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# UCRL LECTURES ON NUMERICAL ANALYSIS AND APPLIED MATHEMATICS

Lecture IV

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Berkeley, California

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# SOLUTION OF ALGEBRAIC AND TRANSCENDENTAL EQUATIONS

## 1. Introduction

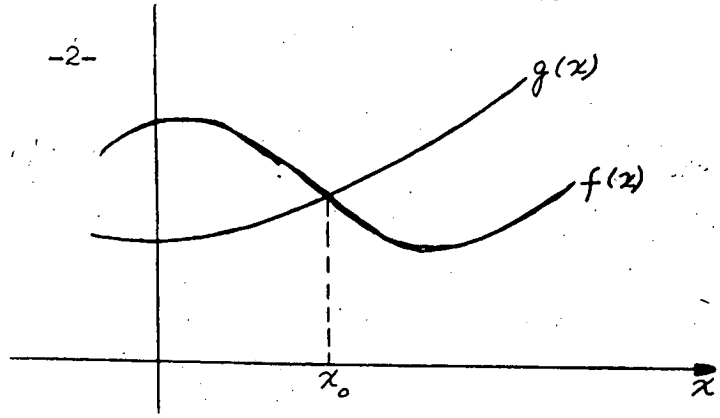
This discussion will cover only the solutions involving real roots of equations, and the methods used here will give solutions of both algebraic equations and transcendental equations.

## 2. Location of the Roots

In order to apply methods which give the solution within some degree of accuracy it is necessary to have an idea of the position of the roots. The following two methods are sufficient for this purpose.

(A) If $f(x)$ is continuous on the interval $a \leq x \leq b$ and if $f(a)$ and $f(b)$ are of opposite signs, then there is at least one real root of $f(x)$ between $a$ and $b$.

(B) If $F(x)$ can be divided into two functions $[-g(x)]$ and $f(x)$, i.e. $\{F(x) = f(x) - g(x)\} = 0$, the method is to plot $f(x)$ and $g(x)$. The abscissa's of the points of intersection of $f(x)$ and $g(x)$ are then the roots of the function $F(x)$, that is, if; $f(x_o) - g(x_o) = 0$, then $F(x_o) = 0$.




## 3. Methods of Solution

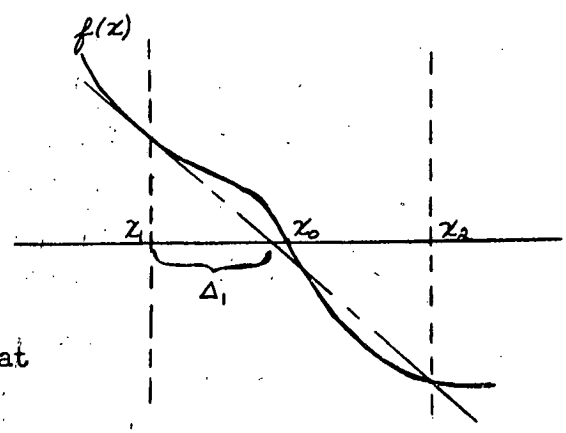
The methods which follow are applicable to transcendental equations, provided that $f(x) = 0$, is continuous in the interval $a \leq x \leq b$, and that the root or roots lie in this interval.

### 3.1 The Method of Repeated Plotting on a Larger Scale.

By plotting $y = f(x)$ it is possible to find two consecutive integers $a_1$ and $b_1$ such that $f(x_o) = 0$ for $a_1 \leq x \leq b_1$. The first approximation for $x_o$ is then $x_o \cong a_1$. The next step of this method is to plot $y = f(x)$ vs. $10x$ on the interval $a_1 \leq x \leq b_1$. If $a_2$ and $b_2$ are two consecutive integers such that $a_2 \leq x_o \leq b_2$, the second approximation for $x_o$ is then $x_o \cong a_1 + .a_2 = a_1.a_2$. By repeating this process, it is possible to determine $x_o$ to the desired degree of accuracy, i.e., $x_o \cong a_1.a_2a_3a_4 \ldots$.

### 3.2 The Method of Interpolation (False Position).

In order to use this method, it is first necessary to obtain a small interval about the desired root $x_o$. This can be found by one of the methods A or B which were outlined previously. Once this interval $(x_1\ x_2)$ is obtained, the assumption that $f(x)$ is linear on this interval, and that $f(x_1) \geq 0$, $f(x_2) \leq 0$ gives:



$$\frac{|f(x_1)|}{\Delta_1} = \frac{|f(x_2)|}{x_2 - (x_1 + \Delta_1)} \quad \text{or} \quad \Delta_1 = \frac{|f(x_1)|\,(x_2 - x_1)}{|f(x_1)| + |f(x_2)|}$$

The first approximation is then:

$$x_o \cong x_1 + \Delta_1 = x_3$$

The second approximation $x_4 = x_3 + \Delta_2$ is found by

$$x_4 = x_3 + \Delta_2 = x_3 + \frac{|f(x_2)|\,(x_3 - x_2)}{|f(x_2)| + |f(x_3)|}$$

and in general $x_n$, the n'th approximation is given by

$$x_n = x_{n-1} + \Delta_n = x_{n-1} + \frac{|f(x_{n-2})|\,(x_{n-1} - x_{n-2})}{|f(x_{n-2})| + |f(x_{n-1})|}$$

### 3.3 The Newton-Raphson Method.

If $f'(x)$ can be evaluated without too much difficulty and if $f'(x)$ never approaches "0" in the interval $a \leq x \leq b$, this method is a good one to use.

It is first necessary to find $a_o$ which is close to $x_o$, that is $a_o - x_o = h_1$ for $h_1$ small, then $f(x_o) = f(a_o + h_1)$ expanded in a Taylor's series becomes

$$f(a_o + h_1) = f(a_o) + h_1 f'(a_o) + \frac{h_1^2}{2} f''(a_o + \theta h_1) \quad \text{for } 0 \leq \theta \leq 1 ,$$

but since $f(x_o) = 0$,

$$f(a_o + h_1) = 0 = f(a_o) + h_1 f'(a_o) \quad \text{assuming "}h_1\text{" is small}$$

from which

$$h_1 = \frac{-f(a_o)}{f'(a_o)}$$

which gives the first approximation, namely

$$a_1 = a_o + h_1 \cong x_o \quad .$$

Succeeding approximations are:

$$a_2 = a_1 + h_2 = a_1 - \frac{f(a_1)}{f'(a_1)} \quad ,$$

$$a_n = a_{n-1} + h_n = a_{n-1} - \frac{f(a_{n-1})}{f'(a_{n-1})} \cong x_o$$

It is worth noting that if $f'(a)$ is large, the first value of $h$, namely, $h_1$ is small and for this reason $a_n$ approaches $x_o$ quite rapidly. However if $f'(a)$ becomes very small, $h$ becomes large and the method fails to produce the desired root "$x_o$".

3.4 The Method of Iteration.

If $\bar{x}$ is an approximate root of the equation $f(x) = 0$, and if $f(x)$ is rewritten in the form; $f(x) = \phi(x) - x = 0$, the first approximation by the method of Iteration is given by:

$$x_1 = \phi(\bar{x}) \ .$$

Succeeding approximations are:

$x_2 = \phi(x_1)$ , $x_3 = \phi(x_2)$ , ... , $x_n = \phi(x_{n-1})$ such that $x_n \cong x_o$ the desired solution.

Note: It is often possible to write $\phi(x)$ in different forms, i.e., for the function $f(x) = \cos x + x - \frac{1}{x} = 0$ , $\phi_1(x) = \frac{1}{x} - \cos x$ , $\phi_2(x) = \frac{1}{\cos x + x}$ , etc. There is no method to tell which $\phi_i(x)$ is the best function to use. However it is usually best to use the $\phi_i(x)$ which is the simplest. If that particular $\phi_i(x)$ does not converge rapidly,

then some other function $\phi_j(x)$ should be tried.

## 4. The Evaluation of an Example by the Previous Methods.

Let $f(x) = e^x - \frac{1}{x} = 0$ and find the solution for values of $x$ between 0 and 1.

By the method A:

$$f(.5) = 1.65 - 2 = -.35$$

$$f(.6) = 1.82 - 1.67 = +.15$$

so that there is a root between .5 and .6.

The following Table has been constructed so that the calculations may be simplified.

| x | f(x) | f'(x) |
|---|---|---|
| .5 | -.35 | 5.65 |
| .51 | -.30 | 5.49 |
| .52 | -.24 | 5.38 |
| .53 | -.19 | 5.26 |
| .54 | -.14 | 5.15 |
| .55 | -.09 | 5.04 |

| x | f(x) | f'(x) |
|---|---|---|
| .56 | -.04 | 4.94 |
| .57 | +.014 | 4.85 |
| .58 | +.06 | 4.76 |
| .59 | +.11 | 4.68 |
| .6 | +.15 | 4.60 |

By the method of Interpolation:

The first approximation is

$$x_3 = .5 + \Delta_1 = .5 + \frac{|f(.5)|(.6 - .5)}{|f(.5)| + |f(.6)|} = .5 + \frac{.35(.1)}{.5} = .57 .$$

The second approximation is

$$x_4 = .57 \quad \frac{|f(.57)|\,(.57 - .6)}{|f(.57)| + |f(.6)|} = \frac{(.014)(-.03)}{.164} + .57 = .569 .$$

Evaluation of $f(x) = e^x - \frac{1}{x} = 0$ by Newton Raphson Method:

Pick $a_o = .5$

$$a_1 = a_o - \frac{f(a_o)}{f'(a_o)} = .5 - \frac{f(.5)}{f'(.5)} = .5 - \frac{(-.35)}{5.65} = .5 + .062 = .56$$

$$a_2 = a_1 - \frac{f(a_1)}{f'(a_1)} = .56 + \frac{.04}{4.94} = .56 + .0081 = .568$$

$$a_3 = .568 - \frac{f(.568)}{f'(.568)} = .568 - \frac{.00466}{.4716} = .568 - .00098 = .5670.$$

The Evaluation of $f(x) = e^x - \frac{1}{x} = 0$ by the method of Iteration:

Let $\phi(x) = e^{-x}$, that is, $x = e^{-x}$. Assuming .5 as the approximate solution, the first approximation is:

$$x_1 = \phi(.5) = e^{-.5} = .607$$

$$x_2 = \phi(.607) = e^{-.607} = .545$$

$$x_3 = \phi(.545) = .580$$

$$x_4 = \phi(.58) = .560$$

$$x_5 = \phi(.560) = .571$$

$$x_6 = \phi(.571) = .565$$

$$x_7 = \phi(.565) = .568$$

$$x_8 = \phi(.568) = .567$$

$$x_9 = \phi(.567) = .567$$ which is correct to three decimals.

## 5. Solutions of Simultaneous Transcendental Equations.

The method of Iteration and the Newton Raphson method can be extended to "n" equations in "n" unknowns.

Consider the case of two equations and two unknowns. In order to find roots, $\bar{x}$ and $\bar{y}$ such that $f(\bar{x}, \bar{y}) \equiv 0$, $g(\bar{x}, \bar{y}) \equiv 0$ by the Newton Raphson method, the two functions are first expanded in a power series.

$$f(x_o + \Delta_x, y_o + \Delta_y) = f(x_o, y_o) + f_x(x_o, y_o)\Delta_x + f_y(x_o, y_o)\Delta_y + \cdots$$

$$g(x_o + \Delta_x, y_o + \Delta_y) = g(x_o, y_o) + g_x(x_o, y_o)\Delta_x + g_y(x_o, y_o)\Delta_y + \cdots .$$

The assumption that

$$x_o + \Delta_x = \bar{x}$$

$$y_o + \Delta_y = \bar{y}$$

gives the two expressions

$$\Delta_x^o = \frac{1}{\Delta^o} \begin{vmatrix} -f(x_o, y_o) & f_y(x_o, y_o) \\ -g(x_o, y_o) & g_y(x_o, y_o) \end{vmatrix} \cong \Delta_x$$

$$\Delta_y^o = \frac{1}{\Delta^o} \begin{vmatrix} f_x(x_o, y_o) & -f(x_o, y_o) \\ g_x(x_o, y_o) & -g(x_o, y_o) \end{vmatrix} \cong \Delta_y$$

where

$$\Delta^o = \begin{vmatrix} f_x(x_o, y_o) & f_y(x_o, y_o) \\ g_x(x_o, y_o) & g_y(x_o, y_o) \end{vmatrix}$$

such that $x_o + \Delta_x^o \cong \bar{x}$ , $y_o + \Delta_y^o \cong \bar{y}$ which is the expression for the first approximation. Succeeding approximations are

$$\bar{x} \cong x_{n+1} = x_n + \Delta_x^n = x_n + \frac{1}{\Delta^n} \begin{vmatrix} -f(x_{n-1}, y_{n-1}) & f(x_{n-1}, g_{n-1}) \\ -g(x_{n-1}, y_{n-1}) & g_y(x_{n-1}, y_{n-1}) \end{vmatrix}$$

$$\bar{y} \cong y_{n+1} = y_n + \Delta_y^n = y_n + \frac{1}{\Delta^n} \begin{vmatrix} f_x(x_{n-1}, y_{n-1}) & -f(x_{n-1}, y_{n-1}) \\ g_x(x_{n-1}, y_{n-1}) & -g(x_{n-1}, y_{n-1}) \end{vmatrix}$$

Solution of Simultaneous Transcendental Equations by the Method of Iteration:

Consider again the case of two equations with two unknowns, $f(x, y) = 0$ and $g(x, y) = 0$. By rewriting the two equations in the form

$$f(x, y) = x - \phi(x, y) = 0 \quad \text{and}$$
$$g(x, y) = y - \theta(x, y) = 0$$

or

$$x = \phi(x, y), \qquad y = \theta(x, y) .$$

Then by using $x_o$ and $y_o$ as values near $\bar{x}$ and $\bar{y}$, the desired roots, the first approximation is

$$x_1 = \phi(x_o, y_o) \qquad y_1 = \theta(x_o, y_o)$$

$$x_2 = \phi(x_1, y_1) \qquad y_2 = \theta(x_1, y_1)$$

and the n'th approximation is

$$\bar{x} \cong x_n = \phi(x_{n-1}, y_{n-1}) ; \qquad \bar{y} \cong y_n = \theta(x_{n-1}, y_{n-1}) .$$

## 6. Bibliography

6.1 Pipes, "Applied Mathematics for Engineers and Physicists".

6.2 Sokolnikoff, I. S. and E. S., "Higher Mathematics for Engineers and Physicists".

6.3 Margenau and Murphy, "The Mathematics of Physics and Chemistry".