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QUASI-PHOTON AND QUASI-ELECTRON DISPERSION RELATIONS*

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ABSTRACT

We establish dispersion relations for the elementary excitations of an isotropic, quantum, electron gas by the equation-of-motion technique. These results are accurate to first order in the random phase approximation.

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1. INTRODUCTION

This paper is principally concerned with the problem of determining dispersion relations for quasi-photons and quasi-electrons which are elementary excitations of electron gas, by the equation-of-motion technique. Particularly useful accounts appear in the work of Suhl and Werthamer [1,2] and Chappell et al. [3,4] We will base our work upon these.

The model that we shall choose for specific application will be that of a gas of electrons imbedded in a smeared out positive background such that there is no net charge. The electrons interact with themselves and the radiation field. No external fields (e.g. magnetic) are applied to the system. Our system is thus isotropic. The assumption of periodic boundary conditions insures homogeneity. We shall apply the thermodynamic limit in which $N \to \infty$, $V \to \infty$, $N/V = n$, where N is the total number of electrons, V is the volume of the sytem and n is the constant density. Our calculations will be performed in such a way that results will be correct to first order in the random phase approximation (herein after designated RPA). This approximation was originally developed by Bohm and Pines. [5,6]

The basic method that we will use in determing dispersion relations for the above model is discussed in the following section. Section 3 and 4 are devoted to obtaining a suitable Hamiltonian. This Hamiltonian is employed to obtain a dispersion relation for photons in section 5 and electrons in section 6.

2. METHOD

The problem of many particles interacting with a quantized field can be conveniently formulated interms of a second quantized Hamiltonian. Such a Hamiltonian will be only partially diagonal in operators that represent the particles (x) and the radiation field (y). It will be of the form:

$$H = \hbar \sum_{\gamma} \omega_{\gamma} x_{\gamma}^{+} x_{\gamma} + H'(x_{\gamma}, x_{\gamma}^{+}, y_{\delta}, y_{\delta}^{+}) + \hbar \sum_{\delta} \omega_{\delta} y_{\delta}^{+} y_{\delta} , \qquad (2.1)$$

where ω_{γ} and ω_{δ} are angular frequencies, and H' is the interaction part of the Hamiltonian. Our problem is to find a linear combination of operators, as yet unspecified,

$$X_{\gamma} = L\,(x_{\gamma}^{j}), \qquad (2.2)$$

and

$$Y_{\delta} = L\,(y_{\delta}^{j}), \qquad (2.3)$$

which obey, at least approximately

$$\left[X_{\gamma} \quad H \right] = \hbar\, \Omega_{\gamma} X_{\gamma} , \qquad (2.4)$$

and

$$\left[Y_{\delta}, H \right] = \hbar\, \Omega_{\delta} Y_{\delta} . \qquad (2.5)$$

If this can be done, X_{γ} (X_{γ}^{+}) and Y_{δ} (Y_{δ}^{+}) are annihilation (creation) operators for elementary excitations (quasi-particles or quasi-photons respectively) of the system of energy $\hbar\,\Omega_{\gamma}$ and $\hbar\Omega_{\delta}$. Consequently, if $|0>$ is the ground state of the system, then $|\gamma>$ ($= X_{\gamma}^{+}|0>$) and $|\delta>$ ($= Y_{\delta}^{+}|0>$) are excited states. Let us focus our attention on the problem finding X_{γ}. The prescription for a doing this is the following: We begin by writing the equation-of-motion for x_{γ}.

$$\left[x_\gamma, H \right] = \hbar \omega_\gamma x_\gamma + \left[x_\gamma, H' \right]. \qquad (2.6)$$

Were H' omitted from Eq. (2.1), Eq. (2.6) would be of the form (2.4) and we would have $X_\gamma = x_\gamma$ ($\Omega_\gamma = \omega_\gamma$). However, this is not the case. The commutator of x_γ with H' introduces new operators which are, in general various products of x, x^+, y, and y^+. We write equations-of-motion for these operator products which many produce new term that are still more complicated products, and for which we must also write equations-of-motion. We are thus led to establish a chain of equations-of-motion which turn out to be coupled through common operator terms. Usually, the chain will not terminate of its own accord and some approximation must be employed to obtain a finite set of linked equations-of-motion (we will use the RPA). After termination , the set of operator products that appear in this chain, along with x_γ, are the $X_\gamma{}^j$ we seek. Having found the $X_\gamma{}^j$, we insert (2.2) with its unknown set of coefficients into (2.4). We thereupon demand that the resulting coefficient factors for each $X_\gamma{}^j$ term vanish. This requirement leads to a dispersion relation for allowed Ω_γ. We repeat the entire procedure to determine the dispersion relation corresponding to Ω_δ.

The complete Hamiltonian for the model system mentioned in the introduction is obtained in the next section.

3. FULL HAMILTONIAN

The Hamiltonian for a system of electrons and radiation in a uniform

positive background is given by[7]

$$H = H_R + H_I^1 + H_I^2 + H_P \tag{3.1}$$

where

$$H_R = \int_V \left\{ 2\pi c^2 \underset{\sim}{P}_T^2(\underset{\sim}{r}, t) + \frac{\left[\text{curl } \underset{\sim}{A}(\underset{\sim}{r}, t) \right]^2}{8\pi} \right\} d\underset{\sim}{r} , \tag{3.2}$$

$$H_I^1 = -\frac{1}{c} \int \underset{\sim}{j}(\underset{\sim}{r}, t) \cdot \underset{\sim}{A}(\underset{\sim}{r}, t)\, d\underset{\sim}{r} , \tag{3.3}$$

$$H_I^2 = \frac{e^2}{2mc^2} \int \psi^+(\underset{\sim}{r}, t)\, \psi(\underset{\sim}{r}, t)\, \underset{\sim}{A}^2(\underset{\sim}{r}, t)\, d\underset{\sim}{r}, \tag{3.4}$$

$$H_P = \frac{\hbar^2}{2m} \int \underset{\sim}{\nabla} \psi^+(\underset{\sim}{r}, t) \cdot \underset{\sim}{\nabla} \psi(\underset{\sim}{r}, t)\, d\underset{\sim}{r} +$$

$$\frac{e^2}{2} \int \frac{\psi^+(\underset{\sim}{r}, t)\, \psi^+(\underset{\sim}{r}', t)\, \psi(\underset{\sim}{r}', t)\, \psi(\underset{\sim}{r}, t)\, d\underset{\sim}{r}}{|\underset{\sim}{r} - \underset{\sim}{r}'|} . \tag{3.5}$$

Here div $\underset{\sim}{A}(\underset{\sim}{r}, t) = 0$ so we are in the Coulomb gauge. The partial current is

$$\underset{\sim}{j}(\underset{\sim}{r}, t) = \frac{-\hbar e}{2mi} \left\{ \psi^+(\underset{\sim}{r}, t)\, \underset{\sim}{\nabla} \psi(\underset{\sim}{r}, t) - \left[\underset{\sim}{\nabla} \psi^+(\underset{\sim}{r}, t) \right] \right.$$

$$\left. \times \psi(\underset{\sim}{r}, t) \right\} . \tag{3.6}$$

The vector potential is expanded in turns of plane wave and polarization states, $\lambda = 1,2$.

$$\underset{\sim}{A}(\underset{\sim}{r}, t) = \left(\frac{2\pi \hbar c}{V} \right)^{\frac{1}{2}} \sum_{k\lambda} k^{-\frac{1}{2}}\, \underset{\sim}{e}_{k\lambda} \left[a_{k\lambda}(t) + a^+_{-k\lambda}(t) \right] \exp[i\underset{\sim}{k} \cdot \underset{\sim}{r}]. \tag{3.7}$$

The polarization vector enjoys the following useful properties[8]

$$\underset{\sim}{k} \cdot \underset{\sim}{e}_{k\lambda} = 0,$$

$$\underset{\sim}{e}_{k\lambda} \cdot \underset{\sim}{e}_{q\mu} = \delta_{\lambda\mu},$$

$$\underset{\sim}{e}_{k\lambda} = \underset{\sim}{e}_{-k\lambda} .$$

The transverse part of the momentum cononically conjugate to $\underset{\sim}{A}$ is given by the expansion

$$\underset{\sim}{P}_T(\underset{\sim}{r},t) = -\frac{i}{4\pi c}\left(\frac{2\pi\hbar c}{V}\right)^{\frac{1}{2}} \sum_{k\lambda} k^{\frac{1}{2}}\, \underset{\sim}{e}_{k\lambda}[a_{k\lambda}(t) - a^{+}_{-k\lambda}(t)] \exp[i\underset{\sim}{k} \cdot \underset{\sim}{r}]. \tag{3.8}$$

Subscript k represents the vector $\underset{\sim}{k}$ in these expressions. The k sum is over all vector $\underset{\sim}{k}$. ψ may be expanded in a set of plane wave and spin states

$$\psi(\underset{\sim}{r},t) = V^{-\frac{1}{2}} \sum_{p\alpha'} c_{p\alpha}(t)\, \chi_{\alpha} \exp[i\underset{\sim}{p} \cdot \underset{\sim}{r}]. \tag{3.9}$$

Here, χ_{α} is one of a set of spin states.[9] The sum is understood to be over all wave vectors $\underset{\sim}{p}$. The p subscript on the Fermi operator represents the vector $\underset{\sim}{p}$. The Boson(a) and Fermi (c) operators satisfy the usual equal-time commutation relations

$$[a_{k\lambda}, a^{+}_{\ell\mu}] = \delta_{k\ell}\, \delta_{\lambda\mu} ,$$
$$[a_{k\lambda}, a_{\ell\mu}] = [a^{+}_{k\lambda}, a^{+}_{\ell\mu}] = 0, \tag{3.10}$$

and

$$[c_{p\alpha}, c^{+}_{q\beta}]_{+} = \delta_{pq}\, \delta_{\alpha\beta},$$

$$[c_{p\alpha}, c_{q\beta}]_{+} = [c^{+}_{p\alpha}, c^{+}_{q\beta}]_{+} = 0. \tag{3.11}$$

Inserting Eqs. (3.7), (3.8) and (3.9) into (3.2), (3.3), (3.4) and (3.5), we obtain (time dependence of operators has been suppressed)

$$H = H_R + H_I^1 + H_I^2 + H_P, \tag{3.1}$$

$$H_R = \hbar c \sum_{k\lambda} k\, a^+_{k\lambda}\, a_{k\lambda}, \tag{3.12}$$

$$H_I^1 = \sum_{p\alpha}\sum_{k\lambda} A_k\, \underset{\sim}{p} \cdot \underset{\sim}{e}_{k\lambda}\, c^+_{p+k,\alpha}\, c_{p\alpha}(a_{k\lambda} + a^+_{-k\lambda}), \tag{3.13}$$

$$H_I^2 = \sum_{p\alpha}\sum_{k\lambda}\sum_{\ell\mu} B_{k\ell\lambda\mu}\, c^+_{p+k+\ell,\alpha}\, c_{p\alpha}(a_{k\lambda}+a^+_{-k\lambda})(a_{\ell\mu}+a^+_{-\ell\mu}), \tag{3.14}$$

$$H_P = \sum_{p\alpha} E_p\, c^+_{p\alpha}\, c_{p\alpha} + \frac{1}{2V}\sum_{k}\sum_{pq}\sum_{\alpha\beta} V_k\, c^+_{p\alpha} c^+_{q+k,\beta} c_{q\beta} c_{p+k,\alpha}, \tag{3.15}$$

where,

$$A_k = \frac{\hbar e}{mc}\left(\frac{2\pi\hbar c}{kV}\right)^{1/2},$$

$$B_{k\ell\lambda\mu} = \frac{\pi\hbar e^2}{mcV}(k\ell)^{-\frac{1}{2}}\, \underset{\sim}{e}_{k\lambda} \cdot \underset{\sim}{e}_{\ell\mu},$$

$$\epsilon_p = \frac{\hbar^2 p^2}{2m}$$

$$V_k = \begin{cases} 4\pi^2/k^2 = \int_V d\underset{\sim}{r}'\, \exp[i\underset{\sim}{k} \cdot \underset{\sim}{r}]/|\underset{\sim}{r} - \underset{\sim}{r}'|, & k \neq 0 \\ 0, & k = 0 \end{cases}.$$

Setting $V_k = 0$, when $k=0$ enables one to take into account the cancellation of the positive background. H_I^1 describes processes in which an electron of momentum $\hbar\, \underset{\sim}{p}$ with spin α is annihilated along with

a photon of momentum $\hbar \underset{\sim}{k}$ and polarization λ (or a photon of momentum $-\hbar \underset{\sim}{k}$ and polarization λ is created) , while an electron is created of momentum $\hbar$ ($\underset{\sim}{p} + \underset{\sim}{k}$) and spin α. H_I^2 involves scattering processes of electrons with photons in which an electron of momentum $\hbar$ ($\underset{\sim}{p} + \underset{\sim}{k} + \underset{\sim}{\ell}$) and spin index α is created in the medium while one of momentum $\hbar \underset{\sim}{p}$ and spin α is destroyed. The four allowed photon processes which go along with this are the annihilation (creation) of a photon of momentum $\hbar \underset{\sim}{k}$ ($-\hbar \underset{\sim}{k}$) and polarization λ with the annihilation (creation of a photon of momentum $\hbar \underset{\sim}{\ell}$ ($-\hbar \underset{\sim}{\ell}$) and polarization μ. The Coulomb term in H_P in involes electron-electron scattering processes in which electrons of momentum $\hbar$ ($\underset{\sim}{p} + \underset{\sim}{k}$) and $\hbar \underset{\sim}{q}$ with spins α and β respectively are scattered to momentums $\hbar \underset{\sim}{p}$ and $\hbar$ ($\underset{\sim}{q} + \underset{\sim}{k}$) and spins α and β.

Section 4 is concerned with a simiplification of this Hamiltonian so that later equations-of-motion will not be unwieldy.

4. SIMPLIFIED HAMILTONIAN

It is not necessary, or for our purposes desirable, to work with the full Hamiltonian, Eqs. (3.1) and (3.12) through (3.15). In fact, to first order in the RPA, it will be possible to incorporate the most important effect of the interaction term H_I^2 into the problem by modifying the frequency, kc, in H_R . The modified frequency will be just the Bogolyubov quasi-photon frequency ($\omega_k = [(kc)^2 + \omega_P^2]^{\frac{1}{2}}$, ω_P is the plasma frequency). The procedure used will follow that of Chappell, et, al.[3] The remaining term in H_I^2, involving electron-photon scattering processes, will be dropped in this order of approximation. The primary effect of the Coulomb term in H_P will be seen to be the replacement of E_p

the kinetic energy, with the Hartree Fock energy of $\epsilon_{p\alpha}$.[10] We will thereby omit screening effects in this order of approximation.[1,2]

To begin, we need only consider the commutator of $a_{k\lambda}$ with H_I^2 and H_R.[3] We get

$$[a_{k\lambda}, H_R + H_I^2] = \hbar kc + 2 \sum_p \sum_\alpha \sum_\ell \sum_\mu B_{k\ell\lambda\mu}\, c^+_{p-k+\ell,\alpha}\, c_{p\alpha}\, (a_{\ell\mu} + a^+_{-\ell\mu}). \qquad (4.1)$$

The first term on the right-hand-side follows from H_R and the second from H_I^2. Rather then write new equations-of-motion for the two operators introduced by this last term, $c^+_{p-k+\ell,\alpha} c_{p\alpha} a_{\ell\mu}$ and $c^+_{p-k+\ell,\alpha} c_{p\alpha} a^+_{-\ell\mu}$, we replace the product of two Fermi operators with its RPA:

$$c_{p-k+\ell,\alpha}\, c_{p\alpha} \approx \langle c^+_{p\alpha}\, c_{p\alpha} \rangle \quad \delta_{k\ell}. \qquad (4.2)$$

This approximation selects that term for which momentum and spin subscripts agree on the Fermi annihilation and creation operators. The resulting product is replaced with its ground state average $\langle \quad \rangle$. We do not know what the actual ground states are, therefore, we make the further approximation that

$$N_{p\alpha} = \langle c^+_{p\alpha}\, c_{p\alpha} \rangle \approx [\exp(\beta E_p) + 1]^{-1} = N_p, \qquad (4.3)$$

$$\beta = 1/k_B T.$$

We see that N_p is the expectation value for the non-interacting gas. Thus, we are left with

$$[a_{k\lambda}, H_R + H_I^2]_{RPA} = \hbar \left(kc + \frac{\omega_p^2}{2kc}\right) a_{k\lambda} - \frac{\hbar\omega_p^2}{2kc}\, a^+_{-k\lambda}. \qquad (4.4)$$

in which

$$\omega_p^2 = 4\pi Ne^2/m V,$$

the plasma frequency. Equation (4.4) is linear in the Boson annihilation and creation operators. It is not of the form (2.5). We will need the commutator for $a^{+}_{-k\lambda}$ in order to obtain a complete set of equations-of-motion. This is readily found by taking the Hermetian conjugate of Eq. (4.4) and replacing $\underset{\sim}{k}$ with $-\underset{\sim}{k}$. We find

$$[a^{+}_{-k\lambda}, H_R + H_I^{\,2}]_{RPA} = -\hbar\left(kc + \frac{\omega_p^{\,2}}{2kc}\right) a^{+}_{-k\lambda} - \frac{\hbar\omega_p^{\,2}}{2kc}\, a_{k\lambda} . \tag{4.5}$$

Since our set of equations-of-motion involves only $a_{k\lambda}$ and $a^{+}_{-k\lambda}$, it is closed. We are led to construct an operator of the form

$$X_{k\lambda} = u_{k\lambda}\, a_{k\lambda} + v_{k\lambda}\, a^{+}_{-k\lambda}, \tag{4.6}$$

and to demand that

$$[X_{k\lambda}, H] = \hbar\omega_{k\lambda}\, X_{k\lambda}. \tag{4.7}$$

This leads to the requirement that

$$\left(kc + \frac{\omega_p^{\,2}}{2kc} - \omega_{k\lambda}\right) u_{k\lambda} - \frac{\omega_p^{\,2}}{2kc}\, v_{k\lambda} = 0, \tag{4.8}$$

$$-\frac{\omega_p^{\,2}}{2kc}\, u_{k\lambda} + \left(kc + \frac{\omega_p^{\,2}}{2kc} + \omega_{k\lambda}\right) v_{k\lambda} = 0 \tag{4.9}$$

The determinant of the coefficients must vanish for a non-trivial solution. This condition yields

$$\omega_{k\lambda} = \pm\, \omega_k,$$

$$\omega_k = [(kc)^2 + \omega_p{}^2]^{\frac{1}{2}}, \tag{4.10}$$

which is the renormalized frequency, replacing kc. There are two values of $v_{k\lambda}$ (corresponding to the positive and negative sign in (4.10) respectively)

$$v^1_{k\lambda} = \frac{\omega_k - kc}{\omega_k + kc}\, u^1_{k\lambda}, \tag{4.11}$$

and

$$v^2_{k\lambda} = \frac{\omega_k + kc}{\omega_k - kc}\, u^2_{k\lambda}\,. \tag{4.12}$$

Thus the $X_{k\lambda}$ are

$$X^1_{k\lambda} = \frac{u^1_{k\lambda}}{\omega_k + kc}\, [(\omega_k + kc)\, a_{k\lambda} + (\omega_k - kc)\, a^+_{-k\lambda}], \tag{4.13}$$

and

$$X^2_{k\lambda} = \frac{u^2_{k\lambda}}{\omega_k - kc}\, [(\omega_k - kc)\, a_{k\lambda} + (\omega_k + kc)\, a^+_{-k\lambda}]. \tag{4.14}$$

Taking the Hermetian conjugate of $X^1_{-k\lambda}$, we find

$$(X^1_{-k\lambda})^+ = \frac{u^{1\,*}_{-k\lambda}}{\omega_k + kc}\, [(\omega_k - kc)\, a_{k\lambda} + (\omega_k + kc)\, a^+_{-k\lambda}], \tag{4.15}$$

so that $(X^1_{-k\lambda})^+$ is identical with $X^2_{k\lambda}$ provided

$$u^{2}_{k\lambda} = \frac{\omega_k - kc}{\omega_k + kc} \quad u^{1\,*}_{-k\lambda} . \tag{4.16}$$

We now identify $X^1_{k\lambda}$ $(X^2_{k\lambda})$ with $b_{k\lambda}(b^+_{-k\lambda})$ and further note that the transformation [from the pair $a_{k\lambda}$ $(a^+_{k\lambda})$ to $b_{k\lambda}$ $(b^+_{k\lambda})$] specified by Eqs. (4.13) and (4.14) with the subsidiary condition (Eq . 4.16) will be canonical if

$$[b_{k\lambda}, b^+_{\ell\mu}] = \delta_{k\ell}\, \delta_{\lambda\mu} . \tag{4.17}$$

Using this condition and Eq. (4.16), we find

$$|u^{1}_{k\lambda}|^2 = \frac{(\omega_k + kc)^2}{4\omega_k kc} . \tag{4.18}$$

Thus, to within an arbitrary phase factor, we have

$$b_{k\lambda} = \tfrac{1}{2} \{ [(\frac{\omega_k}{kc})^{\frac{1}{2}} + (\frac{kc}{\omega_k})^{\frac{1}{2}}]\, a_{k\lambda} + [(\frac{\omega_k}{kc})^{\frac{1}{2}} - (\frac{kc}{\omega_k})^{\frac{1}{2}}]\, a^+_{-k\lambda} \} . \tag{4.19}$$

Here $b_{k\lambda}$ $(b^+_{k\lambda})$ is called the quasi-photon annihilation (creation) operator. It is derived in a more direct fashion in appendix A. We have adopted the more lengthy approach because of the opportunity to introduce the equation-of-motion technique at an earlier and simpler stage in this work. Equation (4.19) and its Hermetian conjugate define the Bogoliubov transformation. The Hamiltonian corresponding to H_R and H_I^2 may now be expressed

$$H'_R = H_R + H_{I\,RPA}^2 = \sum_{k\lambda} \omega_k\, b^+_{k\lambda}\, b_{k\lambda} . \tag{4.20}$$

There will be no problem expressing H_J^1 in terms of $b_{k\lambda}$ and $b^+_{k\lambda}$ since

$$\underset{\sim}{A}(\underset{\sim}{r},t) = \left(\frac{2\pi\hbar c^2}{V}\right)^{\frac{1}{2}} \sum_{k\lambda} \omega_k^{-\frac{1}{2}} \underset{\sim}{e}_{k\lambda} \exp[i\,\underset{\sim}{k}\cdot\underset{\sim}{r}]\,(b_{k\lambda} + b^+_{-k\lambda}) . \tag{4.21}$$

Now let us consider the simplification of H_P, Eq. (3.15). The commutator of $c_{p\alpha}$ with H_P is

$$[c_{p\alpha}, H_P] = E_p\, c_{p\alpha} - V^{-1} \sum_{qk\beta} V_k\, c^+_{q+k,\beta}\, c_{p+k,\alpha}\, c_{q\beta} . \tag{4.22}$$

The last term on the right-hand-side of Eq. (4.22) is linearized by selecting that term for which the momentum and spin subscripts of an annihilation and creation operator agree (the RPA). The corresponding product is replaced with its interaction free ground state average, i.e.,

$$c^+_{q+k,\beta}\, c_{p+k,\alpha}\, c_{q\beta} \approx n_{p+k}\, c_{p\alpha}\, \delta_{qp}\, \delta_{\alpha\beta} .$$

Thus, we obtain

$$[c_{p\alpha}, H_P]_{RPA} = \epsilon_{p\alpha}\, c_{p\alpha} , \tag{4.23}$$

with

$$\epsilon_{p\alpha} = \epsilon_p = E_p - \sum_k V_k\, n_{p+k} , \tag{4.24}$$

The Hartree-Fock energy, and $n_{p+k} = N_{p+k}/V$. We see that we may use for H_P.[11]

$$H_{p\,RPA} = \sum_{p\,\alpha} \epsilon_{p\alpha}\, c^{+}_{p\alpha}\, c_{p\alpha} \,. \tag{4.25}$$

(An alternate approach is given in Appendix A.) We now have a system of quasi-electrons with effective energy $\epsilon_{p\alpha}$. Hence, a better approximation for N_p is

$$N_p \approx [\exp(\beta\hbar\epsilon_p) + 1]^{-1} \,. \tag{4.26}$$

Since ϵ_p depends on N_p, this entails solving a set of simultaneous equations for N_p. With the help of Eqs. (4.20) and (4.25) and using (4.19) and its Hermitian conjugate (or Eq. 4.21) to eliminate $a^{+}_{k\lambda}$ from H_1^{1}, we express the approximate hamiltonian as

$$H_{RPA} = \sum_{p\,\alpha} \epsilon_{p\alpha}\, c^{+}_{p\alpha}\, c_{p\alpha} + \sum_{p\,\alpha} \sum_{k\,\lambda} A'_k\, \underset{\sim}{p} \cdot \underset{\sim}{e}_{k\lambda}\, c^{+}_{p+k,\alpha}\, c_{p\alpha}\, (b_{k\lambda} + b^{+}_{-k\lambda})$$

$$+ \hbar \sum_{k\,\lambda} \omega_k\, b^{+}_{k\lambda}\, b_{k\lambda} \,, \tag{4.27}$$

where

$$A'_k = \frac{\hbar e}{m} \left(\frac{2\pi\hbar}{\omega_k V} \right)^{\frac{1}{2}}$$

depends only on the magnitude of $\underset{\sim}{k}$. We see that this Hamiltonian is diagonal except for the middle term. Sections 5 and 6, which follow, are devoted to obtaining dispersion relations for quasi-photons and quasi-electrons with the help of the above expression for H.

5. QUASI-PHOTON DISPERSION RELATION

The first equation-of-motion that we will need in this calculation is

$$[b_{k\lambda}, H] = \hbar\omega_k \, b_{k\lambda} + \sum_{p\,\alpha} A'_k \, \underset{\sim}{p} \cdot \underset{\sim}{e}_{k\lambda} \, c^{+}_{p-k,\alpha} \, c_{p\alpha} \, . \tag{5.1}$$

This leads us to construct

$$[c^{+}_{p-k,\alpha} \, c_{p\alpha}, H] = (\epsilon_{p\alpha} - \epsilon_{p-k,\alpha}) \, c^{+}_{p-k,\alpha} \, c_{p\alpha} + \sum_{\ell\,\mu} A'_\ell \, \underset{\sim}{e}_{\ell\mu} \cdot [\underset{\sim}{p} c^{+}_{p-k,\alpha} \, c_{p-\ell,\alpha}$$

$$- (\underset{\sim}{p} - \underset{\sim}{k}) \, c^{+}_{p-k+\ell,\alpha} c_{p\alpha}] \, [b_{k\lambda} + b^{+}_{-k\lambda}] \quad . \tag{5.2}$$

In order to simplify Eq. (5.2), we resort to the usual RPA:

$$[c^{+}_{p-k,\alpha} \, c_{p\alpha}, H]_{RPA} = (\epsilon_{p\alpha} - \epsilon_{p-k,\alpha}) \, c^{+}_{p-k,\alpha} \, c_{p\alpha} +$$

$$(N_{p-k,\alpha} - N_{p\alpha}) \sum_{\mu} A'_k \, \underset{\sim}{p} \cdot \underset{\sim}{e}_{k\mu} \, (b_{k\mu} + b^{+}_{-k\mu}) \tag{5.3}$$

to first order. From the last term we find that we also require

$$[b^{+}_{-k\mu}, H] = -\hbar\omega_k \, b^{+}_{-k\mu} - \sum_{p\,\alpha} A'_k \, \underset{\sim}{p} \cdot \underset{\sim}{e}_{k\mu} \, c^{+}_{p-k,\alpha} \, c_{p\alpha} \tag{5.4}$$

This completes the required set of equations-of-motion. We construct $X_{k\lambda}$ in the form:

$$X_{k\lambda} = u_{k\lambda} \, b_{k\lambda} + v_{k\lambda} \, b^{+}_{-k\lambda} + \sum_{p\,\alpha} w_{kp\alpha\lambda} \, c^{+}_{p-k,\alpha} \, c_{p\alpha} \, . \tag{5.5}$$

and insert it in

$$[X_{k\lambda}, H] = \hbar\Omega_{k\lambda} X_{k\lambda} \tag{5.6}$$

to obtain

$$u_{k\lambda}\hbar\omega_k b_{k\lambda} - v_{k\lambda}\hbar\omega_k b^+_{-k\lambda} + (u_{k\lambda} - v_{k\lambda}) \sum_{p\,\alpha} A'_k \underset{\sim}{p} \cdot \underset{\sim}{e}_{k\lambda} c^+_{p-k,\alpha} c_{p\alpha}$$

$$+ \sum_{p\,\alpha} (\epsilon_{p\alpha} - \epsilon_{p-k,\alpha}) v_{kp\alpha\lambda} c^+_{p-k,\alpha} c_{p\alpha} + \sum_{p\,\alpha} \sum_{\mu} A'_k \underset{\sim}{e}_{k\mu} \cdot \underset{\sim}{p} v_{kp\alpha\lambda} X$$

$$(N_{p-k,\alpha} - N_{p\alpha})(b_{k\mu} + b^+_{-k\mu}) = \hbar\Omega_{k\lambda}(u_{k\lambda} b_{k\lambda} + v_{k\lambda} b^+_{-k\lambda} + \sum_{p\,\alpha} v_{kp\alpha\lambda} c^+_{p-k,\alpha} c_{p\alpha}). \tag{5.7}$$

The coefficient of $c^+_{p-k,\alpha} c_{p\alpha}$ vanishes if

$$v_{kp\alpha\lambda} = \frac{(u_{k\lambda} - v_{k\lambda}) A'_k \underset{\sim}{p} \cdot \underset{\sim}{e}_{k\lambda}}{\hbar\Omega_{k\lambda} + \epsilon_{p-k,\alpha} - \epsilon_{p\alpha}}, \tag{5.8}$$

and those of $b_{k\lambda}$ and $b^+_{-k\lambda}$ if

$$(\hbar\omega_k - E_{k\lambda} - \hbar\Omega_{k\lambda}) u_{k\lambda} + E_{k\lambda} v_{k\lambda} = 0 \tag{5.9}$$

$$E_{k\lambda} u_{k\lambda} + (\hbar\omega_k - E_{k\lambda} + \hbar\Omega_{k\lambda}) v_{k\lambda} = 0, \tag{5.10}$$

where

$$E_{k\lambda}(\Omega_{k\lambda}) = \sum_{p\,\alpha} \frac{(\Lambda_k')^2\,(\underset{\sim}{p}\cdot\underset{\sim}{e}_{k\lambda})^2(N_{p\alpha}-N_{p-k,\alpha})}{i\hbar\Omega_{k\lambda} + \epsilon_{p-k,\alpha} - \epsilon_{p\alpha}} \quad . \tag{5.11}$$

In obtaining Eq. (5.11), we have used the fact that since the medium is isotropic, we can set

$$\underset{\sim}{p}\cdot\underset{\sim}{e}_{k\lambda}\,\underset{\sim}{p}\cdot\underset{\sim}{e}_{k\mu} = (\underset{\sim}{p}\cdot\underset{\sim}{e}_{k\lambda})^2\ \delta_{\lambda\mu} \tag{5.12}$$

in the summand of the original sum over $p\alpha\mu$. The expression on the right-hand-side of Eq. (5.11) is but the leading (and most important) term in a lengthy expression given by Glass and Chappell.[4] The extra terms they obtain are smaller and describe absorptive effects corresponding to photon-electron scattering processes we have dropped from our approximate Hamiltonian. Returning to the calculation, we get a dispersion relation from Eqs. (5.9) and (5.10)[12]

$$D_R(\Omega_{k\lambda}) = \Omega_{k\lambda}^2 - \omega_k^{\,2} + \frac{2\omega_k}{\hbar}\ E_{k\lambda}(\Omega_{k\lambda}) = 0. \tag{5.13}$$

We now turn our attention to the calculation of a dispersion relation for quasi-electrons.

6. QUASI-ELECTRON DISPERSION RELATION

We will have need of the following definitions in our calculation[13]

$$X^1_{p\alpha} = c_{p\alpha}, \tag{6.1}$$

$$X^2_{p-k,k\alpha\lambda} = c_{p-k,\alpha}\,(b_{k\lambda}+b^+_{-k\lambda}\ ) , \tag{6.2}$$

$$X^3_{p-k,k\alpha\lambda} = c_{p-k,\alpha}(b_{k\lambda}-b^+_{-k\lambda}). \tag{6.3}$$

The first three equations-of-motion involving these operators will be

$$[X^1_{p\alpha},H] = \epsilon_{p\alpha} X^1_{p\alpha} + \sum_{k\lambda}{}' A'_k \, \underset{\sim}{p} \cdot \underset{\sim}{e}_{k\lambda} X^2_{p-k,k\alpha\lambda} , \tag{6.4}$$

$$[X^2_{p-k,\alpha\lambda},H] = \epsilon_{p-k,\alpha} X^2_{p-k,k\alpha\lambda} + \sum_{\ell\mu}{}' A'_\ell \, (\underset{\sim}{p}-\underset{\sim}{k}) \cdot \underset{\sim}{e}_{\ell\mu} \, c_{p-k-\ell,\alpha}(b_{k\lambda}+b^+_{-k\lambda})(b_{\ell\mu}+b^+_{-\ell\mu})$$

$$+ \hbar\omega_k X^3_{p-k,k\alpha\lambda} , \tag{6.5}$$

$$[X^3_{p-k,k\alpha\lambda},H] = \epsilon_{p-k,\alpha} X^3_{p-k,k\alpha\lambda} + \sum_{\ell\mu} A'_\ell \, (\underset{\sim}{p}-\underset{\sim}{k}) \cdot \underset{\sim}{e}_{\ell\mu} c_{p-k-\ell,\alpha}(b_{k\lambda}+b^+_{-k\lambda})(b_{\ell\mu}-b^+_{-\ell\mu})$$

$$+ \hbar\omega_k X^2_{p-k,k\alpha\lambda} + 2 \sum_{q\beta}{}' A'_k \, \underset{\sim}{q} \cdot \underset{\sim}{e}_{k\lambda} \, c_{p-k} c^+_{q-k,\beta} c_{q\beta} . \tag{6.6}$$

The RPA consists of picking out terms in the sums for which momentum and polarization subscripts agree on Boson annihilation-creation operator pairs (or momentum and spin subscripts for Fermi annihilation-creation operator pairs). All other terms are dropped. Ground state averages are taken as usual, i.e.,

$$(b_{k\lambda}+b^+_{-k\lambda})(b_{\ell\mu}+b^+_{-\ell\mu}) \underset{\text{RPA}}{=} (1+2\,R_{k\lambda})\,\delta_{-k\ell}\,\delta_{\lambda\mu} ,$$

$$(b_{k\lambda}+b^{+}_{-k\lambda})\ (b_{\ell\mu}+b^{+}_{-\ell\mu}) \underset{RPA}{=} -\delta_{-k\ell}\ \delta_{\lambda\mu},$$

$$c_{p-k,\alpha}\ c^{+}_{q-k,\beta}\ c_{q\beta} \underset{RPA}{=} (1-N_{p-k,\alpha}\ c_{p\alpha})\ \delta_{pq}\ \delta_{\alpha\beta},$$

where

$$R_{k\lambda} = \langle b^{+}_{k\lambda}\ b_{k\lambda} \rangle \approx [\exp(\beta\hbar\omega_k) - 1]^{-1} = R_k\ . \tag{6.7}$$

In obtaining the above we have observed that $R_{k\lambda}=R_{-k\lambda}$. Thus, assuming that the occupation numbers are large, we have for our system of equations-of-motion

$$[X^1_{p\alpha},H] = \epsilon_{p\alpha}\ X^1_{p\alpha} + \sum_{k\ \lambda} A'_k\ \underset{\sim}{p}\cdot\underset{\sim}{e}_{k\lambda}\ X^2_{p-k,k\alpha\lambda}, \tag{6.4}$$

$$[X^2_{p-k,k\alpha\lambda},H]_{RPA} = \epsilon_{p-k,\alpha}\ X^2_{p-k,k\alpha\lambda} + 2\ A'_k\ \underset{\sim}{p}\cdot\underset{\sim}{e}_{k\lambda}\ R_{k\lambda} X^1_{p\alpha} + \hbar\omega_k\ X^3_{p-k,k\alpha\lambda}, \tag{6.8}$$

$$[X^3_{p-k,k\alpha\lambda},H]_{RPA} = \epsilon_{p-k,\alpha}\ X^3_{p-k,k\alpha\lambda} + \hbar\omega_k\ X^2_{p-k,k\alpha\lambda} - 2\ A'_k\ \underset{\sim}{p}\cdot\underset{\sim}{e}_{k\lambda} N_{p-k,\alpha} X^1_{p\alpha}. \tag{6.9}$$

We choose X to be of the form

$$X_{p\alpha} = u_{p\alpha}\ X^1_{p\alpha} + \sum_{k\ \lambda} v_{pk\alpha\lambda}\ X^2_{p-k,k\alpha\lambda} + \sum_{k\ \lambda} w_{pk\alpha\lambda}\ X^3_{p-k,k\alpha\lambda}\ , \tag{6.10}$$

and insert it into

$$[X_{p\alpha}, H] = \hbar\Omega_{p\alpha} X_{p\alpha} \cdot \tag{6.11}$$

We thereby obtain

$$(u_{p\alpha}\, \epsilon_{p\alpha} + 2 \sum_{k\lambda} v_{pk\alpha\lambda} A'_k \underset{\sim}{p} \cdot \underset{\sim}{e}_{k\lambda} R_{k\lambda} - 2 \sum_{k\lambda} v_{pk\alpha\lambda} A'_k \underset{\sim}{p} \cdot \underset{\sim}{e}_{k\lambda} N_{p-k,\alpha} -$$

$$u_{p\alpha}\, \hbar\Omega_{p\alpha}) X^1_{p\alpha} + \sum_{k\lambda} (u_{p\alpha} A'_k \underset{\sim}{p} \cdot e_{k\lambda} + v_{pk\alpha\lambda}\, \epsilon_{p-k,\alpha} + \hbar\omega_k\, v_{pk\alpha\lambda} -$$

$$v_{pk\alpha\lambda}\, \hbar\Omega_{p\alpha}) X^2_{p-k,k\alpha\lambda} + \sum_{k\lambda} (v_{pk\alpha\lambda}\, \hbar\omega_k + v_{pk\alpha\lambda} \epsilon_{p-k,\alpha} + \hbar\Omega_{p\alpha} v_{pk\alpha\lambda})\, X^3_{p-k,k\alpha\lambda} = 0. \tag{6.12}$$

We require that the coefficient of each operator must vanish. Hence, we find the dispersion relation[14]

$$D_p(\hbar\Omega_{p\alpha}) = \hbar\Omega_{p\alpha} - \epsilon'_{p\alpha}(\Omega_{p\alpha}) = 0, \tag{6.13}$$

where

$$\epsilon'_{p\alpha} = E_p - \sum_k V_{pk\alpha}(\Omega_{p\alpha})\, n_{p-k,\alpha} - \sum_k U_{pk\alpha}(\Omega_{p\alpha})\, \rho_k, \tag{6.14}$$

with $\rho_k = R_k / V$,

$$V_{pk\alpha}(\Omega_{p\alpha}) = V_k + 4\pi\hbar^4 \left(\frac{c}{m}\right)^2 \frac{p^2 - (\underset{\sim}{p} \cdot \underset{\sim}{k})^2/k^2}{(\hbar\Omega_{p\alpha} - \epsilon_{p-k,\alpha})^2 - (\hbar\omega_k)^2}, \tag{6.15}$$

and

$$U_{pk\alpha}(\Omega_{p\alpha}) = 4\pi\hbar^4\left(\frac{e}{m}\right)^2 \frac{[p^2-(\underset{\sim}{p}\cdot\underset{\sim}{k})^2/k^2]}{\hbar\omega_k} \frac{\hbar\Omega_{p\alpha}-\epsilon_{p-k,\alpha}}{(\hbar\Omega_{p\alpha}-\epsilon_{p-k,\alpha})^2-(\hbar\omega_k)^2}, \tag{6.16}$$

$V_{pk\alpha}$ is an effective electron-electron interaction potential. It arises from a virtual exchange of quasi-photons between electrons. If we set $\Omega_{p\alpha} = \epsilon_{p\alpha}$, we obtain a potential which is analogous to that used by Bardeen, Cooper, and Schrieffer in their treatment of an electron-phonon system.[15] It can be derived by the path integral technique of Feynman.[16] $U_{pk\alpha}$ is due to processes in which a single electron exchanges quasi-photons with the surrounding quasi-photon gas.

In conclusion, we see that the equation-of-motion technique, coupled with the RPA, has proved to be extremely useful in calculating dispersion relations for single excitations of a plasma.

APPENDIX A. RPA HAMILTONIAN

The purpose of this appendix is to achieve the essential results of the section 4 in a simpler fashion. We begin by expressing $\underset{\sim}{A}$ and $\underset{\sim}{P}_T$ in terms of a set of transverse coordinates and momenta

$$\underset{\sim}{A} = 2c \left(\frac{\pi}{V}\right)^{\frac{1}{2}} \sum_{k\,\lambda} q_{k\lambda}(t)\, \underset{\sim}{e}_{k\lambda} \exp[i\underset{\sim}{k} \cdot \underset{\sim}{r}] \,, \tag{A.1}$$

$$\underset{\sim}{P}_T = \frac{1}{2c(\pi V)^{\frac{1}{2}}} \sum_{k\,\lambda} p_{k\lambda}(t)\, \underset{\sim}{e}_{k\lambda} \exp[i\, \underset{\sim}{k} \cdot \underset{\sim}{r}]. \tag{A.2}$$

The coefficients are operators which have the property

$$q_{k\lambda}(t) = q^{+}_{-k\lambda}(t), \tag{A.3}$$

$$p_{k\lambda}(t) = p^{+}_{-k\lambda}(t), \tag{A.4}$$

since $\underset{\sim}{A}$ and $\underset{\sim}{P}_T$ are both real. Substitution of Eqs. (A.1) and (A.2) with (3.9) into Eqs. (3.2) through (3.4) yields

$$H = H_R + H_I^{\,1} + H_I^{\,2} + H_P, \tag{3.1}$$

$$H_R = \frac{1}{2} \sum_{k\,\lambda} [p_{k\lambda} p_{-k\lambda} + (kc)^2\, q_{k\lambda}\, q_{-k\lambda}], \tag{A.5}$$

$$H_I^{\,1} = A \sum_{p\alpha} \sum_{k\lambda} \underset{\sim}{p} \cdot \underset{\sim}{e}_{k\lambda}\, c^{+}_{p+k,\alpha}\, c_{p\alpha}\, q_{k\lambda}, \tag{A.6}$$

$$H_I^{\,2} = A \sum_{p\alpha} \sum_{k\ell} \sum_{\lambda\mu} B'_{k\ell\lambda\mu}\, c^{+}_{p+k+\ell,\alpha}\, c_{p\alpha}\, q_{k\lambda}\, q_{\ell\mu}, \tag{A.7}$$

where

$$A = \frac{2\pi e}{m} \left(\frac{\pi}{V} \right)^{\frac{1}{2}} ,$$

and

$$B'_{k\ell\lambda\mu} = \frac{2\pi e^2}{mV} \, \underset{\sim}{e}_{k\lambda} \cdot \underset{\sim}{e}_{\ell\mu} \, .$$

H_p is given by Eq. (3.15) as before. We focus our attention on H_I^2 which is written in the form

$$H_I^2 = \sum_{p\alpha} \sum_{\substack{k,\ell \\ k_j \neq \ell}} \sum_{\lambda \, \mu} B'_{k\ell\lambda\mu} \, c_{p+k+\ell,\alpha} \, c_{p\alpha} \, q_{k\lambda} \, q_{\ell\mu}$$

$$+ \frac{2\pi e^2}{mV} \sum_{p\alpha} \sum_{k\lambda} c^+_{p\alpha} c_{p\alpha} q_{k\lambda} q_{-k\lambda} + \frac{2\pi e^2}{mV} \sum_{p\alpha} \sum_{k\lambda} (\langle c^+_{p\alpha} c_{p\alpha} \rangle - \langle c^+_{p\alpha} c_{p\alpha} \rangle) \, q_{k\lambda} \, q_{-k\lambda} . \tag{A.8}$$

The last term on the right-hand-side does not contribute. We denote the first term in Eq. (A.8) by U and write

$$H_I^2 = U + \frac{2\pi e^2}{mV} \, \delta N \sum_{k\lambda} q_{k\lambda} q_{-k\lambda} + \frac{2\pi e^2}{mV} \, N \sum_{k\lambda} q_{k\lambda} \, q_{-k\lambda} , \tag{A.9}$$

where

$$\delta N = \sum_{p \, \alpha} (c^+_{p\alpha} \, c_{p\alpha} - \langle c^+_{p\alpha} \, c_{p\alpha} \rangle) \tag{A.10}$$

is the fluctuation in the total number of electrons. It is negligible and can be dropped considering the order of magnitude to which we are working.[1,2] U was obtained by Bohm and Pines in their investigation into the collective modes of a Coulomb plasma.[6] Its omission gives an expression for H_I^2 which is correct in the RPA :

$$H_{I\,RPA}^{2} = \frac{1}{2}\,\omega_P^{2} \sum_{k\lambda} q_{k\lambda}\, q_{-k\lambda} \quad . \tag{A.11}$$

We can now combine H_R with H_I^{2} to obtain H_R'.

$$H_R' = \frac{1}{2} \sum_{k\,\lambda} \{p_{k\lambda} p_{-k\lambda} + [(kc)^2 + \omega_P^{2}]\; q_{k\lambda}\, q_{-k\lambda}\} \; . \tag{A.12}$$

so that now[17]

$$H = H_R + H_I^{1} + H_P. \tag{A.13}$$

The usual transformation may be made upon this Hamiltonian

$$q_{k\lambda} = (\hbar/2\omega_k)^{\frac{1}{2}}\; (b_{k\lambda} + b^{+}_{-k\lambda}) \; , \tag{A.14}$$

$$p_{k\lambda} = -i(\hbar\omega_k/2)^{\frac{1}{2}}\; (b_{k\lambda} - b^{+}_{-k\lambda}), \tag{A.15}$$

where we have used $\omega_k = [(kc)^2 + \omega_P^{2}]^{\frac{1}{2}}$ in place of kc. Insertion of Eqs. (A.14) and (A.15) into (A.1) and (A.2) leads to Eq. (4.21) for $\underset{\sim}{A}$ and

$$\underset{\sim}{P}_I = \frac{-i}{2c} \left(\frac{\hbar}{2\kappa V}\right)^{\frac{1}{2}} \sum_{k\lambda} \omega_k^{\frac{1}{2}}\, \underset{\sim}{e}_{k\lambda}\, \exp[i\, \underset{\sim}{k} \cdot \underset{\sim}{r}]\; (b_{k\lambda} - b^{+}_{-k\lambda}). \tag{A.16}$$

Insertion of Equations (A.14) and (A.15) in (A.13) leads us directly to Eq. (4.27) for H. If we equate the usual transformation to the corresponding transformation above, one has

$$(\hbar/2kc)^{\frac{1}{2}} \; (a_{k\lambda}+a^{+}_{-k\lambda}) = (\hbar/2\omega_k)^{\frac{1}{2}} \; (b_{k\lambda}+b^{+}_{-k\lambda}), \tag{A.17}$$

$$(\hbar kc/2)^{\frac{1}{2}} \; (a_{k\lambda}+a^{+}_{-k\lambda}) = (\hbar\omega_k/2)^{\frac{1}{2}} \; (b_{k\lambda}+b^{+}_{-k\lambda}). \tag{A.18}$$

Solving for $b_{k\lambda}$ and $b^{+}_{k\lambda}$, we obtain Eq. (4.16) and its Hermetian conjugate.

We repeat the same procedure with H_P. H_P is given by

$$H_P = \sum_{p\,\alpha} E_p \; c^{+}_{p\alpha} \; c_{p\alpha} - \frac{1}{2V} \sum_{k} \sum_{p} \sum_{q} \sum_{\alpha} \sum_{\beta} V_k \; c^{+}_{p\alpha} \; c_{q\beta} c^{+}_{q+k,\beta} \; c_{p+k,\alpha}. \tag{A.19}$$

This expression differs from Eq. (3.15) in that we have used the anti-commutation rules to interchange the two middle Fermi operators of the last term. We have for the product of the four Fermi operators of the last term[18]

$$c^{+}_{p\alpha} \; c_{q\beta} c^{+}_{q+k,\beta} c_{p+k,\alpha} \underset{RPA}{=} c^{+}_{p\alpha} c_{p\alpha} c^{+}_{p+k,\alpha} c_{p+k,\alpha} \; \delta_{pq} \; \delta_{\alpha\beta}$$

$$\approx (N_{p\alpha} c^{+}_{p+k,\alpha} c_{p+k,\alpha} + N_{p+k,\alpha} c^{+}_{p\alpha} c_{p\alpha}) \; \delta_{pq} \; \delta_{\alpha\beta}.$$

Hence, we have for (A.19)

$$H_P = \sum_{p\,\alpha} E_p \; c^{+}_{p\alpha} - \frac{1}{2V} \sum_{k} \sum_{p\,\alpha} V_k \; N_{p\alpha} \; c^{+}_{p+k,\alpha} \; c_{p+k,\alpha}$$

$$- \frac{1}{2V} \sum_{k} \sum_{p\,\alpha} V_k \; N_{p+k,\alpha} \; c^{+}_{p\alpha} \; c_{p\alpha} \; . \tag{A.20}$$

A change of index from $\underset{\sim}{p}$ to $\underset{\sim}{p}-\underset{\sim}{k}$ in the $\underset{\sim}{p}$ sum of the middle term in Eq. (A.20) is followed a change of $\underset{\sim}{k}$ to $-\underset{\sim}{k}$ in k sum. We obtain Eq. (A.25) on collecting the remaining terms. Thus, we have established Eq. (A.27) which is the RPA Hamiltonian.

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8. The unit vectors $\underset{\sim}{k}/k$, $\underset{\sim}{e}_{k1}$ and $\underset{\sim}{e}_{k2}$ form a right-handed system. The last property follows from the requirement that $\underset{\sim}{A}$ be real. Note that $\sum_{\lambda} \underset{\sim}{e}_{k\lambda} \underset{\sim}{e}_{k\lambda} = 1 - \underset{\sim}{k}\underset{\sim}{k}/k^2$.
9. The spin state obey the usual property $\chi^{+}_{\alpha} \chi_{\beta} = \delta_{\alpha\beta}$.
10. To be defined shortly .
11. Subscript α on $\epsilon_{p\alpha}$ may be dropped. However, we will retain it in the text for the sake of generality.
12. The allowed frequencies in D_R are poles of the Fourier transform of the double-time, temperature dependent, Green's function, $\langle\langle \underset{\sim}{A}(\underset{\sim}{r}, t); \underset{\sim}{A}(\underset{\sim}{r}', \tau')\rangle\rangle$. A reference is M. J. Smith, Ph.D. dissertation , Howard University, 1969.
13. It is extremely helpful to use compact notation in making these caculations.
14. It is not hard to show that the allowed energies in D_P are poles belonging to the Fourier transform of the double-time, temperature dependent, Green's

function, $\langle\langle \psi(\underset{\sim}{r}, t) ; \psi(\underset{\sim}{r}', t') \rangle\rangle$.

15. J. Bardeen, L. N. Cooper and J.R. Schrieffer, Phys. Rev., 108, 1175-1204 (1957).

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17. H_p is given by Eq. (4.25) as we will shortly show.

18. One notes that $\delta N_{p\alpha} \equiv c^{+}_{p\alpha} c_{p\alpha} - N_{p\alpha} \approx 0$. Therefore, $\delta N_{p\alpha} \delta N_{p+k,\alpha} \approx 0$. This gives the last term plus a c number, $-N_{p\alpha} N_{p+k,\alpha} \delta_{pq} \delta_{\alpha\beta}$, which we have omitted since its effect is merely to renormalize H_p thus leaving our equations-of-motion unaffected. The α subscript on $H_{p\alpha}$ may be dropped if Eqs.(4.3) or (4.26) are used.

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