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UCRL LECTURES ON NUMERICAL ANALYSIS AND APPLIED MATHEMATICS

Lecture I

Glen Culler

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Berkeley, California

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INTERPOLATION

1. Introduction

The functions arising in applied mathematics fall into two categories--functions of a continuous variable and functions of a discontinuous variable. The subject matter of "The Calculus of Finite Differences" and "Interpolation" establishes criterions and methods for choosing functions in one category to represent given functions in the other. The purpose of this presentation is to give an elementary treatment of differences, difference tables, the construction of polynomial tables, interpolation formulas and interpolating polynomials, hoping in this way to present material useful to the computer.

2. Differences and Difference Tables

The most common differences used in interpolation formulas are the Forward, Central, and Backward Differences. The usual definitions of these are:

Forward Difference: $\Delta_h f(x) = f(x + h) - f(x)$

Central Difference: $\delta_h f(x) = f(x + h/2) - f(x - h/2)$

Backward Difference: $\nabla_h f(x) = f(x) - f(x - h)$.

The relationship of these three is evident from the construction of a difference table. Let a tabulated function be given; for example,

x	$f(x)$
x_o	$f(x_o)$
x_1	$f(x_1)$
x_2	$f(x_2)$
x_3	$f(x_3)$.

Form a new column consisting of the numbers $f(x_1) - f(x_o)$, $f(x_2) - f(x_1)$, ... , $f(x_{n+1}) - f(x_n)$, ... Repeat this process to obtain higher differences.

Example:

x	f(x)	1st Diff.	2nd Diff.	3rd Diff.	4th Diff.	5th Diff.
-2	7					
		-2				
-1	5		3			
		1		-1		
0	6		2		-2	
		3		-3		-4
1	9		-1		-6	
		2		-9		
2	11		-10			
		-8				
3	3					

Replacing the numbers in this table by their names according to the previous definitions, we get a table illustrating the relationship the relationship of these differences.

x	f(x)	1st Diff.			2nd Diff.		
-2	7=f(-2)						
		f(-2)	f(-3/2)	f(-1)			
-1	5=f(-1)				$^2f(-2)$	$^2f(-1)$	$^2f(0)$
		f(-1)	f(-1/2)	f(0)			
0	f(0)				$^2f(-1)$	$^2f(0)$	$^2f(1)$
		f(0)	f(1/2)	f(1)			
1	f(1)				$^2f(0)$	$^2f(1)$	$^2f(2)$
		f(1)	f(3/2)	f(2)			
2	f(2)						

An alternative name for the forward difference is "descending difference" since $\Delta^n f(k)$ when n = 1, 2, ... descends diagonally. Similarly, the backward difference is sometimes called an "ascending difference".

3. Factorial Polynomials and Stirling's Numbers

To illustrate the uses of differences factorial polynomials and Stirling's numbers will be introduced. Using these, there are simple ways of tabulating polynomials and of finding polynomials which approximate a tabulated function.

A. Factorial Polynomials:

1. Descending factorials: $x^{(n)} = x(x-1)(x-2)(x-3)(\ldots)(x-n+1)$
2. Central factorials: $x^{[n]} = x(x+\frac{n}{2}-1)(x+\frac{n}{2}-2)(\ldots)(x+\frac{n}{2}-(n-1))$

   This may be equivalently written $x^{(n)} = x(x+\frac{n}{2}-1)^{n-1}$.
3. Ascending factorials: $x^{(-n)} = x(x+1)(x+2)(\ldots)(x+n-1)$.

When applying factorial polynomials to functions tabulated with an interval h = a constant, it suffices to assume $h = 1$ and extend the results to arbitrary h by making use of $x^{(n)} = y_h^{(n)}/h^n$, where

$$y_h^{(n)} = y(y + h)(y + 2h)(\ldots)(y + (n - 1)h) \quad \text{and} \quad x = y/h \quad .$$

Lemma 1: $\dfrac{\Delta_h y_h^{(n)}}{h} = n\, y_h^{(n-1)}$.

This is, of course, analogous to the rule for the derivative of y^n.

Proof: (1)

$$\Delta_h x_h^{(n)} = (x + 1)(x)(x - 1)(\ldots)(x - n) - x(x - 1)(\ldots)(x - n + 1)$$

$$= x(x - 1)(x - 2)(\ldots)(x - n)\left[x + 1 - (x - n + 1)\right]$$

$$= n\, x_h^{(n-1)} \quad , \quad \text{where } h = 1.$$

For $h \neq 1$, (2)

$$\Delta_h \frac{y_h^{(n)}}{h^n} = n \frac{y_h^{(n-1)}}{h^{n-1}}$$

Hence,

$$\frac{\Delta_h y_n^{(n)}}{h} = n\, y_h^{(n-1)}$$

B. Stirling's Numbers:

When the polynomial $x^{(n)}$ is written as a polynomial in x with descending exponents, the coefficients are Stirling's numbers of order n.

$$x^{(n)} = S_o^n x^n + S_1^n x^{n-1} + \ldots + S_{n-1}^n x \quad ,$$

Definition: S^{n}_{i-1} is the i'th Stirling number of order n.

These numbers can be easily computed from the recursion formula

$$S^{n+1}_{i} = S^{n}_{i} - n\, S^{n}_{i-1} \quad .$$

Lemma 2: Every polynomial with real coefficients has an unique representation as a sum of factorial polynomials with decreasing orders.

Proof: Let $f(x) = a_o x^n + a_1 x^{n-1} + \ldots + a_n$ be a given polynomial with a_i real and $a_o \neq 0$. Suppose $f(x) = c_o x^{(n)} + c_1 x^{(n-1)} + \ldots + c_n$ and determine c_i $(i = 1, \ldots, n)$ by expanding the factorials according to the definition of Stirling's numbers. Equating a_i to the coefficient of x^{n-i} in this expansion, one gets:

$$c_o = \frac{a_o}{S^{n}_{o}} \; ; \quad c_1 = a_1 - a_o S^{n}_{1} \; ;$$

$$c_2 = a_2 - a_1 S^{n-1}_{1} + a_o(S^{n-1}_{1} S^{n}_{1} - S^{n}_{2}) \; ; \quad \ldots \; ; \text{ etc.}$$

By definition $S^{n}_{o} = 1$ for every n, so $c_o = a_o$ and all other c_i are uniquely determined by these equations.

<u>Theorem</u>: The n'th difference of an n'th degree polynomial is constant.

Proof: Let $f(x) = a_o x^n + a_1 x^{n-1} + \ldots + a_n$. By lemma 2 this can be rewritten $f(x) = c_o x^{(n)} + c_1 x^{(n-1)} + \ldots + c_n$ and lemma 1 is now applicable. Hence,

$$\Delta_1 f(x) = n\, c_o x^{(n-1)} + (n-1)\, c_1 x^{(n-2)} + \ldots + c_{n-2} \; ;$$

$$\Delta_1^2 f(x) = n(n-1)\, c_o x^{(n-2)} + (n-1)(n-2) c_1 x^{(n-3)} + \ldots + c_{n-2}$$

$$\overline{\cdots} \qquad \overline{\cdots} \qquad \overline{\cdots} \qquad \overline{\cdots}$$

$$\Delta_1^n f(x) = n!\; c_o = \text{a constant as desired.}$$

Equations for c_i $(i = 0, \ldots, n)$ are obtained by setting $x = 0$ in the previous system.

$$c_o = \frac{\Delta_1^n f(0)}{n!}$$

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$$c_i = \frac{\Delta_1^{n-i} f(0)}{(n-i)!}$$

Hence $$f(x) = \frac{\Delta_1^n f(0)}{n!} x^{(n)} + \frac{\Delta_1^{n-1} f(0)}{(n-1)!} x^{(n-1)} + \ldots\ldots + f(0) .$$

Using the theorem it is only necessary to compute the first values of a polynomial function since all other values can be found from the differences.

Example: Tabulation of $e^2 x^3 + \pi^2 x^2 + 1 = f(x)$.

x	$f(x)$	$\Delta_1 f(x)$	$\Delta_1^2 f(x)$	$\Delta_1^3 f(x)$
0	1			
		$e^2 + \pi^2$		
1	$e^2 + \pi^2 + 1$		$6e^2 + 2\pi^2$	
		$7e^2 + 3\pi^2$		$6e^2$
2	$8e^2 + 4\pi^2 + 1$		$12e^2 + 2\pi^2$	
		$19e^2 + 5\pi^2$		$6e^2$
3	$27e^2 + 9\pi^2 + 1$		$18e^2 + 2\pi^2$	
		$37e^2 + 7\pi^2$		$6e^2$
4	$64e^3 + 16\pi^2 + 1$		$24e^2 + 2\pi^2$	
		$61e^2 + 9\pi^2$		$6e^2$
5	etc.		$30e^2 + 2\pi^2$	
		etc.		etc.
6			etc.	

Conversely, if the tabulation of an n'th degree polynomial is given, then the coefficients can be determined from the tabulation. Let $f(x) = a_o x^n + a_1 x^{n-1} + \ldots + a_n$, then

$$a_i = \sum_{k=0}^{i} S_k^{n-i+k} \frac{\Delta^{n-i+k}}{(n-i\ k)!} f(0) .$$

This formula is derived by equating the coefficients of like powers in

$$a_o x^n + a_1 x^{n-1} + \ldots + a_n = f(x) = \frac{\Delta_1^n}{n!} f(0)\, x^{(n)} + \ldots + \Delta_1^1 f(0)\, x^{(1)} + f(0) .$$

Hence, if some function has uniformly small n'th differences, one can assume the n - 1 difference to be constant and deduce an n-1 degree polynomial (from the formula above) as an approximation. This method sometimes yields polynomials of lower degree than the Lagrange polynomial, but which fit the given function uniformly as well.

4. Interpolation Formulas

Derivations:

Newton's Divided Differences

Let $f(a_o)$, $f(a_1)$, ... $f(a_n)$ be known function values.

Definition:

1. $f(x, a_o) = \dfrac{f(x) - f(a_o)}{x - a_o}$

2. $f(x, a_o, a_1) = \dfrac{f(x, a_o) - f(a_o, a_1)}{x - a_1}$

... ...

n+1. $f(x, a_o, a_1, \ldots a_n) = \dfrac{f(x, a_o, a_1, \ldots a_{n-1}) - f(a_o, \ldots, a_n}{x - a_n}$

These may be rewritten as

1. $f(x) = f(a_o) + (x - a_o)f(x, a_o)$

2. $f(x, a_o) = f(a_o, a_1) + (x - a_1) f(x, a_o, a_1)$

... ...

n + 1. $f(x, a_o, \ldots, a_{n-1}) = f(a_o, \ldots, a_n) + (x - a_n)f(x, a_o \ldots, a_n)$

Combining these equations one gets a formula for f(x),

$$f(x) = f(a_o) + (x - a_o)f(a_o, a_1) + (x - a_o)(x - a_1)f(a_o, a_1, a_2) + \ldots$$
$$+ (x - a_o)(\ldots)(x - a_{n-1})f(a_o, a_1, a_2, \ldots, a_n)$$
$$+ (x - a_o)(\ldots)(x - a_n) f(x, a_o, \ldots, a_n) . \qquad (1)$$

Let $R(x) = (x - a_o)(\ldots)(x - a_n) f(x, a_o, \ldots a_n)$. Then

$f(x) = F(x) + R(x)$ where $F(x)$ is the n'th degree polynomial (2) formed by the first n + 1 terms above. Particular choices for $a_o, \ldots a_n$, transform F(x) into Newton's, Stirling's, Gauss's and Everett's interpolation formulas and R(x) becomes respectively their error terms.

When f(x) has continuous derivatives of orders 1, 2, ..., n + 1 the error term can be written more conveniently. Let

$D(z) = f(z) - F(z) - (z - a_o)(\ldots)(z - a_n) f(x, a_o, \ldots, a_n)$ where x is some fixed number ($x \neq a_i$) at which we wish to evaluate R(x). At $z = a_i$ $(i = 0, \ldots, n)$; $D(a_i) = f(a_i) - F(a_i)$; but from (2), $f(a_i) = F(a_i)$ so $D(a_i) = 0$ $(i=0,\ldots,n)$. Consider $z = x$,

$D(x) = f(x) - F(x) - R(x)$ but from (2) $f(x) = F(x) + R(x)$ hence $D(x) = 0$. Since $a_o < x < a_n$, $D(z)$ vanishes at $n + 2$ points in the interval $a_o \leq z \leq a_n$. By Rolle's theorem, the first derivative must then vanish at $n + 1$ points in this interval, the second derivatives vanishes at n points, etc. This proves that the $n+1$ derivative of $D(z)$ must vanish at (at least) one point say ξ, such that $a_o \leq \xi \leq a_n$.

$$D^{(m+1)}(z) = f^{(m+1)}(z) - (m+1)!\, f(x, a_o \ldots, a_n)$$

Therefore, $D^{m+1}(\xi) = 0 = f^{(m+1)}(\xi) - (m+1)!\, f(x, a_o, \ldots a_n)$

$$f(x, a_o, \ldots a_n) = \frac{f^{(m+1)}(\xi)}{(m+1)!} .$$

Hence $R(x)$ may be written

$$R(x) = \frac{(x - a_o)(\ldots)(x - a_n)\, f^{(m+1)}(\xi)}{(m+1)!} \qquad \text{for some } \xi \ (\xi \text{ depending on } x)$$

$a_o \leq \xi \leq a_n$.

Substituting this in (1)

$$f(x) = f(a_o) + (x - a_o)\, f(a_o, a_1) + \ldots + (x - a_o)(\ldots)(x - a_{n-1})\, f(a_o, \ldots a_n)$$

$$+ \frac{(x - a_o)(\ldots)(x - a_n)\, f^{n+1}(\xi)}{(n+1)!}$$

This is Newton's interpolation formula with divided differences.

Formulas:

1. Newton's Formula.

Put $a_i = a_o + ih$, $i = 0, 1, 2, \ldots n.$

$x = a_o + mh$, m is not necessarily an integer.

Then

$$f(a_o, a_1) = \frac{f(a_o) - f(a_1)}{a_o - a_1} = \frac{\Delta_h f(a_o)}{h}$$

$$f(a_o, a_1, a_2) = \frac{f(a_o, a_1) - f(a_1, a_2)}{a_o - a_2} = \frac{\Delta_h f(a_o) - \Delta_h f(a_1)}{-2h^2}$$

$$= \frac{\Delta^2 f(a_o)}{2h^2}$$

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$$f(a_o, a_1, \ldots, a_j) = \frac{\Delta^j f(a_o)}{j!\, h^j}$$

and

$$x - a_o = mh$$

$$x - a_j = a_o + mh - a_j = (m - j)h$$

thus,

$$f(x) = f(a_o) + m\Delta f(a_o) + \frac{m(m-1)}{2!}\Delta^2 f(a_o) + \ldots +$$

$$+ \frac{m(m-1)(\ldots)(m-n+1)}{n!}\Delta^n f(a_o)$$

$$+ \frac{m(m-1)(\ldots)(m-n)}{(n+1)!} f^{n+1}(\xi)$$

2. Gauss's formula of even order.

Put
$$\left.\begin{aligned} a_{2\nu} &= \nu \\ a_{2\nu-1} &= -\nu \\ \frac{1}{(-1)!} &= 0 \end{aligned}\right\} \text{ and stop at order } 2k.$$

$$f(x) = \sum_{0}^{k} \frac{x^{[2\nu]-1}}{(2\ \ -1)!} \Delta^{2\nu-1} f(-\nu) \quad \frac{(x+\nu)\, x^{[2\nu]-1}}{(2\ \ )!} \Delta^{2\nu} f(-\nu) + R$$

where $R = \dfrac{x^{[2k+2]-1}}{(2k+1)!} f^{(2k+1)}(\xi)$.

For the formula of odd order use the same substitution and stop at order $2k-1$.

3. Stirling's Formula.

Let $a_{2\nu} = -\nu$

$a_{2\nu-1} = \nu$. Stop at order $2k$.

Let $\square f(x) = \dfrac{f(x+\frac{1}{2}) + f(x-\frac{1}{2})}{2}$.

Taking the mean of the formula resulting from this substitution and Gauss's formula of even order gives Stirling's formula:

$$f(x) = \sum_{0}^{k} \left[\frac{x^{[2\nu]-1}}{(2\nu-1)!} \square \delta^{2\nu-1} f(0) + \frac{x^{[2\nu]}}{2\nu!} \delta^{2\nu} f(0) \right] + R$$

$$R = \frac{x^{[2k+2]-1}}{(2k+1)!} f^{(2k+1)}(\xi)$$

Aitken's Process:

Let $x_o \quad y_o$

$x_1 \quad y_1$

$x_2 \quad y_2$ be a tabulated function. Denote the desired $y(x)$ by $P_{o,1,\ldots,n}(x)$, where the subscripts designate the points used and the number of indices minus one is the degree of the interpolating polynomial.

Steps:

1. $$y(x) = P_{o,1}(x) = \frac{1}{x_1 - x_o} \begin{vmatrix} y_o & x_o - x \\ y_1 & x_1 - x \end{vmatrix},$$

2. $$y(x) = P_{o,2}(x) = \frac{1}{x_2 - x_o} \begin{vmatrix} y_o & x_o - x \\ y_2 & x_1 - x \end{vmatrix},$$

3. $$y(x) = P_{o,1,2}(x) = \frac{1}{x_2 - x_1} \begin{vmatrix} P_{o,1}(x) & x_1 - x \\ P_{o,2}(x) & x_2 - x \end{vmatrix},$$

etc., using $$y(x) = P_{o,1,2,3,\ldots,i} = \frac{1}{x_i - x_{i-1}} \begin{vmatrix} P_{o,1\ldots i-2,i-1} & x_{i-1} - x \\ P_{o,1\ldots i-2,\, i} & x_i - x \end{vmatrix}$$

This process is especially useful for two reasons: determinants of order two are easily calculated on a computing machine and after each operation a new approximation to the desired value is obtained; inverse interpolation is affected by just reversing the roles of x and y in the formulas.

5. Interpolation Polynomials.

Lagrange Interpolation Polynomials.

Suppose the following $n + 1$ points are given,

$(x_o, f(x_o))$; $(x_1, f(x_1))$; ... ; $(x_n, f(x_n))$. We wish to find a polynomial $P(x)$ such that $P(x_i) = f(x_i)$, $(i=0,1,\ldots,n)$. Denote

$(x - x_i)(x - x_j)(\ldots)(x - x_k)$ by $\prod_{i,j\ldots k}(x)$.

If h is one of the integers $i, j, \ldots, k$, then $\prod_{i,j\ldots,k}(x_h) = 0$;

if h is not one of $i, j,\ldots,k$, the $\prod_{i,j\ldots,k}(x_h) \neq 0$.

Let

$$P(x) = \frac{\prod_{1,2,3\ldots n}(x)}{\prod_{1,2,3\ldots n}(x_o)} f(x_o) + \frac{\prod_{2,3\ldots,n,o}(x)}{\prod_{2,3,\ldots,n}(x_1)} f(x_1) + \ldots + \frac{\prod_{o,1,2\ldots n-1}(x)}{\prod_{o,1,2\ldots n-1}(x_n)} f(x_n).$$

Then

$P(x_i) = f(x_i)$, $(i=0,1,\ldots,n)$, as desired. Such a $P(x)$ is called the Lagrange interpolating polynomial.

Theorem:

There exists one and only one polynomial $P(x)$ of n'th degree such that $P(x_i) = f(x_i)$ for $i = 0, 1, \ldots, n$.

Proof:

By the fundamental of algebra every n'th degree polynomial has exactly n roots. Let $P_1(x)$ and $P_2(x)$ be two polynomials of n'th degree, both satisfying the conditions above. The polynomial $Q(x) = P_1(x) - P_2(x)$ is at most n'th degree, but $P_1(x) = P_2(x)$ for $n + 1$ numbers $x_o, x_1, \ldots, x_n$.

Hence $Q(x)$ has more than n roots. This contradicts the first statement unless $P_1(x) = P_2(x)$ for all x.

Tables of Lagrange interpolation coefficients are published and can be used to compute values of $P(x)$ for approximation purposes.

Unfortunately, putting higher degree Lagrange polynomials through more points of a given function does not assure that the approximation is uniformly improved. A counter-example is worked out in Steffensen's "Interpolation" showing that an n'th degree polynomial, which passes through $n + 1$ evenly distributed points of $\frac{1}{1+x^2}$, (in the interval $-5 \leq x \leq 5$) has an error term greater than $\frac{(1.8)^n}{451}$ at some point in the interval. However, Weierstrass's theorem for polynomial approximation establishes that:

If $f(x)$ is a continuous function throughout any finite interval (a, b), then there exists a sequence of polynomials $P_n(x)$ which converge uniformly to $f(x)$ in (a, b).

6. Bibliography

1.1 Jordan, "Calculus of Finite Differences".

1.2 Milne-Thomson, "Calculus of Finite Differences".

1.3 Steffenson, "Interpolation".

1.4 Milne, "Numerical Calculus".