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Jonathan D. Young

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LEAST EIGENVALUE APPROXIMATION FOR A PROPER STURM-LIOUVILLE SYSTEM BY USE OF CUBIC SPLINES

JONATHAN D. YOUNG*

Abstract. The method of Stodola and Viannello for approximating the least eigenvalue for a proper Sturm-Liouville system is numerically implemented by using cubic splines as successive approximations for a corresponding eigenfunction.

1. Introduction. We consider a differential equation

(1) $$py'' + p'y' + (q + r\lambda)y = 0$$

subject to boundary conditions

(2) $$y(0) = 0 \qquad y(1) = 0,$$

where p, p', q and r are continuous on $[0, 1]$, with p and r strictly positive and q nonpositive thereon. The values of λ for which there are nontrivial solutions are real, distinct, and positive [1], [2]. Hence, there is a least eigenvalue $\underline{\lambda}$ and a corresponding nontrivial solution, called an eigenfunction, unique except for a multiplicative constant.

Transposing the term involving λ in (1) we obtain

(3) $$py'' + p'y' + qy = -\lambda ry.$$

Multiplication by y and integration yields

(4) $$\int_0^1 [pyy'' + p'yy' + qy^2]dx = -\lambda \int_0^1 ry^2 dx$$

*Lawrence Radiation Laboratory, University of California, Berkeley, California 94720.

or

(5) $$\lambda = -\int_0^1 [pyy'' + p'yy' + qy^2]dx / \int_0^1 ry^2 dx$$

for any nontrivial solution y corresponding to λ.

For any function z having piecewise continuous second derivatives on $[0, 1]$ and such that $z(0) = 0$ and $z(1) = 0$ we define

(6) $$R(z) \equiv -\int_0^1 [pzz'' + p'zz' + qz^2]dx / \int_0^1 rz^2 dx.$$

We plan to compute consecutive cubic splines s_k defined on $[0, 1]$ so that

(7) $$\lim_{k \to \infty} R(s_k)$$

exists, and such that for sufficiently large k, we may accept $R(s_k)$ and s_k, respectively, as approximations to $\underline{\lambda}$ and an eigenfunction therefor.

2. Stodola-Vianello method. The Stodola-Vianello method is a process with successive approximations z_k converging, except in very unlikely circumstances, to an eigenfunction corresponding to the least eigenvalue.

As a first guess, we may take any function z_1 having piecewise continuous second derivatives on $[0, 1]$ and vanishing at 0 and 1, then compute $R(z_1)$ by (6).

To obtain an approximation z_k from z_{k-1} we solve

(8) $$pz_k'' + p'z_k' + qz_k = -rz_{k-1}$$

subject to

$$z_k(0) = 0 \qquad z_k(1) = 0,$$

and we then compute $R(z_k)$.

As the process converges [2] we have for sufficiently large k, say k = n, approximations

$$R(z_n) \quad \text{for } \underline{\lambda}$$

and z_n for an eigenfunction corresponding thereto.

Except in the most simple examples, analytic solution of (8) and integration in (6) become impractical and it is necessary to use numerical methods. We elect to use cubic splines for these processes.

3. Numerical solution by use of cubic splines. We shall parallel the process above replacing the approximations z_k by cubic splines s_k defined on [0, 1] with uniformly spaced interior knots therein. The linear space [4] of cubic splines having knots:

$$0 = x_1, x_2, \cdots, x_m = 1$$

and vanishing at 0 and 1 has dimension m. A convenient basis for this space consists of the cubic splines, t_j, $j = 1, \cdots m$ defined by

$$(9) \qquad \begin{array}{llll} t_1(x_i) = 0 & \text{all } i & t_1'(0) = 1 & t_1'(1) = 0 \\ t_j(x_i) = \delta_{ij} & & t_j'(0) = 0 & t_j'(1) = 0 \text{ for } j = 2, \cdots m-1 \\ t_m(x_i) = 0 & \text{all } i & t_m'(0) = 0 & t_m'(1) = -1 \end{array}$$

It may be readily verified that for any cubic spline s in the space, there is a unique set of coefficient α_j such that

$$(10) \qquad s = \Sigma\, \alpha_j t_j,$$

and further that very such linear combination of the t_j is in the space.

For our first approximation, we select an initial set α_{1j}. In the absence of other information $\alpha_{1j} = 1$ for $j = 1, \cdots m$ may be used.

(The choice $\alpha_{1j} = 0$ for all j must be avoided since it leads to a trivial solution.) With the choice of α_{1j} we have

$$(11) \qquad \left.\begin{array}{l} s_1(x_i) = \Sigma\, \alpha_{1j} t_j(x_i) \\ s_1'(x_i) = \Sigma\, \alpha_{1j} t_j'(x_i) \\ s_1''(x_i) = \Sigma\, \alpha_{1j} t_j''(x_i) \end{array}\right\} \; i = 1, \cdots m$$

from which we can compute $R(s_1)$ by (6), performing the integrations numerically. (Since the integrands involved are likely to be of at least sixth degree in x on each subinterval, we used seven interior points in each subinterval between knots with an open end Newton-Cotes quadrature formula [3] exact for sixth-degree polynimials.)

From an approximation s_{k-1}, we obtain the set α_{kj} by solving the linear system

$$(12) \qquad \sum_j [\{pt_j''(x_i) + p't_j(x_i) + qt_j(x_i)\}\, \alpha_{kj} = -\, rs_{k-1}(x_i)] \text{ for } i = 1, \cdots m.$$

Then, as before, we compute s_k, s_k', and s_k'' as in (11) and compute $R(s_k)$.

The process may be terminated when

$$(13) \qquad \text{(a)} \qquad |R(s_k) - R(s_{k-1})| < \epsilon \qquad \text{(specified)},$$

in which case $R(s_k)$ is accepted as an approximation to $\underline{\lambda}$ and s_k is accepted as an approximate eigenfunction; or when

$$(14) \qquad \text{(b)} \qquad |R(s_k) - R(s_{k-1})| > |R(s_{k-1}) - R(s_{k-2})|,$$

in which we conclude the process is not converging; or when

$$(15) \qquad \text{(c)} \qquad k \geq K \qquad \text{(specified)},$$

in which case we conclude convergence is impractically slow.

4. Error estimates. The limit of the cubic spline approximation process is that cubic spline, in the space considered, which best approximates an eigenfunction. Consequently even with acceptable convergence there is, in general, residual error.

Now if an accepted approximation s_n were actually an eigenfunction with $R(s_n)$ as the eigenvalue, then certainly we should have

$$e(x) = ps_n''(x) + p's_n'(x) + \{q + rR(s_n)\}s_n(x) = 0, \quad \text{for all } x \in [0, 1]. \tag{16}$$

Otherwise, the magnitude of the $e(x)$ indicate the extent of residual error. However, these magnitudes are affected by the multiplicative constant which appears in s_n and by the magnitude of $R(s_n)$, hence we define a relative measure of residual error on $[0, 1]$ by

$$\mu = \frac{\int_0^1 [ps_n'' + p's_n' + \{q + rR(s_n)\} s_n]^2 dx}{\int_0^1 [R(s_n)\ s_n]^2\ dx}. \tag{17}$$

Other measures of the residual error could be devised, but the one given above has the merit of simplicity and is strongly indicative of the validity of an approximation.

The numerical examples in the next section illustrate the process in more detail. A computer code, STROBS, has been written which performs the computation involved for any value of m from 3 up to 51. The user need only append a subroutine which computes p, p', q, and r for any given $x \in [0, 1]$. The functions must meet the requirements mentioned in the Introduction, and of course p' must be the derivative of p. A source listing of this code is available from the author.

5. Numerical examples.

Example 1. $\quad y'' + \lambda y = 0$

$$y(0) = 0 \qquad y(1) = 0$$

($p = 1$, $p' = 0$, $q = 0$, $r = 1$ meet requirements). We may obtain

analytically

$$\underline{\lambda} = \pi^2 = 9.8696044 \quad \text{and} \quad y = c \sin \pi x, \ c \neq 0.$$

Results for numerical approximation with $m = 3, 5, 11$, and 21 are given below. In all cases we began with $\alpha_{1j} = 1$, $j = 1, \cdots m$ and tested convergence with

$$\epsilon = 0.00000001.$$

All results were normalized so that they had for initial slope

$$s'(0) = 1$$

and compared with the normalized eigenfunction $f = \frac{1}{\pi} \sin \pi x$

For $m = 3$

x	s_3	ϵ	f	s_3-f
0.	0.	0.	0.	0.
0.5	0.3333333	-.70588235	0.3183099	0.0150234
1.0	0.	0.	0.	0.

$\underline{\lambda}_3 = 9.8823529 \qquad \mu_3 = 0.1091090 \qquad n = 3$

For $m = 5$

x	s_5	ϵ	f	s_5-f
0.	0.	0.	0.	0.
0.25	0.2255921	-.1166332	0.2250791	0.0005130
0.5	0.3190350	-.1648970	0.3183099	0.0007251
0.75	0.2255921	-1166332	0.2250791	0.0005130
1.0	0.	0.	0.	0.

$\underline{\lambda}_5 = 9.8697063 \qquad \mu_5 = 0.0242813 \qquad n = 3$

For $m = 11$ (in view of symmetry, table is abbreviated)

x	s_{11}	ϵ_i	f	$s_{11}-f$
0	0.	0.	0.	0
0.1	0.0983685	-.0080197	0.0983632	0.0000053
0.2	0.2575319	-.0209763	0.2575181	0.0000138
0.4	0.3027468	-.0246488	0.3027307	0.0000161
0.5	0.3183268	-.0259132	0.3183099	0.0000169

$\underline{\lambda}_{11} = 9.8696047$ $\mu_{11} = 0.0037125$ $n = 7$

For $m = 21$

x	s_{21}	ϵ_i	f	$s-f$
0.	0.	0.	0.	0.
0.05	0.0497948	-.0010179	0.0497946	0.0000002
0.01	0.0983635	-.0020094	0.0983632	0.0000003
0.15	0.1445101	-.0029492	0.1445097	0.0000004
0.2	0.1870984	-.0038136	0.1870979	0.0000005
0.25	0.2250796	-.0045815	0.2250791	0.0000005
0.3	0.2575186	-.0052346	0.2575181	0.0000005
0.35	0.2836166	-.00575804	0.2836162	0.0000004
0.4	0.3027311	-.0061401	0.3027307	0.0000004
0.45	0.3143913	-0.0063726	0.3143910	0.0000003
0.5	0.3183103	-.0064507	0.3183099	0.0000004

$\underline{\lambda}_{21} = 9.8696044$ $\mu_{21} = 0.0009217$ $n = 7$

For $m = 26$

$\underline{\lambda}_{26} = 9.869044$ $\mu_{26} = 0.0005894$ $n = 7$

For $m = 51$

$\underline{\lambda}_{51} = 9.86960440$ $\mu_{51} = 0.0001474$ $n = 10$

Example 2. $y'' - 2y' + \lambda y = 0$

$y(0) = 0 \quad y(1) = 0$

Coefficients do not meet requirements; however, multiplying through by e^{-2x} we obtain [2]

$$e^{-2x}y'' + 2e^{-2x}y' + e^{-2x}\lambda y = 0.$$

With $p = e^{-2x}$, $p' = -2e^{-2x}$, $q = 0$, and $r = e^{2x}$, all requirements are met.

The exact value of $\underline{\lambda}$ is $\pi^2 + 1 = 10.869044$ and the corresponding eigenfunction is $\underline{y} = ce^x \sin \pi x$, $c \neq 0$. Summary of results is given below with $s - f$ taken at greatest magnitude and $f = \frac{1}{\pi} e^x \sin \pi x$.

m	n	λ_m	μ_m	$s-f$
3	3	11.0288523	0.0744190	-0.0664710
5	11	10.8782494	0.0227613	-0.0303412
11	14	10.8699230	0.0037356	-0.0052968
21	11	10.8696110	0.0009370	-0.0013379
26	11	10.8696100	0.0006000	-0.0008571

Example 3. $(1+x)y'' + y' + \{(x^2-x) + (2.-x)\lambda\} y = 0$

$y(0) = 0 \quad y(1) = 0$

With $p = 1 + x$, $p' = 1$, $q = x^2-x$, and $r = 2.-x$, all requirements are met. Summary of numerical results is given below

m	n	λ_m	μ_m
3	11	9.7334643	0.4725730
5	11	9.4385179	0.1369278
11	11	9.4231721	0.0232826

m	n	λ_m	μ_m
21	9	9.4228171	0.0058777
26	9	9.4228032	0.0037662
51	11	9.4227942	0.0009434

6. Conclusion. The use of cubic splines provides a convenient numerical implementation of the Stodola-Vianello process. Some approximation error does occur, arising from the numerical solution of (8) and from the numerical integration. This approximation error is substantially reduced as the number of knots is increased, the limit on this being the "size" of the linear system, (12), one is willing to solve. From our experience, increasing the number of knots, after a certain point, does not seem to appreciably increase the number of iterations required to obtain a reasonable approximation.

A considerable advantage afforded by the cubic spline approximation is that the value for the approximating spline s_n and those of its first and second derivative are known at all the knots, and the value $s_n(x)$ can be readily computed for any $x \in [0, 1]$ by interpolating on a cubic segment between two knots adjacent to x.

The error estimate we offer is suggestive and can best be interpreted as a relative measure of how well our approximations for the eigenfunction and eigenvalue satisfy the differential equation.

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