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Poor Man's Derivation of Coherent and Incoherent Scattering From Nuclei*

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A derivation is presented of the coherent and incoherent contributions to the cross sections for particle scattering or particle production off nuclei. The results are illustrated with a simple "square-well", and a discussion of the effects of particle correlations is presented.[1]

I. INTRODUCTION

When calculating particle scattering or production from a nucleus one often estimates the coherent contribution to the cross section with $A^2F^2(q^2)\frac{d\sigma_s}{dq}$ and the incoherent contribution with $A(1 - F^2(q^2))\frac{d\sigma_s}{dq}$ In these expressions A = atomic number, $F^2(q^2)$ is the nuclear form factor squared, and $\frac{d\sigma_s}{dq}$ is the differential cross section with respect to momentum transfer "q" for the interaction with a single nucleon. This prescription is a black magic conundrum to all too many physicists, especially since it can be derived using the Born approximation and amply demonstrated with simple "square-well nuclei."

II. THE NUCLEAR POTENTIAL IN THE BORN APPROXIMATION

Let $V(r)$ be the potential a scattered particle sees when scattering from a single free nucleon. (We neglect PCT spin, statistics, etc.)

The full nucleus will present to the scattered particle a potential

$$U(r,R_1,R_2 \ldots R_A) = U(r,R_A) = \int V(r-R) \sum_{i=1}^{A} \delta(R-R_i) d^3R \ ;$$

"r" is the coordinate of the scattered particle and $R_1, R_2 \ldots R_A$, denoted collectively as R_A, are the coordinates of the nucleons in the nucleus.

In the Born approximation the scattering amplitude from this potential is given by

$$f_A(q) = -\frac{m}{2\pi h^2} \int < f | e^{-ik'\cdot r} \left(\sum_{j=1}^{A} V(r-R)\delta(R-R_j) d^3R \right) e^{ik\cdot r} | 0 > d^3r \prod_{k=1}^{A} d^3R_k,$$

where $f_A(q)$ is the scattering amplitude from a nucleus of atomic number A, $\vec{k}$ and $\vec{k}'$ are the initial and final wave vectors of the scattered particle, $\vec{q} = (\vec{k} - \vec{k}')$ is the momentum transfer to the nucleus, $|0>$ and $|f>$ are the initial and final states of the nucleus, and m is the reduced mass of the scattered particle.

To evaluate this expression set $u = r - R$ and integrate over d^3u

$$f_A(q) = -\frac{m}{2\pi h^2} \int < f | e^{iq\cdot(u+R)} V(u) \sum_{j=1}^{A} \delta(R-R_j) | 0 > d^3R d^3u \prod_{k=1}^{A} d^3R_k \ ,$$

$$= -\frac{m}{2\pi h^2} \tilde{V}(q) \int < f | e^{iq\cdot R} \sum_{j} \delta(R-R_j) | 0 > d^3R \prod_{k}^{A} d^3R_k \ .$$

In this expression $\tilde{V}(q) = \int e^{iq\cdot u} V(u) d^3u$ and

$-\frac{m}{2\pi h^2}\tilde{V}(q) = f_s(q)$ is the scattering amplitude from a single nucleon. The differential cross section for interaction with a single nucleon is $\frac{d\sigma_s}{dq} = |f_s(q)|^2$. Upon integrating out the delta function we have finally

$$f_A(q) = f_s(q) \int < f | \sum_{j=1}^{A} e^{iq\cdot R_j} | 0 > \prod_{k=1}^{A} d^3R_k .$$

III. COHERENT SCATTERING

Coherent scattering occurs when the final nuclear state is the same as the initial one. Only in this case do all particles of the nucleus contribute to the interaction cross section. Setting $< f| = < 0|$ we have

$$f_A(q) = f_s(q) \sum_{j=1}^{A} \int < 0 | e^{iq\cdot R_j} | 0 > \prod_{k=1}^{A} d^3R_k ,$$

$$= f_s(q)\, A \int < 0 | e^{iq\cdot R_1} | 0 > \prod_{k=1}^{A} d^3R_k ,$$

$$= f_s(q)\, A\, F\,(q^2).$$

$F(q^2)$ is the nuclear form factor and we note that it is independent of the subscript of R_i in the integral. Squaring the last expression gives the coherent nuclear cross section

$$\frac{d\sigma_c}{dq} = A^2F^2(q^2)\,\frac{d\sigma_s}{dq}\ .$$

IV. COHERENT AND INCOHERENT SCATTERING FOR A SQUARE-WELL NUCLEUS

A very simple model is sufficient to exhibit the change that occurs when the final state is the same as the initial one. Represent a nucleus of atomic number two as a square well with two identical fermions (bosons or uncorrelated particles would do just as well). If $\psi_1, \psi_2 \ldots \psi_n$ are single-particle eigenfunctions of the well with energies $E_1 < E_2 \ldots < E_n$ we construct a ground state for the two particles $|0> = \frac{1}{\sqrt{2}}\,[\psi_1(R_1)\psi_2(R_2) - \psi_2(R_1)\psi_1(R_2)]$. ($R_1$ and R_2, the position vectors of particles one and two, will not be written out in what follows. In any product of wave functions, the wave on the left will be understood to have argument R_1 and the wave on the right argument R_2.) An excited state is formed by replacing either ψ_1 or ψ_2 with a higher energy state, ψ_n with $n > 2$.

By substituting our ground state into the formula for $f_A(q)$ we exhibit coherent scattering from the well

$$f_A(q) = f_2(q) = f_s(q)\int \frac{1}{\sqrt{2}}\,(\psi_1\psi_2 - \psi_2\psi_1)^\dagger[e^{iq\cdot R_1} + e^{iq\cdot R_2}]\,\times$$

$$\frac{1}{\sqrt{2}}\,(\psi_1\psi_2 - \psi_2\psi_1)d^3R_1 d^3R_2$$

The orthogonality of ψ_1 and ψ_2 reduces this expression to

$$f_2(q)=f_s(q)[\frac{1}{2}\int\psi_1^+(2e^{iq\cdot R_1})\psi_1 d^3R_1 + \frac{1}{2}\int\psi_2^+(2e^{iq\cdot R_2})\psi_2 d^3R_2]$$

In the limit as $q \to 0$ the square bracket term approaches 2, i.e., just the number of particles in the well. Writing this term as $2F(q^2)$ the scattering amplitude becomes

$$f_2(q) = 2f_s(q)F(q^2).$$

Proceeding in the same way but replacing the final state with the excited two-particle wave function $< f| = \frac{1}{\sqrt{2}}(\psi_1\psi_3-\psi_3\psi_1)^\dagger$. The expression for $f_2(q)$ reduces to

$$f_2(q) = f_s(q)\int\frac{1}{2}\psi_3(2e^{iq\cdot R_1})\psi_2 d^3R_1.$$

Notice:

1) We have half as many terms now as we had previously for coherent scattering.

2) As $q \to 0$, the integral goes to 0.

3) The maximum value the integral can take for any choice of q is 1. This results from no more than the orthogonality of the wave functions ψ_n. In this case, when the final state is excited, the only term which survives the integration over $d^3R_1 d^3R_2$ is that term which sandwiches the exponential $e^{iq\cdot R_i}$ between the excited wave $\psi_3(R_i)$ and the wave it replaced in the ground state $\psi_2(R_i)$. The subscript of R_i must be the same in ψ_3,ψ_2 and $e^{iq\cdot R_i}$. Explicitly, the term

$$\int \psi_1^+(R_1)\psi_3^+(R_2)e^{iq\cdot R_1}\psi_1(R_1)\psi_2(R_2)d^3R_1 d^3R_2 = 0.$$

Only the term

$$\int \psi_3^+(R_1)\psi_1^+(R_2)e^{iq\cdot R_1}\psi_2(R_1)\psi\ (R_2)d^3R_1 d^3R_2$$

survives. Quite definitely the scattering involves only one of the single-particle wave functions of the composite nuclear state - only that wave function which changes in the final state.

From this simple example it is seen that any interaction which transfers a quantum number to a nucleus cannot proceed coherently. Such interactions by transferring quantum numbers "pick out" and change one of the nuclear wave functions.

More information can be extracted from this simple model. To estimate the maximum possible contribution of the incoherent interaction to the cross section, we sum the expression for $|f_A(q)|^2$ over all final states, first by replacing ψ_1 and then ψ_2 with higher energy single-particle wave functions ψ_k. The only requirement is that $k \neq 1,2$. The summation extends over free particle states. Upon replacing ψ_2 in the final state we have

$$\frac{d\sigma_I}{dq}(2) = \Sigma|f_A(q)|^2 = \sum_{k\neq 1,2}^{\infty} |f_s(q)|^2 \int \psi_k^+ e^{iq\cdot R}\psi_2 d^3R|^2 ,$$

where $\frac{d\sigma_I}{dq}(2)$ is the contribution to the incoherent cross section $\frac{d\sigma_I}{dq}$ from wave function "2".

Since $e^{iq\cdot R}$ is a unitary operator and the states ψ_k are a complete set $\sum_k^{\infty}|\int \psi_k^+ e^{iq\cdot R}\psi_2 d^3R|^2 = 1$. This implies that

$$\frac{d\sigma_I}{dq}(2) = \frac{d\sigma_s}{dq}\left\{1 - \left|\int \psi_2^+ e^{iq\,R}\psi_2 d^3R\right|^2 - \left|\int \psi_1^+ e^{iq\,R}\psi_2 d^3R\right|^2\right\}.$$

Similarly by replacing ψ_1 in the final state we have

$$\frac{d\sigma_I}{dq}(1) = \frac{d\sigma_s}{dq}\left[1 - \left|\int \psi_1^+ e^{iq\cdot R}\psi_1 d^3R\right|^2 - \left|\int \psi_2^+ e^{iq\cdot R}\psi_1\right|^2\right].$$

For small q, $\left|\int \psi_2^+ e^{iq\cdot R}\psi_1 d^3R\right|^2 \ll \left|\int \psi_2^+ e^{iq\cdot R}\psi_2 d^3R\right|^2$

$\sim \left|\int \psi_1^+ e^{iq\cdot R}\psi_1 d^3R\right|^2$. We can therefore approximate the incoherent differential cross section with

$$\frac{d\sigma_I}{dq} = \frac{d\sigma_I(1)}{dq} + \frac{d\sigma_I(2)}{dq} \approx \frac{d\sigma_s}{dq}\left[2 - \left|\int \psi_2^+ e^{iq\cdot R}\psi_2 d^3R\right|^2\right.$$

$$\left. - \left|\int \psi_1^+ e^{iq\cdot R}\psi_1 d^3R\right|^2\right].$$

But

$$F^2(q^2) = \frac{1}{4}\left|\int \psi_2^+ e^{iq\cdot R}\psi_2 d^3R\right|^2 + \frac{1}{4}\left|\int \psi_1^+ e^{iq\cdot R}\psi_1 d^3R\right|^2$$

$$+ \frac{1}{2}\left(\int \psi_1^+ e^{iq\cdot R}\psi_1 d^3R\right)^* \left(\int \psi_2^+ e^{iq\cdot R}\psi_2 d^3R\right),$$

and for small q

$$F^2(q^2) \sim \frac{1}{2}\left|\int \psi_2^+ e^{iq\cdot R}\psi_2 d^3R\right|^2 + \frac{1}{2}\left|\int \psi_1^+ e^{iq\cdot R}\psi_1 d^3R\right|^2.$$

Therefore we may write $\frac{d\sigma_I}{dq} \leq \frac{d\sigma_s}{dq}\ 2[1 - F^2]$. "Two" is of course the atomic number of the "nucleus" we have been using. The same example carried out for a well with "A" particles in it would have yielded

$$\frac{d\sigma_I}{dq} \leq \frac{d\sigma_s}{dq} A(1 - F^2(q^2)).$$

V. CORRELATIONS

A more formal derivation will give a better insight into these formulae and uncover a new term. We start with the expression for $f_A(q)$ and compute the differential scattering cross section from all final states. (For a given q^2 this means of course we are off the energy shell!)

$$\frac{d\sigma}{dq} = \Sigma |f_A(q)|^2 = |f_s(q)|^2 \sum_f < f| \sum_{j=1}^{A} e^{iq\cdot R_j} |0>^* < f| \sum_{k=1}^{A} e^{iq\cdot R_k}$$

$$|0> \prod_{\ell=1}^{A} d^3R_\ell \prod_{m=1}^{A} d^3R_m .$$

Since the states $< f|$ are a complete set this becomes

$$\frac{d\sigma}{dq} = \frac{d\sigma_s}{dq} \int < 0| \sum_{j=1}^{A} e^{-iq\cdot R_j} \sum_{k=1}^{A} e^{iq\cdot R_k} |0> \prod_{\ell=1}^{A} d^3R_\ell ,$$

$$= \frac{d\sigma_s}{dq} \int <0| \sum_{j=1}^{A} e^{-iq\cdot R_j} e^{iq\cdot R_j} + \sum_{\substack{k,\ell \\ k\neq\ell}}^{A} e^{-iq\cdot R_k} e^{iq\cdot R_\ell} |0> \prod_{m=1}^{A} d^3R_m .$$

The first sum of the integral gives the number "A". The second sum has $A(A-1)$ terms, and as usual the integration is independent of which pair of subscripts k,ℓ $(k \neq \ell)$ we use. The integral over the second sum becomes

$$A(A-1) \int \langle 0 | e^{-iq\cdot R_1} e^{iq\cdot R_2} | 0 \rangle \prod_{m=1}^{A} d^3R_m .$$

To analyze this term let us define $P_1(R)$ as the probability distribution for one of the particles of the nucleus. $P_1(R)$ depends upon the nuclear state but is independent of which particle we choose to study. If the positions of the particles were not correlated with each other, the probability of finding particle 1 at R_1 and particle 2 at R_2 would be $P_2(R_1,R_2) = P_1(R_1)P_1(R_2)$. But we know that the positions are correlated. Fermi statistics prohibits identical particles from being at the same place at the same time, whereas Bose statistics allows such a case. So we modify $P_2(R_1,R_2)$ and write: $P_2(R_1,R_2) = P_1(R_1)P_1(R_2)(1 + C(R_1,R_2))$, where $C(R_1,R_2)$ is the correlation correction.[2] Note that the integral of $P_1(R)$ over the nuclear volume must be 1. Also the integral

$$\int_{\substack{\text{nuc}\\ \text{vol}}} P_2(R_1,R_2) d^3R_1 d^3R_2 = 1 .$$

This implies that $\int P_1(R_1)P_2(R_2)C(R_1,R_2)d^3R_1 d^3R_2 = 0.$
(If this is confusing, write the wave function for two identical fermions in a square well and explicitly exhibit $P_1(R)$, $P_2(R_1,R_2)$, and

$C(R_1,R_2)$. Compare this with the case for two bosons.) With these definitions we can re-express the nuclear form factor

$$F(q^2) = \int < 0| e^{iq\cdot R_1} |0 > \prod_{k=1}^{A} d^3R_k = \int e^{iq\cdot R_1} P_1(R_1) d^3R_1 .$$

Similarly, the integral we are interested in becomes

$$A(A-1) \int P_1(R_1)P_1(R_2)(1+C(R_1,R_2))e^{-iq\cdot R_1} e^{iq\cdot R_2} \, d^3R_1 d^3R_2$$

$$= A(A-1)F^2(q^2) + A(A-1) \int P_1(R_1)P_1(R_2)C(R_1R_2)e^{-iq\cdot(R_1-R_2)} \times d^3R_1 d^3R_2 .$$

With these results we rewrite our cross section

$$\frac{d\sigma}{dq} = A \frac{d\sigma_s}{dq} + A(A-1)F^2(q^2) \frac{d\sigma_s}{dq} + A(A-1) \frac{d\sigma_s}{dq} \int P_1(R_1)P_2(R_2)C(R_1R_2)$$

$$\times \, e^{-iq\cdot(R_1-R_2)} d^3R_1 d^3R_2 \quad .$$

Rearranging this

$$\frac{d\sigma}{dq} = A^2F^2(q^2) \frac{d\sigma_s}{dq} + A(1 - F^2(q^2)) \frac{d\sigma_s}{dq}$$

$$+ A(A-1) \frac{d\sigma_s}{dq} \int P_1(R_1)P_2(R_2)C(R_1,R_2)e^{-iq\cdot(R_1-R_2)} d^3R_1 d^3R_2 .$$

The first term is the coherent cross section which we had before. The

other terms result from excited final states. $A(1 - F^2(q^2)) \frac{d\sigma_s}{dq}$ is the conventional expression for the incoherent contribution. The integral is the correlation addition to this term. We note that as $q \to 0$ the integral goes to zero also, as it must if it originates from excited final nuclear states.

We can get a "feeling" for the correlation integral by assuming $P_1(R)$ is a constant over the nuclear volume V: $P_1(R) = \frac{1}{V}$. Also, assume $C(R_1,R_2) = C(|R_1-R_2|)$ and that $C(|R_1-R_2|) = 0$ if $|R_1-R_2| > \rho$, where ρ is some correlation distance. Substituting into the correlation integral we have

$$A(A-1)\frac{1}{V^2} \int_{\substack{\text{nuc}\\ \text{vol}}} C(|R_1-R_2|)e^{iq\cdot(R_1-R_2)} d^3R_1 d^3R_2 ,$$

$$= A(A-1)\frac{1}{V^2} \int C(|u|)e^{iq\cdot u} d^3u d^3R_1 ,$$

where $u = R_1 - R_2$. Integrating over d^3R_1 gives just V. The integral over d^3u is just the integral over the correlation volume defined by ρ: $V_\rho = \frac{4}{3}\pi\rho^3$. The integral may be considered a form factor over this volume so that we can write the result as $A(A-1)\frac{V_\rho}{V} F_\rho(q)$, where $F_\rho(q)$ is the "correlation form factor". Explicitly

$$F_\rho(q) = \frac{1}{V_\rho} \int_{V_\rho} C(|u|)e^{iq\cdot u} d^3u .$$

With this definition, we again rewrite our result as

$$\frac{d\sigma}{dq} = A^2F^2(q^2)\,\frac{d\sigma_s}{dq} + A(1 - F^2(q^2))\,\frac{d\sigma_s}{dq} + A(A-1)\,\frac{V_\rho}{V}F_\rho(q)\,\frac{d\sigma_s}{dq} \quad .$$

Rearranging this, the right-hand side becomes

$$A^2(F^2(q^2) + \frac{V_\rho}{V}\,F_\rho(q))\,\frac{d\sigma_s}{dq} + A(1 - F^2(q^2) - \frac{V_\rho}{V}\,F_\rho(q))\,\frac{d\sigma_s}{dq}$$

The correlation factor $\frac{V_\rho}{V}\,F_\rho(q)$ may be considered as a modification of the nuclear form factor. Unlike the nuclear form factor, however, it goes to zero as q goes to zero. Also, it is not squared in the final result, and though it is real it may be positive or negative.

VI. SUMMING OVER FINAL STATES

In the preceding arguments, as we have been summing over all final states, we can rigorously claim that our results for the incoherent cross sections are only upper limits. We should like to do some written handwaving to support the statement that in many experimental situations the formulae obtained are not only upper limits, but also in fact are reasonable approximations. This can be done because experiments determine the energy transfers to a nucleus only within experimental errors, and because we expect that the amplitudes summed will be sharply peaked at certain final nuclear states determined by q^2.

For example, an uncertainty in q^2 of 5% for a 5-BeV electron scattering at an angle of 2 deg. leaves the energy of the final nucleus undetermined to within 35 MeV, sufficient to excite many nuclear levels. In addition, we expect that the amplitude for a nuclear excitation will peak at that final state which most closely matches the frequency of the exponential, $e^{iq\cdot r}$, in the integral $\int < f|e^{iq\cdot r}|0 > d^3r$.

For high q^2, where the exponential would go through many oscillations in integrating over the nuclear volume, the final state $< f|$ most probably will contain a free nucleon. We can write $< f| = < f'|e^{-ik\cdot r}$, where k is the wave vector of the free nucleon and $< f'|$ is a nuclear state with A-1 nucleons. The amplitude becomes $\int < f'|e^{ik\cdot r}e^{iq\cdot r}|0 > d^3r$. This is sharply peaked at $k = q$ for q^2 sufficiently greater than the nuclear level spacing. In such a situation with the given experimental error it is not at all unreasonable to estimate the cross section for exciting $< f'|e^{-ik\cdot r}$ with a sum over all final states.

For smaller q^2, when we expect the nucleus to stay together, a semiclassical argument shows the interesting result that the nuclear excitation energy is of the order of the energy of recoil of the nucleus. This energy is often well within experimental error and again our summing over final states is eminently reasonable. The nuclear recoil energy is $q^2/2m_A$, where m_A is the mass of the nucleus. The interaction range of the scattered particle and the nucleus is of the order of the nuclear radius "R". The interaction time "τ" is R/v, where v is the relative velocity. The average force the nucleus sees in this time is "F" $\sim q/\tau = qv/R$. We calculated the distance "D" that the nucleus

moves in this force in order for it to pick up its recoil energy

$$\frac{q^2}{2m_A} = F \times D = \frac{qv}{R} \times D, \text{ and therefore } D \approx qR/(2m_A v).$$

The nucleus becomes excited because the force it sees is not constant over its volume. For simplicity, assume the force changes only along its own direction. The magnitude of this change is estimated as F/R. So the excitation energy is of the order of

$$(F/R) \times R \times D = (qv/R^2) \times R \times (qR/2m_A v) = q^2/2m_A,$$

just the nuclear recoil energy.

* Work performed under the auspices of the U.S. Atomic Energy Commission.

1. A rich man's reference to this discussion is M.L. Goldberger and K.M. Watson, Collision Theory (John Wiley and Sons, Inc., New York, 1964).

2. The notation is the same as that found in Goldberger and Watson, Ref. 1.

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