UNIVERSITY OF CALIFORNIA
Radiation Laboratory
Berkeley, California

UCRL 2029

ELECTRICAL ENGINEERING REVIEW COURSE

LECTURE XIV
June 16, 1952
E. Martinelli

(Notes by: A. Bermingham, W. Salsig, Jr.)

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TRANSMISSION LINES

Parallel transmission lines will be discussed, but the principals outlined below are applicable to other forms.

Parallel transmission lines have low losses so are generally considered lossless.

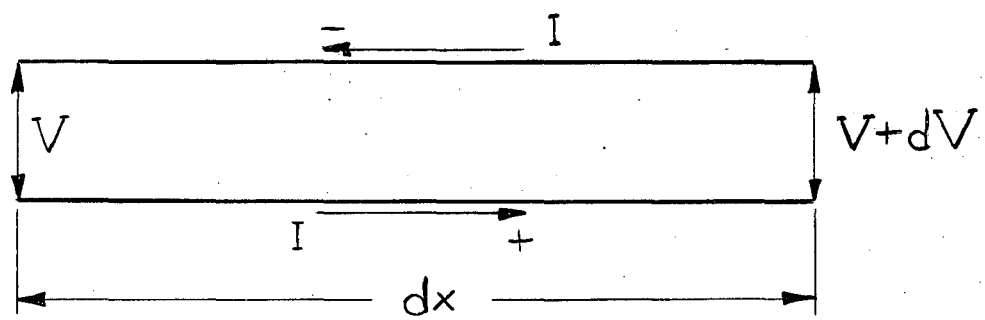

FIG. 1

The change in voltage in length of line dx, Fig. 1, is due to the inductance of the line. The inductance per unit length times the rate of change of current equals the change in voltage:

$$- L \frac{di}{dt} = V \tag{1}$$

The voltage driving the current I across the short length of line dx is dV. The rate of change of voltage down the line is:

$$\frac{\partial V}{\partial x} = -L \frac{\partial I}{\partial t} \tag{2}$$

Part of the current I goes to charge up C, the capacity per unit length of the line:

$$\frac{1}{C} \frac{\partial I}{\partial x} = - \frac{\partial V}{\partial t} \tag{3}$$

Therefore, there will be less current at the load end of the line by the amount of charging done on the line.

The two simultaneous partial differential equations above for I and V on transmission lines are called Telegraphers Equations. The parameters L and C in the usual case are constant. For example, a coaxial line is of constant diameter and

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the two conductors are parallel with uniform spacing.

If current and voltage vary sinusoidally with time, it is possible to add up all the steps of the step voltage and current to get the desired result as follows:

$$V = V_o(x)e^{jwt} \tag{4}$$

where $V_o(x)$ is the maximum voltage at the point x, e equals the base of hyperbolic logarithms, j is the square root of -1, w is the generating frequency, radians per second, and t is time.

$$I = I_o(x)e^{jwt} \tag{5}$$

where $I_o(x)$ is the current at point x equals zero. V_o and I_o are interdependent and related as in equations (2) and (3).

The solution of four equations (2) to (5) is:

$$\frac{dV_o}{dx} = jwLI_o \tag{6}$$

$$\frac{di}{dx} = - jwCV_o \tag{7}$$

Taking the second derivative of equations (6),

$$\frac{d^2V_o}{dx^2} = - jwL \frac{di_o}{dx}$$

$$\frac{di_o}{dx} = \frac{d^2V_o}{dx^2} \Big/ - jwL$$

and substituting this into equation (7),

$$\frac{d^2V_o}{dx^2} \Big/ - jwL = - jwCV_o$$

$$\frac{d^2V_o}{dx^2} + w^2LCV_o = 0 \tag{8}$$

we get an equation of simple harmonic motion. The solution to this equation is:

$$V_o = e^{\pm jw\sqrt{LC}\,x} \tag{9}$$

For the general case, the voltage on a transmission line may be represented by:

$$V_o = A\, e^{j(wt \pm w\sqrt{LC}\,x)} \tag{10}$$

which is a sinusoidal wave, the direction of which is determined by the plus or minus sign. This may also be written as:

$$V_o = A \cos(wt - \frac{x}{c}) \tag{11}$$

where c is the velocity of the wave:

$$c = \frac{1}{\sqrt{LC}} \tag{12}$$

For ordinary transmission lines, i.e., air filled parallel lines or coaxial cable, c is very nearly the velocity of light. L (electromagnetic units) and C (electrostatic units) are geometric factors involving the same parameters and all the parameters cancel out leaving only the velocity of light. Since c equals the velocity of light except for some losses dependent primarily upon the speed of light in the dielectric material used, then these formulas can be used as a means of measuring the dielectric constant.

For a transmission line which is infinitely long, V/I is its impedance Z. In order to find Z, we note that the derivative of equation (10) is:

$$\frac{dV_o}{dx} = \pm jw\sqrt{LC}\, V_o \tag{13}$$

Substituting equation (13) into (6), we get:

$$\pm jw\sqrt{LC}\, V_o = -jwLI_o$$

$$\frac{V_o}{I_o} = \frac{-jwL}{\pm jw\sqrt{LC}} = \pm\sqrt{\frac{L}{C}} = Z_o \tag{14}$$

which is the characteristic impedance of an infinitely long transmission line. For a coaxial line,

$$Z_o = 138 \log_{10} \frac{r_o}{r_i} \tag{15}$$

If r_o, the outer conductor radius if 7/8" and r_i, the inner conductor radius is 3/8", then

$$Z_o = 138 \log_{10} 2.33 = 50.2 \text{ ohms}$$

The impedance of ordinary air dielectric coaxial lines is usually 50 to 70 ohms and not more than 80 to 100 ohms. If the line is filled with a dielectric material, then losses can be as high as 125 to 200 ohms. Television transmission lines, using parallel conductors, can have a characteristic impedance of as much as 1000 ohms.

For a finite length of transmission line, the impedance of a wave moving to the right is by convention:

$$+ Z_o = \frac{V}{I} \tag{16}$$

and for a wave moving to the left is:

$$- Z_o = \frac{V}{I} \tag{17}$$

The plus or minus sign come from equation (12). This is interpreted to mean that

V and I are in phase for waves traveling to the right and out of phase for waves traveling to the left.

Suppose a current is flowing to the right along a line to a load. At the load, part of this current will be reflected to the left. By convention, these conditions can be represented by:

$$\frac{V_+}{I_+} = Z_o \tag{18}$$

$$\frac{V_-}{I_-} = -Z_o \tag{19}$$

$$\frac{V_L}{I_L} = Z_L \tag{20}$$

At the load, the sum of V_+ and V_- must be V_L, the voltage across the load. Therefore:

$$V_+ + V_- = V_L \tag{21}$$

and by the same reasoning,

$$I_+ + I_- = I_L \tag{22}$$

From equation (18) to (22) we get:

$$\frac{V_+}{Z_o} - \frac{V_-}{Z_o} = \frac{V_L}{Z_L} \tag{23}$$

Thus there will be a resultant standing wave pattern on the transmission line.

The reflection coefficient is $\frac{V_-}{V_+} = \frac{\text{amplitude of reflected wave}}{\text{amplitude of incident wave}}$ (24)

Using equation (21) and (23)

$$\frac{V_-}{V_+} = \frac{Z_L - Z_o}{Z_L + Z_o} = \frac{K - 1}{K + 1} \tag{25}$$

where

$$K = \frac{Z_L}{Z_o} = \text{S.W.R.} = \text{standing wave ratio} \tag{26}$$

If the voltage along a line be considered as a function of the distance, then:

$$V(x) = \left[V_+ e^{-j\omega x/c}\right]\left[1 + \frac{V_-}{V_+} e^{2j\omega x/c}\right] \tag{27}$$

where the voltage as a function of x, V(x), is a rotary vector in the Vx plane. V_+ and V_- are likewise such vectors and the 2 in the exponent of e indicates that the V_- vector rotates twice as fast as the V_+ vector.

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Equation (27) can be expressed as:

$$V(x) = V_+ e^{-j\omega x/c} + V_- e^{j\omega x/c}$$

and represented as in Fig. 2

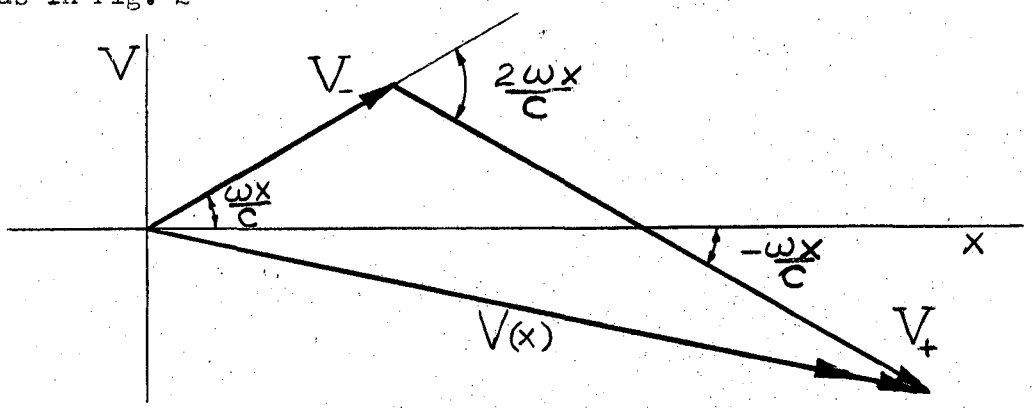
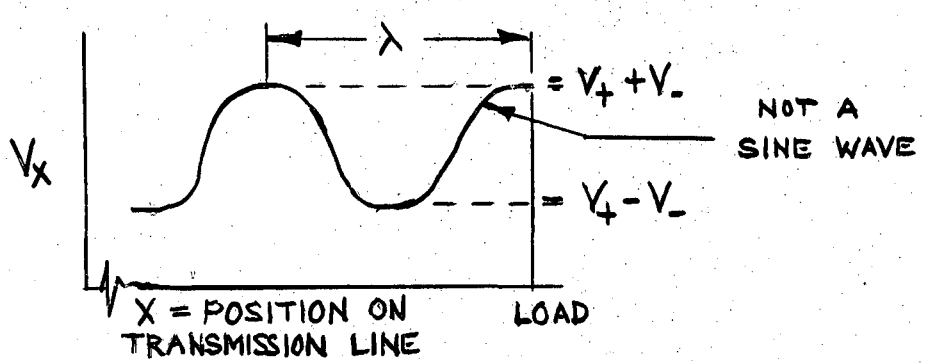


FIG. 2

Note that at θ - 180°, $V(x) = V_+ - V_-$. Thus, the magnitude of the voltage down a transmission line is not a simple function.

Suppose the loac impedance, Z_L, is a pure resistance. Then V_- and V_+ are in phase at the load:

FIG 3



$$\text{Standing Wave Ratio} = \frac{1 + \frac{K-1}{K+1}}{1 - \frac{K-1}{K+1}} = K = \frac{V_+ + V_-}{V_+ - V_1} \tag{28}$$

One can have high standing wave ratios (2 or 3 to 1) with small power loss. Since power is proportional to the square of the voltage, for $V_-/V_+ = 1/3$, (Standing Wave Ratio of 2) one loses only $(1/3)^2 = 1/9$ of the power at a reflection.

Determine the impedance, Z_x, at any point x on a transmission line. From equation (27), remembering that $\omega/c = 2\pi/\lambda$:

$$V_x = V_+ e^{-2\frac{\pi X}{\lambda}} \left[1 + \frac{V_-}{V_+} e^{2j(2\pi \frac{X}{\lambda})} \right] \tag{29}$$

Differentiate with respect to X, equate to:

$$\frac{\partial V}{\partial X} = -j\omega LI, \text{ (Equation 6)}$$

$$\frac{V_x}{I_x} = Z_x = \frac{Z_o\left[1 + \frac{V_-}{V_+}e^{2j(\frac{2\pi X}{\lambda})}\right]}{\left[1 - \frac{V_-}{V_+}e^{2j(\frac{2\pi X}{\lambda})}\right]} \tag{30}$$

Two important cases are:

(1) When $\frac{2\pi X}{\lambda} = \frac{\pi}{2}$ (i.e., $x = \frac{\lambda}{4}$, A Quarter Wave Line)

$$2j(\frac{2\pi X}{\lambda}) = -1$$

$$\text{And } Z\frac{\lambda}{4} = Z_o\frac{(1 - \frac{V_-}{V_+})}{(1 + \frac{V_-}{V_+})} = \frac{Z_o}{K} \tag{31}$$

(2) And when $\frac{2\pi X}{\lambda} = \pi$ ($X = \frac{\lambda}{2}$, and Half Wave Line)

$$Z\frac{\lambda}{2} = Z_oK \tag{32}$$

It is apparent that the impedance varies as one moves along a transmission line. It is thus important where one ties in a generator or load. One can consider a transmission line as a transformer, the voltage transformation being a function of the distance from a minimum voltage point. The transmission line acts like a pure resistance at the minimum voltage position. One can determine the wave position on a line by finding the minimum voltage point.

Consider a transmission line with shorted end:

$$Z_x = Z_o\frac{(1 + \frac{V_-}{V_+}e^{2j(\frac{2\pi X}{\lambda})})}{(1 - \frac{V_-}{V_+}e^{2j(\frac{2\pi X}{\lambda})})} \tag{33}$$

For shorted end $V_x - 0$, and Z_L must therefore = 0; I(Z_o = real number)

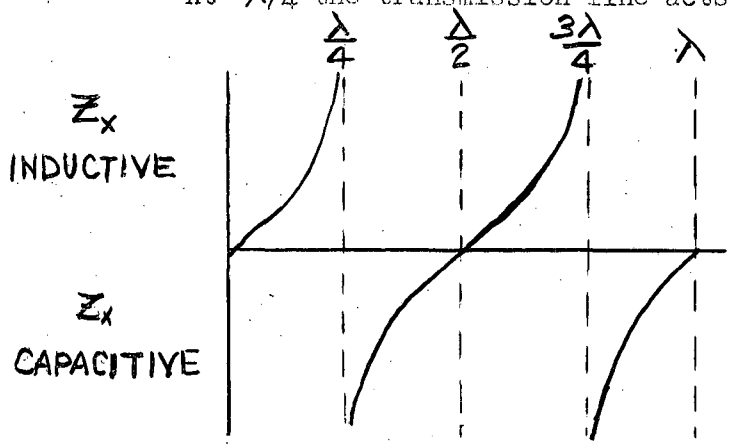
$$\frac{V_-}{V_+} = \frac{Z_L - Z_o}{Z_L + Z_o} = -1 \tag{34}$$

Substituting into expression for Z_x:

$$Z_x = Z_o\left(\frac{1 - e^{2j(\frac{2\pi X}{\lambda})}}{1 + e^{2j(\frac{2\pi X}{\lambda})}}\right) = jZ_o \operatorname{Tan}(\frac{2\pi X}{\lambda}) \tag{35}$$

When X $\lambda/4$ Z_x is inductne

At $\lambda/4$ the transmission line acts like a transformer of infinite turns ratio



RESONANCES
At $\lambda/4$, $3\lambda/4$, $5\lambda/4$ . . .

FIG 4

For a transmission line with open end:

$$\frac{V_-}{V_+} = +1$$

$$Z_x = jZ_o \cot\left(\frac{2\pi x}{\lambda}\right) \tag{36}$$

Although one normally uses these equation only when working on apparatus with frequencies of at least kilocycles, they are also true for 60 cycle circuits. At 60 cycles λ is the order of 3,000 miles. On long power transmission lines, such as the Boulder Dam installation, the affect becomes noticable and the voltage and current do not strictly obey Ohm's law.

At high frequencies, the center conductor of a transmission line can be supported on quarter wave stub lines instead of insulators.

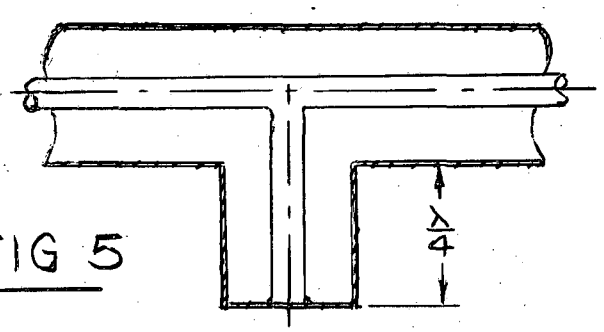
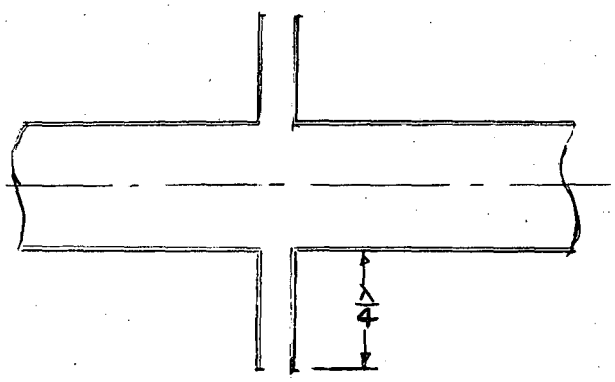


FIG 5

System can be made to work over a relatively broad band, e.g., 9-11 mc.

The stub can be made shorter (less than $\lambda/4$) by filling it with a dielectric material. The foreshortening is proportional to the square root of the dielectric constant.

A "choke joint", which will pass high frequency currents but is an open circuit for low frequency power, is made by using an open ended stub line:



Choke Joint

FIG 6