UNIVERSITY OF CALIFORNIA
Radiation Laboratory
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ELECTRICAL ENGINEERING REVIEW COURSE

LECTURE XI
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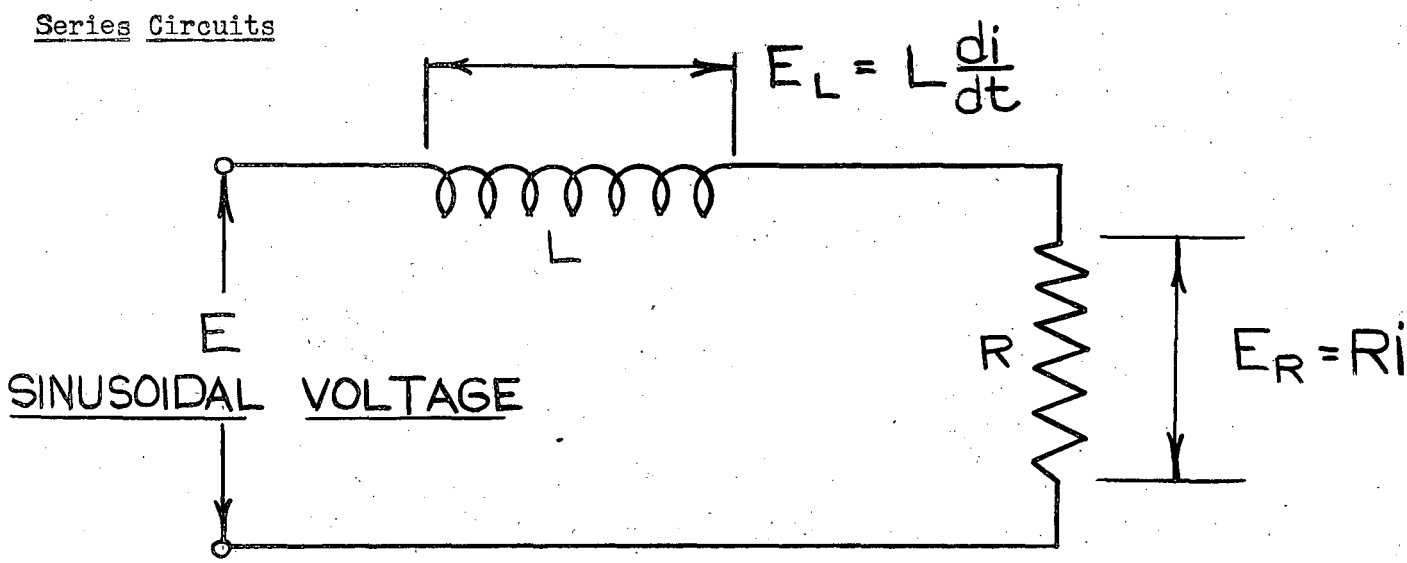
Single-Phase Alternating-Current Circuits

Complex Impedance

Series Circuits



Differential Equation

$$L\frac{di}{dt} + Ri = E$$

Now

$E = E_o e^{j\omega t}$ Actual voltage is real part of this

Where

$e^{j\omega t} = \cos \omega t + j \sin \omega t$

ω = Rotational Frequency = $2\pi f$

f = Frequency-cycles per second

Steady State Condition

$i = i_o e^{j\omega t}$ $\qquad$ i_o = Complex - not necessarily real only

$$\frac{di}{dt} = j\omega i_o e^{j\omega t}$$

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Therefore

$$L\frac{di}{dt} + Ri = E$$

$$Lj\omega i_o e^{j\omega t} + Ri_o e^{j\omega t} = E_o e^{j\omega t}$$

$$j\omega i_o L + Ri_o = E_o$$

$$i_o \left[R + j\omega L\right] = E_o$$

Now

Z = Impedance

$$Z = R + j\omega L$$

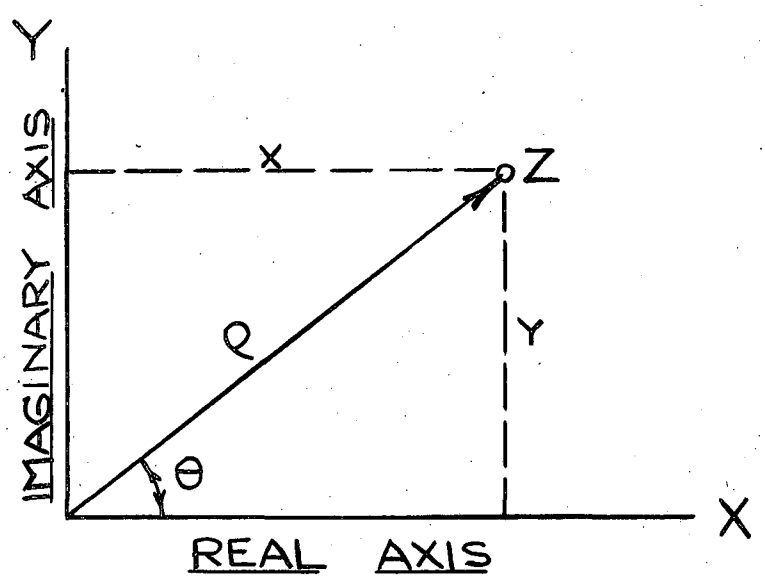
$$i_o Z = E_o$$

Solving For i_o

$$i_o = \frac{E_o}{Z} = \frac{E_o}{R + j\omega L}.$$

COMPLEX NUMBER

A complex number $Z = X + jY$ consists of a real part X, and an imaginary part jY. The value of the number Z can be represented on the "complex plane" by a point having the coordinates X and Y. The length ρ of the radius vector from O to Z is called the "modulus", and the angle Θ made by the radius vector with the X axis is called the "argument".



Look At

$$Z = \rho e^{j\theta}$$
$$= \rho\left[\cos\theta + j\sin\theta\right]$$
$$= \rho\cos\theta + j\rho\sin\theta$$
$$= X + jY$$

$$\rho = \sqrt{X^2 + Y^2} = |Z|$$

$$\tan\Theta = \frac{Y}{X}$$

Multiplication of Impedances with Complex Notation

$$Z_1 = X_1 + jY_1 = \rho_1 e^{j\theta_1}$$

$$Z_2 = X_2 + jY_2 = \rho_2 e^{j\theta_2}$$

$$Z_1 Z_2 = \rho_1 \rho_2 e^{j(\theta_1 + \theta_2)}$$

Returning To

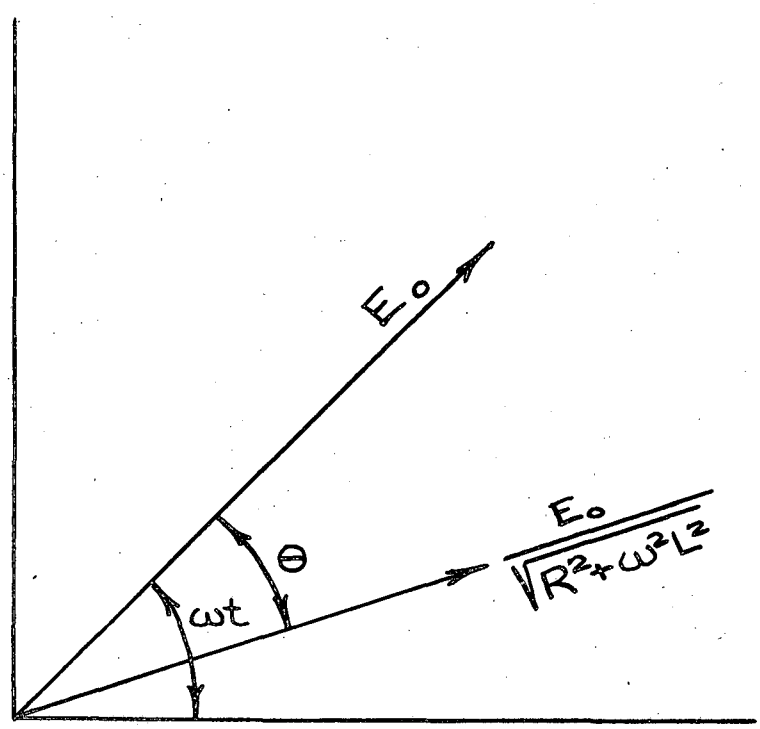
$$i_o = \frac{E_o}{R + j\omega L}$$

$$R + j\omega L = \sqrt{R^2 + \omega^2 L^2}\, e^{j\theta}$$

$$\tan \theta = \frac{\omega L}{R}$$

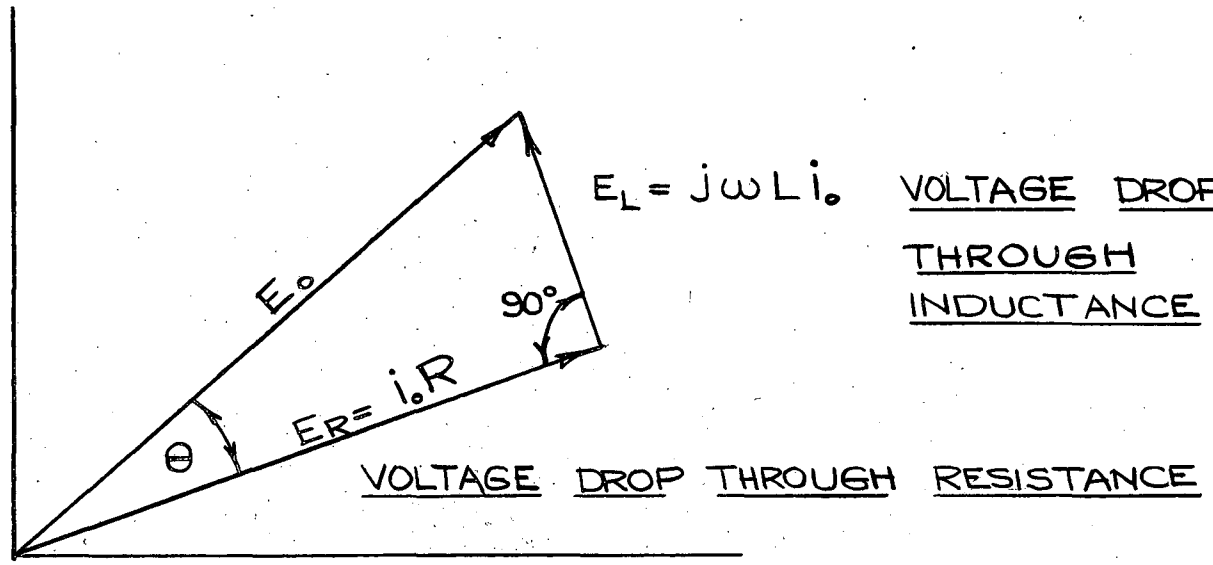
$$i_o = \frac{E_o}{\sqrt{R^2 + \omega^2 L^2}}\, e^{-j\theta}$$

$$i = i_o e^{j\omega t} = \frac{E_o}{\sqrt{R^2 + \omega^2 L^2}}\, e^{j(\omega t - \theta)}$$



EVERYTHING IS SPINNING AROUND THE ORIGIN

VOLTAGE LEADS CURRENT



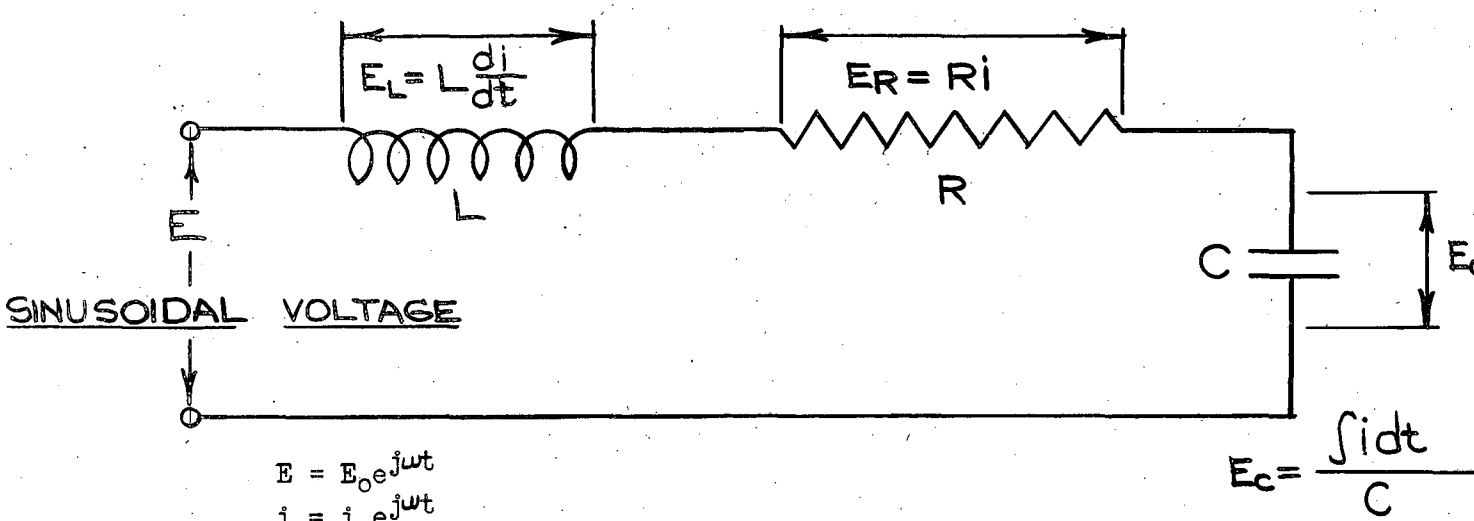
Also

$$j\omega L i_o = \omega L i_o e^{j\frac{\pi}{2}}$$

Since

$$e^{j\frac{\pi}{2}} = \cos\frac{\pi}{2} + j\sin\frac{\pi}{2} = 0 + j = j$$

SERIES CIRCUIT INCLUDING L, R, AND C



$$E = E_o e^{j\omega t}$$
$$i = i_o e^{j\omega t}$$

Now

$$\int i\,dt = \int i_o e^{j\omega t} dt = \frac{1}{j\omega} i_o e^{j\omega t}$$

Differential Equation

$$L\frac{di}{dt} + Ri + \frac{\int i\,dt}{C} = E$$

$$Lj\omega i_o e^{j\omega t} + Ri_o e^{j\omega t} + \frac{i_o e^{j\omega t}}{j\omega C} = E_o e^{j\omega t}$$

$$Lj\omega i_o + Ri_o + \frac{i_o}{j\omega C} = E_o$$

$$i_o\left[R + j\left(\omega L - \frac{1}{\omega C}\right)\right] = E_o$$

Impedance

$$Z = R + j(\omega L - \frac{1}{\omega C})$$

It Is Interesting To Look At Current

$$i_o = \frac{E_o}{Z} = \frac{E_o}{\sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2}} e^{-j\theta}$$

$$\text{Tan } \theta = \frac{\omega L - \frac{1}{\omega C}}{R}$$

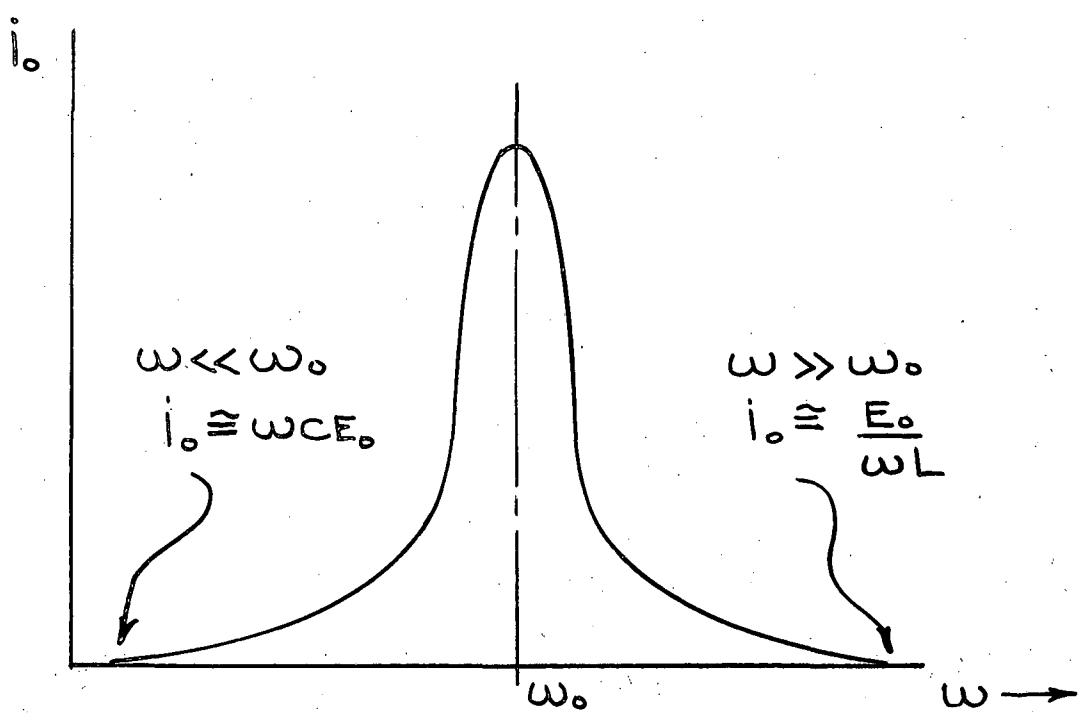
Now Suppose The Resistance (R) Is Small, And The Rotational Frequency (ω) Is Such That

$$\omega L - \frac{1}{\omega C} = 0 \qquad \text{or} \qquad \omega_o = \frac{1}{\sqrt{LC}} \qquad \text{Resonant Frequency}$$

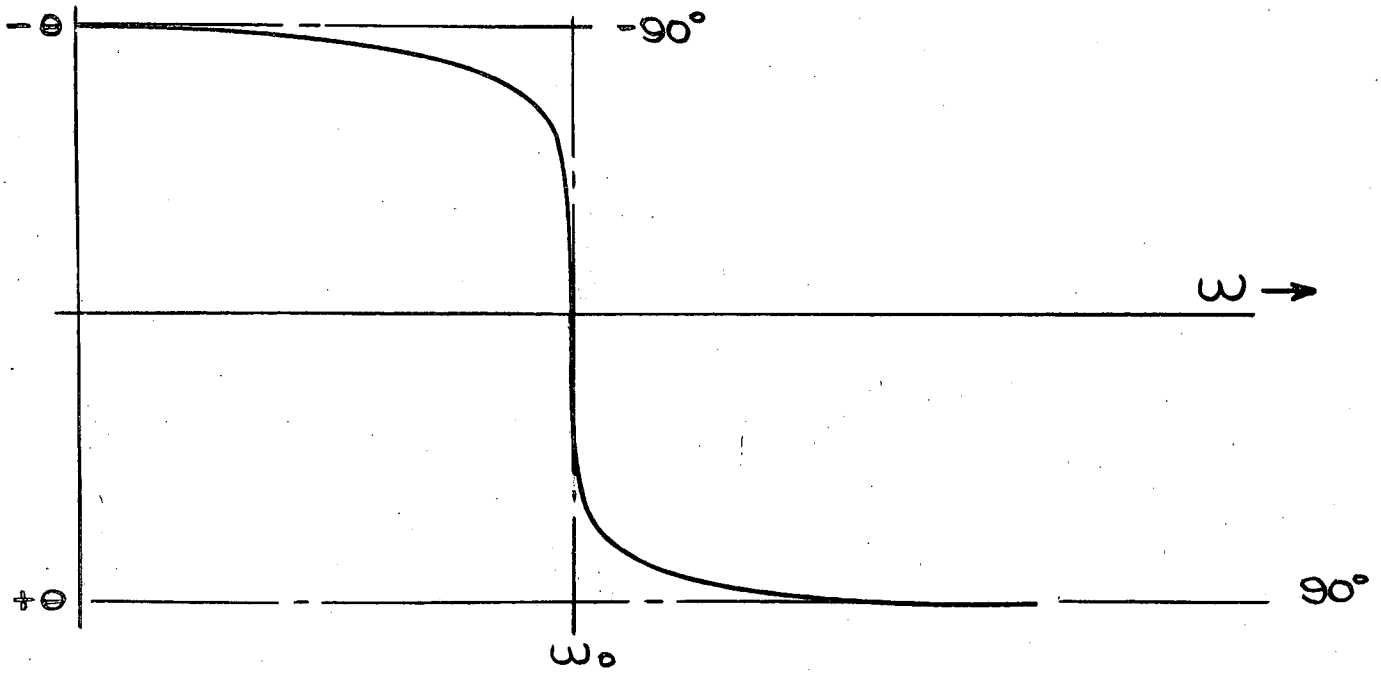
Then

$$|Z| = \sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2} = \sqrt{R^2 + 0^2} = \sqrt{R^2} = R$$

With Small Value Of R Current Can Be Large



Phase Angle (θ) Between Voltage And Current



"Q" Of A Circuit Or Element

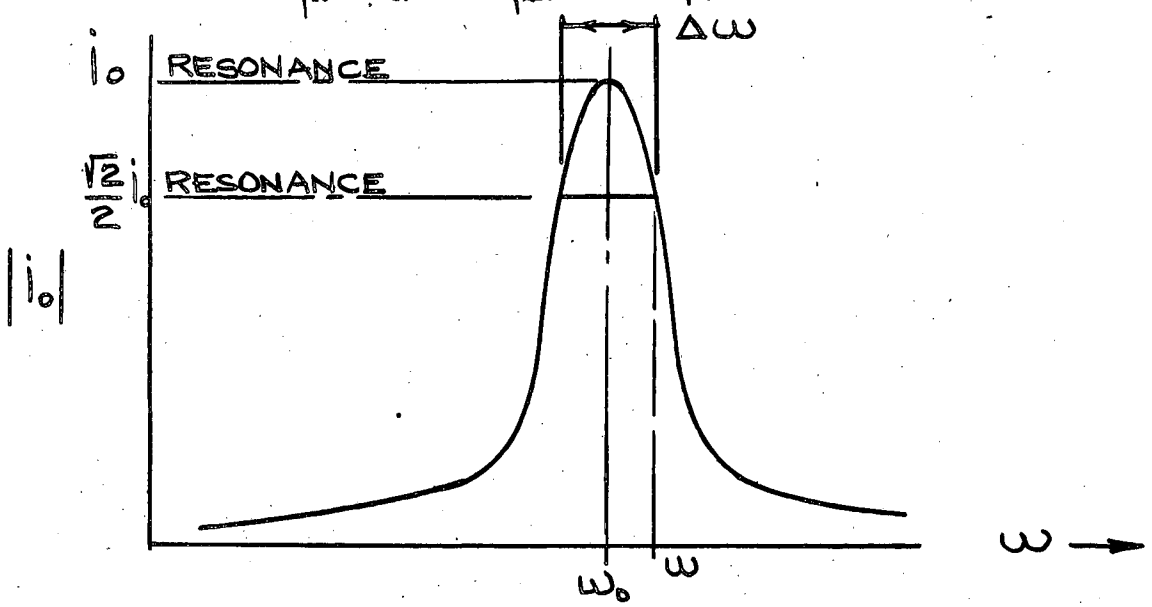
$$|i_o| = \frac{E_o}{\sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2}}$$

If

$$(\omega L - \frac{1}{\omega C}) = R$$

Then

$$|i_o| = \frac{E_o}{\sqrt{R^2 + R^2}} = \frac{E_o}{\sqrt{2R^2}} = \frac{E_o}{\sqrt{2}R} = (\frac{\sqrt{2}}{2})(\frac{E_o}{R}) = \frac{\sqrt{2}}{2} i_o \quad \text{resonance}$$



Also

$$\omega L - \frac{1}{\omega C} = R \qquad \omega_o = \frac{1}{\sqrt{LC}}$$

$$\omega - \frac{1}{\omega CL} = \frac{R}{L} \qquad \omega^2_o = \frac{1}{LC}$$

$$\omega - \frac{\omega_o^2}{\omega} = \frac{R}{L}$$

$$\omega^2 - \omega_o^2 = \frac{R\omega}{L} \qquad (\omega - \omega_o)(\omega + \omega_o) = \frac{R\omega}{L}$$

When the Resonance is fairly sharp so that the peak of the curve is approximately symmetrical about the resonant frequency (ω_o)

Then

$$\Delta\omega \sim 2(\omega - \omega_o)$$

Also For Sharp Resonance

$$\omega \sim \omega_o \quad \text{so} \quad (\omega + \omega_o) \sim 2\,\omega_o$$

Therefore

$$(\omega - \omega_o)(2\,\omega_o) = \frac{R\omega_o}{L} \qquad 2(\omega - \omega_o)(\omega_o) = \frac{R\omega_o}{L}$$

$$\Delta\omega(\omega_o) = \frac{R\omega_o}{L}$$

$$\Delta\omega = \frac{R}{L}$$

$$\frac{\Delta\omega}{\omega_o} = \frac{R}{\omega_o L} = \frac{1}{Q}$$

Or

$$Q = \frac{\omega_o L}{R}$$

"Q" is sometimes termed the "figure of merit" of a circuit, but is usually referred to simply as the "Q" of a circuit or element.

Power Dissipated In This Circuit

Power = (Ei)(Real Parts)

$$E = E_o \cos \omega t \qquad |Z| = \sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2}$$

$$i = \frac{E_o}{|Z|} \cos(\omega t - \theta)$$

$$P = \frac{E_o^2}{|Z|} \cos(\omega t - \theta)(\cos \omega t)$$

$$= \frac{E_o^2}{|Z|} \left[\cos \omega t \cos \theta + \sin \omega t \sin \theta\right] \left[\cos \omega t\right]$$

$$= \frac{E_o^2}{|Z|} \left[\cos^2 \omega t \cos \theta + \sin \omega t \cos \omega t \sin \theta\right]$$

Now Average Values Of

$$\cos^2\omega t = 1/2$$
$$\sin\omega t \cos\omega t = 0$$

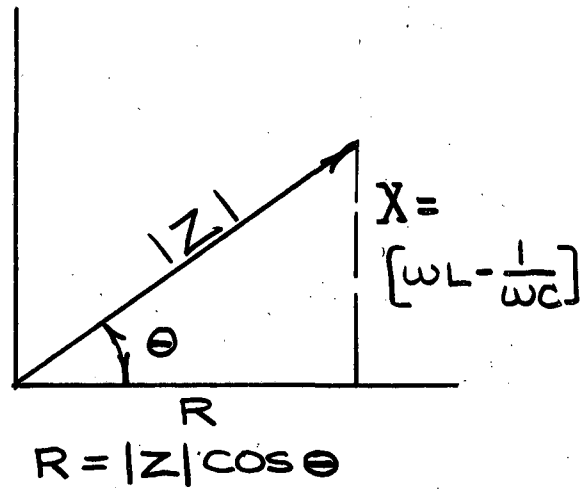
Therefore

$$P = \frac{E_o^{\,2}}{2|Z|}\cos\theta$$

$\cos\theta$ = Power Factor

Also

$$P = \frac{E^2_{rms}}{|Z|}\cos\theta$$



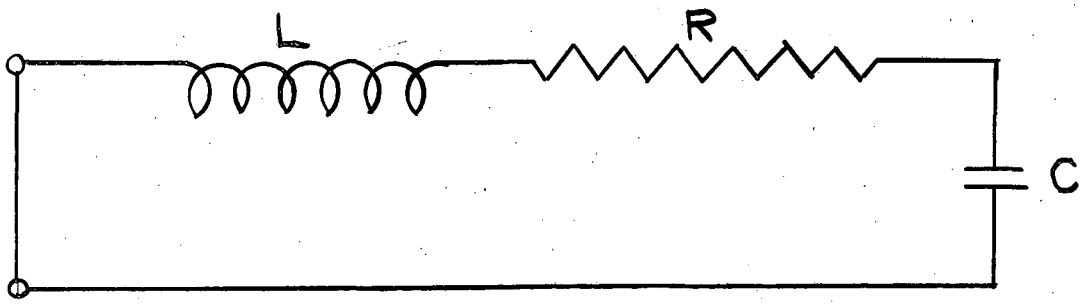
Expressing This In Terms Of The Current

$$E^2_{rms} = I^2_{rms}|Z|^2$$

$$P = I^2_{rms}|Z|\cos\theta$$

$$P = I^2_{rms}\,R$$

Free Oscillation Of This System



Charge Is Initially Stored In The Condenser

$$L\frac{di}{dt} + Ri + \frac{Q}{C} = 0$$

$$i = \frac{dQ}{dt}$$

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = 0$$

Rearranging The Equation

$$\frac{d^2Q}{dt^2} + \frac{R}{L}\frac{dQ}{dt} + \frac{Q}{CL} = 0$$

Let

$$2b = \frac{R}{L} \qquad a^2 = \frac{1}{CL} \qquad b = \frac{R}{2L}$$

Then

$$\frac{d^2Q}{dt^2} + 2b\frac{dQ}{dt} + a^2Q = 0$$

In This Case The Auxiliary Equation Is

$$m^2 + 2bm + a^2 = 0$$

And The Roots Are

$$m = -b \pm \sqrt{b^2 - a^2}$$

Oscillation Will Take Place When The Roots Of The Auxilliary Equation Are Imaginary

When

$$b^2 < a^2$$

Denote

$$b^2 - a^2 \text{ by } -\alpha^2$$

So That

$$m = -b \pm j\alpha$$

Solution Of Equation

$$Q = e^{-bt}(A\cos\alpha t + B\sin\alpha t)$$

Initial Conditions

$$Q = Q_o \qquad t = 0 \qquad \frac{dQ}{dt} = 0$$

When t = 0, Constants A and B Can Be Evaluated

$$Q = Q_o e^{-bt}(\cos\alpha t - \frac{b}{\alpha}\sin\alpha t)$$

Now

$$\alpha = \sqrt{a^2 - b^2} = \sqrt{\frac{1}{CL} - (\frac{R}{2L})^2}$$

When

$$\frac{1}{LC} \ggg \frac{R}{2L}$$

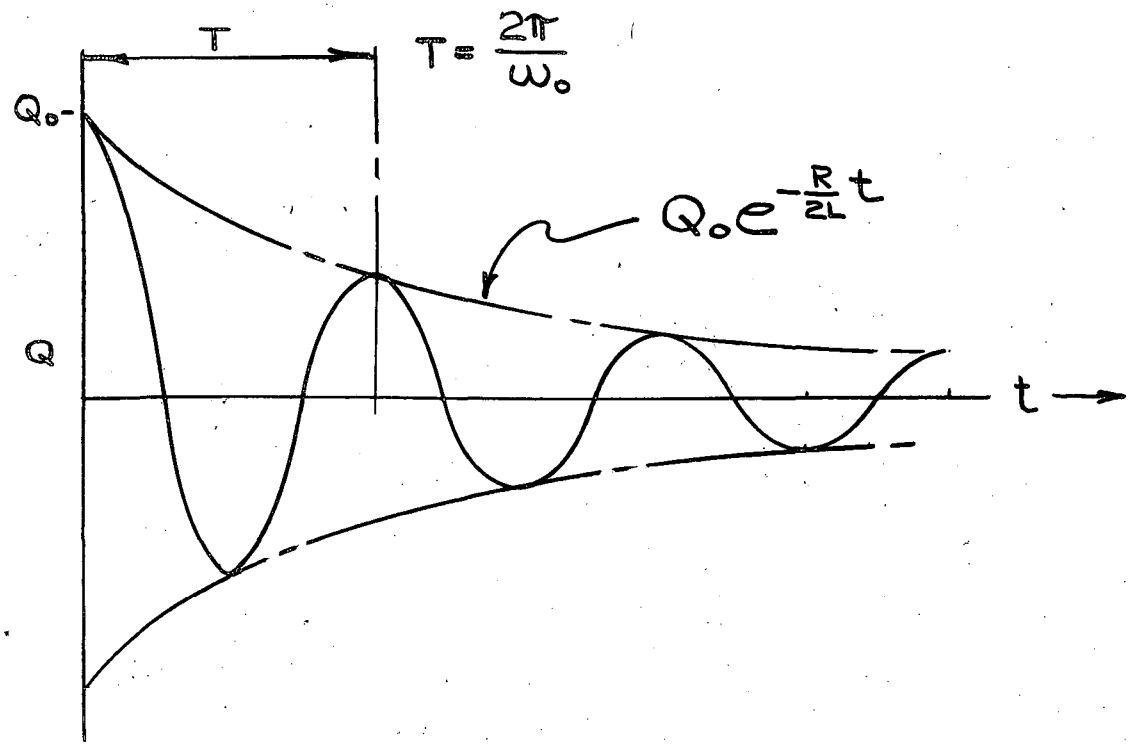
$$a \ggg b$$

Then

$$\alpha = \sqrt{\frac{1}{LC}} = \omega_o = a$$

$$\frac{b}{\alpha} = \frac{b}{a} = 0$$

Equation

$$Q = Q_o e^{-\frac{R}{2L}t} (\cos\omega_o t + 0 \sin\omega_o t)$$

$$Q = Q_o \cos\omega_o t (e^{-\frac{R}{2L}t})$$