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UCRL LECTURES ON NUMERICAL ANALYSIS AND APPLIED MATHEMATICS

Lecture V

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EQUATION THEORY

1. Introduction

The following subject matter is devoted to the principal methods available for finding the real and complex roots of both algebraic and transcendental equations. Although the roots may be determined exactly in some cases, in general the methods employed will yield the roots only within a predetermined standard of accuracy.

2. Preliminary Theorems for Approximate Location of the Roots.

2.1 If $f(x)$ is continuous in the real interval $a < x < b$ and if $f(a) < 0$ and $f(b) > 0$ [or $f(a) > 0$ and $f(b) < 0$] then $f(c) = 0$ where $a < c < b$, a,b,c real numbers.

2.2 Descartes' law of signs for polynomials.

2.21 $f(x)$ has n or $n - 2k$ positive real roots if there are n variations in sign of the terms of $f(x)$, n and k are non-negative integers.

2.22 $f(x)$ has n or $n - 2k$ negative real roots if there are n variations in sign of the terms of $f(-x)$, n and k are non-negative integers.

2.3 Sturm's theorem for polynomials. Given a polynomial, $f(x)$, without multiple roots, it is possible to find a finite sequence of functions, $g_1(x)$, $g_2(x)$, ... , $g_n(x)$, such that if $v(x)$ equals the number of variations in sign of these functions, then $v(a) - v(b)$ equals the number of real roots in the interval (ab).

3. Formulas for Roots of Polynomials of Degree $n \leqq 4$.

3.1 $n = 1$ If $ax + b = 0$ then $x = -\frac{b}{a}$

3.2 $n = 2$ If $ax^2 + bx + c = 0$ then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

3.3 $n = 3$ If $ax^3 + bx^2 + cx + d = 0$

then

$$x_1 = \sqrt[3]{A} + \sqrt[3]{B}$$

$$x_2 = \omega \sqrt[3]{A} + \omega^2 \sqrt[3]{B}$$

$$x_3 = \omega^2 \sqrt[3]{A} + \omega \sqrt[3]{B}$$

where $\omega^3 = 1$ and

$$A = -\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}, \qquad B = -\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}$$

$$p = \frac{c}{a} - \frac{b^2}{3a^2}, \qquad q = \frac{d}{a} - \frac{bc}{3a^2} + \frac{2b^3}{27a^3}$$

3.4 $n = 4$ If $x^4 + ax^3 + bx^2 + cx + d = 0$

then

$$x = \frac{-\frac{a}{2} \mp \sqrt{\frac{a^2}{4} - b + y} \pm \sqrt{\left(\frac{a}{2} \pm \sqrt{\frac{a^2}{4} - b + y}\right)^2 - 4\left(\frac{y}{2} \pm \sqrt{-d + \frac{y^2}{4}}\right)}}{2}$$

where y is such that

$$y^3 - by^2 + (ac - 4d)y + 4bd - a^2d - c^2 = 0 \quad .$$

This implies that

$$\left(x^2 + \frac{a}{2}x + \frac{y}{2}\right)^2 = (ex + f)^2$$

where $e^2 = \frac{a^2}{4} - b + y$ and $f^2 = -d + \frac{y^2}{4}$.

Remark: If $n > 4$ then it is impossible to give a formula indicating a <u>finite</u> number of operations to be performed on the coefficients of the polynomial thus extracting the roots.

4. Horner's Method for Obtaining the Real Roots of Polynomials of Arbitrary Degree.

4.1 Determine r, an integer, so that $f(r)$ and $f(r+1)$ have opposite signs and values different from zero.

Then $r < x < r+1$.

4.2 Obtain the next decimal place of this root by reducing all the roots by r.

$$f(x) = f(r) + (x - r)\, f'(r) + \dots + \frac{(x-r)^n}{n!}\, f^{(n)}(r)$$

If $y = x - r$

then $\bar{f}(y) = f(y + r) = f(r) + f'(r)y + \dots + \frac{f^{(n)}(r)}{n!}\, y^n$.

4.3 Multiply these roots by 10 , $z = 10\,y$.

$$\bar{\bar{f}}(z) = 10^n f(r) + 10^{n-1} f'(r)\, z + \dots + \frac{f^{(n)}(r)}{n!}\, z^n \quad .$$

4.4 Determine $\bar{r}$, an integer, so that $\bar{\bar{f}}(r)$ and $\bar{\bar{f}}(\bar{r}+1)$ have opposite signs. This gives the second decimal place. Continuation of this process will give the root accurate to as many decimal places as are desired.

4.5 Determination of the coefficients of $\bar{f}(y)$.

If $f(x) = A_n x^n + A_{n-1} x^{n-1} + \dots + A_1 x + A_o$ and

$$\bar{f}(y) = B_n y^n + B_{n-1} y^{n-1} + \dots + B_1 y + B_o \quad \text{then}$$

$$B_n = \frac{f^{(n)}(r)}{n!} = \frac{n!\, A_n}{n!} = A_n$$

$$B_{n-1} = \frac{f^{(n-1)}(r)}{(n-1)!} = n A_n r + A_{n-1}$$

$$\vdots$$

$$B_1 = f'(r) = n A_n r^{n-1} + (n-1) A_{n-1} r^{n-2} + \dots + 2 A_2 r + A_1$$

$$B_o = f(r) = A_n r^n + \dots + A_1 r + A_o \quad .$$

This suggests an easy method for calculating the coefficients, B_i.

Example: $x^3 + 2x - 5 = 0$ $\qquad f(2) = -1$, $\quad f(3) = 16$

so $r = 2$.

1	0	-2	-5	$\lfloor 2$
1	$\frac{2}{2}$	$\frac{4}{2}$	$\frac{4}{-1}$	$= f(2)$
1	$\frac{2}{4}$	$\frac{8}{10}$	$= f'(2)$	
1	$\frac{2}{6}$	$= \frac{f''(2)}{2}$		

$$\bar{f}(y) = y^3 + 6y^2 + 10y - 1 = 0 \quad \text{where } y = x - 2$$

$$\bar{\bar{f}}(z) = z^3 + 60z^2 + 1000z - 1000 = 0 \quad \text{where } z = 10(x - 2) .$$

5. Root-square Method for Polynomials.

5.1 Polynomials with real and unequal roots

$$f(x) = x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_n = (x - r_1)(x - r_2) \dots (x - r_n) .$$

It may be assumed that

$$|r_1| > |r_2| > \dots > |r_n| .$$

Now the coefficients of a polynomial are elementary symmetric functions of the roots. Thus,

$$a_1 = -\sum_{i=1}^{n} r_i \qquad a_2 = \sum_{\substack{i,j=1 \\ i<j}}^{n} r_i r_j , \dots ,$$

$$a_n = (-1)^n \prod_{i=1}^{n} r_i .$$

If $g(x) = (x + r_1^m)(x + r_2^m) \dots (x + r_n^m)$

$$= x^n + b_1 x^{n-1} + \dots + b_n$$

then $b_1 = \sum_{i=1}^{n} r_i^m , \dots , b_n = \prod_{i=1}^{n} r_i^m .$

If $m > N$, then within a prescribed decimal place accuracy,

$b_1 = r_1^m$ and $|r_1| = \sqrt[m]{b_1}$

$b_2 = r_1^m r_2^m$ and $|r_2| = \sqrt[m]{b_2/b_1}$.

In general $|r_i| = \sqrt[m]{\dfrac{b_i}{b_1 b_2 \dots b_{i-1}}}$.

Since if $r_1 > r_2$ then $\lim_{n\to\infty}\left(\frac{r_2}{r_1}\right)^n = 0$. Now if $m = 2$,

then $(x^n + a_2 x^{n-2} + \dots)^2 = (a_1 x^{n-1} + \dots)^2$.

Letting $y = -x^2$

$$g(y) = y^n + b_1 x^{n-1} + \dots + b_n = (y + r_1^2)(y + r_2^2) \dots (y + r_n^2)$$

and $b_i = a_i^2 - 2 a_{i+1} a_{i-1} + 2 a_{i+2} a_{i-2} + \dots$.

This gives a simple method to calculate the new coefficients and by repeated application of the process m can be made sufficiently large.

It can be established that m is sufficiently large whenever the new coefficients are the squares of the previous ones within the degree of accuracy required.

If $b_i = a_i^2$ for all i then

$$|r_i| = \sqrt[m]{\frac{a_i}{a_1 \dots a_{i-1}}} = |\overline{r_i}| = \sqrt[2m]{\frac{a_i^2}{(a_1 \dots a_{i-1})^2}}$$

5.2 Polynomials with complex roots.

If the roots are real, then the new coefficients will all be positive; thus, negative coefficients indicate complex roots. Also the presence of complex roots may be ascertained by Descartes' and Sturm's theorems.

Let the roots be $r_1, \dots, r_{j-1}, s \pm t_i, r_{j+2}, \dots, r_n$

$$|r_j| = |r_{j+1}| = \sqrt{s^2 + t^2}, \qquad r_j = |r_j| e^{i\theta}$$

$$f(x) = (x - r_1) \dots (x - r_{j-1})(x - |r_j| e^{i\theta})(x - |r_j| e^{-i\theta}) \dots (x - r_n)$$

$$g(y) = (y + r_1^m) \dots (y + |r_j^m| e^{mi\theta})(y + |r_j^m| e^{-mi\theta}) \dots (y + r_n^m) .$$

For $m > N$

$$g(y) = y^n + r_1^m y^{n-1} + r_1^m r_2^m y^{n-2} + \dots + 2 r_1^m r_2^m \dots r_{j-1}^m |r_j^m| \cos m\theta y^{n-j}$$

$$+ r_1^m r_2^m \dots r_{j-1}^m |r_j^{2m}| y^{n-j-1} + \dots .$$

$$\cos m\theta = \frac{b_j}{2 r_1^m \dots r_{j-1}^m |r_j|^m} \qquad |r_j| = \sqrt[2m]{\frac{b_{j+1}}{b_1 b_2 \dots b_{j-1}}}$$

$$\cos m\theta = 2\sqrt{\frac{b_j}{b_1 \dots b_{j-1} b_{j+1}}} .$$

5.3 Polynomials with real roots, some equal.

$$f(x) = x^n + a_1 x^{n-1} + \dots + a_n = (x - r_1) \dots (x - r_j)(x - r_{j+1}) \dots (x - r_n).$$

$$r_j = r_{j+1} \qquad |r_1| > |r_2| > \dots > |r_j| = |r_{j+1}| > \dots > |r_n|$$

If $m > N$

$$g(y) = y^n + r_1^m y^{n-1} + \dots + r_1^m \dots r_{j-1}^m y^{n-j+1} + 2 r_1^m \dots r_{j-1}^m r_j^m x^{n-j}$$

$$+ r_1^m \dots r_{j-1}^m r_j^{2m} y^{n-j-1} + \dots .$$

Usually upon doubling m, the coefficients are the squares of the original ones. In this case, however, the new coefficient is half the square of the old one.

$$a_j = 2\, r_1^m \cdots r_{j-1}^m\, r_j^m$$

$$b_j = 4\, r_1^{2m} \cdots r_{j-1}^{2m}\, r_j^{2m} - 2r_1^{2m} \cdots r_j^{2m} = 2r_1^{2m} \cdots r_j^{2m}$$

$$\left| r_j \right| = \sqrt[2m]{\frac{b_{j+1}}{b_1 \cdots b_{j-1}}}$$

6. Extension of the Root-square Method to Transcendental Functions.

The success of this method is based upon the fact that the coefficients of the polynomial are elementary symmetric functions of the roots. (If all coefficients are divided by the coefficient of x^o, then the new coefficients are elementary functions of the reciprocals of the roots.)

Thus in order to apply this method to a transcendental function, it will be necessary that

(a) the function have a power series expansion within a circle of convergence having infinite radius, and

(b) the function may be represented as a product of linear factors.

For these functions, the power series may be approximated by a polynomial and the root-square method applied to this polynomial.

7. Solution by Reversal of Power Series.

If $y = a_o + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$

then $x = b_1 (y - a_o) + b_2 (y - a_o)^2 + \cdots$ provided $a_1 \neq 0$.

By substituting x in the series for y, rearranging terms and equating coefficients, the b_i's may be calculated.

$$b_1 = \frac{1}{a_1}, \qquad b_2 = -\frac{a_2}{a_1^3}, \qquad b_3 = \frac{2a_2^2 - a_1 a_3}{a_1^5}$$

$$b_n = \frac{1}{n!} \frac{d^{n-1}}{dx^{n-1}} \left[\left\{ f(x) \right\}^n \right]_{x=0} \qquad \text{where } f(x) = x(y - a_o) .$$

nb_n = coefficient of x^{n-1} in $\left\{ f(x) \right\}^n$. These results were obtained by Lagrange. Finally,

$$x = -\frac{a_o}{a_1} - \frac{a_o^2 a_2}{a_1^3} - \frac{a_o^3(2a_2^2 - a_3 a_1)}{a_1^5} + \cdots .$$

If $a_1 = 0$, then x can be solved as a series in fractional powers of $y - a_o$, the Puiseux series.

If $a_2 \neq 0$ then two series are obtained.

If $y = a_o + a_n x^r + a_{n+1} x^{n+1} + \cdots$ then let

$$z^n = (y - a_o) = \left[\sum_{k=0}^{\infty} c_k x^k \right]^n = a_n x^n + a_{n+1} x^{n+1} +$$

In general one obtains n different series

$$z_r = \sqrt[n]{y - a_o} = \sum_{k=o}^{\infty} c_{kr} x^k \quad \text{where } r = 1, 2, \ldots, n .$$

Reversing these series; n series are obtained

$$x_r = \sum_{k=0} b_{kr} (y - a_o)^{k/n} \quad \text{where } r = 1, 2, \ldots, n .$$

8. Bibliography

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