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S-MATRIX DERIVATION OF THE MOTION OF AN ELECTRON IN A CLASSICAL ELECTROMAGNETIC FIELD*

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ABSTRACT

It is shown how the motion of an electron in a given external classical electromagnetic field is derived from S-matrix principles.

I. INTRODUCTION

The motion of an electron in a classical electromagnetic field is well understood on the basis of quantum field theory. If radiative corrections are ignored then the solution is that given by Feynman in his "theory of positions." If S-matrix theory is to be acceptable as the fundamental form of quantum theory, replacing theories based on microscopic space-time assumptions, then it must be able to reproduce this important result.

This result follows, in fact, quite easily from S-matrix principles. Indeed, the derivation is trivial. Nonetheless it seems worthwhile to set it down explicitly. For, in the first place, it had not been generally recognized, even by S-matrix theorists, that the result comes out so easily.[1] In the second place, there have been previous unsuccessful efforts to derive the result.[1] In the third place, it is useful for didactic reasons to show how a well-known and important result of the older formalism comes out of the new approach: the basic ideas of the S-matrix dynamics are nicely illustrated in this simple example. In the fourth place, the derivation shows how classical fields fit naturally into the S-matrix framework.

II. CLASSICAL FIELDS AND COHERENT STATES

The work of Glauber,[2] Sudarshan,[3] Klauder,[4] Rohrlich,[5] and others makes it clear that classical fields are represented in quantum mechanics by "coherent states." A coherent state is a coherent superposition of states with different numbers of quanta, in which the amplitudes corresponding to the different numbers of quanta are adjusted in a particularly convenient way. The coherent states of the photon field can be labelled by the complex positive-frequency solutions $A_\mu^+(x)$ of the free field equations

$$\Box A_\mu^+(x) = 0 \tag{2.1a}$$

subject to the constraint[6]

$$\partial^\mu A_\mu^+(x) = 0. \tag{2.1b}$$

The amplitudes of the components with different numbers of photons are adjusted so that the states $|A_\mu^+(x)\rangle$ are right eigenstates of the annihilation operator part of the field operator $A_{op\mu}(x)$:

$$A_{op\mu}^+(x)|A_\mu^+(x)\rangle = A_\mu^+(x)|A_\mu^+(x)\rangle. \tag{2.2a}$$

Correspondingly, the conjugate states $\langle A_\mu^+(x)|$ are left eigenstates of the creation operator part of the field $A_{op}(x)$:

$$\langle A_\mu^+(x)|A_{op\mu}^-(x) = \langle A_\mu^+(x)|A_\mu^-(x). \tag{2.2b}$$

Here $A_{op\mu}^-(x) \equiv [A_{op\mu}^+(x)]^\dagger$ is the creation operator part of $A_{op\mu}(x)$, and $A_\mu^-(x) \equiv [A_\mu^+(x)]^*$ is a complex negative-frequency solution of the free-field equations. Note that

$$\langle A_\mu{}^+(x)|A_{op\mu}(x)|A_\mu{}^+(x)\rangle = A_\mu(x)\,\langle A_\mu{}^+(x)|A_\mu{}^+(x)\rangle\ , \quad (2.3)$$

where

$$A_\mu(x) \equiv A_\mu{}^+(x) + A^-(x)$$

$$= 2\mathrm{Re}\ A_\mu{}^+(x) \quad (2.4)$$

is real. That is, the expectation value of $A_{op\mu}(x)$ in the state $|A^+(x)\rangle$ is the real field $A_\mu(x)$ defined in (2.4).

The problem of computing the motion of a particle in an external field is, from the S-matrix point of view, the problem of computing the matrix element

$$\langle\psi_f|S[A_\mu{}^+(x)]|\psi_i\rangle \equiv \langle A_\mu{}^+(x)\psi_f|S|A_\mu{}^+(x)\psi_i\rangle, \quad (2.5)$$

where ψ_i and ψ_f represent the initial and final states of the particle. We shall consider the specific case in which this particle is a spin-$\frac{1}{2}$ particle.

III. THE PHOTON-FERMION VERTEX

It is convenient to represent the S matrix for a spin-$\frac{1}{2}$ particle by the covariant two-component formalism.[7] This eliminates spurious complications associated with redundant components. The connection between the usual S matrix

$$S_{ab} = \bar{u}_a(k_f)\, T\, u_b(k_i) \qquad (a,b = \pm\tfrac{1}{2}) \tag{3.1}$$

and the covariant M function $M_\alpha^{\ \beta}$ is

$$M = (v_f \cdot \sigma)^{\frac{1}{2}}\, S (v_i \cdot \tilde{\sigma})^{\frac{1}{2}}. \tag{3.2}$$

Here $v_i^{\ \mu} \equiv p_i^{\ \mu}/m_i$ and $v_f^{\ \mu} \equiv p_f^{\ \mu}/m_f$ are the components of the initial and final covariant velocities. And

$$(v \cdot \sigma) \equiv v^\mu \sigma_\mu \equiv v^0 + \underset{\sim}{v} \cdot \underset{\sim}{\sigma}\,, \tag{3.3a}$$

and

$$(v \cdot \tilde{\sigma}) = v^\mu \tilde{\sigma}_\mu \equiv v^0 - \underset{\sim}{v} \cdot \underset{\sim}{\sigma}\,, \tag{3.3b}$$

where $\underset{\sim}{\sigma}$ is the usual Pauli matrix. If the 4-by-4 matrix T is written, in the Weyl representation, as

$$T = \begin{pmatrix} T^{++} & T^{+-} \\ T^{-+} & T^{--} \end{pmatrix}, \tag{3.4a}$$

then (3.2) becomes

$$M = T^{++} + (v_f \cdot \sigma)\, T^{--} (v_i \cdot \tilde{\sigma}) + (T^{+-})(v_i \cdot \tilde{\sigma}) + (v_f \cdot \sigma)\, T^{-+}. \tag{3.4b}$$

For example, if T is

$$T^{\mu} = \gamma^{\mu} = \begin{pmatrix} 0 & \sigma_{\mu} \\ \tilde{\sigma}_{\mu} & \end{pmatrix}, \tag{3.5a}$$

then M is

$$M_{\mu} \equiv \sigma_{\mu}(v_i \cdot \tilde{\sigma}) + (v_f \cdot \sigma)\,\tilde{\sigma}_{\mu}. \tag{3.5b}$$

If T is

$$T_{\mu\nu} = \sigma_{\mu\nu} = \frac{i}{2}(\gamma_{\mu}\gamma_{\nu} - \gamma_{\nu}\gamma_{\mu}), \tag{3.6a}$$

then M is

$$M_{\mu\nu} \equiv \frac{i}{2}[(\sigma_{\mu}\tilde{\sigma}_{\nu} - \sigma_{\nu}\tilde{\sigma}_{\mu}) + (v_f \cdot \sigma)(\tilde{\sigma}_{\mu}\sigma_{\nu} - \tilde{\sigma}_{\nu}\sigma_{\mu})(v_i \cdot \tilde{\sigma})]. \tag{3.6b}$$

If T is

$$T_I = I \tag{3.7a}$$

then M is

$$M_I \equiv I + (v_f \cdot \sigma)(v_i \cdot \tilde{\sigma}). \tag{3.7b}$$

The demand of invariance under proper Lorentz transformations and space reflections requires that[8]

$$M = (a \cdot \sigma)(v_i \cdot \tilde{\sigma}) + (v_f \cdot \sigma)(a \cdot \tilde{\sigma}) + (b \cdot \sigma)(v_i \cdot \tilde{\sigma}) - (v_f \cdot \sigma)(b \cdot \tilde{\sigma}), \tag{3.8}$$

where a is some vector of the problem and b is a pseudovector.

For the photon-fermion vertex the vector a and the pseudovector b must be linear in the photon field A. Two possible forms are

$$a^{\mu} = e\, A^{\mu} \tag{3.9a}$$

and

$$b^{\mu} = g \sum_{\nu,\lambda,\rho} A_{\nu}\, q_{\lambda}\, Q_{\rho}\, \epsilon^{\nu\lambda\rho\mu}$$

$$\equiv g[AqQ]^{\mu} = g[Aqv_i]^{\mu}, \tag{3.9b}$$

where

$$q \equiv p_i - p_f, \tag{3.10a}$$

$$Q \equiv (p_i + p_f)/2m. \tag{3.10b}$$

Equation (3.9b) gives the only possible pseudovector. Equation (3.9a) gives the only vector a that is linear in A and not quadratic in the momenta. The only other independent vectors of the problem that are linear in A, and that contribute to (3.8), are

$$a^{\mu} = h(Q\cdot A)\, Q^{\mu} \tag{3.11a}$$

and

$$a^{\mu} = h'(q\cdot A)\, Q^{\mu}. \tag{3.11b}$$

These same two forms (3.11a) and (3.11b) with q^{μ} in place of Q^{μ} give no contribution to (3.8) because of the identity

$$(v \cdot \sigma)(v \cdot \tilde{\sigma}) = v^2 = 1. \tag{3.12}$$

[Here we have assumed that $m_i = m_f = m$, so that $q = p_i - p_f = (v_i - v_f)m$.]

The insertion of (3.9a) into (3.8) gives

$$M = eA^{\mu}M_{\mu} \equiv eA \cdot M, \tag{3.13a}$$

where M_{μ} is defined in (3.5b). The insertion of (3.9b) into (3.8) gives (for $q^2 = 0$, $m_i = m_f = m$)

$$M = g(q^{\mu}A^{\nu} - q^{\nu}A^{\mu})M_{\mu\nu} = 2gq^{\mu}A^{\nu}M_{\mu\nu}, \tag{3.13b}$$

where $M_{\mu\nu}$ is defined by (3.6b). The derivation of (3.13b) is given in Appendix A.

The insertion of (3.11a) into (3.8) gives

$$M = h(Q \cdot A)M_I, \tag{3.13c}$$

where M_I is defined by (3.7b). However, it is shown in Appendix A that (3.13c) is a linear combination of (3.12a) and (3.13b), on mass shell. The form (3.11b) does not contribute because $q \cdot A = 0$. Thus if we write

$$J_{\mu} = eM_{\mu} + 2gq^{\nu}M_{\nu\mu}, \tag{3.14}$$

then the photon-fermion vertex function (on-mass-shell) is

$$\Gamma = A^{\mu}J_{\mu}. \tag{3.15}$$

This is a 2-by-2 matrix in the spin space of the fermion. Replacing the suppressed arguments one has

$$\Gamma(q;\, p_f, p_i) \;=\; A^{\mu}(q)\, J_{\mu}(p_f, p_i), \tag{3.16}$$

where

$$A^{\mu}(q) \;=\; \int A^{\mu}(x)\, e^{iqx}\, d^4x. \tag{3.17}$$

We also define

$$V(q;\, p_i, p_f) \;=\; -i\Gamma(-q;\, p_i, p_f)\, \times\, (2\pi)^4\, \delta^4(p_i - q - p_f). \tag{3.18}$$

This function $V(q;\, p_i, p_f)$ represents the connected part of the S matrix for photon-fermion scattering, on mass shell. (Some of the momenta must be complex if the mass-shell and conservation-law constraints are both satisfied. S-matrix theory is characterized by the use of functions in unphysical regions in which the momenta are complex, but still satisfy the mass-shell and conservation-law constraints.) The minus sign in from of q in Γ of (3.18) arises from the fact that q is minus the momentum-energy of the incident photon, which means that if $A^{\mu}(x)$ is the incident photon field then $-q$ is the appropriate argument of (3.17). The $-i$ in (3.18) is conventional; it makes our e the same as the usual charge on the electron.

IV. SCATTERING OF AN ELECTRON IN AN EXTERNAL FIELD

The calculation is carried out in the approximation where radiative corrections are ignored. That is, only the contributions corresponding to tree graphs are included. A typical tree graph is shown in Fig. 1.

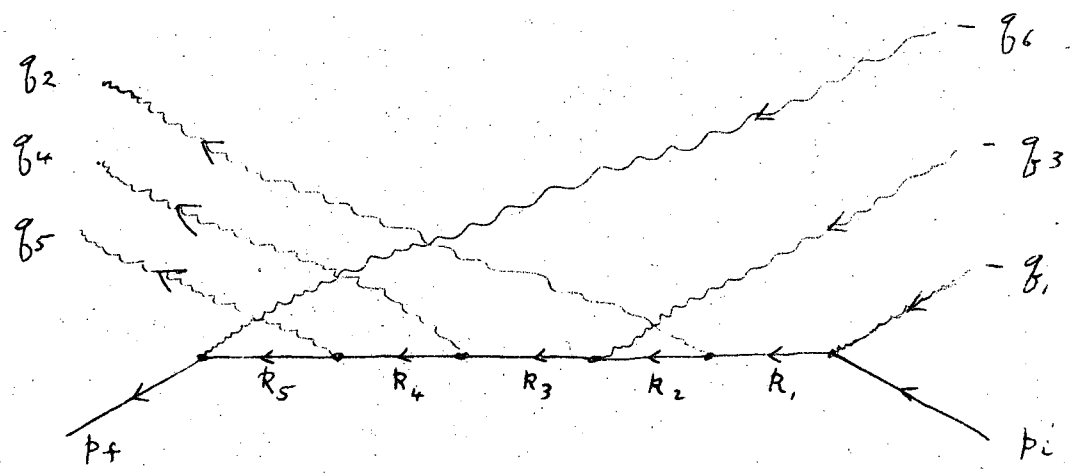

Fig. 1. A typical tree graph associated with the scattering of a particle by an external field.

Multiple application of the pole-factorization theorem[9] gives for the contribution to M corresponding to Fig. 1 the expression

$$M(p_f;\, q_6, \cdots q_1;\, p_i) = V_6 \prod_{j=1}^{5} \left(\frac{i}{k_j^{\,2} - m_j} \right) V_j \,, \qquad (4.1)$$

where

$$V_j = V(q_j;\ k_j, k_{j-1}). \tag{4.2}$$

Here $k_6 = p_f$ and $k_0 = p_i$.

The pole-factorization theorem gives only the residues of the poles. Thus it does not determine the function completely: An arbitrary entire function can be added to (4.1) without disturbing the residues. One simple way to add an entire function would be to augment the V's by entire functions that vanish on the mass shell.

In the S-matrix or dispersion approach the ambiguity regarding the entire function is to be resolved by conditions on the asymptotic behavior of the functions. In this example one sees clearly how asymptotic conditions become, in effect, conditions on the off-mass-shell part of the vertex functions V. Thus the asymptotic conditions play in S-matrix theory a role similar to that played by the choice of the interaction in field theory.

A fundamental difference between S-matrix theory and field theory lies in the approach to the problem of determining the asymptotic behaviors. In field theory one imposes microscopic space-time conditions, which have asymptotic implication in momentum space. In S-matrix theory one examines the problem directly in momentum space, and tries to satisfy the conditions imposed by unitarity, and general analyticity requirements: No a priori conditions are imposed on the microscopic space-time structure--the asymptotic behavior is not presumed to be generated by any particular form, or even by any conceivable form, of local interaction.

The problem of determining the correct asymptotic behavior is a very deep and fundamental one, which lies far outside the scope of this note. But in regard to asymptotic behavior the minimal coupling has a preferred status: it is linear in the momenta, whereas all other polynomial couplings are at least cubic. The a priori recommendation of the minimal coupling in S-matrix theory is thus essentially the same as in field theory: in terms of powers of the momenta it is the simplest possible coupling--it gives the gentlest possible asymptotic behavior, within the polynomial framework.

To obtain the contribution of M(Fig. 1) to $\langle A_\mu^+(x)\psi_f|S|A_\mu^+(x)\psi_i\rangle$ one must fold it into the initial and final wave functions. This gives

$$S_6 = \int \psi_f^*(p_f) \frac{d^4p_f}{(2\pi)^4} (2\pi)\, \delta^+(p_f^2 - m^2)$$

$$\times\ \psi_i(p_i) \frac{d^4p_i}{(2\pi)^4} (2\pi)\, \delta^+(p_i^2 - m^2)$$

$$\times \prod_{j=1}^{6} \frac{d^4q_j}{(2\pi)^4} (2\pi)\, \delta(q^2)$$

$$\times\ M(p_f;\, q_b, \cdots q_1;\, p_i). \tag{4.3}$$

This contribution S_6 contains the contribution of Fig. 1, together with the $(2^6 - 1)$ similar contributions corresponding to the other two orientations (forward or backward) of the six photon legs.

The effect of choosing the coherent initial state $|A_\mu^+(x)\rangle$ is to couple each incoming line to the same initial photon wave function $A_\mu^+(x)$. And the effect of choosing the coherent final state $\langle A_\mu^+(x)|$ is to couple each outgoing photon lines to the same final photon wave function $A^-(x) \equiv [A_\mu^+(x)]^*$. Since the two parts of $A(x) \equiv A^+(x) + A^-(x)$ correspond to positive and negative frequencies, one can join them together and let the sign of q^0 determine the part of $A(x)$ that is used.

To obtain the full scattering, without radiative corrections, one must sum over all the tree diagrams. This gives

$$\langle\psi_f|S[A_\mu^+(x)]|\psi_i\rangle = \langle A_\mu^+(x)\psi_f|S|A_\mu^+(x)\psi_i\rangle = \sum_{n=0}^{\infty} S_n, \qquad (4.4)$$

where S_n for $n > 0$ is given by the obvious generalization of (4.3), and

$$S_0 = \langle\psi_f|S(0)|\psi_i\rangle$$

$$= \int \psi_f^*(p)\, \psi_i(p)\, \frac{d^4p}{(2\pi)^4}\, (2\pi)\, \delta^+(p^2 - m^2). \qquad (4.5)$$

If the minimal coupling is used, then the result (4.4) is just the two-component form of the result obtained by Feynman, on the basis of the Dirac equation and space-time considerations.[10] The plus $i\epsilon$ rule for integrating over the pole singularity comes from the macro-causality condition,[11] which is a purely S-matrix condition.

What has been calculated here is the S-matrix element, not the detailed temporal development. Of course, Feynman's solution is known

to be equivalent to those obtained by solving an equivalent Schroedinger-type equation.[12] So in this simple case one can imagine, if one chooses, that the time development between the preparation and measurement of the particle proceeds according to a Schroedinger-type equation. However, the empirical content of quantum theory resides in the S matrix.[13] Thus this extra bit of imagery adds no empirical content. Whether a microscopic space-time picture can be formed for the complete problem (with radiative corrections) when realistic asymptotic conditions are imposed is not known.

In the above discussion the field A was assumed to be a free field. If one wishes to take into account the sources of the field then the various photon lines can be attached to lines representing the sources. If the pole-factorization theorem is applied now to the intermediate photon lines of the new expanded tree graphs, then the field quantities $A_\mu(q)\, 2\pi\, \delta(q^2)$ get replaced by $J_\mu(q)\, q^{-2}$, where $J_\mu(q)$ is the source function. Notice that questions of normalization that in field theory are resolved by commutation relations are in S-matrix theory resolved by the pole-factorization theorem, the normalization of which is fixed by unitarity.

The pole-factorization theorem is derived from unitarity and the assumed normal analytic structure,[11] without any assumption about the reality of the coupling constant e. But if e were not real then unitarity and the normal analytic structure could not be maintained near a pole singularity, because the contributions from the connected parts of S and $S^\dagger$ become dominant, yet do not cancel, and there are no intermediate states in the unitarity equation that can make up the difference. The reality of e follows, therefore, essentially from unitarity.

V. CONCLUDING REMARKS

The essential point of the above discussion is simply that as long as only tree graphs are considered the S-matrix and field theory solutions are almost trivially equivalent. The only difference is one of viewpoint. In field theory one resolves the ambiguities not resolved by general principles by the choice of space-time Lagrangian; in S-matrix theory one constructs the scattering function by using the Cauchy formula, or some generalization of it, and resolves the ambiguities by asymptotic conditions in momentum space.

The important differences between field theory and S-matrix theory lie at a deeper level. S-matrix theory is more general in that it does not require the asymptotic behavior to correspond to simple microscopic space-time conditions, and it takes as its basic equations not a set of ad hoc differential equations in space and time, but rather the unitarity equations, and the discontinuity equations that follow from them. These equations are expressed in terms of the mass-shell scattering functions and their analytic continuations, and seem to be much more secure than the differential equations. The greater generality of the S-matrix formulation must, however, be restricted by imposing asymptotic conditions in momentum space.

According to the "bootstrap" idea of S-matrix theory the asymptotic properties are closely linked to the analyticity and unitarity properties that hold in the finite part of momentum-energy space: the asymptotic properties cannot simply be specified by ad hoc conditions, such as a choice of Lagrangian, but are instead determined by the requirement that they be compatible with the analyticity and unitarity

requirements. This idea has great aesthetic appeal. It is not yet known how successful it will be. In the meantime the general rule is to use the simplest asymptotic properties that seem to be compatible with analyticity, unitarity, and experiment.

APPENDIX A. THE PAULI TERM

The aim here is to derive (3.13b) from (3.8) and (3.9b). Insertion of (3.9b) into (3.8) gives

$$M = g\{[Aqv_i]\cdot\sigma\,(v_i\cdot\tilde{\sigma}) - (v_f\cdot\sigma)[Aqv_i]\cdot\tilde{\sigma}\}$$

$$= g\{\tfrac{1}{2}[Aqv_i]\cdot\sigma\,(v_i\cdot\tilde{\sigma}) - \tfrac{1}{2}[qAv_i]\cdot\sigma\,(v_i\cdot\tilde{\sigma})$$

$$+ \tfrac{1}{2}(v_f\cdot\sigma)[v_i qA]\cdot\tilde{\sigma} - \tfrac{1}{2}(v_f\cdot\sigma)[v_i Aq]\cdot\tilde{\sigma}\}, \qquad \text{(A.1)}$$

where the antisymmetry of the box product is used. Insertion of the identities[7]

$$[abc]\cdot\sigma = -i(a\cdot\sigma)(b\cdot\tilde{\sigma})(c\cdot\sigma) + i(a\cdot\sigma)(b\cdot c)$$

$$-i(b\cdot\sigma)(a\cdot c) + i(c\cdot\sigma)(a\cdot b) \qquad \text{(A.2a)}$$

and

$$[abc]\cdot\tilde{\sigma} = -i(c\cdot\tilde{\sigma})(b\cdot\sigma)(a\cdot\tilde{\sigma}) + i(a\cdot\tilde{\sigma})(b\cdot c)$$

$$-i(b\cdot\tilde{\sigma})(a\cdot c) + i(c\cdot\tilde{\sigma})(a\cdot b) \qquad \text{(A.2b)}$$

gives

$$M = g\,\tfrac{i}{2}[-(A\cdot\sigma)(q\cdot\tilde{\sigma})(v_i\cdot\sigma)(v_i\cdot\tilde{\sigma}) + (A\cdot\sigma)(q\cdot v_i)(v_i\cdot\tilde{\sigma})$$

$$-(q\cdot\sigma)(A\cdot v_i)(v_i\cdot\tilde{\sigma}) + (v_i\cdot\sigma)(A\cdot q)(v_i\cdot\tilde{\sigma})$$

$$+ (q\cdot\sigma)(A\cdot\tilde{\sigma})(v_i\cdot\sigma)(v_i\cdot\tilde{\sigma}) - (q\cdot\sigma)(A\cdot v_i)(v_i\cdot\tilde{\sigma})$$

$$+ (A\cdot\sigma)(q\cdot v_i)(v_i\cdot\tilde{\sigma}) - (v_i\cdot\sigma)(q\cdot A)(v_i\cdot\tilde{\sigma})$$

$$- (v_f\cdot\sigma)(A\cdot\tilde{\sigma})(q\cdot\sigma)(v_i\cdot\tilde{\sigma}) + (v_f\cdot\sigma)(v_i\cdot\tilde{\sigma})(q\cdot A)$$

Equation (A.3) continued

Equation (A.3) Continued

$$-(v_f \cdot \sigma)(q \cdot \tilde{\sigma})(v_i \cdot A) + (v_f \cdot \sigma)(A \cdot \tilde{\sigma})(v_i \cdot q)$$

$$+ (v_f \cdot \sigma)(q \cdot \tilde{\sigma})(A \cdot \sigma)(v_i \cdot \tilde{\sigma}) - (v_f \cdot \sigma)(v_i \cdot \tilde{\sigma})(A \cdot q)$$

$$+ (v_f \cdot \sigma)(A \cdot \tilde{\sigma})(v_i \cdot q) - (v_f \cdot \sigma)(q \cdot \tilde{\sigma})(v_i \cdot A)]$$

$$= g(q^\mu A^\nu - q^\nu A^\mu)M_{\mu\nu} + R$$

$$= 2g\, q^\mu A^\nu M_{\mu\nu} + R, \tag{A.3}$$

where

$$R = ig[(A \cdot \sigma)(v_i \cdot \tilde{\sigma}) + (v_f \cdot \sigma)(A \cdot \tilde{\sigma})](q \cdot v_i)$$

$$- ig[(q \cdot \sigma)(v_i \cdot \tilde{\sigma}) + (v_f \cdot \sigma)(q \cdot \tilde{\sigma})](A \cdot v_i). \tag{A.4}$$

The second term on the right-hand side of (A.4) vanishes by virtue of (3.12). The first term vanishes because $q \cdot v_i = 0$, which follows from the condition $q^2 = 0$, which implies $v_i \cdot v_f = 1$:

$$(v_i - v_f) \cdot (v_i - v_f) = 0 = 2 - 2(v_i \cdot v_f). \tag{A.5}$$

If the condition $q^2 = 0$ is relaxed then the forms (3.9b) and (3.13b) differ by a multiple of the minimal coupling. More generally, all the coupling constants can become functions of $v_i \cdot v_f$. On-mass-shell the coupling constants are, of course, constants.

To show that (3.13c) is a linear combination of (3.13a) and (3.13b) write (with m now normalized to unity)

$$A^{\mu}M_{\mu} = (A\cdot\sigma)(v_i\cdot\tilde{\sigma}) + (v_f\cdot\sigma)(A\cdot\tilde{\sigma})$$

$$= (A\cdot\sigma)(Q + \tfrac{1}{2}q)\cdot\tilde{\sigma} + (Q - \tfrac{1}{2}q)\cdot\sigma\,(A\cdot\tilde{\sigma})$$

$$= [A\cdot\sigma\;\; Q\cdot\tilde{\sigma} + Q\cdot\sigma\;\; A\cdot\tilde{\sigma}] + \tfrac{1}{2}[A\cdot\sigma\;\; q\cdot\tilde{\sigma} - q\cdot\sigma\;\; A\cdot\tilde{\sigma}]$$

$$= 2A\cdot q + \tfrac{1}{2}[A\cdot\sigma\;\; q\cdot\tilde{\sigma} - q\cdot\sigma\;\; A\cdot\tilde{\sigma}]. \qquad (A.6)$$

Then write (using $q^2 = 0$, which implies $Q^2 = 1$),

$$(Q\cdot A)M_I = (Q\cdot A)(1 + v_f\cdot\sigma\;\; v_i\cdot\tilde{\sigma})$$

$$= (Q\cdot A)[1 + (Q - \tfrac{1}{2}q)\cdot\sigma\;\; (Q + \tfrac{1}{2}q)\cdot\tilde{\sigma}]$$

$$= 2Q\cdot A + \tfrac{1}{2}(Q\cdot A)[Q\cdot\sigma\;\; q\cdot\tilde{\sigma} - q\cdot\sigma\;\; Q\cdot\tilde{\sigma}]. \qquad (A.7)$$

Thus

$$2[A^{\mu}M_{\mu} - (Q\cdot A)M_I]$$

$$= [A\cdot\sigma\;\; q\cdot\tilde{\sigma} - q\cdot\sigma\;\; A\cdot\tilde{\sigma}] - (A\cdot Q)[Q\cdot\sigma\;\; q\cdot\tilde{\sigma} - q\cdot\sigma\;\; Q\cdot\tilde{\sigma}]. \qquad (A.8)$$

Notice that the right-hand side of (A.8), like the right-hand side of (3.9b), vanishes if A is proportional to any linear combination of Q and q.

Equation (3.6b) gives

$$-i(q^{\mu}A^{\nu} - q^{\nu}A^{\mu})M_{\mu\nu} =$$

$$(q^{\mu}A^{\nu} - q^{\nu}A^{\mu})[\sigma_{\mu}\tilde{\sigma}_{\nu} + (Q - \tfrac{1}{2}q)\cdot\sigma\;\; \tilde{\sigma}_{\mu}\sigma_{\nu}(Q + \tfrac{1}{2}q)\cdot\tilde{\sigma}]\,. \qquad (A.9)$$

Using (once again) the identity

$$a \cdot \sigma \; b \cdot \tilde{\sigma} = 2a \cdot b - b \cdot \sigma \; a \cdot \tilde{\sigma}, \tag{A.10}$$

and $q^2 = 1 - Q^2 = 0$, one obtains

$$-i(q^\mu A^\nu - q^\nu A^\mu)M_{\mu\nu} = (q^\mu A^\nu - q^\nu A^\mu)[2\sigma_\mu \; \tilde{\sigma}_\nu + R'], \tag{A.11}$$

where

$$\begin{aligned} R' = \frac{1}{2}\sigma_\mu[Q \cdot \tilde{\sigma} \; q \cdot \sigma - q \cdot \tilde{\sigma} \; Q \cdot \sigma]\tilde{\sigma}_\nu \\ - 2(Q - \tfrac{1}{2}q)_\mu \; \sigma_\nu(Q + \tfrac{1}{2}q) \cdot \tilde{\sigma} \\ - 2(Q - \tfrac{1}{2}q) \cdot \sigma \; \tilde{\sigma}_\mu(Q + \tfrac{1}{2}q)_\nu \\ + 4(Q - \tfrac{1}{2}q)_\mu \; (Q + \tfrac{1}{2}q)_\nu \; . \end{aligned} \tag{A.9}$$

Thus (A.8) and (A.11) give

$$\begin{aligned} 2A^\mu M_\mu - 2(Q \cdot A)M_I - \tfrac{i}{2}(q^\mu A^\nu - A^\mu q^\nu)M_{\mu\nu} \\ = - A \cdot Q[Q \cdot \sigma \; q \cdot \tilde{\sigma} - q \cdot \sigma \; Q \cdot \tilde{\sigma}] + \tfrac{1}{2}[q^\mu A^\nu - A^\mu q^\nu]R' . \end{aligned} \tag{A.10}$$

To show that (A.10) is zero one can write

$$A = (A \cdot Q)Q + B \; . \tag{A.11}$$

It has already been noted that the contribution to A proportional to Q gives a null contribution to the left-hand side of (A.10). Thus it must give a null contribution to the right-hand side. The contribution B satisfies $B \cdot Q = 0$. Thus it is sufficient to show that the second term on the right of (A.10) is zero when A is replaced by B.

The first term in R' when dotted into q on either the right or left vanishes due to the conditions $q^2 = Q \cdot q = 0$. Repeated use of these two conditions gives, for the second term on the right of (A.10), with B in place of A,

$$\frac{1}{2}[q^\mu B^\nu - q^\nu B^\mu][q_\mu \sigma_\nu (Q + \frac{1}{2} q) \cdot \tilde{\sigma} - (Q - \frac{1}{2} q) \cdot \sigma \tilde{\sigma}_\mu q_\nu]$$

$$= -\frac{1}{2}[(Q - \frac{1}{2} q) \cdot \sigma \; q \cdot \tilde{\sigma} + q \cdot \sigma (Q + \frac{1}{2} q) \cdot \tilde{\sigma}] B \cdot q$$

$$= -\frac{1}{2}[Q \cdot \sigma \; q \cdot \tilde{\sigma} + q \cdot \sigma \; Q \cdot \tilde{\sigma}](B \cdot q)$$

$$= -(Q \cdot q)(B \cdot q) = 0. \qquad \text{(A.12)}$$

The calculations have been given because the two-component formalism is not well known: they are prototype calculations.

FOOTNOTES AND REFERENCES

* This work was supported in part by the U.S. Atomic Energy Commission.

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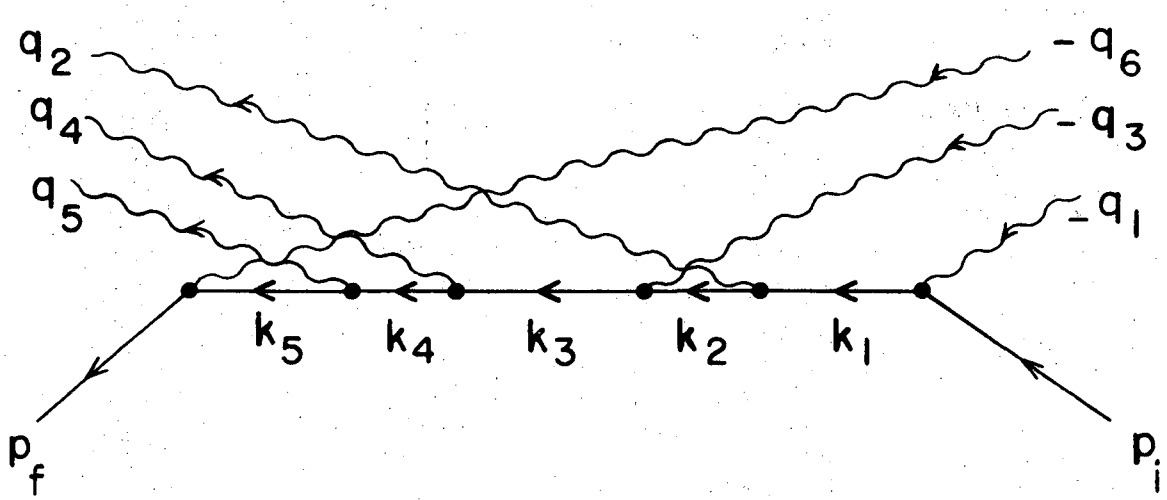

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Fig. 1

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