to the $a^{\alpha+1}$ leg in Eq. (2.2), or, in taking the trace of Eq. (2.2), we should take $|\lambda_{a^\alpha}\rangle \equiv |\lambda_{a^{\alpha+1}}\rangle$. Although $a_n^{\alpha}(a_n^{\alpha+})$ and $a_m^{\alpha+1}$ $(a_m^{\alpha+1^+})$ are two commuting operators defined on two disjointed Hilbert spaces, they actually describe identical states (Fig. 1). More explicitly, in taking the trace of Eq. (2.2), we carry out the following procedures:

(a) Insert Gross and Schwarz's[9] spurion-free propagator to the right of $\langle\lambda_{a^\alpha}|$ state and integrate over d^4k_α, i.e., insert the expression[11]

$$\int d^4k_\alpha \, \mathcal{P}^+(-k_\alpha)\, D(k_\alpha)\, \Omega(k_\alpha)\, \mathcal{P}(k_\alpha)\, \Omega(k_\alpha)$$

$$= \int d^4k_\alpha \int_0^1 dt \left(\frac{t}{t-1}\right)^{-\ell(k_\alpha)-1} (1-t)^{A^+(-k_\alpha)-1} \left(\frac{t}{t-1}\right)^R (1-t)^{A(k_\alpha)-1} \times \Omega(k_\alpha) \tag{2.6}$$

$$= \int d^4k_\alpha \int_0^1 dt\, t^{-\ell(k_\alpha)-1} (1-t)^{\alpha_0-1-\frac{1}{2}k_\alpha^2}\, t^R\, \Omega(k_\alpha) \left(\frac{t-1}{t}\right)^R \Omega^+(-k_\alpha) \times t^R\, \Omega(k_\alpha), \tag{2.7}$$

where

$$\ell(k_\alpha) = \alpha_0 + \frac{1}{2}k_\alpha^{\,2},$$

$$R = \sum_{n=1}^{\infty} n\, a_n^{\alpha+}\, a_n^{\alpha},$$

$$\Omega(k_\alpha) = \exp(a^{\alpha+}|k_\alpha) : \exp(a^{\alpha+}|M_- - I|a^\alpha) :$$

$$\Omega^+(-k_\alpha) = : \exp(a^{\alpha+}|M_-^{\,T} - I|a^\alpha) : \exp(a^\alpha|k_\alpha), \tag{2.8}$$

and the expressions (2.6) or (2.7) to be placed to the right of the state $\langle\lambda_{a^\alpha}|$. From Eqs. (2.7) and (2.8) it can be shown that

$$\langle\lambda_{a^\alpha}|\mathcal{P}^+(-k_\alpha)\, D(k_\alpha)\, \Omega(k_\alpha)\mathcal{P}(k_\alpha)\, \Omega(k_\alpha)$$

$$= \int_0^1 dt\, t^{-\ell(k_\alpha)-1}(1-t)^{\alpha_0-1+\frac{1}{2}k_\alpha^{\,2}} \langle 0|\exp\{(\lambda^*_{a^\alpha} - k_\alpha|M_-^{\,T}\frac{t}{t-1}|a^\alpha)\}. \tag{2.9}$$

(b) Replace the notations $(a^{\alpha+}|$, $(a^{\alpha+1}|$ in Eq. (2.2) by the complex parameters λ^*_α, $\lambda_{\alpha+1}$, i.e.,

$$(a^{\alpha+}| \rightarrow (\lambda^*_\alpha - k_\alpha|[M_-^{\,T}\frac{t}{t-1}]\,, \tag{2.10a}$$

$$(a^{\alpha+1}| \rightarrow (\lambda_{\alpha+1}|. \tag{2.10b}$$

Equation (2.10a) is suggested from Eq. (2.9). It is obtained by commuting $(a^\alpha|$ in Eq. (2.9) to the rightmost of Eq. (2.2) and annihilating on the vacuum.

(c) Set $|\lambda_\alpha) = |\lambda_{\alpha+1})$, $k_\alpha = -k_{\alpha+1}$ and then integrate over the complex λ plane, i.e., insert the integral

$$\int d\left|\frac{\lambda}{\sqrt{2}}\right) d\left|\frac{\lambda^*}{\sqrt{2}}\right) \exp\{-(\lambda^*|\lambda)\}. \tag{2.11}$$

After performing the above steps (a), (b), (c), we get the planar single-loop amplitude (Fig. 2); call it FPL(1):

$$\mathrm{FPL}(1) = \int d^4k_\alpha \int_0^1 dt\, t^{-\ell(k_\alpha)-1} (1-t)^{\alpha_0-1+\frac{1}{2}k_\alpha^2} \int \prod dy_i \{Y_{S+2}\} I, \tag{2.12}$$

where $k_\alpha = -k_{\alpha+1}$, and

$$I = \int d\left|\frac{\lambda}{\sqrt{2}}\right) d\left|\frac{\lambda^*}{\sqrt{2}}\right) \exp\{-(\lambda^*|\lambda) + (\lambda^*|Y|\lambda) + (\lambda^*|F) + (\lambda|E)\}, \tag{2.13}$$

with

$$Y \equiv M_-^T \frac{t}{t-1} p_\alpha(\alpha+2), \quad Y' \equiv M_-^T \frac{t}{t-1}, \tag{2.14a}$$

$$|F\rangle \equiv \sum_{\substack{i=0 \\ (i\neq\alpha,\alpha+1)}}^{S+1} Y'\, p_\alpha(i)|k_i), \tag{2.14b}$$

$$\begin{aligned} |E) &\equiv \sum_{\substack{i=0 \\ (i\neq\alpha,\alpha+1)}}^{S+1} p_{\alpha+1}(i)|k_i) - |k_\alpha), \\ &= \sum_{\substack{i=0 \\ (i\neq\alpha,\alpha+1)}}^{S+1} p_{\alpha+1}(i)|k_i) + p_{\alpha+1}(\alpha+2)|k_{\alpha+1}) + p_{\alpha+1}(\alpha)|k_\alpha), \\ &\equiv \sum_{\substack{i=0 \\ (i\neq\alpha,\alpha+1)}}^{S+1} p_{\alpha+1}\begin{bmatrix} i \\ \alpha \\ \alpha+2 \end{bmatrix}\Bigg| \begin{matrix} k_i \\ k_\alpha \\ k_{\alpha+1} \end{matrix}\Bigg). \end{aligned} \tag{2.14c}$$

We have kept the sum of momenta equal to zero in Eq. (2.14c).

Equation (2.13) would have been a simple Gaussian integral if Y were a c number, as was the case when we neglected the spurion problem. However, now Y is an infinite-dimensional matrix in the oscillator indices m and n, so we have to evalute Eq. (2.13) by the principal-axes method.[12]

B. The Principal-Axes Method

We write Eq. (2.13) as

$$I = \int d\left|\frac{\lambda}{\sqrt{2}}\right) d\left|\frac{\lambda^*}{\sqrt{2}}\right) \exp\left\{-\frac{1}{2}\left((\lambda|,(\lambda^*|\right)\begin{pmatrix} 0 & [I] - [Y]^T \\ [I] - [Y] & 0 \end{pmatrix}\begin{pmatrix} |\lambda) \\ |\lambda^*) \end{pmatrix} + \begin{pmatrix} (E| & 0 \\ 0 & (F| \end{pmatrix}\begin{pmatrix} |\lambda) \\ |\lambda^*) \end{pmatrix}\right\}. \tag{2.15}$$

We regard $\begin{pmatrix} |\lambda) \\ |\lambda^*) \end{pmatrix}$ as a two-component vector; each component itself is an infinite dimensional vector defined in the harmonic oscillator space. To perform the integrations over λ and λ^*, we imagine ourselves to have gone to the principal axes of the first matrix in Eq. (2.15), called the Δ matrix. Then the Δ matrix becomes diagonal and we can do the integrations over λ^*,λ explicitly, since both integrals are simple Gaussian integrals. After having done the λ,λ^* integrations, we can imagine ourselves going back to the original axes of the Δ matrix. Thus the integral of Eq. (2.15) gives the result

$$I = \frac{1}{(\det[\Delta])^{\frac{1}{2}}} \exp\left\{\frac{1}{2}\left((E|,(F|\right)\frac{1}{[\Delta]}\begin{pmatrix} |E) \\ |F) \end{pmatrix}\right., \tag{2.16}$$

where

$$[\Delta] = \begin{pmatrix} 0 & [I] - [Y]^T \\ [I] - [Y] & 0 \end{pmatrix} \equiv [G] - [H], \tag{2.17a}$$

$$[G] = \begin{pmatrix} 0 & [I] \\ [I] & 0 \end{pmatrix}, \tag{2.17b}$$

$$[H] = \begin{pmatrix} 0 & [Y]^T \\ [Y] & 0 \end{pmatrix}, \tag{2.17c}$$

$$[GH] = \begin{pmatrix} [Y] & 0 \\ 0 & [Y]^T \end{pmatrix}. \tag{2.17d}$$

We can formally expand the matrix $\frac{1}{[\Delta]}$ as a power series in the $[GH]$ matrix

$$\frac{1}{[\Delta]} = \frac{1}{[G] - [H]} = \sum_{n=0}^{\infty} [GH]^n [G]$$

$$= \begin{pmatrix} 0 & [I] \\ [I] & 0 \end{pmatrix} + \begin{pmatrix} [Y] & 0 \\ 0 & [Y]^T \end{pmatrix} [G] + [GH]^2 [G] + \cdots . \tag{2.18}$$

Hence Eq. (2.16) becomes

$$I = \frac{1}{(\det[\Delta])^{\frac{1}{2}}} \exp\left\{ \frac{1}{2} \sum_{n=0}^{\infty} \left((E|, (F| \right) [GH]^n \begin{pmatrix} |F) \\ |E) \end{pmatrix} \right\} \tag{2.19a}$$

$$= \frac{1}{(\det[\Delta])^{\frac{1}{2}}} \exp\left\{ (E|F) + (E|Y|F) + (E|Y^2|F) + \cdots \right\}. \tag{2.19b}$$

We now need to calculate Eq. (2.19b), order by order, in the [GH] matrix. Some useful identities are

$$M_{-}x|k_i) = (1 - x)|k_i) - |k_i), \tag{2.20a}$$

$$M_{-}^{T}x|k_i) = \left(\frac{x}{x - 1}\right)|k_i), \tag{2.20b}$$

$$\sum_i Yx|k_i) = \sum_i \frac{1}{1 - \frac{1}{fx}}|k_i), \quad \left(\sum_i k_i=0\right), \tag{2.21a}$$

$$\sum_i Y^{T}x|k_i) = \sum_i f(1 - x)|k_i), \quad \left(\sum_i k_i=0\right), \tag{2.21b}$$

where

$$f \equiv p_{\alpha}(\alpha + 2)\frac{t}{t - 1} . \tag{2.22}$$

Equations (2.21a) and (2.21b) suggest that we can define one projective operator Q:

$$Q^{-1}(x) = \frac{1}{1 - \frac{1}{fx}}, \qquad Q(x) = \frac{1}{f(1 - \frac{1}{x})} . \tag{2.23}$$

This projective operator enables us to obtain the nth-order term in the [GH] matrix. To see this, we express $|E)$, $|F)$, $Y|F)$, $Y^{T}|E)$ in terms of the projective operator $Q^{\pm}$. From Eqs. (2.14), (2.21), and (2.23), we have

$$|F) = \sum_{\substack{i=0 \\ (i\neq\alpha,\alpha+1)}}^{S+1} Q^{-1} \frac{1}{p_{\alpha+1}(i)}|k_i), \tag{2.24a}$$

$$|E) = \sum_{\substack{i=0 \\ (i\neq\alpha,\alpha+1)}}^{S+1} p_{\alpha+1}\begin{bmatrix} i \\ \alpha \\ \alpha+2 \end{bmatrix} \Bigg| \begin{matrix} k_i \\ k_\alpha \\ k_{\alpha+1} \end{matrix}\Bigg) , \tag{2.24b}$$

$$Y|F) = \sum_{\substack{i=0 \\ (i\neq\alpha,\alpha+1)}}^{S+1} Q^{-1} \quad Q^{-1}\left(\frac{1}{p_{\alpha+1}(i)}\right)|k_i), \tag{2.25a}$$

$$Y^T|E) = \sum_{\substack{i=0 \\ (i\neq\alpha,\alpha+1)}}^{S+1} \frac{1}{Q\left(\dfrac{1}{p_{\alpha+1}\begin{bmatrix} i \\ \alpha \\ \alpha+2 \end{bmatrix}}\right)} \Bigg| \begin{matrix} k_i \\ k_\alpha \\ k_{\alpha+1} \end{matrix}\Bigg) . \tag{2.25b}$$

We can thus represent the [GH] matrix of Eq. (2.17d) as an operator acting on the vector $\begin{pmatrix} |F) \\ |E) \end{pmatrix}$. From Eqs. (2.17d), (2.24), and (2.25) we get

$$[GH]_{op} = \begin{pmatrix} [Y] & 0 \\ 0 & [Y]^T \end{pmatrix} = \begin{pmatrix} Q^{-1}(\ ) & 0 \\ 0 & \dfrac{1}{Q\dfrac{1}{(\ )}} \end{pmatrix} . \tag{2.26}$$

It is easily checked that the GH operator defined on the new vector $\begin{pmatrix} |F)_1 \\ |E)_1 \end{pmatrix} \equiv [GH]\begin{pmatrix} |F) \\ [E) \end{pmatrix}$ has a form identical to Eq. (2.26). It follows that

$$[GH]_{op}{}^n = \begin{pmatrix} Q^{-n}(\ ) & 0 \\ 0 & \dfrac{1}{Q^n\dfrac{1}{(\ )}} \end{pmatrix} . \tag{2.27}$$

Hence, from Eqs. (2.19a), (2.27), and (2.24), the nth-order term is

$$I_n \equiv \exp\left\{\frac{1}{2}\left((E|,(F|\right)[GH]^n \begin{pmatrix} |F) \\ |E) \end{pmatrix}\right\}$$

$$= \prod_{\substack{i,j=0 \\ (i,j\neq\alpha,\alpha+1)}}^{S+1} \left\{ y_i - \hat{p}_{\alpha+1}^{-1}\, Q^{n+1}\, \hat{p}_{\alpha+1} \begin{bmatrix} y_j \\ y_\alpha \\ y_{\alpha+2} \end{bmatrix} \right\}^{-k_i k_{[j,\alpha,\alpha+1]}} , \qquad (2.28)$$

where

$$\hat{p}_{\alpha+1}(i) \equiv \frac{1}{p_{\alpha+1}(i)} = p(\alpha,\alpha+1,\alpha+2,i). \qquad (2.29)$$

In obtaining Eq. (2.28), we have used the fact that

$$\sum_i k_i = \sum_j k_j = k_\alpha + k_{\alpha+1} = 0$$

to move the operator $\hat{p}_{\alpha+1}$, Q to the y_j side. The conservation of momenta guarantees the projective invariance of Eq. (2.28).

We can define the projective operator R, which will be interpreted as the projective operator for going around the loop, as

$$R \equiv \hat{p}_{\alpha+1}^{-1}\, Q\, \hat{p}_{\alpha+1}. \qquad (2.30)$$

We have

$$R(y_{\alpha+1}) = y_{\alpha+1}, \qquad (2.31a)$$

$$R(y_{\alpha+2}) = y_\alpha, \qquad (2.31b)$$

hence Eq. (2.28) becomes

$$I_n = \prod_{\substack{i,j=0 \\ (i,j\neq\alpha,\alpha+1)}}^{S+1} \{y_i - R^{n+1}(y_j)\}^{-k_i\cdot k_j} \prod_{\substack{i=0 \\ (i\neq\alpha,\alpha+1)}}^{S+1} \left\{\frac{y_i - R^{n+1}(y_\alpha)}{y_i - R^{n}(y_\alpha)}\right\}^{-k_i\cdot k_\alpha} . \quad (2.32)$$

Thus Eq. (2.19) gives

$$I = \frac{1}{(\det[\Delta])^{\frac{1}{2}}} \prod_{n=0}^{\infty} I_n = \frac{1}{(\det[\Delta])^{\frac{1}{2}}} \prod_{n=0}^{\infty} \prod_{\substack{i,j=0 \\ (i,j\neq\alpha,\alpha+1)}}^{S+1} \{y_i - R^{n+1}(y_j)\}^{-k_i\cdot k_j}$$

$$\times \prod_{\substack{i=0 \\ (i\neq\alpha,\alpha+1)}}^{S+1} \left\{\frac{y_i - R^{+\infty}(y_\alpha)}{y_i - y_\alpha}\right\}^{-k_i\cdot k_\alpha} . \quad (2.33)$$

In order to express our single-loop answer in a manifestly projectively invariant form, we need to express the Koba-Nielsen variables $y_\alpha, y_{\alpha+1}$ associated with the excited legs of the original tree, and the Chan variables t associated with the sewed propagator, in terms of the invariant points x_1, x_2 and the multiplier X of the projective operator R. As defined in Eqs. (1.7) to (1.13), we have

$$0 < X < 1 : R^{+\infty}(z_1) = x_2, \qquad z_1 \neq x_1, \quad (2.34a)$$

$$R^{-\infty}(z_2) = x_1, \qquad z_2 \neq x_2, \quad (2.34b)$$

$$R(z) = \frac{z(x_2 - x_1X) - x_1x_2(1 - X)}{z(1 - X) + (x_2X - x_1)} . \quad (2.35)$$

The explicit form of the variables x_1, x_2, and X corresponding to the projective generator R, defined in Eq. (2.30), can be found by using the identity

$$\hat{p}^{-1}_{\alpha+1}(z) = y_\alpha - \frac{y_\alpha - y_{\alpha+1}}{1 - z\left(\frac{y_{\alpha+1} - y_{\alpha+2}}{y_\alpha - y_{\alpha+2}}\right)} \tag{2.36}$$

to express $R \equiv \hat{p}^{-1}_{\alpha+1} Q\, \hat{p}_{\alpha+1}$ as

$$R(z) = \frac{z[y_\alpha(y_{\alpha+2}-y_{\alpha+1})-fy_{\alpha+1}(y_{\alpha+1}-y_\alpha)]+y_\alpha y_{\alpha+1}(y_{\alpha+1}-y_{\alpha+2})+y_{\alpha+1}y_{\alpha+2}f(y_{\alpha+1}-y_\alpha)}{z[(y_{\alpha+2}-y_{\alpha+1})-f(y_{\alpha+1}-y_\alpha)]+y_{\alpha+1}(y_{\alpha+1}-y_{\alpha+2})+y_{\alpha+2}f(y_{\alpha+1}-y_\alpha)} . \tag{2.37}$$

If we compare Eq. (2.37) with the standard form, Eq. (2.35), we obtain a set of identities

$$a \equiv \frac{t}{t-1}\, d, \qquad t = \frac{a}{a-d}, \tag{2.38a}$$

$$d \equiv \frac{(y_\alpha - y_{\alpha-1})(y_\alpha - y_{\alpha+1})}{(y_{\alpha+1} - y_{\alpha-1})(y_\alpha - y_{\alpha+2})}, \tag{2.38b}$$

$$\ell = \left(\frac{a(y_\alpha - y_{\alpha+1})(y_{\alpha+2} - y_{\alpha+1})}{X(x_1 - x_2)^2}\right)^{\frac{1}{2}}, \tag{2.38c}$$

$$(1 - a) = \ell(1 - X), \tag{2.38d}$$

$$\frac{y_\alpha - ay_{\alpha+1}}{ay_{\alpha+2} - y_{\alpha+1}} = \frac{x_2 - x_1 X}{x_2 X - x_1}, \tag{2.38e}$$

$$y_\alpha y_{\alpha+1} - y_{\alpha+1}y_{\alpha+2}a = x_1 x_2(1 - a). \tag{2.38f}$$

This set of identities enables us to compute the Jacobian factor

$$J \equiv \frac{\partial(t,y_\alpha,y_{\alpha+1})}{\partial(X,x_1,x_2)} . \tag{2.39}$$

In Appendix II, we perform the calculation. The result (in the frame $x_1 = \infty$, $x_2 = 0$) is

$$J = \frac{d}{(a-d)^2}(1-X) . \tag{2.40}$$

In order to go from the set of variables $(t,y_\alpha,y_{\alpha+1})$ to the new set of variables (X,x_1,x_2), we also need to separate out all factors containing the variables y_α, $y_{\alpha+1}$ in $\{Y_{S+2}\}$ in Eq. (2.12). From Eq. (2.4), we have

$$\{Y_{S+2}\} = \prod_{\substack{i,j=0 \\ \left(\substack{i\neq j \\ i,j\neq\alpha,\alpha+1}\right)}}^{S+1} (y_i - y_j)^{-\frac{1}{2}k_i\cdot k_j} \prod_{\substack{i=0 \\ (i\neq\alpha,\alpha+1)}}^{S+1} \left\{\frac{y_i - y_\alpha}{y_i - y_{\alpha+1}}\right\}^{-k_i\cdot k_\alpha} \prod_{\substack{i=0 \\ (i\neq\alpha-1,\alpha,\alpha+1)}}^{S+1} (y_i - y_{i+1})^{\alpha_0-1}$$

$$\times \left\{(y_\alpha - y_{\alpha+1})^{-\alpha_0-1}\left[\frac{(y_{\alpha-1}-y_\alpha)(y_{\alpha+1}-y_{\alpha+2})}{(y_\alpha - y_{\alpha+2})(y_{\alpha-1}-y_{\alpha+1})}\right]^{-\frac{1}{2}k_\alpha^2-1}\right.$$

$$\left.\times\ [(y_\alpha - y_{\alpha+2})(y_{\alpha-1}-y_{\alpha+1})]^{\alpha_0-1}(y_a - y_b)(y_b - y_c)(y_c - y_a)\right\} . \tag{2.41}$$

Now we can extract out all factors involving t, y_α, $y_{\alpha+1}$ in Eq. (2.12). From Eq. (2.12) and (2.41), they are

$$dt\, dy_\alpha\, dy_{\alpha+1} \left(\frac{t}{t-1}\right)^{-\alpha_0 - \frac{1}{2}k_\alpha^2} \frac{1}{t(1-t)} \left\{ \left[\frac{(y_\alpha - y_{\alpha-1})(y_{\alpha+2} - y_{\alpha+1})}{(y_\alpha - y_{\alpha+2})(y_{\alpha-1} - y_{\alpha+1})} \right]^{-\frac{1}{2}k_\alpha^2 - 1} \right.$$

$$\times\ (y_\alpha - y_{\alpha+1})^{-\alpha_0 - 1} (y_\alpha - y_{\alpha+2})^{\alpha_0 - 1} (y_{\alpha-1} - y_{\alpha+1})^{\alpha_0 - 1}$$

$$\left. \times\ (y_a - y_b)(y_b - y_c)(y_c - y_a) \right\} \quad . \tag{2.42}$$

For simplicity, we choose the frame

$$y_a = y_{\alpha+1} = x_1 = \infty, \quad y_b = x_2 = 0, \quad y_c = y_{\alpha+2} = 1. \tag{2.43}$$

This is the frame in which the BHS[1] and ABG[1] planar single-loop formula is expressed; in this frame, the projective operator R reduces to its multiplier X. In the frame of Eq. (2.43), we have, from Eqs. (2.38) and (2.31),

$$y_\alpha \rightarrow X y_{\alpha+2}, \tag{2.44a}$$

$$a \rightarrow X\ , \tag{2.44b}$$

$$d \rightarrow \frac{X - y_{\alpha-1}}{1 - X}\ . \tag{2.44c}$$

Substituting Eqs. (2.44), (2.43), and (2.40) in the expression (2.42), we find that (2.42) is equal to

$$dX[dx_1][dx_2]\ X^{-\ell(k_\alpha)-1} (1 - X)(X - y_{\alpha-1})^{\alpha_0 - 1}, \tag{2.45}$$

where $[dx_1],[dx_2]$ means x_1, x_2 are not to be integrated over. The factor $(1 - X)$ in Eq. (2.45) is the famous linear dependence correction factor[1] to the planar loop amplitude. We emphasize here that, though we obtain this factor $(1 - X)$ in a particular frame, the factor

is actually frame-independent, since it is one minus the multiplier of the projective operator. This fact enables us to find the similar linear-dependence correction factors for the N-loop case.

We finally combine Eqs. (2.12), (2.33), (2.41), and (2.45), thus obtaining the planar single-loop formula (in the frame $x_1 = \infty$, $x_2 = 0$, $y_{\alpha+2} = 1$) (Fig. 3)

$$\mathrm{FPL}(1) = \int d^4k_\alpha \int_0^1 dX\, X^{-\ell(k_\alpha)-1}(1 - X)$$

$$\times \int \frac{\prod_{\substack{i \\ (i\neq\alpha,\alpha+1)}} dy_i[dy_{\alpha+2}][dx_1][dx_2](x_1 - y_{\alpha+2})(y_{\alpha+2} - x_2)(x_2 - x_1)}{\prod_{\substack{i=0 \\ (i\neq\alpha\pm1,\alpha)}}^{S+1} (y_i - y_{i+1})(x_1 - x_2)^2(X - y_{\alpha-1})}$$

$$\times \left\{ \prod_{\substack{i=0 \\ (i\neq\alpha\pm1,\alpha)}}^{S+1} (y_i - y_{i+1})^{\alpha_0}(X - y_{\alpha-1})^{\alpha_0} \right\} \left\{ \frac{1}{(\det[\Delta])^{\frac{1}{2}}} \right\}$$

$$\times \left\{ \prod_{\substack{i=0 \\ (i\neq\alpha,\alpha+1)}}^{S+1} \left(\frac{y_i - x_2}{y_i - x_1} \right)^{-k_i\cdot k_\alpha} \prod_{n=-\infty}^{+\infty} \prod_{\substack{i,j=0 \\ (n=0,i\neq j) \\ (i,j\neq\alpha,\alpha+1)}}^{S+1} [y_i - X^n y_j]^{-\frac{1}{2}k_i\cdot k_j}, \right. \tag{2.46}$$

where $\{y_0, y_1, \cdots y_{\alpha-1}, x_2, x_1, y_{\alpha+2}, \cdots y_{S+1}\}$ is the new set of Koba-Nielsen variables lying on a unit circle. The ordering is

$$0 = x_2 \leq y_{\alpha-1} \leq y_{\alpha-2} \leq \cdots \leq y_{\alpha+2} \leq x_1 = \infty. \tag{2.47}$$

From Eq. (2.38a,b), one can show[13] that $0 \leq X < 1$ (strictly less than unity), hence we have the important inequality

$$x_2 \leq \cdots \leq Xy_{\alpha+2} \leq y_{\alpha-1} \leq y_{\alpha-2} \leq \cdots \leq y_{\alpha+2} \leq X^{-1}y_{\alpha-1} \leq \cdots \leq x_1 . \tag{2.48}$$

Now we would like to go to a general frame $y_a = \infty$, $y_b = 1$, $y_c = 0$ in Eq. (2.46). We note the only factor in Eq. (2.46) which changes when the projective frame is changed is $(X - y_{\alpha-1})$. The unique generalization of this factor to a general projective frame is[14,15]

$$(y_{\alpha-1} - X) \rightarrow \frac{(x_1 - y_{\alpha+2})[y_{\alpha-1} - R(y_{\alpha+2})]}{[x_1 - R(y_{\alpha+2})]}$$

$$= \frac{(x_1 - y_{\alpha+2})(x_2 - y_{\alpha-1}) - X(x_2 - y_{\alpha+2})(x_1 - y_{\alpha-1})}{(x_2 - x_1)} . \tag{2.49}$$

One can easily check that Eq. (2.49) together with $\prod_{i \neq \alpha,\alpha+1} (y_i - y_{i+1})$ is invariant under infinitesimal projective transformation (all variables appear either twice or not at all), and in the case $x_1 = \infty$, $x_2 = 0$, Eq. (2.49) reduces to $(y_{\alpha-1} - X)$.

Hence the planar single-loop amplitude, expressed in a general projective frame, has the form (Fig. 3)

$$\mathrm{FPL}(1) = \int d^4k_\alpha \int_0^1 dX\, X^{-\ell(k_\alpha)-1}(1 - X)$$

$$\times \int \frac{\prod_{\substack{i=0 \\ (i\neq a,b,c,\alpha,\alpha+1)}}^{S+1} dy_i\; dx_2 dx_1 [dy_a][dy_b][dy_c](y_a - y_b)(y_b - y_c)(y_c - y_a)}{\prod_{\substack{i=0 \\ (i\neq\alpha\pm1,\alpha)}}^{S+1} (y_i - y_{i+1})(x_2 - x_1)^2 (x_1 - y_{\alpha+2})[y_{\alpha-1} - R(y_{\alpha+2})] \Big/ [x_1 - R(y_{\alpha+2})]}$$

$$\times \left\{ \prod_{\substack{i=0 \\ (i\neq\alpha\pm1,\alpha)}}^{S+1} (y_i - y_{i+1})^{\alpha_0} \left[\frac{(x_1 - y_{\alpha+2})[y_{\alpha-1} - R(y_{\alpha+2})]}{[x_1 - R(y_{\alpha+2})]} \right]^{\alpha_0} \right\} \left\{ \frac{1}{(\det[\Delta])^{\frac{1}{2}}} \right\}$$

$$\times \left\{ \prod_{\substack{i=0 \\ (i\neq\alpha,\alpha+1)}}^{S+1} \left(\frac{y_i - x_2}{y_i - x_1} \right)^{-k_i\cdot k_\alpha} \prod_{n=-\infty}^{+\infty} \prod_{\substack{i,j=0 \\ (n=0,i\neq j) \\ (i,j\neq\alpha,\alpha+1)}}^{S+1} [y_i - R^n(y_j)]^{-\frac{1}{2}k_i\cdot k_j} \right\}, \qquad (2.50)$$

where the ordering is

$$y_{a-1} \leq \cdots \leq y_{\alpha+3} \leq y_{\alpha+2} \leq x_1 \leq x_2 \leq y_{\alpha-1} \leq y_{\alpha-2} \cdots \leq y_a = \infty,$$

and x_1, x_2 are the two invariant points of R; X is the multiplier of R,

$$R(z) = \frac{z(x_2 - x_1 X) - x_1 x_2 (1 - X)}{z(1 - X) + x_2 X - x_1}, \qquad (2.51)$$

and[16]

$$(\det[\Delta])^{-\frac{1}{2}} = \prod_{n=1}^{\infty} (1 - X^n)^{-4}. \qquad (2.52)$$

III. PLANAR N-LOOP AMPLITUDE

To obtain the planar N-loop amplitude, we consider the tree with 2N excited legs, as shown in Fig. 4. We label each adjacent pair of excited legs by y_α and $y_{\alpha+1}$, which are associated with the α loop. We can directly write down the 2Nth-factorized tree amplitude[4] corresponding to Fig. 4.

$$G_{(Y)}^{(2N)}(a's) = \int\prod dy_i \{Y_{S+2}\} \exp\Bigg\{ \sum_\alpha \Bigg[\sum_{\substack{i=0 \\ (i\neq\alpha,\alpha+1)}}^{S+1} (a^\alpha | p_\alpha(i) | k_i)$$

$$+ \sum_{\substack{i=0 \\ (i\neq\alpha,\alpha+1)}}^{S+1} (a^{\alpha+1} | p_{\alpha+1}(i) | k_i) \Bigg]$$

$$+ \frac{1}{2} \sum_{\substack{\alpha,\beta \\ (\alpha\neq\beta)}} \Big[(a^\alpha | p_\alpha(\beta) M_- p(\alpha+1,\beta+1,\alpha,\beta) M_-^T p_\beta(\alpha) | a^\beta)$$

$$+ (a^{\alpha+1} | p_{\alpha+1}(\beta+1) M_- p(\alpha,\beta,\alpha+1,\beta+1) M_-^T p_{\beta+1}(\alpha+1) | a^{\beta+1}) \Big]$$

$$+ \sum_{\alpha,\beta} (a^\alpha | p_\alpha(\beta+1) M_- p(\alpha+1,\beta,\alpha,\beta+1) M_-^T p_{\beta+1}(\alpha) | a^{\beta+1}) \Bigg\}, \tag{3.1}$$

where, depending on the dots, we have

$$p_\alpha(i) = p(\alpha,\alpha+1,\alpha-1,i),$$

$$p_{\alpha+1}(i) = p(\alpha+1,\alpha,\alpha+2,i) = \frac{p_\alpha(\alpha+2)}{p_\alpha(i)}, \tag{3.2}$$

and when $\alpha = \beta$, the last term becomes[4]

$$(a^{\alpha}|p_{\alpha}(\beta+1)M_{-}p(\alpha+1,\beta,\alpha,\beta+1)M_{-}^{T}p_{\beta+1}(\alpha)|a^{\beta+1}) \rightarrow (a^{\alpha}|p_{\alpha}(\alpha+2)|a^{\alpha+1}). \quad (3.3)$$

In Eq. (3.1) we have distinguished the a^{α} set from the $a^{\alpha+1}$ set, since a^{α} will be replaced by λ_{α}^{*}, and $a^{\alpha+1}$ by λ_{α}.

We now apply the sewing prescriptions (a),(b),(c) of Sec. II A to Eq. (3.1) and simultaneously join the N pairs of excited legs. We make, in Eq. (3.1), the replacements

$$(a^{\alpha}| \rightarrow (\lambda_{\alpha}^{*} - k_{\alpha}|\left[M_{-}^{T}\frac{t_{\alpha}}{t_{\alpha}-1}\right] ,$$

$$(a^{\alpha+1}| \rightarrow (\lambda_{\alpha}|, \quad (3.4)$$

where t_{α} is the propagator variables associated with the α loop. We insert the expression, in Eq. (3.1),

$$\int \prod_{\alpha=\{\mathcal{L}\}} d^4k_{\alpha} \int_0^1 \prod_{\alpha=\{\mathcal{L}\}} dt_{\alpha}\, t_{\alpha}^{-\ell(k_{\alpha})-1}(1-t_{\alpha})^{\alpha_0-1+\frac{1}{2}k_{\alpha}^2}$$

$$\times \int \prod_{\alpha=\{\mathcal{L}\}} d\left|\frac{\lambda_{\alpha}}{\sqrt{2}}\right) \; d\left|\frac{\lambda_{\alpha}^{*}}{\sqrt{2}}\right) \; \exp\left\{-\sum_{\alpha=\{\mathcal{L}\}}(\lambda_{\alpha}^{*}|\lambda_{\alpha})\right\} , \quad (3.5)$$

where the symbol $\{\mathcal{L}\}$ denotes the collection of all labelings of Koba-Nielsen variables associated with the N loops. We then get the planar N-loop amplitude denoted by FPL(N):

$$\mathrm{FPL}(N) = \int \prod_{\alpha=\{\mathcal{L}\}} d^4k_\alpha \int_0^1 \prod_{\alpha=\{\mathcal{L}\}} dt_\alpha \, t_\alpha^{-\ell(k_\alpha)-1} (1 - t_\alpha)^{\alpha_0 - 1 + \frac{1}{2}k_\alpha^2}$$

$$\times \int \prod dy_i \{Y_{S+2}\} I \, , \tag{3.6}$$

where

$$I = \int \prod_{\alpha=\{\mathcal{L}\}} d\left|\frac{\lambda_\alpha}{\sqrt{2}}\right) \, d\left|\frac{\lambda_\alpha^*}{\sqrt{2}}\right) \, \exp\left\{ - \sum_{\alpha=\{\mathcal{L}\}} (\lambda_\alpha^* | \lambda_\alpha) + \sum_{\alpha=\{\mathcal{L}\}} [(\lambda_\alpha^* | F_\alpha)\right.$$

$$+ (\lambda_\alpha | E_\alpha)] + \frac{1}{2} \sum_{\substack{\alpha,\beta=\{\mathcal{L}\} \\ (\alpha\neq\beta)}} [(\lambda_\alpha^* | \bar{D}_{\alpha\beta} | \lambda_\beta^*) + (\lambda_\alpha | \bar{A}_{\alpha\beta} | \lambda_\beta)]$$

$$\left. + \sum_{\alpha,\beta=\{\mathcal{L}\}} (\lambda_\alpha^* | \bar{C}_{\alpha\beta} | \lambda_\beta) \right\} \, , \tag{3.7}$$

and

$$Y'_\alpha = M_-^T \frac{t_\alpha}{t_\alpha - 1} \, , \tag{3.8}$$

$$|F_\alpha) = \sum_{\substack{i=0 \\ (i\neq\alpha,\alpha+1)}}^{S+1} Y'_\alpha \, p_\alpha(i) |k_i) \, , \tag{3.9a}$$

$$|E_\alpha) = \sum_{\substack{i=0 \\ (i\neq\alpha,\alpha+1)}}^{S+1} p_{\alpha+1} \left[\begin{matrix} i \\ \alpha \\ \alpha + 2 \end{matrix} \right| \left| \begin{matrix} k_i \\ k_\alpha \\ k_{\alpha+1} \end{matrix} \right) \, , \tag{3.9b}$$

$$\bar{D}_{\alpha\beta} = Y'_\alpha p_\alpha(\beta) M_- p(\alpha+1,\beta+1,\alpha,\beta) M_-^T p_\beta(\alpha) Y'^T_\beta, \quad \bar{D}_{\alpha\alpha} \equiv 0, \tag{3.10a}$$

$$\bar{A}_{\alpha\beta} = p_{\alpha+1}(\beta+1)M_{-}p(\alpha,\beta,\alpha+1,\beta+1)M_{-}^{T}p_{\beta+1}(\alpha+1), \qquad \bar{A}_{\alpha\alpha} \equiv 0, \quad (3.10b)$$

$$\bar{C}_{\alpha\beta} = Y'_{\alpha}p_{\alpha}(\beta+1)M_{-}p(\alpha+1,\beta,\alpha,\beta+1)M_{-}^{T}p_{\beta+1}(\alpha), \qquad \bar{C}_{\alpha\alpha} = Y'_{\alpha}p_{\alpha}(\alpha+2). \quad (3.10c)$$

For notational convenience, we introduce N-dimensional vectors in the loop number space by defining

$$|\underset{\sim}{F}) \equiv \left(|F_{\alpha}), |F_{\beta}), \cdots |F_{\sigma})\right), \qquad 0 < \alpha < \beta < \cdots < \sigma < \cdots < S+1,$$

$$|\underset{\sim}{E}) \equiv \left(|E_{\alpha}), |E_{\beta}), \cdots |E_{\sigma})\right),$$

$$|\underset{\sim}{\lambda}) \equiv \left(|\lambda_{\alpha}), |\lambda_{\beta}), \cdots |\lambda_{\sigma})\right),$$

$$|\underset{\sim}{\lambda}^{*}) \equiv \left(|\lambda_{\alpha}^{*}), |\lambda_{\beta}^{*}), \cdots |\lambda_{\sigma}^{*})\right). \quad (3.11)$$

Then Eq. (3.7) becomes

$$I = \int d\left|\frac{\underset{\sim}{\lambda}}{\sqrt{2}}\right) \; d\left|\frac{\underset{\sim}{\lambda}^{*}}{\sqrt{2}}\right) \; \exp\{-(\underset{\sim}{\lambda}^{*}|\underset{\sim}{\lambda}) + (\underset{\sim}{\lambda}^{*}|\underset{\sim}{F}) + (\underset{\sim}{\lambda}|\underset{\sim}{E})$$

$$+ \frac{1}{2}(\underset{\sim}{\lambda}^{*}|[\bar{D}]|\underset{\sim}{\lambda}^{*}) + \frac{1}{2}(\underset{\sim}{\lambda}|[\bar{A}]|\underset{\sim}{\lambda}) + (\underset{\sim}{\lambda}^{*}|[\bar{C}]|\underset{\sim}{\lambda})\}, \quad (3.12)$$

where $|\underset{\sim}{E})$, $|\underset{\sim}{F})$ are N-dimensional vectors in the loop-number space and each of these N components itself is an infinite-dimensional vector in the harmonic oscillator space. Similarly, $[\bar{A}]$, $[\bar{D}]$, $[\bar{C}]$ are $N \times N$ matrices in the loop-number space and each matrix element itself is $\infty \times \infty$ dimensional matrix in the harmonic oscillator space.

Analogous to the single-loop case of Eqs. (2.15) through (2.19), we can easily obtain the $[\Delta]$, $[GH]$ matrices by writing Eq. (3.12) as

$$I = \int d\left|\frac{\underset{\sim}{\lambda}}{\sqrt{2}}\right) \; d\left|\frac{\underset{\sim}{\lambda}^*}{\sqrt{2}}\right) \; \exp\left\{-\frac{1}{2}\left(\left(\underset{\sim}{\lambda}\right|,\left(\underset{\sim}{\lambda}^*\right|\right)\begin{pmatrix} -[\bar{A}] & [I] - [\bar{C}]^T \\ [I] - [\bar{C}] & -[\bar{D}] \end{pmatrix}\right.\right.$$

$$\left.\times \begin{pmatrix} \left|\underset{\sim}{\lambda}\right) \\ \left|\underset{\sim}{\lambda}^*\right) \end{pmatrix} + \begin{pmatrix} \left(\underset{\sim}{E}\right| & 0 \\ 0 & \left(\underset{\sim}{F}\right| \end{pmatrix} \begin{pmatrix} \left|\underset{\sim}{\lambda}\right) \\ \left|\underset{\sim}{\lambda}^*\right) \end{pmatrix}. \tag{3.13}$$

Hence

$$[\Delta] = \begin{pmatrix} -[\bar{A}] & [I] - [\bar{C}]^T \\ [I] - [\bar{C}] & -[\bar{D}] \end{pmatrix}, \tag{3.14}$$

$$[GH] = \begin{pmatrix} [\bar{C}] & [\bar{D}] \\ [\bar{A}] & [\bar{C}]^T \end{pmatrix}, \tag{3.15}$$

and

$$I = \frac{1}{(\det[\Delta])^{\frac{1}{2}}} \exp\left\{\frac{1}{2} \sum_{n=0}^{\infty} \left(\left(\underset{\sim}{E}\right|,\left(\underset{\sim}{F}\right|\right)[GH]^n \begin{pmatrix} \left|\underset{\sim}{F}\right) \\ \left|\underset{\sim}{E}\right) \end{pmatrix}\right\}. \tag{3.16}$$

We have thus completed the multiple-factorization formulation of unitary closed multiloops. This formulation can easily be extended to nonplanar and overlapping multiloops.

We now carry out the calculation of the individual terms in the exponent of Eq. (3.16). We first separate out all Koba-Nielsen variables associated with the N loops in Eq. (3.9a) and (3.9b):

$$|F_\alpha) = \sum_{\substack{i=0\\ i\neq\{\mathcal{L},\mathcal{L}+1\}}}^{S+1} Y'_\alpha p_\alpha(i)|k_i) + \sum_{\substack{\beta\\ \beta=\{\mathcal{L}\}\\ (\beta\neq\alpha)}} Y'_\alpha p_\alpha \begin{bmatrix} \beta \\ \beta+1 \end{bmatrix} \Bigg| \begin{matrix} k_\beta \\ k_{\beta+1} \end{matrix} \Bigg)$$

$$\equiv \sum_{\substack{i=0\\ i\neq\{\mathcal{L},\mathcal{L}+1\}}}^{S+1} \sum_{\substack{\beta\\ \beta=\{\mathcal{L}\}}} Q_\alpha^{-1}\, p_\alpha \begin{bmatrix} i \\ \beta \\ \beta+1 \end{bmatrix}_{(\beta\neq\alpha)} \Bigg| \begin{matrix} k_i \\ k_\beta \\ k_{\beta+1} \end{matrix} \Bigg) \quad ,(3.17a)$$

$$|E_\alpha) = \sum_{\substack{i=0\\ i\neq\{\mathcal{L},\mathcal{L}+1\}}}^{S+1} \sum_{\substack{\beta\\ \beta=\{\mathcal{L}\}}} p_{\alpha+1} \begin{bmatrix} i \\ \alpha \\ \alpha+2 \\ \beta \\ \beta+1 \end{bmatrix}_{(\beta\neq\alpha)} \Bigg| \begin{matrix} k_i \\ k_\alpha \\ k_{\alpha+1} \\ k_\beta \\ k_{\beta+1} \end{matrix} \Bigg) , \quad (3.17b)$$

where

$$Q_\alpha^{-1}(x) = \frac{1}{1 - \frac{t_\alpha - 1}{t_\alpha x}} , \qquad Q_\alpha(x) = \frac{1}{\left(\frac{t_\alpha}{t_\alpha - 1}\right)\left(1 - \frac{1}{x}\right)} . \quad (3.18)$$

The zero-order term can be obtained from Eq. (3.17):

$$I_0 \equiv \exp\{(\underset{\sim}{E}|\underset{\sim}{F})\}$$

$$= \prod_{\substack{i,j=0\\ i,j=\{\mathcal{L},\mathcal{L}+1\}}}^{S+1} \prod_{\alpha,\beta,\gamma=\{\mathcal{L}\}} \left\{ \begin{bmatrix} y_i \\ y_\beta \\ y_{\beta+1} \end{bmatrix}_{(\beta\neq\alpha)} - p_\alpha^{-1} Q_\alpha \hat{p}_{\alpha+1} \begin{bmatrix} j \\ \gamma \\ \gamma+1 \\ \alpha \\ \alpha+2 \end{bmatrix}_{(\gamma\neq\alpha)} \right\}^{-k_{[i,\beta,\beta+1]}\cdot k_{[j,\gamma,\gamma+1,\alpha,\alpha+1]}} , \quad (3.19)$$

where $k_\alpha + k_{\alpha+1} = k_\beta + k_{\beta+1} = k_\gamma + k_{\gamma+1} = \sum_i k_i = \sum_j k_j = 0,$

and

$$\hat{p}_{\alpha+1}(i) = \frac{1}{p_{\alpha+1}(i)} = p(\alpha,\alpha+1,\alpha+2,i). \tag{3.20}$$

In obtaining Eq. (3.19), we have used Eqs. (2.20) and (2.21). We also use the fact that the sum of momenta is zero to move all operators to the y_j side.

We can define the projective operator that corresponds to going around the α-loop as

$$R_\alpha(x) = p_\alpha^{-1} Q_\alpha \hat{p}_{\alpha+1}(x), \quad R_\alpha^{-1}(x) = \hat{p}_{\alpha+1}^{-1} Q_\alpha^{-1} p_\alpha(x). \tag{3.21}$$

We have

$$R_\alpha^{-1}(y_\alpha) = y_{\alpha+2}, \tag{3.22a}$$

$$R_\alpha^{-1}(y_{\alpha+1}) = y_{\alpha+1}, \tag{3.22b}$$

and, from Eq. (3.17),

$$|F_\alpha) = \sum_{\substack{i=0 \\ i\neq\{\mathcal{L},\mathcal{L}+1\}}}^{S+1} \hat{p}_{\alpha+1} R_\alpha^{-1} \begin{bmatrix} i \\ \beta \\ \beta+1 \end{bmatrix}_{(\alpha\neq\beta)} \left| \begin{matrix} k_i \\ k_\beta \\ k_{\beta+1} \end{matrix} \right) , \tag{3.23a}$$

$$|E_\alpha) = \sum_{\substack{i=0 \\ i\neq\{\mathcal{L},\mathcal{L}+1\}}}^{S+1} p_{\alpha+1} \begin{bmatrix} i \\ \alpha \\ \alpha+2 \\ \beta \\ \beta+1 \end{bmatrix}_{(\alpha\neq\beta)} \left| \begin{matrix} k_i \\ k_\alpha \\ k_{\alpha+1} \\ k_\beta \\ k_{\beta+1} \end{matrix} \right) . \tag{3.23b}$$

With Eqs. (3.21), (3.22), and (3.23) we can simplify Eq. (3.19) to read

$$I_0 = \prod_{\alpha=\{\mathcal{L}\}} \prod_{\substack{i,j=0 \\ i,j\neq\{\mathcal{L},\mathcal{L}+1\}}}^{S+1} \{y_i - R_\alpha^{\pm}(y_j)\}^{-\frac{1}{2}k_i\cdot k_j}$$

$$\times \prod_{\alpha,\beta=\{\mathcal{L}\}} \prod_{\substack{i=0 \\ i\neq\{\mathcal{L},\mathcal{L}+1\}}}^{S+1} \left\{\frac{y_i - R_\alpha^{\pm}(y_\beta)}{y_i - R_\alpha^{\pm}(y_{\beta+1})}\right\}^{-k_i\cdot k_\beta}_{(\alpha\neq\beta)} \left\{\frac{y_i - R_\alpha(y_\alpha)}{y_i - y_\alpha}\right\}^{-k_i\cdot k_\alpha}$$

$$\times \prod_{\alpha,\beta,\gamma=\{\mathcal{L}\}} \left\{\frac{y_\beta - R_\alpha^{\pm}(y_\gamma)}{y_\beta - R_\alpha^{\pm}(y_{\gamma+1})}\cdot\frac{y_{\beta+1} - R_\alpha^{\pm}(y_{\gamma+1})}{y_{\beta+1} - R_\alpha^{\pm}(y_\gamma)}\right\}^{-\frac{1}{2}k_\beta\cdot k_\gamma}_{\substack{(\beta\neq\alpha)\\(\gamma\neq\alpha)}}$$

$$\times \left\{\frac{y_\beta - R_\alpha(y_\alpha)}{y_\beta - y_\alpha}\cdot\frac{y_{\beta+1} - y_\alpha}{y_{\beta+1} - R_\alpha(y_\alpha)}\right\}^{-k_\alpha\cdot k_\beta}_{(\beta\neq\alpha)} . \tag{3.24}$$

To calculate the first-order term, we note

$$[GH]\begin{pmatrix} |\underset{\sim}{F}) \\ |\underset{\sim}{E}) \end{pmatrix} = \begin{pmatrix} [\bar{C}]|\underset{\sim}{F}) + [\bar{D}]|\underset{\sim}{E}) \\ [\bar{A}]|\underset{\sim}{F}) + [\bar{C}]^T|\underset{\sim}{E}) \end{pmatrix} \equiv \begin{pmatrix} |\underset{\sim}{F})_1 \\ |\underset{\sim}{E})_1 \end{pmatrix} . \tag{3.25}$$

Hence we need to calculate the four quantities

$$[\bar{C}]|\underset{\sim}{F}), \quad [\bar{D}]|\underset{\sim}{E}), \quad [\bar{A}]|\underset{\sim}{F}), \quad (\underset{\sim}{E}|[\bar{C}] \quad . \tag{3.26}$$

We observe that these four quantities are the only possible ways that the state $|\underset{\sim}{\lambda}^*)$ can contract with the state $|\underset{\sim}{\lambda})$, both in the loop-number space and in the harmonic oscillator space. We therefore expect

that there are no more than these four elements that need to be calculated in all higher orders. (This observation is also true for the nonplanar and overlapping multiloop case.) This can be checked by examining the higher-order expansion of Eq. (3.16). We calculate these four elements by using the identities of Eqs. (2.20) and (2.21) and get, from (3.23) and (3.10),

$$\bar{C}_{\alpha\beta}|F_\beta) = \sum_{\substack{i=0 \\ i\neq\{\mathcal{L},\mathcal{L}+1\}}}^{S+1} Q_\alpha^{-1}\, O_{\alpha\beta}^{(1)}\, \hat{p}_{B+1}\, R_\beta^{-1} \begin{bmatrix} i \\ \gamma \\ \gamma+1 \end{bmatrix}_{(\beta\neq\gamma)} \left| \begin{matrix} k_i \\ k_\gamma \\ k_{\gamma+1} \end{matrix} \right) , \quad (3.27a)$$

$$\bar{D}_{\alpha\beta}|E_\beta) = \sum_{\substack{i=0 \\ i\neq\{\mathcal{L},\mathcal{L}+1\}}}^{S+1} Q_\alpha^{-1}\, O_{\alpha\beta}^{(2)} \frac{1}{Q_B(1/p_{\beta+1} \begin{bmatrix} i \\ \beta \\ \beta+2 \\ \gamma \\ \gamma+1 \end{bmatrix}_{(\beta\neq\gamma)})} \left| \begin{matrix} k_i \\ k_\beta \\ k_{\beta+1} \\ k_\gamma \\ k_{\gamma+1} \end{matrix} \right) (\alpha\neq\beta) , \quad (3.27b)$$

$$\bar{A}_{\alpha\beta}|F_\beta) = \sum_{\substack{i=0 \\ i\neq\{\mathcal{L},\mathcal{L}+1\}}}^{S+1} O_{\alpha\beta}^{(3)}\, \hat{p}_{\beta+1}\, R_\beta^{-1} \begin{bmatrix} i \\ \gamma \\ \gamma+1 \end{bmatrix}_{(\beta\neq\gamma)} \left| \begin{matrix} k_i \\ k_\gamma \\ k_{\gamma+1} \end{matrix} \right) , \quad (\alpha\neq\beta), \quad (3.27c)$$

$$\bar{C}_{\alpha\beta}{}^T|E_\beta) = \sum_{\substack{i=0 \\ i\neq\{\mathcal{L},\mathcal{L}+1\}}}^{S+1} O_{\alpha\beta}^{(4)} \frac{1}{Q_\beta(1/p_{\beta+1} \begin{bmatrix} i \\ \beta \\ \beta+2 \\ \gamma \\ \gamma+1 \end{bmatrix}_{(\beta\neq\gamma)})} \left| \begin{matrix} k_i \\ k_\beta \\ k_{\beta+1} \\ k_\gamma \\ k_{\gamma+1} \end{matrix} \right) (\alpha\neq\beta) , \quad (3.27d)$$

where summation over β,γ is understood, and

$$o^{(1)}_{\alpha\beta}(x) = p_\alpha(\beta+1)\left[1 - \frac{p(\alpha+1,\beta,\beta+1)}{1 - \frac{1}{p_{\beta+1}(\alpha)x}}\right], \quad o^{(1)}_{\alpha\alpha}(x) = x, \tag{3.28a}$$

$$o^{(2)}_{\alpha\beta}(x) = p_\alpha(\beta)\left[1 - \frac{p(\alpha+1,\beta+1,\alpha,\beta)}{1 - \frac{1}{p_\beta(\alpha)x}}\right], \quad o^{(2)}_{\alpha\alpha}(x) = 0, \tag{3.28b}$$

$$o^{(3)}_{\alpha\beta}(x) = p_{\alpha+1}(\beta+1)\left[1 - \frac{p(\alpha,\beta,\alpha+1,\beta+1)}{1 - \frac{1}{p_{\beta+1}(\alpha+1)x}}\right], \quad o^{(3)}_{\alpha\alpha}(x) = 0, \tag{3.28c}$$

$$o^{(4)}_{\alpha\beta}(x) = p_{\alpha+1}(\beta)\left[1 - \frac{p(\alpha+1,\beta,\alpha,\beta+1)}{1 - \frac{1}{p_\beta(\alpha+1)x}}\right], \quad o^{(4)}_{\alpha\alpha}(x) = x. \tag{3.28d}$$

By using the identities involving the cross ratios in the Appendix A, it is easy to check that we have

$$\begin{aligned} o^{(1)}_{\alpha\beta}\,\hat{p}_{\beta+1}(x) &= p_\alpha(x), \\ o^{(2)}_{\alpha\beta}\,\hat{p}_{\beta}(x) &= p_\alpha(x), \\ o^{(3)}_{\alpha\beta}\,\hat{p}_{\beta+1}(x) &= p_{\alpha+1}(x), \\ o^{(4)}_{\alpha\beta}\,\hat{p}_{\beta}(x) &= p_{\alpha+1}(x), \end{aligned} \tag{3.29}$$

where $\hat{p}_\beta(x) = \frac{1}{p_\beta(x)}$, $\hat{p}_{\beta+1}(x) = \frac{1}{p_{\beta+1}(x)}$. Or,

$$
\begin{aligned}
o^{(1)}_{\alpha\beta} &= p_\alpha \, \hat{p}^{-1}_{\beta+1} , \\
o^{(2)}_{\alpha\beta} &= p_\alpha \, \hat{p}_\beta^{\,-1} , \\
o^{(3)}_{\alpha\beta} &= p_{\alpha+1} \, \hat{p}^{-1}_{\beta+1}, \\
o^{(4)}_{\alpha\beta} &= p_{\alpha+1} \, \hat{p}_\beta^{\,-1}.
\end{aligned}
\tag{3.30}
$$

Substituting Eq. (3.30) in Eq. (3.27) and using the identity

$$
\hat{p}_\beta^{\,-1}(x) = p_\beta^{\,-1}\left(\frac{1}{x}\right), \quad \hat{p}^{-1}_{\beta+1}(x) = p^{-1}_{\beta+1}\left(\frac{1}{x}\right), \tag{3.31}
$$

and the definitions for $R_\alpha^{\pm}$;

$$
R_\alpha^{\,+}(x) \equiv R_\alpha(x) = p_\alpha^{\,-1} \, Q_\alpha \, \hat{p}_{\alpha+1}(x), \quad R_\alpha^{\,-1}(x) = \hat{p}^{-1}_{\alpha+1} \, Q_\alpha^{\,-1} \, p_\alpha(x), \tag{3.21}
$$

we get

$$
\overline{C}_{\alpha\beta}|F_\rho) = \sum_{\substack{i=0 \\ i\neq\{\mathcal{L},\mathcal{L}+1\}}}^{S+1} \hat{p}_{\alpha+1} \, R_\alpha^{\,-1} \, R_\beta^{\,-1} \begin{bmatrix} i \\ \gamma \\ \gamma+1 \end{bmatrix}_{(\beta\neq\gamma)} \left| \begin{matrix} k_i \\ k_\gamma \\ k_{\gamma+1} \end{matrix} \right) ,
$$

$$
\overline{D}_{\alpha\beta}|E_\beta) = \sum_{\substack{i=0 \\ i\neq\{\mathcal{L},\mathcal{L}+1\}}}^{S+1} \hat{p}_{\alpha+1} \, R_\alpha^{\,-1} \, R_\beta \begin{bmatrix} i \\ \beta \\ \beta+2 \\ \gamma \\ \gamma+1 \end{bmatrix}_{(\beta\neq\gamma)} \left| \begin{matrix} k_i \\ k_\beta \\ k_{\beta+1} \\ k_\gamma \\ k_{\gamma+1} \end{matrix} \right) \quad (\alpha\neq\beta),
$$

Equation (3.32) continued

Equation (3.32) continued.

$$\bar{A}_{\alpha\beta}|F_\beta) = \sum_{\substack{i=0 \\ i\neq\{\mathcal{L},\mathcal{L}+1\}}}^{S+1} p_{\alpha+1} R_\beta^{-1} \begin{bmatrix} i \\ \gamma \\ \gamma+1 \end{bmatrix}_{(\beta\neq\gamma)} \left| \begin{matrix} k_i \\ k_\gamma \\ k_{\gamma+1} \end{matrix} \right) \quad (\alpha\neq\beta)$$

$$\bar{C}_{\alpha\beta}{}^T|E_\beta) = \sum_{\substack{i=0 \\ i\neq\{\mathcal{L},\mathcal{L}+1\}}}^{S+1} p_{\alpha+1} R_\beta \begin{bmatrix} i \\ \beta \\ \beta+2 \\ \gamma \\ \gamma+1 \end{bmatrix}_{(\beta\neq\gamma)} \left| \begin{matrix} k_i \\ k_\beta \\ k_{\beta+1} \\ k_\gamma \\ k_{\gamma+1} \end{matrix} \right) \quad . \tag{3.32}$$

In Eq. (3.32), summation over β,γ is understood.

It is now straightforward to calculate the first-order term. From Eq. (3.25) and (3.32), we get

$$|F_\alpha)_1 = \sum_{\substack{i=0 \\ i\neq\{\mathcal{L},\mathcal{L}+1\}}}^{S+1} \sum_{\substack{\beta,\gamma \\ (\beta\neq\alpha,-)}} \hat{p}_{\alpha+1} R_\alpha^{-1} R_\beta^{\pm} \begin{bmatrix} i \\ \gamma \\ \gamma+1 \end{bmatrix}_{(\beta\neq\gamma)} \left| \begin{matrix} k_i \\ k_\gamma \\ k_{\gamma+1} \end{matrix} \right)$$

$$+ \sum_{\substack{\beta \\ (\beta\neq\alpha)}} \hat{p}_{\alpha+1} R_\alpha^{-1} R_\beta \begin{bmatrix} \beta \\ \beta+2 \end{bmatrix} \left| \begin{matrix} k_\beta \\ k_{\beta+1} \end{matrix} \right), \tag{3.33a}$$

$$|E_\alpha)_1 = \sum_{\substack{i=0 \\ i\neq\{\mathcal{L},\mathcal{L}+1\}}}^{S+1} \sum_{\substack{\beta,\gamma \\ (\beta\neq\alpha,-)}} p_{\alpha+1} R_\beta^{\pm} \begin{bmatrix} i \\ \gamma \\ \gamma+1 \end{bmatrix}_{(\beta\neq\gamma)} \left| \begin{matrix} k_i \\ k_\gamma \\ k_{\gamma+1} \end{matrix} \right)$$

$$+ \sum_{\beta} p_{\alpha+1} R_\beta \begin{bmatrix} \beta \\ \beta+2 \end{bmatrix} \left| \begin{matrix} k_\beta \\ k_{\beta+1} \end{matrix} \right). \tag{3.33b}$$

We observe that the first-order vector $\begin{pmatrix} |\underset{\sim}{F})_1 \\ |\underset{\sim}{E})_1 \end{pmatrix}$ differs from the zero-order vector $\begin{pmatrix} |\underset{\sim}{F}) \\ |\underset{\sim}{E}) \end{pmatrix}$ essentially only by the insertion of the operator $R_\beta^{\pm}$. This, in fact, is a general feature for all higher-order calculation. To see this, we again represent the GH matrix as an operator acting on the vector $\begin{pmatrix} |\underset{\sim}{F}) \\ |\underset{\sim}{E}) \end{pmatrix}$. From Eqs. (3.15), (3.25), (3.23), and (3.32), we see that the operator is

$$[GH]_{\alpha\beta} = \begin{pmatrix} \overline{C} & \overline{D} \\ \overline{A} & \overline{C}^T \end{pmatrix} = \begin{pmatrix} \hat{p}_{\alpha+1} R_\alpha^{-1} \hat{p}_{\beta+1}^{-1} (\ )_\beta & \underset{(\alpha\neq\beta)}{\hat{p}_{\alpha+1} R_\alpha^{-1} R_\beta p_{\beta+1}^{-1} (\ )_\beta} \\ \underset{(\alpha\neq\beta)}{p_{\alpha+1} \hat{p}_{\beta+1}^{-1} (\ )_\beta} & p_{\alpha+1} R_\beta p_{\beta+1}^{-1} (\ )_\beta \end{pmatrix} . \tag{3.34}$$

Again, this GH-operator representation can be used for the vector

$$\begin{pmatrix} |\underset{\sim}{F})_n \\ |\underset{\sim}{E})_n \end{pmatrix} = [GH]^n \begin{pmatrix} |\underset{\sim}{F}) \\ |\underset{\sim}{E}) \end{pmatrix}.$$

The remarkable feature about this GH-operator representation is that $[GH]^n$ can be found by inserting $R_\gamma^{\pm} R_\delta^{\pm} \cdots R_\sigma^{\pm}$ (total number of $R^{\pm}$'s is $n - 1$) in Eq. (3.34). i.e.,

$$[GH]_{\alpha\beta}{}^{n} = \begin{pmatrix} \sum\limits_{\gamma,\cdots\delta=\{\mathcal{L}\}} \hat{p}_{\alpha+1}\, R_{\alpha}{}^{-1}[R^{\pm}]^{n-1}_{\gamma\delta}\, \hat{p}^{-1}_{\beta+1}(\ )_{\beta} & \sum\limits_{\gamma,\cdots,\delta=\{\mathcal{L}\}} \hat{p}_{\alpha+1}\, R_{\alpha}{}^{-1}[R^{\pm}]^{n-1}_{\gamma\delta}\, p^{-1}_{\beta+1}(\ )_{\beta} \\ \sum\limits_{\gamma,\cdots,\delta=\{\mathcal{L}\}} p_{\alpha+1}[R^{\pm}]^{n-1}_{\gamma\delta}\, \hat{p}^{-1}_{\beta+1}(\ )_{\beta} & \sum\limits_{\gamma,\cdots,\delta=\{\mathcal{L}\}} p_{\alpha+1}[R^{\pm}]^{n-1}_{\gamma\delta} R_{\beta}\, p^{-1}_{\beta+1}(\ )_{\beta} \end{pmatrix}, \tag{3.35}$$

where $[R^{\pm}]^{n-1}_{\gamma\delta} \equiv R_{\gamma}^{\pm} R_{\lambda}^{\pm} \cdots R_{\delta}^{\pm}$ (total number = n - 1), with the restriction that $\mathcal{L} \neq \mathcal{L}'$ in products $R_{\mathcal{L}} R^{-1}_{\mathcal{L}'}$ or $R^{-1}_{\mathcal{L}} R_{\mathcal{L}'}$. Now we can directly obtain the nth-order result by sandwiching Eq. (3.35) between

$$\left((E_{\alpha}|,(F_{\alpha}|\right) \quad \text{and} \quad \begin{pmatrix} |F_{\alpha}) \\ |E_{\alpha}) \end{pmatrix} :$$

$$I_n \equiv \exp\left\{\frac{1}{2}\left((\underset{\sim}{E}|(\underset{\sim}{F}|\right)[GH]^n \begin{pmatrix} |\underset{\sim}{F}) \\ |\underset{\sim}{E}) \end{pmatrix}\right\}$$

$$= \prod_{\alpha,\cdots,\beta=\{\mathcal{L}\}} \prod_{\substack{i,j=0 \\ i,j\neq\{\alpha,\alpha+1\}}}^{S+1} \{y_i - [R^{\pm}]^{n+1}_{\alpha\beta}(y_j)\}^{-\frac{1}{2}k_i\cdot k_j}$$

$$\times \prod_{\alpha,\cdots,\beta,\lambda=\{\mathcal{L}\}} \prod_{\substack{i=0 \\ i\neq\{\mathcal{L},\mathcal{L}+1\}}}^{S+1} \left\{\frac{y_i - [R^{\pm}]^{n+1}_{\alpha\beta}(y_{\lambda})}{y_i - [R^{\pm}]^{n+1}_{\alpha\beta}(y_{\lambda+1})}\right\}^{-k_i\cdot k_{\lambda}}_{(\beta\neq\lambda)}$$

$$\times \left\{\frac{y_i - [R^{\pm}]_{\alpha\beta}{}^{n} R_{\lambda}(y_{\lambda})}{y_i - [R^{\pm}]_{\alpha\beta}{}^{n}(y_{\lambda})}\right\}^{-k_i\cdot k_{\lambda}}_{(\beta\neq\lambda,-)}$$

$$\times \prod_{\alpha,\beta,\cdots,\delta,\lambda=\{\mathcal{L}\}} \left\{\frac{y_{\beta} - [R^{\pm}]^{n+1}_{\alpha\delta}(y_{\lambda})}{y_{\beta} - [R^{\pm}]^{n+1}_{\alpha\delta}(y_{\lambda+1})} \cdot \frac{y_{\beta+1} - [R^{\pm}]^{n+1}_{\alpha\delta}(y_{\lambda+1})}{y_{\beta+1} - [R^{\pm}]^{n+1}_{\alpha\delta}(y_{\lambda})}\right\}^{-\frac{1}{2}k_{\beta}\cdot k_{\lambda}}_{\binom{\beta\neq\alpha}{\lambda\neq\delta}}$$

$$\times \left\{\frac{y_{\beta} - [R^{\pm}]_{\alpha\delta}{}^{n} R_{\lambda}(y_{\lambda})}{y_{\beta} - [R^{\pm}]_{\alpha\delta}{}^{n}(y_{\lambda})} \cdot \frac{y_{\beta+1} - [R^{\pm}]_{\alpha\delta}{}^{n}(y_{\lambda})}{y_{\beta+1} - [R^{\pm}]_{\alpha\delta}{}^{n} R_{\lambda}(y_{\lambda})}\right\}^{-k_{\beta}\cdot k_{\lambda}}_{\binom{\delta\neq\lambda,-}{\alpha\neq\beta}}$$

Equation (3.36) continued

Equation (3.36) continued

$$\times \left\{ \frac{R_\beta(y_\beta) - [R^\pm]^{n-1}_{\alpha\delta} R_\lambda(y_\lambda)}{R_\beta(y_\beta) - [R^\pm]^{n-1}_{\alpha\delta}(y_\lambda)} \cdot \frac{y_\beta - [R^\pm]^{n-1}_{\alpha\delta}(y_\lambda)}{y_\beta - [R^\pm]^{n-1}_{\alpha\delta} R_\lambda(y_\lambda)} \right\}^{-\frac{1}{2}k_\beta \cdot k_\lambda}_{\left(\substack{\delta\neq\lambda,-\\ \alpha\neq\beta,+}\right)} , \quad (3.36)$$

where $n \geq 1$.

We note in Eq. (3.36) that the $R_\lambda^{\pm}$ operator occurs in different orders. This is necessary to facilitate the infinite number of cancellations in the terms involving y_λ which lead to the invariant points $R_\lambda^{+\infty}(\ )$ of the projective operators R_λ. In Appendix C we show how the cancellations actually lead to the invariant points.

By combining terms of all orders from Eqs. (3.16), (3.24), (3.36), and(A2.5,6), we thus obtain

$$I = \frac{1}{(\det[\Delta])^{\frac{1}{2}}} \prod_{n=0}^{\infty} I_n = \frac{1}{(\det[\Delta])^{\frac{1}{2}}} \prod_{\alpha,\cdots,\beta=\{\mathcal{L}\}} \prod_{n=0}^{\infty} \prod_{\substack{i,j=0\\ k,j\neq\{\mathcal{L},\mathcal{L}+1\}}}^{S+1}$$

$$\times \{y_i - [R^\pm]^{n+1}_{\alpha\beta}(y_j)\}^{-\frac{1}{2}k_i \cdot k_j} \prod_{\alpha,\cdots,\beta,\lambda=\{\mathcal{L}\}} \prod_{n=0}^{\infty} \prod_{\substack{i,j=0\\ i,j\neq\{\mathcal{L},\mathcal{L}+1\}}}^{S+1}$$

$$\times \left\{ \frac{y_i - [R^\pm]^{n}_{\alpha\beta}(x_\lambda^{(2)})}{y_i - [R^\pm]^{n+1}_{\alpha\beta}(x_\lambda^{(1)})} \right\}^{-k_i \cdot k_\lambda}_{(\beta\neq\lambda)} \prod_{\alpha=\{\mathcal{L}\}} \prod_{\substack{i=0\\ i\neq\{\mathcal{L},\mathcal{L}+1\}}}^{S+1} \left\{ \frac{1}{y_i - y_\alpha} \right\}^{-k_i \cdot k_\alpha}$$

$$\times \prod_{\alpha,\cdots,\delta\beta,\lambda=\{\mathcal{L}\}} \prod_{n=0}^{\infty} \left\{ \frac{x_\beta^{(1)} - [R^\pm]^{n+1}_{\alpha\delta}(x_\lambda^{(1)})}{x_\beta^{(2)} - [R^\pm]^{n+1}_{\alpha\delta}(x_\lambda^{(1)})} \right.$$

Equation (3.37) continued

Equation (3.37) continued

$$\times \left\{ \frac{x_\beta^{(2)} - [R^{\pm}]_{\alpha\delta}{}^{n}(x_\lambda^{(2)})}{x_\beta^{(1)} - [R^{\pm}]_{\alpha\delta}{}^{n}(x_\lambda^{(2)})} \right\}^{-\frac{1}{2}k_\beta \cdot k_\lambda}_{\left(\substack{\alpha\neq\beta \\ \delta\neq\lambda}\right)}$$

$$\times \prod_{\alpha,\beta=\{\mathcal{L}\}} \left\{ \frac{x_\alpha^{(1)} - y_\beta}{y_\alpha - y_\beta} \cdot \frac{y_\alpha - x_\beta^{(1)}}{x_\alpha^{(1)} - x_\beta^{(2)}} \right\}^{-\frac{1}{2}k_\alpha \cdot k_\beta}_{(\alpha\neq\beta)} , \tag{3.37}$$

where $R_{\alpha\delta}^{\pm 0} \equiv I$, and $x_\lambda^{(1)}, x_\lambda^{(2)}$ are the two invariant points of R_λ, i.e.,

$$x_\lambda^{(1)} = R_\lambda{}^{-\infty}(z_1) = y_{\lambda+1}, \quad z_1 \neq x_\lambda^{(2)},$$

$$x_\lambda^{(2)} = R_\lambda{}^{+\infty}(z_2), \quad z_2 \neq x_\lambda^{(1)}. \tag{3.38}$$

We note that all factors in Eq. (3.37) which contain the variable y_α, $\alpha = \{\mathcal{L}\}$ are cancelled by similar factors in $\{Y_{S+2}\}$, Eq. (3.6)

$$\{Y_{S+2}\} = (y_a - y_b)(y_b - y_c)(y_c - y_a) \prod_{\substack{i,j=0 \\ i\neq j \\ i,j\neq\{\mathcal{L},\mathcal{L}+1\}}}^{S+1} (y_i - y_j)^{-\frac{1}{2}k_i \cdot k_j}$$

$$\times \prod_{\substack{i=0 \\ i\neq\{\mathcal{L},\mathcal{L}+1\}}}^{S+1} \prod_{\alpha=\{\mathcal{L}\}} \left\{ \frac{y_i - y_\alpha}{y_i - x_\alpha^{(1)}} \right\}^{-k_i \cdot k_\alpha} \{x_\alpha^{(1)} - y_\alpha\}^{k_\alpha \cdot k_\alpha}$$

Equation (3.39) continued

Equation (3.39) continued

$$\times \prod_{\alpha,\beta=\{\mathcal{L}\}} \left\{ \frac{(x_\alpha^{(1)} - x_\beta^{(2)})(y_\alpha - y_\beta)}{(x_\alpha^{(1)} - y_\beta)(y_\alpha - x_\beta^{(2)})} \right\}^{-\frac{1}{2}k_\alpha \cdot k_\beta}_{(\alpha\neq\beta)} \prod_{\substack{i=0 \\ i\neq\{\mathcal{L}\pm 1,\mathcal{L}\} \\ (n=0,i\neq j)}}^{S+1} (y_i - y_{i+1})^{\alpha_0 - 1}$$

$$\times \prod_{\alpha=\{\mathcal{L}\}} \{(y_\alpha - y_{\alpha-1})^{-\frac{1}{2}k_\alpha^2 - 1} (x_\alpha^{(1)} - y_\alpha)^{-\alpha_0 - 1 - k_\alpha^2} (y_{\alpha+2} - x_\alpha^{(1)})^{-\frac{1}{2}k_\alpha^2 - 1}$$

$$\times (x_\alpha^{(1)} - y_{\alpha-1})^{\alpha_0 + \frac{1}{2}k_\alpha^2} (y_{\alpha+2} - y_\alpha)^{\alpha_0 + \frac{1}{2}k_\alpha^2} \}, \tag{3.39}$$

where we choose the ordering $y_i \geq y_j$ if $i < j$. By combining Eq. (3.37) with (3.39), we get

$$\{Y_{S+2}\}I = \frac{1}{(\det[\Delta])^{\frac{1}{2}}} \prod_{\alpha,\cdots,\beta=\{\mathcal{L}\}} \prod_{n=0}^{\infty} \prod_{\substack{i,j=0 \\ i,j\neq\{\mathcal{L},\mathcal{L}+1\} \\ (n=0,i\neq j)}}^{S+1}$$

$$\times \{y_i - [R^\pm]_{\alpha\beta}{}^n(y_j)\}^{-\frac{1}{2}k_i \cdot k_j}$$

$$\times \prod_{\alpha,\cdots,\beta,\lambda=\{\mathcal{L}\}} \prod_{n=0}^{\infty} \prod_{\substack{i,j=0 \\ i,j\neq\{\mathcal{L},\mathcal{L}+1\}}}^{S+1} \left\{ \frac{y_i - [R^\pm]_{\alpha\beta}{}^n(x_\lambda^{(2)})}{y_i - [R^\pm]_{\alpha\beta}{}^n(x_\lambda^{(1)})} \right\}^{-k_i \cdot k_\lambda}_{(\beta\neq\lambda)}$$

$$\times \prod_{\alpha,\cdots,\delta,\beta,\lambda=\{\mathcal{L}\}} \prod_{n=0}^{\infty} \left\{ \frac{x_\beta^{(2)} - [R^\pm]_{\alpha\delta}{}^n(x_\lambda^{(2)})}{x_\beta^{(2)} - [R^\pm]_{\alpha\delta}{}^n(x_\lambda^{(1)})} \cdot \frac{x_\beta^{(1)} - [R^\pm]_{\alpha\delta}{}^n(x_\lambda^{(1)})}{x_\beta^{(1)} - [R^\pm]_{\alpha\delta}{}^n(x_\lambda^{(2)})} \right\}^{-\frac{1}{2}k_\beta \cdot k_\lambda}_{\left(\substack{\alpha\neq\beta \\ \delta\neq\lambda}\right)}$$

$$\times \prod_{\substack{i=0 \\ i\neq\{\mathcal{L}\pm 1,\mathcal{L}\}}}^{S+1} (y_i - y_{i+1})^{\alpha_0 - 1} \left\{ \prod_{\alpha=\{\mathcal{L}\}} \left[\frac{(y_\alpha - y_{\alpha-1})(y_{\alpha+2} - x_\alpha^{(1)})}{(x_\alpha^{(1)} - y_{\alpha-1})(y_{\alpha+2} - y_\alpha)} \right]^{-\frac{1}{2}k_\alpha^2 - 1} \right.$$

Equation (3.40) continued

$$\times \; (x_\alpha^{(1)} - y_{\alpha-1})^{\alpha_0-1}(y_{\alpha+2} - y_\alpha)^{\alpha_0-1}(x_\alpha^{(1)} - y_\chi)^{-\alpha_0-1}(y_a - y_b)$$

$$\times \; (y_b - y_c)(y_c - y_a) . \qquad (3.40)$$

We note that the last factor in the brace { } of Eq. (3.40) is identical (except for the product sign $\prod_{\alpha=\{\mathcal{L}\}}$) to those factors in the single-loop case, Eqs. (2.41) and (2.42), which are cancelled when we transform from the set of variables $(t_\alpha, y_\alpha, y_{\alpha+1})$ to the new set of variables $[X_\alpha, x_\alpha^{(1)}, x_\alpha^{(2)}]$ [where $X_\alpha, x_\alpha^{(1)}, x_\alpha^{(2)}$ are the multiplier and the two invariant points of the projective operator R_α corresponding to going around the α loop].

In strict analogy to the single-loop case of Eqs. (2.41) through (2.45), (2.46), (2.49), and (2.50), we can choose the particular frame $x_\alpha^{(1)} = \infty$, $x_\alpha^{(2)} = 0$, $y_{\alpha+2} = 1$ and compute the Jacobian factor $J_\alpha \equiv \dfrac{\partial(t_\alpha, y_\alpha, y_{\alpha+1})}{\partial(X_\alpha, x_\alpha^{(1)}, x_\alpha^{(2)})}$ for the α loop. We then eliminate $t_\alpha, y_\alpha, y_{\alpha+1}$ in favor of $X_\alpha, x_\alpha^{(1)}, x_\alpha^{(2)}$, and we obtain the frame-independent linear dependence correction factor $(1 - X_\alpha)$. We can then go back to a general frame [see Eqs. (2.46), (2.49), and (2.50)]; and, in particular, to the frame $x_\beta^{(1)} = \infty$, $x_\beta^{(2)} = 0$, $y_{\beta+2} = 1$, where β refers to the β loop $(\beta \neq \alpha)$. Again, in this frame we can compute the Jacobian factor $J_\beta = \dfrac{\partial(t_\beta, y_\beta, y_{\beta+1})}{\partial(X_\beta, x_\beta^{(1)}, x_\beta^{(2)})}$ and eliminate $t_\beta, y_\beta, y_{\beta+1}$ in favor of $X_\beta, x_\beta^{(1)}, x_\beta^{(2)}$, so obtaining the linear dependence correction factor $(1 - X_\beta)$ for the β loop. Since the overall Jacobian factor for N loops is

$$J \equiv \frac{\partial(t_\alpha, y_\alpha, y_{\alpha+1}, t_\beta y_\beta, y_{\beta+1}, \cdots)}{\partial(X_\alpha, x_\alpha^{(1)}, x_\alpha^{(2)}, X_\beta, x_\beta^{(1)}, x_\beta^{(2)}, \cdots)}$$

$$= \prod_{\alpha=\{\mathcal{L}\}} \frac{\partial(t_\alpha, y_\alpha, y_{\alpha+1})}{\partial(X_\alpha, x_\alpha^{(1)}, x_\alpha^{(2)})} = \prod_{\alpha=\{\mathcal{L}\}} J_\alpha, \qquad (3.41)$$

thus the overall linear dependence correction factor for N loops is

$$\prod_{\alpha=\{\mathcal{L}\}} (1 - X_\alpha), \qquad (3.42)$$

where X_α is the multiplier of the projective operator R_α going around the α loop.

We now put everything together by combining Eqs. (3.6), (3.40), and (3.42), and we finally get the planar N-loop formula (Fig. 5),

$$\mathrm{FPL}(N) = \int \prod_{\alpha=\{\mathcal{L}\}} d^4k_\alpha \int \prod_{\alpha=\{\mathcal{L}\}} dX_\alpha \, X_\alpha^{-\ell(k_\alpha)-1} (1 - X_\alpha)$$

$$\times \int \frac{\prod_{\substack{i=0 \\ i\neq a,b,c}}^{S+1} dy_i \prod_{\substack{\alpha=\{\mathcal{L}\} \\ [\mathcal{L},\mathcal{L}+1\}}} dx_\alpha^{(1)} dx_\alpha^{(2)} [dy_a][dy_b][dy_c](y_a-y_b)(y_b-y_c)(y_c-y_a)}{\prod_{\substack{i=0 \\ i\neq\{\mathcal{L}\pm1,\mathcal{L}\}}}^{S+1} (y_i-y_{i+1}) \prod_{\alpha=\{\mathcal{L}\}} (x_\alpha^{(2)}-x_\alpha^{(1)})^2 \left\{\frac{(x_\alpha^{(1)}-y_{\alpha+2})[y_{\alpha-1}-R_\alpha(y_{\alpha+2})]}{[x_\alpha^{(1)}-R_\alpha(y_{\alpha+2})]}\right\}}$$

$$\times \prod_{\substack{i=0 \\ i\neq\{\mathcal{L}\pm1,\mathcal{L}\}}}^{S+1} (y_i - y_{i+1})^{\alpha_0} \prod_{\alpha=\{\mathcal{L}\}} \left\{\frac{(x_\alpha^{(1)} - y_{\alpha+2})[y_{\alpha-1} - R_\alpha(y_{\alpha+2})]}{[x_\alpha^{(1)} - R_\alpha(y_{\alpha+2})]}\right\}^{\alpha_0}$$

Equation (3.43) continued

Equation (3.43) continued

$$\times \left\{ \frac{1}{(\det[\Delta])^{\frac{1}{2}}} \right\}$$

$$\times \prod_{\substack{\text{all } \alpha,\beta,\cdots=\{\mathcal{L}\}}} \prod_{n=0}^{\infty} \left\{ \prod_{\substack{i,j=0 \\ i,j\neq\{\mathcal{L},\mathcal{L}+1\} \\ (n=0,\, i\neq j)}}^{S+1} \{y_i - [R^{\pm}]_{\alpha\beta}{}^{n}(y_j)\}^{-\frac{1}{2}k_i\cdot k_j} \right.$$

$$\times \prod_{\substack{i=0 \\ i\neq\{\mathcal{L},\mathcal{L}+1\}}}^{S+1} \left[\frac{y_i - [R^{\pm}]_{\alpha\beta}{}^{n}(x_\lambda^{(2)})}{y_i - [R^{\pm}]_{\alpha\beta}{}^{n}(x_\lambda^{(1)})} \right]^{-k_i\cdot k_\lambda} \quad (\beta\neq\lambda)$$

$$\left. \times \left[\frac{x_\beta^{(1)} - [R^{\pm}]_{\alpha\delta}{}^{n}(x_\lambda^{(1)})}{x_\beta^{(2)} - [R^{\pm}]_{\alpha\delta}{}^{n}(x_\lambda^{(1)})} \cdot \frac{x_\beta^{(2)} - [R^{\pm}]_{\alpha\delta}{}^{n}(x_\lambda^{(2)})}{x_\beta^{(1)} - [R^{\pm}]_{\alpha\delta}{}^{n}(x_\lambda^{(2)})} \right]^{-\frac{1}{2}k_\beta\cdot k_\lambda}_{\left(\substack{\alpha\neq\beta \\ \delta\neq\lambda}\right)} \right\} , \quad (3.43)$$

where y_a, y_b, y_c are any three fixed variables of the set of $S + 2$ variables $\left\{y_i, i\neq\{\mathcal{L},\mathcal{L}+1\},\ i = 0,1,\cdots S+1;\ x_\alpha^{(1)}, x_\alpha^{(2)}; \alpha=\{\mathcal{L}\}\right\}$. The ordering of these $S + 2$ Koba-Nielsen variables is shown in Fig. (6a). The projective operator $R_\alpha{}^{\pm}$, which corresponds to going around the α loop, is defined by its multiplier X_α, and its two invariant points $x_\alpha^{(1)}$, $x_\alpha^{(2)}$:

$$R_\alpha{}^{\pm}(z) = \frac{z[x_\alpha^{(2)} - x_\alpha^{(1)}]X_\alpha{}^{\pm} - x_\alpha^{(1)}\, x_\alpha^{(2)}(1 - X_\alpha{}^{\pm})}{z(1 - X_\alpha{}^{\pm}) + x_\alpha^{(2)}\, X_\alpha{}^{\pm} - x_\alpha^{(1)}} .$$

The notation $[R^{\pm}]_{\alpha\beta}{}^{n}$ represents

$$[R^{\pm}]_{\alpha\beta}{}^{n} \equiv R_{\alpha}{}^{\pm}R_{\lambda}{}^{\pm}\cdots R_{\beta}{}^{\pm},$$

with total number of $\{\alpha,\lambda,\cdots\beta\}$ equal to n; in the particular case in which $n = 0$, $[R^{\pm}]_{\alpha\beta}{}^{0} = I$. It is implied that $\mathcal{L} \neq \mathcal{L}'$ in product $R_{\mathcal{L}}R_{\mathcal{L}'}^{-1}$, or $R_{\mathcal{L}}^{-1}R_{\mathcal{L}'}$. The divergent factor $(\det[\Delta])^{-\frac{1}{2}}$ will be shown in the Sciuto three-reggeon formulation to give

$$(\det[\Delta])^{-\frac{1}{2}} = \prod_{\{\bar{R}\}}(1 - X_{\bar{R}})^{-4}, \tag{3.44}$$

where $\{\bar{R}\}$ denotes the projective group elements generated by $R_{\alpha}{}^{\pm}$, $\alpha = \{\mathcal{L}\}$, i.e., it contains terms like $R_{\alpha}{}^{\pm n}R_{\beta}{}^{\pm m}\cdots$; $\alpha,\beta,\cdots = \{\mathcal{L}\}$, and $n,m,\cdots = 0,1,\cdots\infty$. The symbols $X_{\bar{R}}$ denote multipliers of the projective group elements.

We see that the planar N-loop formula, Eq. (3.43), is hardly different from the planar single-loop formula, Eq. (2.50). We interpret various factors in Eq. (3.42) as follows:

(1) The volume element and the factors before $(\det[\Delta])^{-\frac{1}{2}}$, together with $\prod_{i\neq j}\{y_i - y_j\}^{-\frac{1}{2}k_i\cdot k_j}$, is projectively invariant. It is symmetrical with respect to the $(S + 2 - 2N)$ external legs and also symmetrical with respect to the N loops.

(2) The factor $\{y_i - [R^{\pm}]_{\alpha\beta}{}^{n}(y_j)\}^{-\frac{1}{2}k_i\cdot k_j}$ describes all "lines" connecting the external y_i leg with the external y_j leg and which go round the N loops a total number of n times (in either direction). The restriction that $\mathcal{L} \neq \mathcal{L}'$ in the product $R_{\mathcal{L}}R_{\mathcal{L}'}^{-1}$ or $R_{\mathcal{L}}^{-1}R_{\mathcal{L}'}$ implies that a line does not go successively round the same loop in opposite

directions. The $n = 0$ component describes the line connecting the external legs y_i with y_j without surrounding any of the N loops.

(3) The factor

$$\left\{\frac{y_i - [R^{\pm}]_{\alpha\beta}{}^{n}(x_\lambda^{(2)})}{y_i - [R^{\pm}]_{\alpha\beta}{}^{n}(x_\lambda^{(1)})}\right\}^{-k_i\cdot k_\lambda}_{(\beta\neq\lambda)}$$

describes all "lines" connecting the external y_i leg with the center points $x_\lambda^{(1)}, x_\lambda^{(2)}$ of the λ loop and which go round the loops a total of n times. The final loop surrounded must not be the λ loop.

(4) The last factor in Eq. (3.43)

$$\left[\frac{x_\beta^{(1)} - [R^{\pm}]_{\alpha\delta}{}^{n}(x_\lambda^{(1)})}{x_\beta^{(1)} - [R^{\pm}]_{\alpha\delta}{}^{n}(x_\lambda^{(2)})} \cdot \frac{x_\beta^{(2)} - [R^{\pm}]_{\alpha\delta}{}^{n}(x_\lambda^{(2)})}{x_\beta^{(2)} - [R^{\pm}]_{\alpha\delta}{}^{n}(x_\lambda^{(1)})}\right]^{-\frac{1}{2}k_\beta\cdot k_\lambda}_{\left(\substack{\alpha\neq\beta\\ \delta\neq\lambda}\right)},$$

describes the "lines" connecting the β loop with the λ loop, and going round the loops a total of n times. The first loop surrounded must not be the β loop, the last must not be the λ loop. The $n = 0$ component describes the lines directly connecting the β loop with the λ loop without going around any of the other $N - 2$ loops [and the $(S + 2 - 2N)$ external legs].

(5) The divergent determinant factor, Eq. (3.44), describes all "closed lines" going around the N loops. The lines are not distinguished by their overall directions or by the point at which they begin.

To conclude, we see that we have a mathematically exact expression for the "rubber-band" (or fish-net) model with any number of holes cut in it. Our planar N-loop formula is manifestly projectively

invariant, symmetrical with respect to the N loops and the $S + 2 - 2N$ external scalar legs, and hence, manifestly dual. The extension[17] to nonplanar, overlapping or nonorientable (or both) N-loop diagrams is straightforward. In fact, we can almost guess the exact formula for them, although the details of the proof will be mathematically more complicated. They will be given in the subsequent paper(s).

IV. THE N-LOOP AMPLITUDE IN THE FORMALISM OF SCIUTO

The calculation of the N-loop amplitude via Sciuto three-reggeon vertex functions parallels the previous calculation, except that duality and projective invariance are not obtained until the end.

The sequence of manipulations will be as follows: we first insert one set of intermediate states $|\lambda_\alpha\rangle\langle\lambda_\alpha|$ in the upper portion of each loop; we then notice that the resulting amplitude consists of the vacuum expectation value of a product of propagators and vertices (which now depend on λ_α); by contracting over these propagators and modified vertices, we will mix the various λ's, thereby obtaining a Gaussian integral much like before; finally, we perform the integral by going to the principal axes.

In writing the amplitude, we shall express the base line of operators in terms of "a" operators, "b" operators will denote the upper portion of each loop. Into an ordinary multiperipheral tree, we shall insert N loops, each denoted by L^α (see Fig. 7),

$$\begin{aligned} FPL(N) &= \langle 0|_a V_a^{\,S} D_a^{\,S} V_a^{S-1} D_a^{S-1} \cdots V_a^{\alpha+2} D_a^{\alpha+2} L^\alpha \\ &\times D_a^{\,\alpha} V_a^{\alpha-1} D_a^{\alpha-1} \cdots V_a^{\,2} D_a^{\,2} V_a^{\,1} |0\rangle_a, \end{aligned} \tag{4.1}$$

where

$$L^\alpha \equiv \langle 0|_b W_{aba}^{\alpha+1} D_a^{\alpha+1} D_b^{\,\alpha} \bar{W}_{aba}^{\alpha} |0\rangle_b,$$

$$W_{aba}^{\alpha+1} \equiv \exp[(k_{\alpha+1}|a^+) + (a^+|b)_+ + (\pi_{\alpha+1}|b)]$$

$$\times \exp[(k_{\alpha+1}|a) + (a|b)_-],$$

Equation (4.2) continued

Equation (4.2) continued

$$\bar{W}^{\alpha}_{aba} \equiv \exp[(k_\alpha|a^+) + (a^+|b^+)_-] \ \exp[(\pi_\alpha|b^+) + (k_\alpha|a) + (a|b^+)_+],$$

$$V_a^{\ i} \equiv \exp(k_i|a^+) \exp(k_i|a) \ ,$$

$$D_a^{\ i} \equiv \int_0^1 dx_i \ x_i^{R_a - \alpha(\pi_i^{\ 2})-1} (1 - x_i)^{-c} \ , \quad \alpha(\pi_i^{\ 2}) \equiv \frac{1}{2}\pi_i^{\ 2} - \frac{1}{2} m^2,$$

$$k_\alpha \equiv -k_{\alpha+1}. \tag{4.2}$$

(Notice that we have dropped all harmonic oscillator indices. Also, linear dependences and the $(1 - z)^{R-\alpha}$ factors have been omitted, since they will become critical only at a later stage of our work.)

We now insert coherent states and contract over "b" operators:

$$\langle 0|_b \ W^{\alpha+1}_{aba} \ D_a^{\alpha+1} \ D_b^{\ \alpha}|\lambda_\alpha\rangle\langle\lambda_\alpha|\bar{W}^{\alpha}_{aba}|0\rangle_b$$

$$= \int_0^1 du_\alpha \ \exp\{(a^+|k_{\alpha+1}) + (a^+|M_+ u_\alpha|\lambda_\alpha)\}$$

$$\times \ \exp\{(a|k_{\alpha+1}) + (a|M_- u_\alpha|\lambda_\alpha)\} D_a^{\alpha+1} \ \exp\{(a^+|k_\alpha) + (a^+|M_-|\lambda_\alpha^*)\}$$

$$\times \ \exp\{(a|k_\alpha) + (a|M_+|\lambda_\alpha^*)\} \ \exp\{(\pi_{\alpha+1}|u_\alpha|\lambda_\alpha) - (\pi_\alpha|\lambda_\alpha^*)\} \ u_\alpha^{-\alpha(k_\alpha^{\ 2})-1}$$

$$\times \ (1 - u_\alpha)^{-c}. \tag{4.3}$$

We shall find it very convenient to redefine our momenta:

$$|\bar{k}_{\alpha+1}) \equiv |k_{\alpha+1}) + M_+ u_\alpha |\lambda_\alpha) ,$$

$$|\bar{k}'_{\alpha+1}) \equiv |k_{\alpha+1}) + M_- u_\alpha |\lambda_\alpha),$$

$$|\bar{k}_\alpha) \equiv |k_\alpha) + M_- |\lambda^*_\alpha) ,$$

$$|\bar{k}'_\alpha) \equiv |k_\alpha) + M_+ |\lambda^*_\alpha),$$

$$|\bar{k}_i), |\bar{k}'_i) \equiv |k_i) \quad \text{if } i \neq \alpha \text{ or } \alpha + 1 ,$$

$$\bar{V}^i \equiv \exp(\bar{k}_i | a^+) \exp(\bar{k}'_i | a). \tag{4.4}$$

In the new notation, our amplitude reads

$$\mathrm{FPL}(N) = \prod_{\alpha=\{\mathcal{L}\}} \left\{ \int_0^1 du_\alpha \int d\left|\frac{\lambda_\alpha}{\sqrt{2}}\right) \int d\left|\frac{\lambda^*_\alpha}{\sqrt{2}}\right) \int d^4 k_\alpha \right.$$

$$\times [u_\alpha^{-\alpha(k_\alpha^2)-1} (1 - u_\alpha)^{-c}][\exp-(\lambda^*_\alpha|\lambda_\alpha)]$$

$$\left. \times [\exp\{(\pi_{\alpha+1}|u_\alpha|\lambda_\alpha) - (\pi_\alpha|\lambda^*_\alpha)\}] \langle 0|_a \bar{V}_a^S D_a^S \cdots \bar{V}_a^2 D_a^2 \bar{V}_a^1 |0\rangle_a \right\}. \tag{4.5}$$

[$\{\mathcal{L}\}$ represents the set of all loop indices.] Notice now that the vertices $\bar{V}_a^{\alpha+1}$ and $\bar{V}_a^{\alpha}$ contain the loop variables.

It is now a simple matter to contract over the "a" oscillators, thereby leaving a pure c-number expression:

$$T \equiv \langle 0|_a \bar{V}_a^S D_a^S \cdots \bar{V}_a^2 D_a^2 \bar{V}_a^1 |0\rangle_a$$

$$= \prod_{i=2}^{S} \int_0^1 dx_i \, x_i^{-\alpha(\pi_i^2)-1} (1 - x_i)^{-c} \exp\left\{ \sum_{i>j}^{S} (\bar{k}_j | x_{j+1,i} | \bar{k}'_i) \right\}, \tag{4.6}$$

where $x_{j+1,i} \equiv y_i\, y_j^{-1}$, $y_i \equiv \prod_{\ell=2}^{i} x_\ell$, $y_1 \equiv 1$, $y_0 = 0$. It is not hard to separate the coefficients of $\lambda\lambda$, $\lambda\lambda^*$, etc.:

$$\bar{T} \equiv \exp\left\{\sum_{i>j}^{S} (\bar{k}_j|x_{j+1,i}|\bar{k}'_i)\right\} = \prod_{\alpha,\beta=\{\mathcal{L}\}} \exp\left\{(\lambda_\alpha|[A_{\alpha\beta}]|\lambda_\beta)\right.$$

$$\left. + (\lambda_\alpha|[B_{\alpha\beta}]|\lambda^*_\beta) + (\lambda^*_\alpha|[C_{\alpha\beta}]|\lambda_\beta) + (\lambda^*_\alpha|[D_{\alpha\beta}]|\lambda^*_\beta)\right\}$$

$$\times \prod_{\gamma=\{\mathcal{L}\}} \exp\left\{(\lambda_\gamma|E_\gamma) + (\lambda^*_\gamma|F_\gamma)\right\} \prod_{i>j}^{S} \exp\left\{(k_i|x_{j+1,i}|k_j)\right\}, \tag{4.7}$$

where

$$[A_{\alpha\beta}] = u_\alpha\, M_-^T y_{\alpha+1}\, y_{\beta+1}^{-1}\, M_+\, u_\beta \ (\alpha > \beta), \ = 0 \ \text{otherwise},$$

$$[B_{\alpha\beta}] = u_\alpha\, M_-^T y_{\alpha+1}\, y_\beta^{-1}\, M_- (\alpha \geq \beta), = 0 \ \text{otherwise},$$

$$[C_{\alpha\beta}] = M_+\, y_\alpha\, y_{\beta+1}^{-1}\, M_- (\alpha > \beta), = 0 \ \text{otherwise},$$

$$[D_{\alpha\beta}] = M_+\, y_\alpha\, y_\beta^{-1}\, M_- (\alpha > \beta), = 0 \ \text{otherwise},$$

$$|E_\alpha) = \sum_{j=1}^{\alpha} u_\alpha\, M_-^T\, y_{\alpha+1}\, y_j^{-1}|k_j) + \sum_{j=\alpha+2}^{S} u_\alpha\, M_+\, y_j\, y_{\alpha+1}^{-1}|k_j)$$

$$+ u_\alpha|\pi_{\alpha+1}),$$

$$|F_\alpha) = \sum_{j=1}^{\alpha-1} M_+\, y_\alpha\, y_j^{-1}|k_j) + \sum_{j=\alpha+1}^{S} M_-^T\, y_j\, y_\alpha^{-1}|k_j) - |\pi_\alpha). \tag{4.8}$$

We shall symmetrize these matrices as follows:

$$[\bar{A}] = ([A] + [A]^T), \quad [\bar{D}] = ([D] + [D]^T), \quad [\bar{C}] = ([B] + [C]^T). \tag{4.9}$$

As before, we now can perform the integration via the principal-axes method:

$$\prod_{\alpha=\{\mathcal{L}\}} \int d\left|\frac{\lambda_\alpha}{\sqrt{2}}\right) \; d\left|\frac{\lambda_\alpha^*}{\sqrt{2}}\right) \; \bar{T}$$

$$= \prod_{\alpha=\{\mathcal{L}\}} \int d\left|\frac{\lambda_\alpha}{\sqrt{2}}\right) \; d\left|\frac{\lambda_\alpha^*}{\sqrt{2}}\right) \; \exp\left\{ \frac{1}{2}\left(\left(\lambda_\alpha\right|\left(\lambda_\alpha^*\right|\right) \right.\right.$$

$$\left.\left. \times \; [\Delta] \begin{pmatrix} |\lambda_\alpha) \\ |\lambda_\alpha^*) \end{pmatrix} + \left(\left(\underset{\sim}{E}\right|\left(\underset{\sim}{F}\right|\right) \begin{pmatrix} |\lambda_\alpha) \\ |\lambda_\alpha^*) \end{pmatrix} \right\}$$

$$\times \prod_{i>j}^{S} \exp\{(k_j|x_{j+1,i}|k_j)\} = (\det[\Delta])^{-\frac{1}{2}} \exp\left\{ \frac{1}{2}\left(\left(\underset{\sim}{E}\right|\left(\underset{\sim}{F}\right|\right)[\Delta]^{-1} \right.\right.$$

$$\left.\left. \times \begin{pmatrix} |\underset{\sim}{E}) \\ |\underset{\sim}{F}) \end{pmatrix} \right\} \prod_{i>j}^{S} \exp\{(k_j|x_{j+1,i}|k_i)\}, \tag{4.10}$$

where

$$[\Delta] = \begin{pmatrix} -[\bar{A}] & [1] - [\bar{C}] \\ [1] - [\bar{C}]^T & -[\bar{D}] \end{pmatrix} \equiv [G] - [H],$$

where

$$[G] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad [H] = \begin{pmatrix} [\bar{A}] & [\bar{C}] \\ [\bar{C}]^T & [\bar{D}] \end{pmatrix}.$$

Now that the integrations have been performed, we are at the stage where the projective transformations R_α must be chosen. When the matrices contained in $[\Delta]$ are contracted on $|\underset{\sim}{E})$ and $|\underset{\sim}{F})$, the action is equivalent to a projective transformation because of the action of M_+ and M_-. Once the projective transformations R_α have been identified, it is a simple task to reformulate $[\Delta]$ entirely in terms of these operators. Fortunately, as we shall see in the section concerning the singularity structure of the N-loop amplitude, the choice for the projective transformation is almost forced on us:

$$R_\alpha = y_\alpha \begin{pmatrix} 1 & -x_{\alpha+1} \\ 1 & -x_{\alpha+1}(1 - u_\alpha) \end{pmatrix} y_\alpha^{-1} ,$$

$$= y_\alpha B_\alpha y_\alpha^{-1} . \tag{4.11}$$

At this point, we mention that the contraction of the binomial matrices M_+ and M_- upon variables leaves a <u>residue term</u> in the following fashion:

$$M_- x|k) = ([1 - x] - [1])|k). \tag{4.12}$$

The extra "1" left over from the contraction with M_- shall be called the residue term. <u>Though it plays a critical role later in the paper, we shall drop all residue terms for the present</u>. When such terms are dropped, it is not hard to reexpress $[\Delta]$ entirely in terms of R and an auxiliary operator K which shall vanish at the end of the calculation:

$$\left((E|\ (F|\right)[\Delta]^{-1}\begin{pmatrix}|\underset{\sim}{E})\\ |\underset{\sim}{F})\end{pmatrix} = \sum_{m=0}^{\infty}\left((\underset{\sim}{F}|\ (\underset{\sim}{E}|\right)[HG]^{m}\begin{pmatrix}|\underset{\sim}{E})\\ |\underset{\sim}{F})\end{pmatrix}$$

$$\cong \sum_{r=0}^{\infty}\ \sum_{\substack{\alpha,\beta,\gamma,\delta=\{\mathcal{L}\}\\ (\beta\neq\gamma)\\ (\alpha\neq\delta)}}\ \sum_{\substack{i,j,\ell,m=1\\ (i\neq\alpha)\\ (j\neq\beta+1)\\ (\ell\neq\gamma+1)\\ (m\neq\delta)}}^{S+1}\begin{pmatrix}\dfrac{(k_i|}{Ky_\alpha^{-1}y_i} & \dfrac{(k_j|}{K^{-1}\dfrac{1}{B_\beta y_\beta^{-1}y_j}}\end{pmatrix}_{\alpha\beta}$$

$$\times\begin{pmatrix}KB_\alpha y_\alpha^{-1}y_\gamma K^{-1}(\ ) & KB_\alpha y_\alpha^{-1}y_\delta B_\delta^{-1}\dfrac{1}{K(\ )}\\[2ex] K^{-1}\dfrac{1}{y_\beta^{-1}y_\gamma K^{-1}(\ )} & K^{-1}\dfrac{1}{y_\beta^{-1}y_\delta B_\delta^{-1}\frac{1}{K}(\ )}\end{pmatrix}^{r}_{\alpha\beta,\gamma\delta}$$

$$\times\begin{pmatrix}KB_\gamma y_\gamma^{-1}y_\ell|k_\ell)\\[2ex] K^{-1}y_\delta y_m^{-1}|k_m)\end{pmatrix}_{\gamma\delta}, \tag{4.13}$$

where $K(z) = 1 - \frac{1}{z}$, $K^{-1}(z) = \frac{1}{1-z}$, $K^{-1} \neq \frac{1}{K}$. (Notice that B and K operate to the right.)

Though awkward in appearance, this combination of R's and K's simplifies immensely when the contractions are actually performed. Fortunately, $[HG]$ retains the same form regardless of the order of the expansion, which guarantees that the K's will vanish in the nth-order calculation if they vanish for the first. The second-order matrix, for example, is

$$[HG]^2_{\alpha\beta,\gamma\delta} = \sum_{\substack{\rho,\sigma,\tau,\zeta \\ \xi,\mu,\nu,\lambda=\{\mathcal{L}\} \\ (\sigma\neq\alpha,\sigma\neq\gamma) \\ (\zeta\neq\alpha,\mu\neq\gamma) \\ (\xi\neq\beta,\nu\neq\beta) \\ (\tau\neq\gamma,\nu\neq\delta)}}$$

$$\left(\begin{array}{c|c} KB_\alpha y_\alpha^{-1} y_\rho B_\rho y_\rho^{-1} y_\gamma K^{-1} & KB_\alpha y_\alpha^{-1} y_\tau B_\tau y_\tau^{-1} y_\gamma B_\gamma^{-1} \frac{1}{K} \\ + KB_\alpha y_\alpha^{-1} y_\sigma B_\sigma^{-1} y_\sigma^{-1} y_\gamma K^{-1} & + K^{-1} \dfrac{1}{y_\beta^{-1} y_\nu B_\nu y_\nu^{-1} y_\delta B_\delta^{-1} \frac{1}{K}} \\ \hline K^{-1} \dfrac{1}{y_\beta^{-1} y_\xi B_\xi y_\xi^{-1} y_\gamma K^{-1}} & K^{-1} \dfrac{1}{y_\beta^{-1} y_\nu B_\nu y_\nu^{-1} y_\delta B_\delta^{-1} \frac{1}{K}} \\ + K^{-1} \dfrac{1}{y_\alpha^{-1} y_\mu B_\mu^{-1} y_\mu^{-1} y_\gamma K^{-1}} & + K^{-1} \dfrac{1}{y_\beta^{-1} y_\lambda B_\lambda^{-1} y_\lambda^{-1} B_\delta \frac{1}{K}} \end{array}\right)_{\alpha\beta,\gamma\delta}$$

At this point, when the contractions over all indices are about to be performed, we shall make our second simplifying assumption: <u>by dropping all loop momenta terms, we shall restore conservation of momentum</u>. (As we shall later see, our two assumptions will cancel each other.) By restoring conservation of momentum, we are allowed to make projective transformations at will:

$$\left(\left(\underset{\sim}{F}\right| \left(\underset{\sim}{E}\right|\right) \begin{pmatrix} \left|\underset{\sim}{E}\right) \\ \left|\underset{\sim}{F}\right) \end{pmatrix} = \sum_{\alpha=\{\mathcal{L}\}} \sum_{i,j} \left(k_j \middle| \frac{KB_\alpha y_\alpha^{-1} y_j}{K y_i y_\alpha^{-1}} \middle| k_i\right)$$

$$+ \left(k_j \middle| \frac{K^{-1} y_\alpha y_i^{-1}}{K^{-1} \dfrac{1}{B_\alpha y_\alpha^{-1} y_j}} \middle| k_i\right)$$

$$\longrightarrow \prod_{\alpha=\{\mathcal{L}\}} \prod_{i,j} \left(K y_\alpha^{-1} y_i - K B_\alpha y_\alpha^{-1} y_j\right)^{-k_i k_j/2}$$

$$\times \left(K^{-1} \frac{1}{B_\alpha y_\alpha^{-1} y_j} - K^{-1} y_\alpha y_i^{-1}\right)^{-k_i k_j/2}$$

$$= \prod_{\alpha=\{\mathcal{L}\}}^{n} \prod_{i,j} \left(y_i - y_\alpha B_\alpha y_\alpha^{-1} y_j\right)^{-k_i k_j/2} \left(y_j - y_\alpha B_\alpha^{-1} y_\alpha^{-1} y_i\right)^{-k_i k_j/2} .$$

(Notice that we have summed over all harmonic oscillator states, and have projectively manipulated the contents of the parentheses because momentum is conserved among the k's.) Though we have exhibited only the zeroth-order calculation explicitly, the higher-order contractions proceed in exactly the same manner, yielding terms such as

$$\prod_{\alpha,\beta=\{\mathcal{L}\}} \prod_{i,j} (y_i - R_\alpha R_\beta y_j)^{-k_i k_j/2} (y_i - R_\alpha R_\beta^{-1} y_j)^{-k_i k_j/2}$$

$$\times (y_i - R_\alpha^{-1} R_\beta y_j)^{-k_i k_j/2} (y_i - R_\alpha^{-1} R_\beta^{-1} y_j)^{-k_i k_j/2} \tag{4.14}$$

($\alpha \neq \beta$ if R_α and R_β^{-1} are juxtaposed). In general, all terms may be expressed simply as

$$\prod_{\alpha,\beta=\{\mathcal{L}\}} \prod_{i,j} \prod_{m=0}^{\infty} \left(y_i - (R^{\pm})^{(m)}_{\alpha\beta}\, y_j\right)^{-k_i k_j/2} . \tag{4.15}$$

(Again, R and R^{-1} are never juxtaposed; also, $i \neq j$ when $m = 0$.) Notice that we have almost derived the predicted result.

In our haste to derive the above result, we have neglected several critical factors, which we shall now investigate:

(1) We have neglected all terms associated with the loop momenta in $|E)$ and $|F)$.

(2) We have neglected all residue terms coming from contractions over M_+ and M_-.

(3) We have neglected the term $\exp(\sum_{i>j}^{S} [k_i|x_{j+1,i}|k_j)]$.

(4) The variables associated with k_α and $k_{\alpha+1}$ (the loop momenta) are <u>not</u> the invariant points of the projective transformation R_α.

In short, there are an infinite number of terms which do not agree with the result found in the previous calculation with the N-factorized tree. But, as we shall see, all these infinite deviations

cancel in a most fortunate manner, term for term, until only the invariant points are left.

Though the results of the cancellation are quite elegant, the details are quite tedious and involved, especially since several types of cancellations occur within each larger one. Since the details are presented in the Appendix, only a short outline of the procedure is presented here.

The cancellations occur because the residue terms, <u>which occur only with loop momenta, mimic the original main term when all contractions are performed, except that the residue terms occur with one less power of R than the main term, and they occur with the opposite-signed exponent</u>. In other words, for every expression containing n R's,

$$\prod_{i,j=1}^{S+1} (y_i - R_\alpha R_\beta \cdots R_\rho y_j)^{-k_i k_j/2} , \tag{4.16}$$

we have a series of residue terms of degree n - 1 which occur with only loop momenta and with an exponent equal in magnitude but opposite in sign:

$$\prod_{i=1}^{S+1} \prod_{\tau=\{\mathcal{L}\}} (y_i - R_\alpha R_\beta \cdots R_\pi y_\tau)^{+k_i k_\tau/2} . \tag{4.17}$$

In general, terms involving the loop momentum factors occurring in the main term of order n cancel with the residue terms arising from the n + 1st-order main term. <u>This cancellation is exact, to all orders, and for an arbitrary number of loops</u>. We are left with one

uncancelled term involving R^n, where n approaches infinity. It turns out in fact that the variable y_α (which is <u>not</u> the invariant point of R_α) becomes replaced by $R_\alpha^n y_\alpha$, where n approaches infinity. In the limit, however, this expression becomes arbitrarily close to the invariant point. (We remark here that a second series of cancellations occurs when the R's are projectively moved from one side of the expression to the other. Details are left to the Appendix.)

Though the details are quite involved, the answer is quite elegant:

$$\exp\Big(\sum_{i>j}^{S} \big(k_j|x_{j+1,i}|k_i\big)\Big) \exp\left\{-\frac{1}{2}\Big(\big(\underset{\sim}{E}\big| \ \big(\underset{\sim}{F}\big|\Big) [\Delta]^{-1} \begin{pmatrix} |\underset{\sim}{E}) \\ |\underset{\sim}{F}) \end{pmatrix}\right\}$$

$$= \prod_{\substack{m=0 \\ \alpha,\beta=\{\mathcal{L}\}}}^{\infty} \prod_{\substack{i,j=0 \\ (i\neq\alpha,\alpha+1) \\ (j\neq\beta,\beta+1) \\ (m=0,i\neq j)}}^{S+1} \Big(w_i - (R^{\pm})^{(m)}_{\alpha\beta}\ w_j\Big)^{-k_i k_j/2}$$

$$\times \prod_{\alpha=\{\mathcal{L}\}} \left\{(y_\alpha - w_{\alpha+1})(w_\alpha - y_{\alpha+1})\Big(\frac{y_\alpha}{w_\alpha}\Big)\right\}^{-k_\alpha k_{\alpha+1}/2}$$

$$\times \prod_{i>j}^{S} (-y_j)^{-k_i k_j}, \tag{4.18}$$

where $w_i \equiv y_i, \quad (i \neq \alpha, \alpha + 1)$

$$w_\alpha \equiv x_\alpha^{(2)} \equiv R_\alpha^{\infty}(z_1), \quad z_1 \neq w_{\alpha+1}$$

$$w_{\alpha+1} \equiv x_\alpha^{(1)} \equiv R_\alpha^{-\infty}(z_2), \quad z_2 \neq w_\alpha$$

$$R_{\alpha\beta} = \delta_{\alpha\beta} \quad \text{if} \quad m = 0.$$

Now that the final answer is within reach, we remark that the total effect of linear dependences upon the N-loop planar amplitude is to modify $D_{\alpha+1}$:

$$D_{\alpha+1} \rightarrow \int_0^1 dx_{\alpha+1}\, x_{\alpha+1}^{R-\alpha(\pi_{\alpha+1}^2)-1} (1 - x_{\alpha+1})^{-c}$$

$$\times \left\{ \frac{1 - x_{\alpha+1}}{1 - x_{\alpha+1}[1 - u_\alpha(1 - x_\alpha)]} \right\} \quad . \tag{4.19}$$

So actually, whenever $x_{\alpha+1}$ appears, what is really meant is $\bar{x}_{\alpha+1}$, where

$$\bar{x}_{\alpha+1} = \frac{x_{\alpha+1}(1 - x_{\alpha+1})}{\{1 - x_{\alpha+1}[1 - u_\alpha(1 - x_\alpha)]\}} \quad . \tag{4.20}$$

The choice for the propagator variables is displayed in Fig. 7 ; notice that all $(1 - z)^R$ factors coming from the Sciuto vertex have been added in explicitly. Thus whenever $x_{\alpha+1}$ appears during the contraction over oscillator states (as in $y_{\alpha+1}$ or in R_α), it should be replaced by $\bar{x}_{\alpha+1}$. Also, u_α should be replaced by $u_\alpha(1 - x_\alpha)$ as in Fig. 7.

In summary, we find

$$\mathrm{FPL}(N) = \prod_{\alpha=\{\mathcal{L}\}} \left\{ \int d\left|\frac{\lambda_\alpha}{\sqrt{2}}\right) \quad d\left|\frac{\lambda_\alpha^*}{\sqrt{2}}\right) \int d^4 k_\alpha \right.$$

$$\times \exp\left[-\frac{1}{2}\left((\lambda_\alpha| \ (\lambda_\alpha^*|) \ [\Delta] \begin{pmatrix} |\lambda_\alpha) \\ |\lambda_\alpha^*) \end{pmatrix} + \left((\underset{\sim}{E}| \ (\underset{\sim}{F}| \right) \begin{pmatrix} |\lambda_\alpha) \\ |\lambda_\alpha^*) \end{pmatrix} \right]$$

$$\times \int_0^1 du_\alpha \left(u_\alpha (1 - x_\alpha) \right)^{-\alpha(k_\alpha^2)-1} - (1 - u_\alpha)^{-c}$$

$$\left. \times (1 - u_\alpha)^{-\alpha(\pi_{\alpha+2}^2)} \left[\frac{1 - x_{\alpha+1}}{1 - x_{\alpha+1}[1 - u_\alpha(1 - x_\alpha)]} \right]^{-\alpha(\pi_{\alpha+1}^2)} \right\}$$

$$\times \prod_{i=2}^{S} \left\{ \int_0^1 dx_i \ x_i^{-\alpha(\pi_i^2)-1} (1 - x_i)^{-c} \right\}$$

$$\times \prod_{i>j}^{S} \exp[(k_j | x_{j+1,i} | k_i)] \tag{4.21}$$

$$= \left\{ \int \prod_{\alpha=\{\mathcal{L}\}} d^4 k_\alpha \int_{R_1} \prod_{\beta=\{\mathcal{L}\}} dX_{R_\beta} \int_{R_2} \prod_{i=2}^{S} dw_i \right\}$$

$$\times \left\{ \prod_{\alpha=\{\mathcal{L}\}} (1 - X_{R_\alpha}) \prod_{\{\bar{R}\}} (1 - X_{\bar{R}})^{-4} \prod_{\beta=\{\mathcal{L}\}} X_{R_\beta}^{-\alpha(k_\beta^2)-1} \right\}$$

Equation (4.22) continued

Equation (4.22) continued

$$\times \left\{ \prod_{n=0}^{\infty} \prod_{\substack{i,j=0 \\ (i\neq\alpha,\alpha+1) \\ (\delta\neq\beta,\beta+1) \\ (n=0,i\neq j)}}^{S+1} \prod_{\alpha,\beta=\{\mathcal{L}\}} [w_i - (R^{\pm})^{(n)}_{\alpha\beta}\, w_j]^{-k_i k_j/2} \right\}$$

$$\times \prod_{\substack{i=0 \\ (i\neq\{\mathcal{L}\},\{\mathcal{L}\}-1,\{\mathcal{L}\}+1)}} (w_i - w_{i+1}) \prod_{\alpha=\{\mathcal{L}\}} \left[\frac{\left(R_\alpha(w_{\alpha-1}) - w_{\alpha+2}\right)(x_\alpha^{(1)} - w_{\alpha-1})}{[x_\alpha^{(1)} - R_\alpha(w_{\alpha-1})]} \right]^{-\frac{m^2}{2}}$$

$$\left\{ \frac{(w_0 - w_1)(w_1 - w_{S+1})(w_{S+1} - w_0)[x_\alpha^{(1)} - R_\alpha(w_{\alpha-1})]}{\prod_{\substack{i=0 \\ (i\neq\{\mathcal{L}\},\{\mathcal{L}\}-1,\{\mathcal{L}\}+1)}}^{S+1} (w_i - w_{i+1}) \prod_{\alpha=\{\mathcal{L}\}} [(R_\alpha(w_{\alpha-1})-w_{\alpha+2}](x_\alpha^{(1)} - x_\alpha^{(2)})^2 (x_\alpha^{(1)} - w_{\alpha-1})]} \right\}, \tag{4.22}$$

where $w_0 = \infty = w_{S+2}$, $(R^{\pm})^{(0)}_{\alpha\beta} \equiv \delta_{\alpha\beta}$

$w_1 = 1$,

$w_{S+1} = 0$,

$\{\bar{R}\}$ = set of all closed loops,

$(R^{\pm})^{(n)}_{\alpha\beta}$ = set of all open loops,

$w_\alpha = x_\alpha^{(2)}$, $w_{\alpha+1} = x_\alpha^{(1)}$.

(see next section for the determinant calculation)

We have made use of the following identities:

$$R_\alpha^{\infty}(z_1) \equiv w_\alpha = y_\alpha \, \bar{x}_{\alpha+1} \, x_{\alpha+1}^{-1}, \qquad z_1 \neq w_{\alpha+1},$$

$$R_\alpha^{-\infty}(z_2) \equiv w_{\alpha+1} = y_\alpha \, x_{\alpha+1}, \qquad z_2 \neq w_\alpha,$$

$$X_{R_\alpha} = \frac{u_\alpha(1 - x_\alpha)x_{\alpha+1}}{(1 - x_{\alpha+1})[1 - x_{\alpha+1}(1 - u_\alpha(1 - x_\alpha))]},$$

$$\frac{\partial(w_\alpha, w_{\alpha+1}, X_{R_\alpha})}{\partial(x_\alpha, x_{\alpha+1}, u_\alpha)} = \frac{y_{\alpha-1}(1 - x_\alpha)\, x_{\alpha+1}(w_{\alpha+1} - w_\alpha)}{[1 - x_\alpha(1 - u_\alpha(1 - x_\alpha))]^2(1 - x_{\alpha+1})}$$

and

$$(w_{\alpha+1} - w_{\alpha+2})(w_\alpha - w_{\alpha-1}) - X_{R_\alpha}(w_{\alpha+1} - w_{\alpha-1})(w_\alpha - w_{\alpha+2})$$

$$= \frac{[R_\alpha(w_{\alpha-1}) - w_{\alpha+2}](w_\alpha - w_{\alpha+1})(w_\alpha - w_{\alpha-1})}{[w_\alpha - R_\alpha(w_{\alpha-1})]}$$

$$= u_\alpha(1 - x_\alpha)(1 - x_{\alpha+1})^{-1} \, y_{\alpha-1}^{-1}(1 - x_{\alpha+1})^{-1} \, (1 - x_\alpha)^{-1}(1 - x_{\alpha+2})^{-1}.$$

The value of the multiplier, unfortunately, lies between 0 and ∞. But because of the relationship

$$X_{R_\alpha} = \frac{y_\alpha - x_\alpha^{(2)}}{y_\alpha - x_\alpha^{(1)}},$$

we see immediately that the invariant points are equal when $X_{R_\alpha} = 1$. It is not hard to show that the invariant points are strictly ordered in the regions $0 < X_{R_\alpha} < 1$ and $1 < X_{R_\alpha} < \infty$, but they have reverse

orderings in these regions. Because the imaginary part of the amplitude remains unchanged if we discard part of the region of integration, we shall adopt the branch where $0 < X_{R_\alpha} < 1$. The region R_2 is the same as in the previous section; R_1 is determined implicitly by conditions on the w's (all X's lie between 0 and 1).

V. SINGULARITY STRUCTURE OF THE N-LOOP AMPLITUDE

In the Introduction, the language of projective transformations was employed to great advantage; particularly important was the fact that multiplication by w in the single loop corresponds to a projective transformation that has been diagonalized. The factor w appearing in the singularity of the single loop can be seen to be projectively invariant once we see that w is actually the "multiplier" of the original transformation R:

$$\prod_{n=1}^{\infty} (1 - w^n)^{-4} = \prod_{n=1}^{\infty} (1 - X_R{}^n)^{-4} = \prod_{n=1}^{\infty} (1 - X_{(R^n)})^{-4}. \quad (5.1)$$

In a sense, there is a singularity factor for each of the closed paths one can make around the interior point of the dual diagram. Since the N-loop amplitude contains N projective transformations, Mandelstam has conjectured that its singularity is simply

$$\prod_{\{\bar{R}\}} (1 - X_{\bar{R}})^{-4}, \quad (5.2)$$

where $\bar{R}$ represents the set of all distinct products one can form from N projective transformations and their powers, i.e., there is a separate singularity for each of the infinitely many topologically distinct closed paths through N interior points of the dual diagram. The word "distinct" demands considerable clarification.

Notice that in the single-loop amplitude terms like w^{-1} do not occur, meaning that the direction one takes around this interior point does not matter. Second, notice that the point at which one decides to enter the closed loop makes no difference, meaning that we mustn't overcount by including both $P_1P_2P_3$ and $P_3P_1P_2$. These two properties are guaranteed by the properties of the multiplier itself:

$$X_P = X_{P^{-1}}, \qquad 0 \leq X \leq 1 , \tag{5.3}$$

$$X_{PQ} = X_{QP}. \tag{5.4}$$

Simply stated, one mustn't include inverses of previously counted terms, and one mustn't include their cyclic permutations.

A transformation R belongs to the equivalence class of Q if R has a decomposition given by cyclic permutations on the decomposition of P or inverses of such cyclic permutations.

Clearly, the singularity structure can now be given as

$$\prod_{\{\overline{P}\}} (1 - X_{\overline{P}})^{-4}, \tag{5.5}$$

where $\{\overline{P}\}$ is taken over different equivalence classes.

Unfortunately, present mathematical techniques are not powerful enough to prove this conjecture to all orders.[3] Instead, a power expansion will be made on both the determinant and the multiplier to show their exact equivalence to fourth order.

Fortunately, there is a small class of determinants which can be diagonalized and shown to have this structure. More complicated determinants, however, must be power expanded.

Consider the case where $N = 1$,

$$\det^{-\frac{1}{2}}[\Delta] = \det^{-1}(1 - u_\alpha M_-^T x_{\alpha+1} M_-)$$

$$= \det^{-1}(1 - u_\alpha M_- M_+ x_{\alpha+1} M_-)$$

$$= \det^{-1}\left(1 - u_\alpha M_- \frac{1}{1 - x_{\alpha+1}} M_+ \frac{-x_{\alpha+1}}{1 - x_{\alpha+1}}\right)$$

$$= \det^{-1}\left(1 - \frac{1}{1 - x_{\alpha+1}} M_+ \frac{-x_{\alpha+1} u_\alpha}{1 - x_{\alpha+1}} M_-\right)$$

$$= \det^{-1}\left\{1 - \left(\frac{1}{1 - x_{\alpha+1}(1 - u_\alpha)}\right) M_+ \left(\frac{u_\alpha x_{\alpha+1}}{1 - x_{\alpha+1}(1 - u_\alpha)}\right)\right\}$$

$$= \det^{-1}\left\{1 - \frac{u_\alpha x_{\alpha+1}}{[1 - x_{\alpha+1}(1 - u_\alpha)]^2} M_+\right\} . \tag{5.6}$$

But, as we shall show,

$$\det^{-1}(1 - xM_+) = \prod_{n=1}^{\infty}(1 - y^n)^{-4},$$

where

$$y = [1 - 2x \pm (1 - 4x)^{\frac{1}{2}}](2x)^{-1}. \tag{5.7}$$

Now we let

$$\Phi_{R_\alpha} = \frac{1 - x_{\alpha+1}(1 - u_\alpha)}{(u_\alpha x_{\alpha+1})^{\frac{1}{2}}} . \tag{5.8}$$

But

$$X_{R_\alpha} = \frac{\Phi_{R_\alpha}^{\ 2} - 2 \pm \Phi_{R_\alpha}(\Phi_{R_\alpha}^{\ 2} - 4)^{\frac{1}{2}}}{2} \quad . \tag{5.9}$$

Therefore

$$\det^{-1}[\Delta] = \prod_{n=1}^{\infty} (1 - X_{R_\alpha}^{\ n})^{-4} . \tag{5.10}$$

In other words, the correspondence between the determinant calculation and the expansion of the multiplier is so close that one can write down the relationship by inspection. As an added bonus, we see that the projective transformation is almost determined:

$$\Phi_{R_\alpha} = \frac{1 - x_{\alpha+1}(1 - u_\alpha)}{(u_\alpha\, x_{\alpha+1})^{\frac{1}{2}}} \rightarrow R_\alpha = \begin{pmatrix} 1 & -x_{\alpha+1} \\ 1 & -x_{\alpha+1}(1 - u_\alpha) \end{pmatrix} . \tag{5.11}$$

An obvious generalization of this operator to the N-loop case has been employed throughout the previous calculation.

(Now we must prove the statement made earlier concerning the determinant of M_+. This statement is most readily proved by proceeding backwards, that is, by assuming that the answer has the form

$$\det^{-1}(1 - y) = \prod_{n=1}^{\infty} (1 - y^n)^{-4}$$

$$(\text{where } y \equiv y^n\, \delta_{nm}). \tag{5.12}$$

Now we make similarity transformations:

$$\det(1 - y) = \det(1 - M_- y M_-) = \det\left[1 - (1 - y)M_- \left(\frac{-y}{1 - y}\right)\right]$$

$$= \det[1 - (-y)M_-] = \det[1 - M_-^T(-y)M_- M_-^T]$$

$$= \det[1 - M_-^T(-y)M_+] = \det\left[1 - \left(\frac{y}{1 + y}\right) M_+ \left(\frac{1}{1 + y}\right)\right]$$

$$= \det\left[1 - \left(\frac{y}{(1 + y)^2}\right) M_+\right] \tag{5.13}$$

where we have used

$$M_-^2 = 1, \quad M_+ = M_- M_-^T. \tag{5.14}$$

Therefore

$$\det(1 - xM_+) = \prod_{n=1}^{\infty} (1 - y^n)^4 \tag{5.15}$$

if

$$y = \frac{1 - 2x \pm (1 - 4x)^{\frac{1}{2}}}{2x} \quad \text{for} \quad 0 \leq y \leq 1 \quad \text{QED.}\} \tag{5.16}$$

Now that the single loop has been shown to have this projective character, we proceed to the N-loop case, whose determinant is too complicated to diagonalize. Instead, we shall power expand the determinant:

$$\det(1 - A) = \exp\left\{- \sum_{n=1}^{\infty} \frac{\mathrm{Tr}(A^n)}{n}\right\} . \tag{5.17}$$

This is easily proven if A is first assumed to be diagonal; a similarity transformation generalizes it for the arbitrary case.

Simultaneously, we shall make an expansion of the postulated singularity and show that there is almost a <u>one-to-one</u> correspondence between these two expression.

One last mathematical prerequisite and then the calculation will begin: we still haven't shown how to take the trace of an arbitrary number of binomial matrices. This last remaining tool is obtained if one first understands how to trace over one M_+; induction will give us the trace over an arbitrarily large number of M_+'s. There are several ways of tracing over M_+, the easiest being to simply identify the power expansion as that of the square root:

$$\mathrm{Tr}(uM_+) = \sum_{n=1}^{\infty} \frac{u^n(2n-1)!}{n!(n-1)!} = \frac{1}{2}\left(\frac{1}{(1-4u)^{\frac{1}{2}}} - 1\right) \quad \text{for} \quad 0 \le u \le \frac{1}{4}. \tag{5.18}$$

We can reduce an <u>n</u>th-order trace to $n - 1$ by using the fact that $M_+ = M_- M_-^T$:

$$\mathrm{Tr}(a_1 M_+ a_2 M_+ \cdots a_n M_+)$$

$$= \mathrm{Tr}\left(\frac{-a_1}{1-a_1} M_+ \frac{a_2}{1-a_1} M_+ \cdots M_+ \frac{a_{n-1}}{1-a_n} M_+ \frac{-a_n}{1-a_n}\right).$$

In general, we find

$$\mathrm{Tr}(a_1 M_+ a_2 M_+ \cdots a_n M_+) = \frac{1}{2}\left(\frac{1}{(1-4u)^{\frac{1}{2}}} - 1\right) \tag{5.19}$$

where

$$u \equiv \frac{\prod_{i=1}^{n} a_i}{\left\{1 - \sum_{i=1}^{n} a_i + \sum_{i,j=1}^{n} \epsilon_{ij} a_i a_j - \sum_{i,j,k=1}^{n} \epsilon_{ijk} a_i a_j a_k + \cdots\right\}}$$

$$\epsilon_{i_1 i_2 i_3 \cdots i_k} = 1 \quad \text{if} \quad i_p \geq i_q + 2 \ (p > q), \qquad (i_k - i_1 \neq n - 1),$$

$$= 0 \quad \text{otherwise.}$$

For example, for $n = 4$, we have the following nonzero ϵ's:

$$(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_{13}, \epsilon_{24}).$$

Now we have all the mathematical tools to evaluate all traces to all orders. For future reference, we exhibit the matrices explicitly:

$$[\bar{C}]_{\alpha\beta} = u_\alpha M_+ y_{\alpha+1}^{-1} y_\beta M_+,$$

$$[\bar{A}]_{\alpha\beta} = u_\alpha y_{\alpha+1} (y_{\alpha+1} - y_{\beta+1})^{-1} M_+ u_\beta y_{\beta+1} (y_{\beta+1} - y_{\alpha+1})^{-1},$$

$$[\bar{D}]_{\alpha\beta} = y_\beta (y_\beta - y_\alpha)^{-1} M_+ y_\alpha (y_\alpha - y_\beta)^{-1}, \quad \text{for } (\alpha \neq \beta),$$

$$[\bar{C}]_{\alpha\beta}{}^T = M_+ y_{\beta+1}^{-1} y_\alpha M_+ u_\beta. \tag{5.20}$$

Now we are ready to compare (5.17) with a subclass of terms found in the singularity generated by one transformation R (which in turn may be decomposed):

$$\prod_{n=1}^{\infty}(1 - X_R^{\,n}) = \prod_{n=1}^{\infty}\prod_{m=1}^{\infty} \exp\left(-\frac{X_R^{\,nm}}{m}\right) = \prod_{m=1}^{\infty} \exp\left[-\frac{X_R^{\,m}}{m(1 - X_R^{\,m})}\right]$$

$$= \prod_{m=1}^{\infty} \exp\left\{-\frac{1}{2m}\left[\frac{\Phi_{R^m}}{(\Phi_{R^m}^{\,2} - 4)^{\frac{1}{2}}} - 1\right]\right\} \tag{5.21}$$

where

$$X_R^{\frac{1}{2}} + X_R^{-\frac{1}{2}} = \Phi_R. \tag{5.22}$$

The close similarity between (5.17) and (5.21) allows one to establish a one-to-one correspondence between the two. We shall now take the first trace and derive the first term in the expansion of the singularity:[18]

$$\mathrm{Tr}\left[[\bar{C}]\right] = \mathrm{Tr}\left[[\bar{C}]^T\right]$$

$$= \mathrm{Tr} \sum_{\alpha=1}^{n} (u_\alpha M_+ x_{\alpha+1}^{-1} M_+)$$

$$= \sum_{\alpha=1}^{n} \frac{1}{2}\left[\frac{1}{(1 - 4w_\alpha)^{\frac{1}{2}}} - 1\right]$$

where

$$w_\alpha = \frac{u_\alpha x_{\alpha+1}}{[1 - x_{\alpha+1}(1 - u_\alpha)]^2} .$$

(We have adopted the convention of summing all Greek indices from 1 to n for convenience.) But

$$\Phi_{R_\alpha} = \frac{\mathrm{Tr}\ R_\alpha}{(\det R_\alpha)^{\frac{1}{2}}} = \frac{1 - x_{\alpha+1}(1 - u_\alpha)}{(u_\alpha\ x_{\alpha+1})^{\frac{1}{2}}} .$$

Therefore

$$\mathrm{Tr}([\overline{C}]) = \sum_{\alpha=1}^{n} \frac{1}{2}\left\{\frac{\Phi_{R_\alpha}}{(\Phi_{R_\alpha}{}^2 - 4)^{\frac{1}{2}}} - 1\right\} = \sum_{\alpha=1}^{n} \frac{X_{R_\alpha}}{1 - X_{R_\alpha}} \tag{5.23}$$

(we see that $w_\alpha{}^{-\frac{1}{2}} = \Phi_{R_\alpha}$). The process can be continued indefinitely.

(I) $\frac{1}{2}\ \mathrm{Tr}([\overline{C}]^2)$

$$= \frac{1}{2} \sum_{\alpha,\beta=1}^{n} \mathrm{Tr}(u_\alpha M_+ y_{\alpha+1}^{-1} y_\beta M_+ u_\beta M_+ y_{\beta+1}^{-1} y_\alpha M_+)$$

$$= \frac{1}{4} \sum_{\alpha,\beta=1}^{n} \left[\frac{1}{(1 - 4w_{\alpha\beta})^{\frac{1}{2}}} - 1\right],$$

where

$$w_{\alpha\beta} = \frac{u_\alpha u_\beta y_\alpha y_\beta y_{\alpha+1}^{-1} y_{\beta+1}^{-1}}{(1 - u_\alpha - u_\beta - y_{\alpha+1}^{-1} y_\beta - y_{\beta+1}^{-1} y_\alpha + u_\alpha u_\beta + y_\beta y_{\alpha+1}^{-1} y_\alpha y_{\beta+1}^{-1})^2} .$$

(I') $$\Phi_{R_\alpha R_\beta} = \frac{\mathrm{Tr}(R_\alpha R_\beta)}{[\det(R_\alpha R_\beta)]^{\frac{1}{2}}}$$

$$= (u_\alpha u_\beta x_{\alpha+1} x_{\beta+1})^{-\frac{1}{2}} \left\{ \mathrm{Tr}\ y_\alpha \begin{pmatrix} 1 & -x_{\alpha+1} \\ 1 & -x_{\alpha+1}(1 - u_\alpha) \end{pmatrix} \right.$$

$$\left. \times\ y_\beta y_\alpha{}^{-1} \begin{pmatrix} 1 & -x_{\beta+1} \\ 1 & -x_{\beta+1}(1 - u_\beta) \end{pmatrix} y_\beta{}^{-1} \right\}$$

$$= [1 - y_\beta^{-1} y_{\alpha+1} - y_{\beta+1} y_\alpha^{-1} + x_{\alpha+1} x_{\beta+1} (1 - u_\alpha)(1 - u_\beta)]$$

$$\times \quad (u_\alpha u_\beta x_{\alpha+1} x_{\beta+1})^{-\frac{1}{2}}.$$

Therefore

$$\sum_{\alpha,\beta=1}^{n} \frac{1}{2} \frac{X_{R_\alpha R_\beta}}{1 - X_{R_\alpha R_\beta}} = \sum_{\alpha,\beta=1}^{n} \frac{1}{4} \left\{ \frac{\Phi_{R_\alpha R_\beta}}{(\Phi^2_{R_\alpha R_\beta} - 4)^{\frac{1}{2}}} - 1 \right\} = \frac{\mathrm{Tr}([C]^2)}{2}. \quad (5.24)$$

II) $\frac{1}{3} \mathrm{Tr}([\overline{C}]^3)$

$$= \frac{1}{3} \sum_{\alpha,\beta,\gamma=1}^{n} \mathrm{Tr}(u_\alpha M_+ y_{\alpha+1}^{-1} y_\beta M_+ u_\beta M_+ y_{\beta+1}^{-1} y_\gamma M_+ u_\gamma M_+ y_{\gamma+1}^{-1} y_\alpha M_+)$$

$$= \sum_{\alpha,\beta,\gamma=1}^{n} \frac{1}{6} \left[\frac{1}{(1 - 4w_{\alpha\beta\gamma})^{\frac{1}{2}}} - 1 \right], \quad (5.25)$$

where

$$w_{\alpha\beta\gamma} = u_\alpha u_\beta u_\gamma y_\alpha y_\beta y_\gamma y_{\alpha+1}^{-1} y_{\beta+1}^{-1} y_{\gamma+1}^{-1}$$

$$\times (1 - u_\alpha - u_\beta - u_\gamma - y_{\alpha+1}^{-1} y_\beta - y_{\beta+1}^{-1} y_\gamma - y_{\gamma+1}^{-1} y_\alpha + u_\alpha u_\beta + u_\beta u_\gamma$$

$$+ u_\gamma u_\alpha + u_\alpha y_\gamma y_{\beta+1}^{-1} + u_\beta y_\alpha y_{\gamma+1}^{-1} + u_\gamma y_\beta y_{\alpha+1}^{-1} + y_\beta y_\gamma y_{\alpha+1}^{-1} y_{\beta+1}^{-1}$$

$$+ y_\gamma y_\alpha y_{\beta+1}^{-1} y_{\gamma+1}^{-1}$$

$$+ y_\alpha y_\beta y_{\alpha+1}^{-1} y_{\gamma+1}^{-1} - u_\alpha u_\beta u_\gamma - y_\alpha y_\beta y_\gamma y_{\alpha+1}^{-1} y_{\beta+1}^{-1} y_{\gamma+1}^{-1})^2.$$

$$\text{II}')\quad \Phi_{R_\alpha R_\beta R_\gamma} = \frac{\mathrm{Tr}(R_\alpha R_\beta R_\gamma)}{[\det(R_\alpha R_\beta R_\gamma)]^{\frac{1}{2}}} = (u_\alpha u_\beta u_\gamma x_{\alpha+1} x_{\beta+1} x_{\gamma+1})^{-\frac{1}{2}}$$

$$\times\ \mathrm{Tr}\left\{ y_\alpha y_\gamma^{-1}\begin{pmatrix} 1 & -x_{\alpha+1} \\ 1 & -x_{\alpha+1}(1-u_\alpha) \end{pmatrix} \quad y_\beta y_\alpha^{-1}\begin{pmatrix} 1 & -x_{\beta+1} \\ 1 & -x_{\beta+1}(1-u_\beta) \end{pmatrix} \right.$$

$$\left. \times\ y_\beta^{-1} y_\gamma \begin{pmatrix} 1 & -x_{\gamma+1} \\ 1 & -x_{\gamma+1}(1-u_\gamma) \end{pmatrix} \right\}$$

$$= [1 + x_{\alpha+1}(1-u_\alpha)x_{\gamma+1}y_\gamma y_\beta^{-1} + y_{\beta+1}(1-u_\beta)y_\alpha x_{\alpha+1} y_\gamma^{-1}$$

$$+ x_{\gamma+1}(1-u_\gamma)y_\beta x_{\beta+1} y_\alpha^{-1} - x_{\alpha+1} y_\alpha y_\beta^{-1} - x_{\gamma+1} y_\gamma y_\alpha^{-1}$$

$$-x_{\beta+1} y_\beta y_\gamma^{-1}](u_\alpha u_\beta u_\gamma x_{\alpha+1} x_{\beta+1} x_{\gamma+1})^{-\frac{1}{2}}.$$

Therefore

$$\sum_{\alpha,\beta,\gamma=1}^{n} \frac{1}{3}\, \frac{X_{R_\alpha R_\beta R_\gamma}}{1 - X_{R_\alpha R_\beta R_\gamma}} = \sum_{\alpha,\beta,\gamma=1}^{n} \frac{1}{6}\left\{ \frac{\Phi_{R_\alpha R_\beta R_\gamma}}{(\Phi^2_{R_\alpha R_\beta R_\gamma} - 4)^{\frac{1}{2}}} - 1 \right\}$$

$$= \frac{1}{3}\,\mathrm{Tr}([\bar{C}]^3)\,; \tag{5.26}$$

and also

$$\sum_{\alpha,\beta,\gamma,\delta=1}^{n} \frac{1}{4}\, \frac{X_{R_\alpha R_\beta R_\gamma R_\delta}}{1 - X_{R_\alpha R_\beta R_\gamma R_\delta}} = \sum_{\alpha,\beta,\gamma,\delta=1}^{n} \frac{1}{8}\left\{ \frac{\Phi_{R_\alpha R_\beta R_\gamma R_\delta}}{(\Phi_{R_\alpha R_\beta R_\gamma R_\delta} - 4)^{\frac{1}{2}}} - 1 \right\}$$

$$= \frac{1}{4}\,\mathrm{Tr}([\bar{C}]^4).$$

It is not hard to verify

$$\mathrm{Tr}([\bar{A}][\bar{D}]) = \sum_{\substack{\alpha,\beta=1 \\ (\alpha\neq\beta)}}^{n} \frac{X_{R_\alpha R_\beta}^{-1}}{1 - X_{R_\alpha R_\beta}^{-1}}$$

and

$$\mathrm{Tr}([\bar{A}][\bar{D}][\bar{C}]) = \sum_{\substack{\alpha,\beta,\gamma=1 \\ (\beta\neq\gamma) \\ (\alpha\neq\gamma)}} \frac{X_{R_\alpha R_\beta R_\gamma}^{-1}}{1 - X_{R_\alpha R_\beta R_\gamma}^{-1}} \quad .$$

Though only one of the fourth-order multipliers has been checked for the n-loop case, <u>all</u> such terms have been explicitly checked in the double-loop amplitude. Because of the close similarity between our original double-loop result and the arbitrary case, we can immediately establish the nature of all fourth-order terms:

$$\mathrm{Tr}\left[[\bar{C}][\bar{C}][\bar{A}][\bar{D}]\right] = \sum_{\substack{\alpha,\beta,\gamma,\delta=\{\mathcal{L}\} \\ (\alpha\neq\delta,\gamma\neq\delta)}} \frac{X_{R_\alpha R_\beta R_\gamma R_\delta}^{-1}}{1 - X_{R_\alpha R_\beta R_\gamma R_\delta}^{-1}}$$

and

$$\mathrm{Tr}\left[[\bar{A}][\bar{D}][\bar{A}][\bar{D}]\right] = \sum_{\substack{\alpha,\beta,\gamma,\delta=\{\mathcal{L}\} \\ (\alpha\neq\beta,\beta\neq\gamma,\gamma\neq\delta,\delta\neq\alpha)}} \frac{X_{R_\alpha R_\beta}^{-1} R_\gamma R_\delta^{-1}}{1 - X_{R_\alpha R_\beta}^{-1} R_\gamma R_\delta^{-1}} \quad .$$

Also

$$\mathrm{Tr}\left[[\bar{C}][\bar{A}][\bar{C}]^T[\bar{D}]\right] = \sum_{\substack{\alpha,\beta,\gamma,\delta=\{\mathcal{L}\} \\ (\beta\neq\gamma,\alpha\neq\delta)}} \frac{X_{R_\alpha R_\beta R_\gamma}^{-1} R_\delta^{-1}}{1 - X_{R_\alpha R_\beta R_\gamma}^{-1} R_\delta^{-1}} \quad .$$

Notice that the summation conventions assure us that no R and R^{-1} are ever juxtaposed.

When a careful study of the coefficients is made, all terms expected to appear in a fourth-order calculation do, in fact, appear. The absence of terms for both RRR^{-1} and $RR^{-1}R^{-1}$ indicates that double counting within the same equivalence class does not appear.

Several obvious generalizations are possible when calculating terms to higher orders, but will not be presented for lack of a rigorous justification. Unfortunately, we do not know of any rigorous way of proving the result for all orders.[3] In passing, we mention that the determinant calculation has been performed in several different configurations; all display the predicted result. Of particular interest, however, are the diagrams whereby the trace over one or more loops is possible <u>before</u> one goes to the principle axis. In the amplitude represented by Fig. 8a, where the trace over the central loop is performed first, several <u>infinite</u> classes of terms have been identified. With the help of Dr. J. Scherk, the singularity for the double tadpole amplitude of Fig. 8b has also been shown to have the conjectured behavior for an even wider class of terms.

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APPENDICES

A. Useful Identities

We list useful identities involving cross ratios:

$$P(x,y,z,w) = \frac{(x - z)(y - w)}{(x - w)(y - z)}, \tag{A.1}$$

$$1 - P(x,y,z,w) = P(x,z,y,w), \tag{A.2}$$

$$\frac{1}{P(x,y,z,w)} = P(x,y,w,z), \tag{A.3}$$

$$P(x,y,z,w)\, P(x,y,u,z) = P(x,y,u,w), \tag{A.4}$$

$$P(x,y,z,w) = P(y,x,w,z) = P(z,w,x,y) = P(w,z,y,x), \tag{A.5}$$

$$P(x,y,z,w) = \frac{1}{1 - \frac{1}{P(x,w,z,y)}}. \tag{A.6}$$

We also list identities involving $M_{\pm}$:

$$M_- M_-^{\,T} = M_+, \tag{A.7}$$

$$M_+ M_-^{\,T} = M_-, \tag{A.8}$$

$$M_- M_- = M_0, \tag{A.9}$$

$$M_+ x M_- = \frac{1}{1 - x} M_+ \frac{1}{1 - \frac{1}{x}}, \tag{A.10}$$

$$M_- x M_- = (1 - x) M_- \frac{1}{1 - \frac{1}{x}}, \tag{A.11}$$

$$M_+ x M_-^T = M_- \frac{1}{1 - \frac{1}{x}} M_-^T (1 - x), \tag{A.12}$$

$$M_+ x M_- = M_-^T \frac{1}{x} M_+ \text{ (formally)}, \tag{A.13}$$

$$M_-^T x M_- = M_+ \frac{1}{x} M_+ \text{ (formally)}, \tag{A.14}$$

$$M_- x |) = (1 - x)|) - |), \tag{A.15}$$

$$M_-^T x |) = \frac{1}{1 - \frac{1}{x}} |) \ . \tag{A.16}$$

B. Jacobian Calculation

We show how to compute the Jacobian factor $J = \frac{\partial(t,y_\alpha,y_{\alpha+1})}{\partial(X,x_1,x_2)}$ from the set of identities Eqs. (2.38) and (2.31). We list them here:

$$t = \frac{a}{a - d}, \tag{B.1}$$

$$d = \frac{(y_\alpha - y_{\alpha-1})(y_\alpha - y_{\alpha+1})}{(y_{\alpha+1} - y_{\alpha-1})(y_\alpha - y_{\alpha+2})}, \tag{B.2}$$

$$a = \frac{y_\alpha + y_{\alpha+1}\dfrac{x_2 - x_1X}{x_2X - x_1}}{y_{\alpha+1} + y_{\alpha+2}\dfrac{x_2 - x_1X}{x_2X - x_1}}, \tag{B.3}$$

$$y_\alpha = R(y_{\alpha+2}) = \frac{y_{\alpha+2}(x_2 - x_1X) - x_1x_2(1 - X)}{y_{\alpha+2}(1 - X) + x_2X - x_1}. \tag{B.4}$$

From Eq. (B.1), we get

$$J = \frac{d}{(a - d)^2}\,\frac{\partial(a,y_\alpha,y_{\alpha+1})}{\partial(X,x_1,x_2)}, \tag{B.5}$$

where we have used the theorem that, if two rows are identical, then the determinant is zero. We now take derivatives of a, Eq. (B.3), with respect to X,x_1,x_2 and evaluate the Jacobian in the frame $x_1 = \infty$, $x_2 = 0$, $y_{\alpha+2} = 1$; we get $(y_{\alpha+1} = x_1)$

$$J = \frac{d}{(a - d)^2} \begin{vmatrix} \frac{\partial y_\alpha}{\partial x_1} & \frac{\partial y_\alpha}{\partial x_2} \\ \frac{\partial y_{\alpha+1}}{\partial x_1} & \frac{\partial y_{\alpha+1}}{\partial x_2} \end{vmatrix} . \tag{B.6}$$

We then use Eq. (B.4) to take derivatives of y_α with respect to x_2 and evaluate the result in the frame $x_1 = \infty$, $x_2 = 0$; we get Eq. (2.40):

$$J = \frac{d}{(a - d)^2} (1 - X). \tag{B.7}$$

In passing, we note that the method we have used for the calculation of the Jacobian can also be used for the nonplanar case. We will show this in subsequent paper(s).

C. Elimination of $y_{\mathcal{L}^*}$

We show how the infinite number of cancellations involving y_α beautifully occur in the reasoning which leads from Eq. (3.36) to Eq. (3.37). We first consider the factors raised to the power $-k_i \cdot k_\lambda$ in Eq. (3.36). They can be written as

$$I_n^{(e\ell)} \equiv \left\{ \frac{y_i - [R^{\pm}]^{n+1}_{\alpha\beta}(y_\lambda)}{y_i - [R^{\pm}]^{n+1}_{\alpha\beta}(y_{\lambda+1})} \right\}_{(\beta\neq\lambda)} \left\{ \frac{y_i - [R^{\pm}]_{\alpha\beta}{}^{n}\, R_\lambda(y_\lambda)}{y_i - [R^{\pm}]_{\alpha\beta}{}^{n}(y_\lambda)} \right\}_{(\beta\neq\lambda,-)}$$

$$= \frac{1}{\{y_i - [R^{\pm}]^{n+1}_{\alpha\beta}(y_{\lambda+1})\}_{(\beta\neq\lambda)}} \left\{ \frac{y_i - [R^{\pm}]^{n+1}_{\alpha\beta}(y_\lambda)}{y_i - [R^{\pm}]_{\alpha\delta}{}^{n}\, R^{\pm}_\lambda(y_\lambda)} \right\}$$

$$\times \left\{ \frac{y_i - [R^{\pm}]^{n-1}_{\alpha\delta}\, R_\lambda{}^{-1}(y_\lambda)}{y_i - [R^{\pm}]_{\alpha\beta}{}^{n}(y_\lambda)} \right\} \{y_i - [R^{\pm}]_{\alpha\beta}{}^{n}\, R_\lambda(y_\lambda)\}$$

$$= \frac{1}{\{y_i - [R^{\pm}]^{n+1}_{\alpha\beta}(y_{\lambda+1})\}_{(\beta\neq\lambda)}} \left\{ \frac{y_i - [R^{\pm}]^{n+1}_{\alpha\beta}(y_\lambda)}{y_i - [R^{\pm}]_{\alpha\beta}{}^{n}(y_\lambda)} \right\}$$

$$\times \left\{ \frac{y_i - [R^{\pm}]^{n-1}_{\alpha\delta}\, R_\lambda{}^{-1}(y_\lambda)}{y_i - [R^{\pm}]_{\alpha\delta}{}^{n}\, R_\lambda{}^{-1}(y_\lambda)} \right\} . \qquad \text{(C.1)}$$

Hence the infinite product over n gives

$$I^{(e\ell)} \equiv \prod_{n=0}^{\infty} I_n^{(e\ell)} = \prod_{n=0}^{\infty} \frac{1}{\{y_i - [R^{\pm}]_{\alpha\beta}^{n+1}(y_{\lambda+1})\}_{(\beta\neq\lambda)}} \frac{1}{(y_i - y_\lambda)}$$

$$\times \lim_{N\to\infty} \left\{ \frac{y_i - [R^{\pm}]_{\alpha\beta}^{N+1}(y_\lambda)}{y_i - [R^{\pm}]_{\alpha\beta}^{N} R_\lambda^{-1}(y_\lambda)} \right\}$$

$$= \prod_{n=0}^{\infty} \frac{1}{\{y_i - [R^{\pm}]_{\alpha\beta}^{n+1}(y_{\lambda+1})\}_{(\beta=\lambda)}} \left\{ \frac{1}{y_i - y_\lambda} \right\}$$

$$\times \lim_{N\to\infty} \{y_i - [R^{\pm}]_{\alpha\beta}^{N+1}(y_\lambda)\}_{(\beta=\lambda,-)} \quad . \tag{C.2}$$

But the last factor in Eq. (C.2) can be written as

$$\{y_i - [R^{\pm}]_{\alpha\beta}^{N+1}(y_\lambda)\}_{(\beta\neq\lambda)} \{y_i - [R^{\pm}]_{\alpha\beta}^{N} R_\lambda(y_\lambda)\}_{(\beta\neq\lambda)}$$

$$\times \{y_i - [R^{\pm}]_{\alpha\beta}^{N-1} R_\lambda^{2}(y_\lambda)\}_{(\beta\neq\lambda)} \cdots \{y_i - [R^{\pm}]_{\alpha\beta}^{N-M} R_\lambda^{M}(y_\lambda)\}_{(\beta\neq\lambda)} \cdots$$

$$\times \{y_i - R_\lambda^{+N+1}(y_\lambda)\} , \tag{C.3}$$

where we have used $\mathcal{L} \neq \mathcal{L}'$ in the product $R_{\mathcal{L}} R_{\mathcal{L}'}^{-1}$. As $N \to \infty$, it is clear that either $(N - M) \to \infty$, or M must tend to ∞, in Eq. (C.3). If M tends to infinity, the expression $R_\lambda^{M}(y_\lambda)$ becomes $x_\lambda^{(2)}$, which is one of the invariant points of R_λ. If $(N - M)$ tends to infinity, it can be shown[7] that $[R^{\pm}]_{\alpha\beta}^{N-M}(z)$ is independent of z. We may therefore set z equal to $x_\lambda^{(2)}$ in this case. The expression Eq. (C.3) is thus equal to

$$\prod_{n=0}^{\infty} \{y_i - [R^{\pm}]_{\alpha\beta}{}^{n}(x_\lambda^{(2)})\}_{(\beta\neq\lambda)} \quad . \tag{C.4}$$

The second invariant point of R_λ is $y_{\lambda+1}$ [see Eq. (3.22b)]; let us denote it $x_\lambda^{(1)}$ ($= R_\lambda^{-\infty}(z)$, $z \neq x_\lambda^{(2)}$). Combining Eqs. (C.4) with (C.2), we get the desired result,

$$I^{(e\ell)} = \prod_{n=0}^{\infty} \left\{ \frac{y_i - [R^{\pm}]_{\alpha\beta}{}^{n}(x_\lambda^{(2)})}{y_i - [R^{\pm}]_{\alpha\beta}^{n+1}(x_\lambda^{(1)})} \right\} \left\{ \frac{1}{y_i - y_\lambda} \right\} \quad . \tag{C.5}$$

Similar arguments can be applied to the factors raised to the power $(-k_\beta \cdot k_\lambda)$ in Eq. (3.36), called this factor $I_n^{(\ell\ell)}$. We get

$$I^{(\ell\ell)} \equiv \prod_{n=0}^{\infty} I_n^{(\ell\ell)} = \prod_{n=0}^{\infty}$$

$$\left\{ \frac{x_\beta^{(2)} - [R^{\pm}]_{\alpha\delta}{}^{n}(x_\lambda^{(2)})}{x_\beta^{(1)} - [R^{\pm}]_{\alpha\delta}{}^{n}(x_\lambda^{(2)})} \cdot \frac{x_\beta^{(1)} - [R^{\pm}]_{\alpha\delta}^{n+1}(x_\lambda^{(1)})}{x_\beta^{(2)} - [R^{\pm}]_{\alpha\delta}^{n+1}(x_\delta^{(1)})} \right\}_{\binom{\beta\neq\alpha}{\delta\neq\lambda}}$$

$$\times \left\{ \frac{(x_\beta^{(1)} - y_\lambda)(x^{(1)} - y_\beta)}{(y_\beta - y_\lambda)(x_\beta^{(2)} - x_\lambda^{(1)})} \right\}_{(\beta\neq\lambda)} \quad . \tag{C.6}$$

The cancellation process, which occurs between loop momenta terms and binomial residue terms, is now carried out with the second method, which uses Sciuto three-reggeon vertices.

The calculation, beginning with the first term in the expansion of (4.13), is straightforward:

$$[\overline{C}]|\underset{\sim}{E}) = \sum_{\alpha,\beta=\{\mathcal{L}\}} \sum_{\substack{j=0 \\ j\neq\beta+1}}^{S+1} u_\alpha M_-^T y_{\alpha+1} y_\beta^{-1} M_- u_\beta \frac{|k_j)}{1 - y_{\beta+1}^{-1} y_j}$$

$$= \sum_{\alpha,\beta=\{\mathcal{L}\}} \sum_{\substack{j=0 \\ j\neq\beta+1}}^{S+1} \frac{u_\alpha |k_j)}{1 - \dfrac{y_{\alpha+1}^{-1} y_\beta}{1 - \dfrac{u_\beta}{1 - y_{\beta+1}^{-1} y_j}}}$$

$$- \sum_{\alpha,\beta=\{\mathcal{L}\}} u_\alpha \frac{|k_\beta)}{1 - y_{\alpha+1}^{-1} y_\beta}$$

$$= \sum_{\alpha,\beta=\{\mathcal{L}\}} \sum_{\substack{j=0 \\ j\neq\beta+1}}^{S+1} KB_\alpha y_\alpha^{-1} y_\beta B_\beta y_\beta^{-1} y_j |k_j)$$

$$- \sum_{\alpha,\beta=\{\mathcal{L}\}} KB_\alpha y_\alpha^{-1} y_\beta |k_\beta). \qquad \text{(C.7)}$$

Obviously, these contractions can be continued indefinitely. The end result is quite elegant:

$$\begin{pmatrix} [\bar{C}] & [\bar{A}] \\ [\bar{D}] & [\bar{C}]^T \end{pmatrix}^n \begin{pmatrix} |\underset{\sim}{E}) \\ |\underset{\sim}{F}) \end{pmatrix} = \sum_{\substack{j=0 \\ (j\neq\sigma+1)}}^{S+1} \sum_{\substack{i=0 \\ (i\neq\sigma)}}^{S+1} [\bar{H}]^n \begin{pmatrix} |\bar{E}_j) \\ |\bar{F}_i) \end{pmatrix}_\sigma$$

$$- \sum_{\substack{\mu=\{\mathcal{L}\},\{\mathcal{L}\}+1 \\ (\mu\neq\sigma+1)}} \sum_{\substack{\nu=\{\mathcal{L}\},\{\mathcal{L}\}+1 \\ (\nu\neq\sigma)}} [\bar{H}]^{n-1} \begin{pmatrix} |\bar{E}_\mu) \\ |\bar{F}_\nu) \end{pmatrix}_\sigma , \qquad (C.8)$$

where

$$[\bar{H}] \equiv \begin{pmatrix} KB_\alpha y_\alpha^{-1} y_\beta K^{-1} & KB_\alpha y_\alpha^{-1} y_\beta \frac{1}{\bar{K}} \\ K^{-1} \frac{1}{y_\alpha^{-1} y_\beta K^{-1}} & K^{-1} \frac{1}{y_\alpha^{-1} y_\beta B_\beta^{-1} \frac{1}{\bar{K}}} \end{pmatrix}$$

and

$$|\bar{E}_j)_\sigma = KB_\sigma y_\sigma^{-1} y_j |k_j),$$

$$|\bar{F}_j)_\sigma = K^{-1} y_\sigma y_j^{-1} |k_j) \quad . \qquad (C.9)$$

(Notice that $[\bar{H}]$ is the operator reformulation of $[H]^T$.)

At this point, we easily see the cancellations occurring between nth- and n - 1st - order terms involving loop momenta.

In the limit as n goes to infinity, terms involving external momenta accumulate all possible combinations of n R's or less. The loop momenta terms, however, contain only combinations of exactly n R's.

The loop momenta terms, as shown in the argument leading to (A3.4), tend to a definite limit if we invoke the theorem in Ford. Either (a) the combination of n R's contains a factor $R_\alpha^{n_0}$ at the end, where n_0 is large, in which case we approach the invariant point, or (b) it does not contain a factor $R_\alpha^{n_0}$ at the end, in which case we can replace the argument by the invariant point, since such a term is insensitive to changes in the argument. We can combine the loop momenta terms, which now contain the invariant points, with the external momenta terms (only the term $\lim_{n\to\infty} R_\alpha^{\pm n}(z)$ escapes this recombination):

$$\sum_{m=0}^{\infty} \begin{pmatrix} [\bar{C}] & [\bar{A}] \\ [\bar{D}] & [\bar{C}]^T \end{pmatrix}^m \begin{pmatrix} |\underset{\sim}{E}) \\ |\underset{\sim}{F}) \end{pmatrix} = \sum_{m=0}^{\infty} \sum_{\substack{j=0 \\ (j\neq\sigma,\sigma+1)}}^{S+1} [\bar{H}]^m \begin{pmatrix} |\bar{E}'_j) \\ |\bar{F}'_j) \end{pmatrix}_\sigma - \begin{pmatrix} |\bar{E}'_\sigma) \\ |\bar{F}'_{\sigma+1}) \end{pmatrix}_\sigma , \tag{C.10}$$

where

$$|\bar{E}'_j)_\sigma \equiv KB_\sigma y_\sigma^{-1} w_j |k_j),$$

$$|\bar{F}'_j)_\sigma \equiv K^{-1} y_\sigma w_j^{-1} |k_j),$$

and

$$w_i \equiv y_i \quad (i \neq \{\mathcal{L}\}, \{\mathcal{L}\}+1),$$

$$w_\alpha \equiv x_\alpha^{(2)} \equiv R_\alpha^{\infty}(z_1), \quad z_1 \neq x_\alpha^{(1)},$$

$$w_{\alpha+1} \equiv x_\alpha^{(1)} = R_\alpha^{-\infty}(z_2), \quad z_2 \neq x_\alpha^{(2)}.$$

Notice that the first set of cancellations was accomplished before the summation over all oscillator states was taken. The second set of cancellations occurs when we examine terms like

$$[y_i - (R^{\pm})^n w_j]^{-k_i k_j/2} .$$

The previous cancellation insures that w_j contains the invariant points; y_j, however, does not. If we shift $(R^{\pm})^n$ from one side of the equation to the other, we find expressions like $R_\alpha(\infty)$ and $R_\alpha^{-1}(0)$ (∞ and 0 are introduced to complete the momentum conservation in k_i).

Let R_1 and R_2 represent the totality of combinations of R's in the expansion. Then we have

$$\left((F|\ (E|\right)_\rho \sum_{m=0}^{\infty} \sum_{\substack{j=0 \\ (j\neq\rho,\sigma+1)}}^{S+1} [\vec{H}]^m \begin{pmatrix} |\bar{E}'_j) \\ |\bar{F}'_j) \end{pmatrix}_\sigma$$

$$\equiv \left((F|\ (E|\right)_\rho \sum_{\substack{j=0 \\ (j\neq\sigma,\sigma+1)}}^{S+1} \begin{pmatrix} KR_{1\rho\sigma} w_j\, |k_j) \\ K^{-1} \dfrac{|k_j)}{R_{2\rho\sigma}} \end{pmatrix}_\sigma$$

$$= \prod_{\rho,\sigma=\{\mathcal{L}\}} \prod_{\substack{i,j,\ell=0 \\ (j\neq\rho) \\ (i\neq\rho+1) \\ (\ell\neq\sigma,\sigma+1)}} (Ky_\rho^{-1} y_j - KR_{1\rho\sigma} w_\ell)^{-k_j k_\ell/2}$$

$$\times \left(K^{-1} \frac{1}{B_\rho y_\rho^{-1} y_i} - K^{-1} \frac{1}{R_{2\rho\sigma} w_\ell}\right)^{-k_i k_\ell/2}$$

Equation (C.11) continued

Equation (C.11) continued

$$= \prod_{\rho,\sigma=\{\mathcal{L}\}} \prod_{\substack{i,j,\ell=0\\(j\neq\rho)\\(i\neq\rho+1)\\(\rho\neq\sigma,\sigma+1)}} \left(R^{-1}_{1\sigma\rho}y_j - w_\ell\right)^{-k_jk_\ell/2}$$

$$\times \left(R^{-1}_{1\sigma\rho}(0) - w_\ell\right)^{k_{\rho+1}k_\ell/2} \left(R^{-1}_{2\sigma\rho}B_\rho y_\rho^{-1}y_i - w_\ell\right)^{-k_ik_\ell/2}$$

$$\times \left(R^{-1}_{2\sigma\rho}B_\rho y_\rho^{-1}(\infty) - w_\ell\right)^{-k_\rho k_\ell/2} \quad . \tag{C.11}$$

But $R_\alpha(\infty) = y_\alpha$ and $R_\alpha^{-1}(0) = y_{\alpha+1}$. Therefore, we see that a series of counterterms appears with one less degree than R_1 or R_2, making possible a <u>second</u> set of cancellations. After all cancellations are performed, we arrive at the desired result:

$$\prod_{i>j}^{S} \exp\{(k_j|x_{j+1,i}|k_i)\} \exp\{\tfrac{1}{2}\left((\underset{\sim}{E}| \ (\underset{\sim}{F}|\right) [\Delta]^{-1} \begin{pmatrix} |\underset{\sim}{E}) \\ |\underset{\sim}{F}) \end{pmatrix}$$

$$= \prod_{\substack{i,j=0\\(i\neq\alpha,\alpha+1)\\(j\neq\beta,\beta+1)\\(m=0,i\neq j)}}^{S+1} \prod_{m=0}^{\infty} \prod_{\alpha,\beta=\{\mathcal{L}\}} \left(w_i - (R^{\pm})^{(n)}_{\alpha\beta} w_j\right)^{-k_ik_j/2}$$

Equation (C.12) continued

$$\times \prod_{\gamma=\{\mathcal{L}\}} [(y_\gamma - w_{\gamma+1})(w_\gamma - w_{\gamma+1})(y_\gamma w_\gamma^{-1})]^{-k_\gamma k_{j+1}}$$

$$\times \prod_{i>j}^{S+1} (-y_j)^{-k_i k_j} . \qquad (C.12)$$

FOOTNOTES AND REFERENCES

* This work was supported in part by the U.S. Atomic Energy Commission.

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3. After this work was done, we discovered that C. Lovelace and V. Alessandrini also calculated the multiloop amplitude, using the elegant formalism of third Abelian integrals on Riemann surfaces. Of particular importance is Alessandrini's powerful proof of the form of the singularity to all orders.

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10. K. Kikkawa and H. Sato, University of Tokyo preprint UT-36, March 1970.

11. Although $\mathcal{P}^{+}D\Omega\,\mathcal{P}_{\Omega}$ is not gauge-invariant, nevertheless our factorized trees with dots on opposite sides of the excited leg are related to each other by the twist operator. Hence there is no ambiguity arising from the gauge invariance.

12. This was suggested by Professor Mandelstam.

13. We can solve X in terms of t (in the frame $y_{\alpha+1} = x_1 = \infty$, $x_2 = 0$, $y_{\alpha+1} = 1$); from Eq. (2.38a,b,c), we get

$$X = \frac{1}{2(1-t)}\, 1 \pm [1 - 4(1-t)t\, y_{\alpha-1}]^{\frac{1}{2}}.$$

The minus branch gives $0 \leq X < 1$, whereas the plus branch gives $1 \leq X \leq \infty$.

14. This was demonstrated by Professor Mandelstam.

15. This factor also comes out naturally, if we neglect the spurious problem. In this case,

$$X = tP_{\alpha}(A+2) \doteq tP(\alpha,\alpha+1,\alpha-1,\alpha+2)$$

$$= t\,\frac{(y_{\alpha} - y_{\alpha-1})(y_{\alpha+1} - y_{\alpha+2})}{(y_{\alpha} - y_{\alpha+2})(y_{\alpha+1} - y_{\alpha-1})},$$

and $x_1 = y_{\alpha+1}$, $x_2 = y_\alpha$; hence from the factor $(1 - t)^{\alpha_0 - 1}$ one gets this factor.

16. The determinant calculation is given in the Sciuto three-reggeon formulation.

17. M. Kaku and Loh-ping Yu, Unitarization of the Dual Resonance Amplitude. II. Nonplanar N-Loop Amplitude, Lawrence Radiation Laboratory Report UCRL-20055, to be issued.

18. We have adopted the streamlined notation of combining all terms appearing in $[\Delta]$ into four distinct factors by repeated use of the formal identities given in Appendix A. Thus, many of the manipulations presented here are only formal, since the argument often becomes larger than one, and hence the expression doesn't converge. Each step can be rigorously verified, however, if one avoids these formal identities and considers the various values of the argument separately.

FIGURE CAPTIONS

Fig. 1. Doubly factorized tree diagram.

Fig. 2. Planar single-loop diagram; t, y_α, $y_{\alpha+1}$ are three variables associated with the loop.

Fig. 3. Planar single-loop diagram; x_1, x_2, and X are the two invariant points and the multiplier of the projective operator R that goes around the loop. We assign x_1, x_2 to the center of the loop.

Fig. 4. The 2Nth-factorized tree diagram.

Fig. 5. Planar N-loop diagram. It is a rubber band with N holes cut in it.

Fig. 6. (a) The ordering of y_i and $x_\alpha^{(1)}$, $x_\alpha^{(2)}$, $\alpha = \{\mathcal{L}\}$.

(b) Symmetrical ordering of y_i and $x_\alpha^{(1)}$, $x_\alpha^{(2)}$, $\alpha = \{\mathcal{L}\}$ in which $x^{(1)}$, $x^{(2)}$ are the two invariant points of the projective operator $R_\alpha R_\beta R_\gamma \cdots$.

Fig. 7. Planar N-loop diagram. Propagator variables are given explicitly.

Fig. 8. (a) Several infinite classes of singular terms can be isolated by analyzing this diagram.

(b) An even larger class of singularities can be isolated in the double tadpole amplitude.

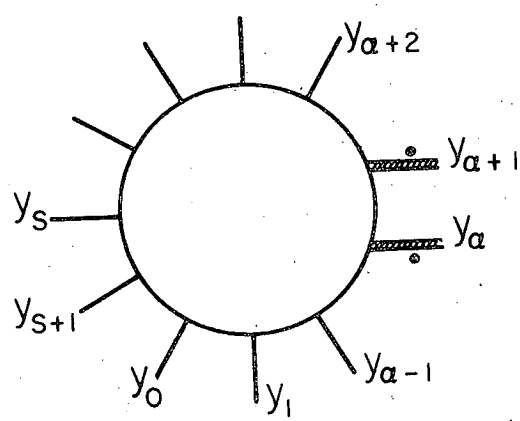

Fig. 1

XBL708-3651

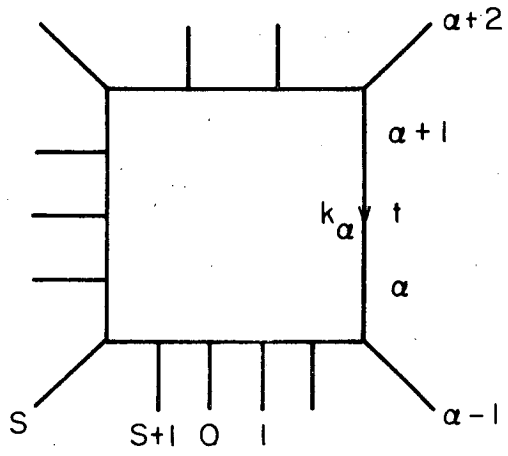

Fig. 2

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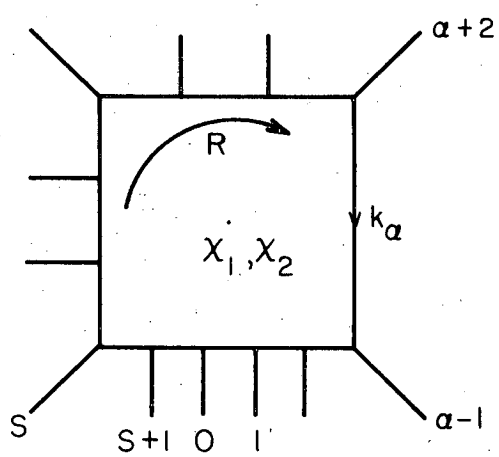

Fig. 3

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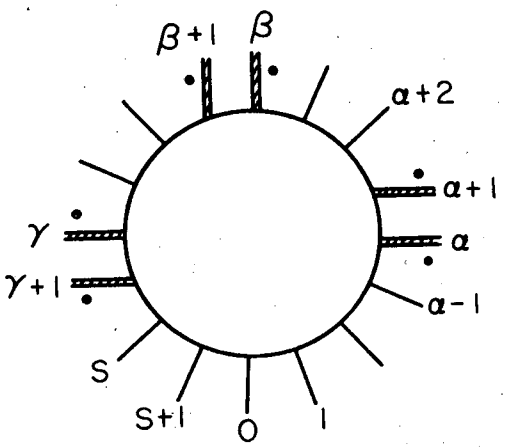

Fig. 4

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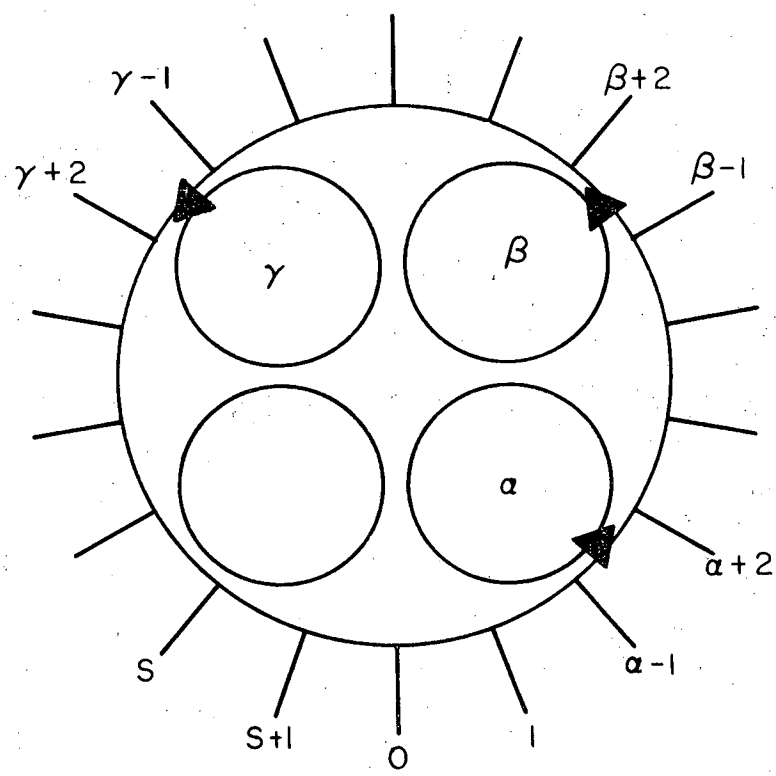

Fig. 5

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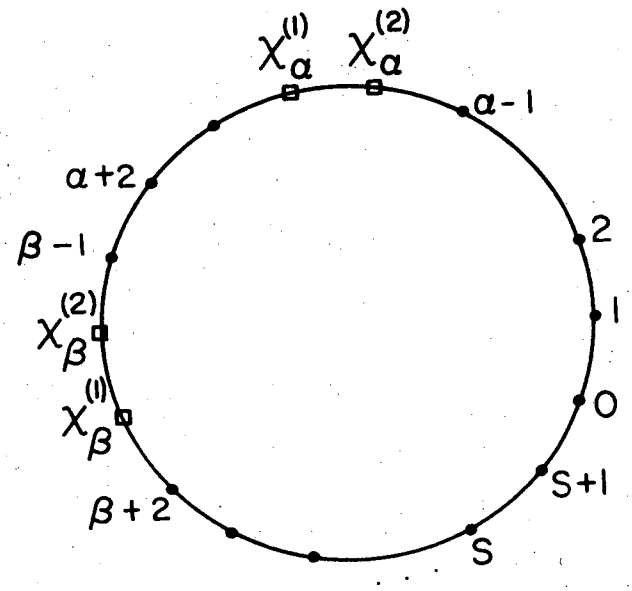

Fig. 6a

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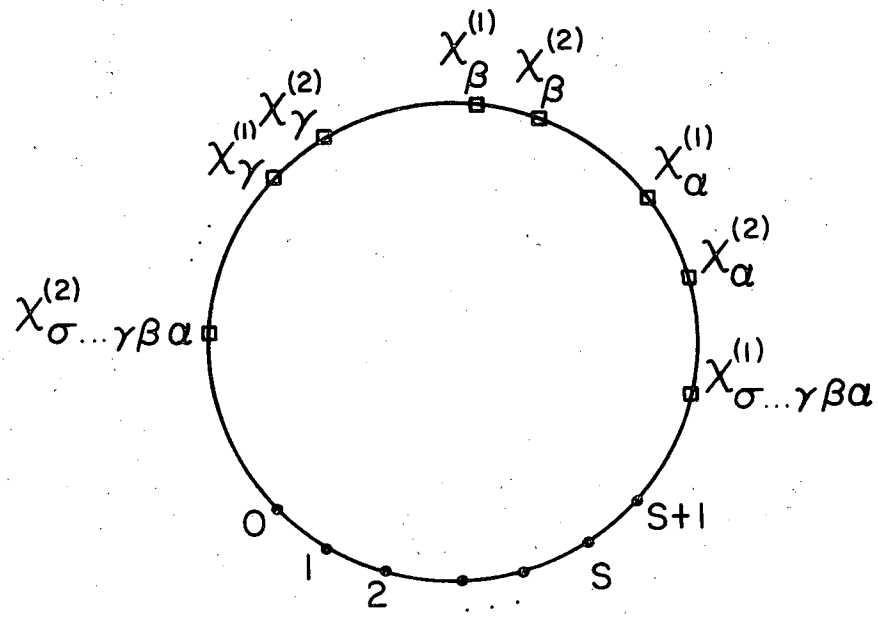

Fig. 6b

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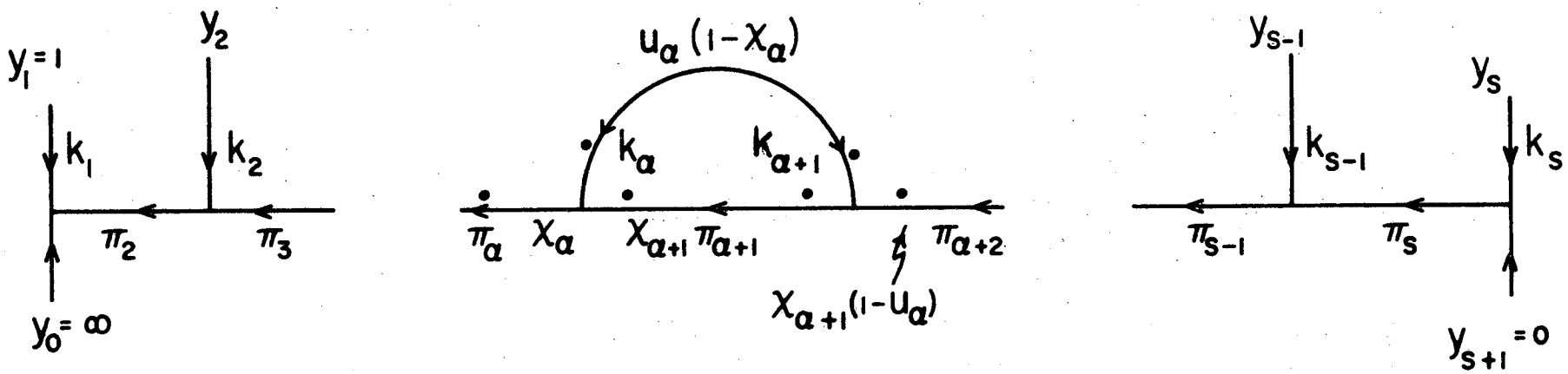

Fig. 7

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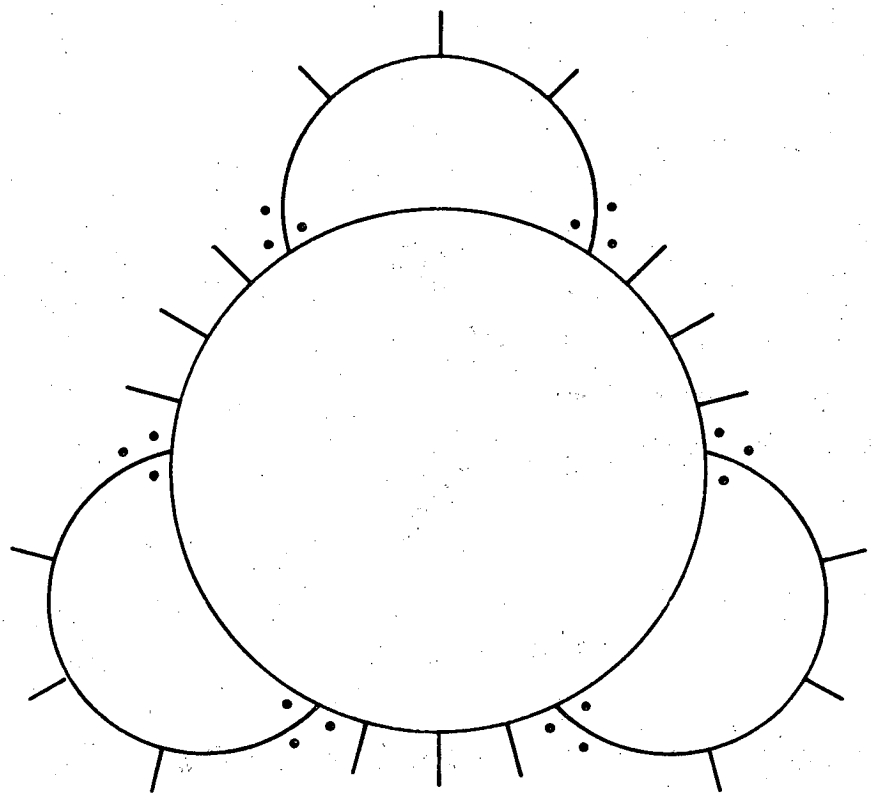

Fig. 8a

XBL708-3648

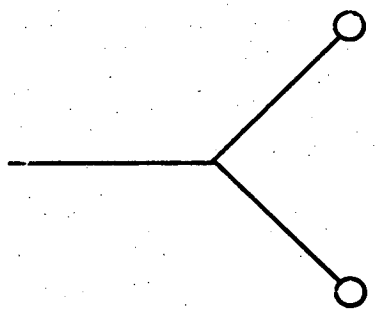

Fig. 8b

XBL 708-3649

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