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UCRL LECTURES ON NUMERICAL ANALYSIS AND APPLIED MATHEMATICS

Lecture VI

Robert Pexton

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DETERMINANTS AND LINEAR EQUATIONS

1. Introduction

Two pertinent problems confronting mathematicians developing a mathematical theory are the attainment of rigor complemented by ease of execution. This is particularly true in the numerical branch of mathematics where extended operations of a similar nature are extremely tedious.

The application of the theory of determinants greatly facilitates the work required to solve simultaneous linear equations. Moreover, this theory has been adopted to electrical computing machines. The advantages as well as the limitations of various methods currently in use will be outlined in this article.

2. Solutions of Simultaneous Linear Equations

Consider the following first degree simultaneous equations in two unknowns:

$ax + by = k$ x and y are unknowns to be determined and a, b, c, d, k and ℓ are constants. (1)

$cx + dy = \ell$ (2)

If we multiply (1) by $-c$ and (2) by a and then sum the resultant products, the unknown x is eliminated and we have

$$(ad - bc)y = a\ell - kc \quad .$$

If $ad - bc \neq 0$ we can solve for y explicitly, i.e.

$$y = \frac{a\ell - kc}{ad - bc} \quad .$$

If we multiply (1) by d and (2) by $-b$ and then sum the resultant products, the unknown y is eliminated and we have

$$(ad - bc)x = dk - b\ell \qquad \text{whence, as above,}$$

$$x = \frac{dk - b\ell}{ad - bc} \quad .$$

Recalling that in analytical geometry the equation of a straight line is $y = mx + b$, we note that equations (1) and (2) can be put in this form. From (1) $y = -\frac{a}{b}x + \frac{k}{b}$ and from (2) $y = -\frac{c}{d}x + \frac{\ell}{c}$.

If (3) $\frac{c}{d} = \frac{a}{b}$, the slopes of the two lines are the same and therefore the lines are parallel or coincident. Consequently, there will be no unique simultaneous solution of equations (1) and (2). Moreover, if we note that equation (3) can be expressed as $ad - bc = 0$, we see why division by $ad - bc = 0$ was excluded previously.

In a similar manner three simultaneous linear equations of the first degree in three unknowns can be solved under certain restrictions. Suppose, for example, we wish to solve the following equations:

$$a_1x + b_1y + c_1z = k_1 \tag{4}$$

$$a_2x + b_2y + c_2z = k_2 \tag{5}$$

$$a_3x + b_3y + c_3z = k_3 . \tag{6}$$

By multiplying (4), (5) and (6) by $b_2c_3 - b_3c_2$, $b_1c_3 - b_3c_1$, and $b_1c_2 - b_2c_1$, respectively, and then summing the products, we find that the y and z terms have been eliminated. The coefficient of x is $a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1$. In a similar manner one can solve for y and z. (Note that their coefficients are exactly the same as the coefficient of x.) Furthermore, if the coefficents of the unknowns differ from zero, we can solve explicitly for x, y, and z.

Again considering equations (4), (5), and (6) from the standpoint of analytic geometry, we see that each equation represents a plane in three-dimensional space. If the coefficient of the unknowns $\neq 0$, the intersection of two planes will be a straight line and the simultaneous intersection of three planes will be a point. But, like our first example, if the coefficient of the unknowns is zero, the planes do not intersect in one unique point, and therefore the equations have no simultaneous solution.

Simultaneous linear equations with more unknowns can be operated on in a similar manner. However, in order to minimize these laborious computations, mathematicians have devised an improved method of operating on linear simultaneous equations by determinants. Let us return to our three equations in three unknowns.

$$a_1x + b_1y + c_1z = k_1$$

$$a_2x + b_2y + c_2z = k_2$$

$$a_3x + b_3y + c_3z = k_3 .$$

A determinant may be defined as the value of a square array of numbers.

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \left[b_2c_3 - c_2b_3 \right] - b_1 \left[a_2c_3 - a_3c_2 \right] + c_1 \left[a_2b_3 - b_2a_3 \right]$$

If the a_i's are replaced by the k_i's we have

$$\begin{vmatrix} k_1 & b_1 & c_1 \\ k_2 & b_2 & c_2 \\ k_3 & b_3 & c_3 \end{vmatrix} = k_1 \left[b_2c_3 - c_2b_3 \right] - b_1 \left[k_2c_3 - c_2k_3 \right] + c_1 \left[k_2b_3 - b_2k_3 \right] .$$

Hence, we can denote how to solve for x by the following symbols:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} x = \begin{vmatrix} k_1 & b_1 & c_1 \\ k_2 & b_2 & c_2 \\ k_3 & b_3 & c_3 \end{vmatrix} .$$

Similarly for y and z :

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} y = \begin{vmatrix} a_1 & k_1 & c_1 \\ a_2 & k_2 & c_2 \\ a_3 & k_3 & c_3 \end{vmatrix}$$

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} z = \begin{vmatrix} a_1 & b_1 & k_1 \\ a_2 & b_2 & k_2 \\ a_3 & b_3 & k_3 \end{vmatrix} .$$

Notice that the determinant operating on the unknowns is always the same whereas the determinants on the right hand side of the equations vary in a definite manner; i.e., the column of coefficients which multiply the particular unknown in the given equations is replaced by a column of constants which are the values of the right hand side of the given equations.

If the determinant operating on the unknowns is not equal to zero, we can divide both sides of the equation by this determinant, thereby obtaining an explicit solution for the unknown. This is known as Cramer's rule.

The simultaneous solutions of n equations in n unknowns involves a determinant of n^2 elements as follows:

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix}$$

The subscripts have the following meaning:

The first subscript denotes the row in which a particular element is located.

The second subscript denotes the column in which a particular element is located.

For example, a_{ij} would be an element in the i 'th row and the j 'th column.

The determinant D may be defined as

$$D = \sum_{i=1}^{n!} (-1)^j \, a_{i_1 1} \, a_{i_2 2} \cdots a_{i_n n} \quad \text{where}$$

$i_1 \ldots i_n$ is an arrangement of $1, 2, \ldots n$ derived from the latter by j successive interchanges.

For our purposes the three most valuable properties of determinants which permits a circumlocution of the expansion definition of a determinant are:

(a) Interchanging any two rows or any two columns is equivalent to changing the sign of the determinant.

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = - \begin{vmatrix} b_1 & a_1 \\ b_2 & a_2 \end{vmatrix} = - \begin{vmatrix} a_2 & b_2 \\ a_1 & b_1 \end{vmatrix} .$$

(b) If any row or any column is multiplied by a constant c, the determinant is multiplied by c.

$$\begin{vmatrix} ca_1 & b_1 \\ ca_2 & b_2 \end{vmatrix} = c \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} .$$

(c) If any row or any column is added to another row or column, the determinant remains unchanged.

$$\begin{vmatrix} a_1 + b_1 & b_1 \\ a_2 + b_2 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} .$$

By operating with these three transformations we can reduce a determinant to a special form designated as a triangular determinant or a diagonal determinant. The value of the determinant is then equal to the product of the diagonal elements.

For illustrative purposes consider a determinant of three simultaneous equations in three unknowns:

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} .$$

Let us operate on row one with the operator (b). Furthermore select our operator to be $c = \frac{1}{a_1}$. Then

$$D = a_1 D_1 = a_1 \begin{vmatrix} 1 & b_1/a_1 & c_1/a_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} .$$

Next let us use a combination of operators (b) and (c) in the following manner:

(1) Multiply column one by $-b_1/a_1$ and add the result to column two.

(2) Multiply column one by $-c_1/a_1$ and add the result to column three.

Under this transformation

$$D = a_1 D_1 = a_1 \begin{vmatrix} 1 & 0 & 0 \\ a_2 & \beta_2 & \gamma_2 \\ a_3 & \beta_3 & \gamma_3 \end{vmatrix} .$$

Similarly we operate on the element in the second row and the second column.

Multiply column (2) by $-\gamma_2/\beta_2$ and add to column three. The

result is

$$D = a_1 \begin{vmatrix} 1 & 0 & 0 \\ a_2 & \beta_2 & 0 \\ a_3 & \beta_3 & \delta_3 \end{vmatrix} .$$

Therefore, $D = a_1 \left[1 \times \beta_2 \times \delta_3 \right]$.

Extension of this type of operation for determinants of higher order is straight forward.

This basic method has been elaborated so that the number of steps required to solve a series of simultaneous linear algebraic equations with the aid of an electrical computing machine is a minimum. In the Transactions of the American Institute for Electrical Engineers, Volume 60, 1941, Professor Prescott Crout has published a method which we shall now explore. This process may also be found in the first chapter of Numerical Calculus by Milne.

3. A Short Determinantal Method for Solving Simultaneous Linear Equations.

Consider the following equations:

$$a_1x + b_1y + c_1z + d_1w = f_1$$

$$a_2x + b_2y + c_2z + d_2w = f_2$$

$$a_3x + b_3y + c_3z + d_3w = f_3$$

$$a_4x + b_4y + c_4z + d_4w = f_4$$

where the a_i, b_i, c_i, d_i, f_i $(i = 1, 2, 3, 4)$ are known constants.

The given matrix has the following form:

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 & f_1 \\ a_2 & b_2 & c_2 & d_2 & f_2 \\ a_3 & b_3 & c_3 & d_3 & f_3 \\ a_4 & b_4 & c_4 & d_4 & f_4 \end{vmatrix} .$$

This matrix is operated on as follows:

$$\begin{array}{lllll}
a_1 & b_1/a_1 & c_1/a_1 & d_1/a_1 & f_1/a_1 \\
a_2 & \left(b_2 - a_2 \dfrac{b_1}{a_1}\right) = A & \left(\dfrac{c_2 - a_2 \dfrac{c_1}{a_1}}{A}\right) = C & \left(\dfrac{d_2 - a_2 \dfrac{d_1}{a_1}}{A}\right) = D & \left(\dfrac{f_2 - a_2 \dfrac{f_1}{a_1}}{A}\right) = F \\
a_3 & \left(b_3 - a_3 \dfrac{b_1}{a_1}\right) = B & \left(c_3 - a_3 \dfrac{c_1}{a_1} - BC\right) = G & \left(\dfrac{d_3 - a_3 \dfrac{d_1}{a_1} - BD}{G}\right) = H & \left(\dfrac{f_3 - a_3 \dfrac{f_1}{a_1} - BF}{G}\right) = J \\
a_4 & \left(b_4 - a_4 \dfrac{b_1}{a_1}\right) = K & \left(c_4 - a_4 \dfrac{c_1}{a_1} - KC\right) = L & \left(d_4 - a_4 \dfrac{d_1}{a_1} - KD - HL\right) = M & \left(\dfrac{f_4 - a_4 \dfrac{f_1}{a_1} - KF - JL}{M}\right) \\
 & & & & = N
\end{array}$$

In a more convenient form we have

$$\begin{vmatrix} a_1 & b_1/a_1 & c_1/a_1 & d_1/a_1 & f_1/a_1 \\ a_2 & A & C & D & F \\ a_3 & B & G & H & J \\ a_4 & K & L & M & N \end{vmatrix} .$$

The solutions are obtained in the following manner

$$w = N$$

$$z = J - Hw$$

$$y = F - Dw - Cz$$

$$x = f_1/a_1 - d_1/a_1(w) - c_1/a_1\ (z) - b_1/a_1(y).$$

This process is equivalent to transforming a matrix into diagonal form.

A continuous check may be obtained by performing similar operations on an extra column, each term of this column being the sum of the terms of the matrix in its particular row. After completing the operations on the original matrix, the sum of the terms to the right of the diagonal term in any particular row will be one less than the value computed in the extra column.

Let the matrix assume the following form with the addition of the checking column

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 & f_1 & g_1 \\ a_2 & b_2 & c_2 & d_2 & f_2 & g_2 \\ a_3 & b_3 & c_3 & d_3 & f_3 & g_3 \\ a_4 & b_4 & c_4 & d_4 & f_4 & g_4 \end{vmatrix} \quad \text{where} \quad \begin{vmatrix} g_1 = a_1 + b_1 + c_1 + d_1 + f_1 \\ g_2 = a_2 + b_2 + c_2 + d_2 + f_2 \\ g_3 = a_3 + b_3 + c_3 + d_3 + f_3 \\ g_4 = a_4 + b_4 + c_4 + d_4 + f_4 \end{vmatrix}.$$

After the operations, the form will be

$$\begin{vmatrix} a_1 & b_1/a_1 & c_1/a_1 & d_1/a_1 & f_1/a_1 & g_1/a_1 \\ a_2 & A & C & D & F & \Delta \\ a_3 & B & G & H & J & \theta \\ a_4 & K & L & M & N & \gamma \end{vmatrix}$$

where

$$\Delta = \frac{g_2 - a_2\, g_1/a_1}{A}$$

$$\theta = \frac{g_3 - a_3\, g_1/a_1 - \Delta B}{G}$$

$$\gamma = \frac{g_4 - a_4\, g_1/a_1 - \Delta K - \theta L}{M} \quad .$$

Hence, we have

$$g_1/a_1 = 1 + b_1/a_1 + c_1/a_1 + d_1/a_1 + f_1/a_1$$

$$\Delta = 1 + C + D + F$$

$$\theta = 1 + H + J$$

$$\gamma = 1 + N \; .$$

If the coefficients of the unknowns in the system of linear equations are of divergent orders of magnitude, a change of variables may be made thus permitting the significant figures to be retained.

$$a_1\, 10^{\alpha} x + b_1\, 10^{\beta} y + c_1\, 10^{\gamma} z + d_1\, 10^{\delta} w = f_1\, 10^{\epsilon}$$

$$a_2\, 10^{\alpha} x + b_2\, 10^{\beta} y + c_2\, 10^{\gamma} z + d_2\, 10^{\delta} w = f_2\, 10^{\epsilon}$$

$$a_3\, 10^{\alpha} x + b_3\, 10^{\beta} y + c_3\, 10^{\gamma} z + d_3\, 10^{\delta} w = f_3\, 10^{\epsilon}$$

$$a_4\, 10^{\alpha} x + b_4\, 10^{\beta} y + c_4\, 10^{\gamma} z + d_4\, 10^{\delta} w = f_4\, 10^{\epsilon} \; .$$

Let

$$\bar{x} = 10^{\alpha} x$$

$$\bar{y} = 10^{\beta} y$$

$$\bar{z} = 10^{\gamma} z$$

$$\bar{w} = 10^{\delta} w \quad .$$

Then writing the matrix

10^{α}	10^{β}	10^{γ}	10^{δ}	10^{ϵ}
a_1	b_1/a_1	c_1/a_1	d_1/a_1	f_1/a_1
a_2	A	C	D	F
a_3	B	G	H	J
a_4	K	L	M	N

the solutions are

$$w = N\left(\frac{10^{\epsilon}}{10^{\delta}}\right) \qquad y = (F - Dw - Cz)\left(\frac{10^{\epsilon}}{10^{\beta}}\right)$$

$$z = (J - Hw)\left(\frac{10^{\epsilon}}{10^{\gamma}}\right) \qquad x = (f_1/a_1 - d_1/a_1(w) - c_1/a_1(z) - b_1/a_1(y))\left(\frac{10^{\epsilon}}{10^{\alpha}}\right)$$

Let us consider the case where the coefficients are complex numbers. For concreteness we shall employ four simultaneous equations in four unknowns, the procedure being easily adapted to larger simultaneous systems.

$$z_{11} f_1(w) + z_{12} f_2(w) + z_{13} f_3(w) + z_{14} f_4(w) = z_{15}$$

$$z_{21} f_1(w) + z_{22} f_2(w) + z_{23} f_3(w) + z_{24} f_4(w) = z_{25}$$

$$z_{31} f_1(w) + z_{32} f_2(w) + z_{33} f_3(w) + z_{34} f_4(w) = z_{35}$$

$$z_{41} f_1(w) + z_{42} f_2(w) + z_{43} f_3(w) + z_{44} f_4(w) = z_{45} .$$

We recall from the theory of complex numbers that if $z = x + iy$ then $\bar{z} = x - iy$, $z\bar{z} = |z|^2$

$$|z|^2 = x^2 + y^2 \qquad \frac{1}{z} = \frac{1}{z} \cdot \frac{\bar{z}}{z} = \frac{\bar{z}}{|z|^2} .$$

Hence, we can replace division by a complex member by multiplication.

Our primary matrix will be as follows:

$$\begin{vmatrix} z_{11} & z_{12} & z_{13} & z_{14} & z_{15} \\ z_{21} & z_{22} & z_{23} & z_{24} & z_{25} \\ z_{31} & z_{32} & z_{33} & z_{34} & z_{35} \\ z_{41} & z_{42} & z_{43} & z_{44} & z_{45} \end{vmatrix} .$$

We operate on this matrix with the same sequence of steps employed in the solution of simultaneous equations with real coefficients.

Therefore,

(see next page)

$$
\begin{vmatrix}
z_{11} & z_{12}\dfrac{\bar{z}_{11}}{|z_{11}|^2} & z_{13}\dfrac{\bar{z}_{11}}{|z_{11}|^2} & z_{14}\dfrac{\bar{z}_{11}}{|z_{11}|^2} & z_{15}\dfrac{\bar{z}_{11}}{|z_{11}|^2} \\
 & = w_{12} & & & \\
z_{21} & z_{22} - z_{21}w_{12} & (z_{23} - z_{21}w_{13})\dfrac{\bar{w}_{22}}{|w_{22}|^2} & (z_{24} - z_{21}w_{14})\dfrac{\bar{w}_{22}}{|w_{22}|^2} & (z_{25} - z_{21}w_{15})\dfrac{\bar{w}_{22}}{|w_{22}|^2} \\
 & = w_{22} & = w_{23} & = w_{24} & = w_{25} \\
z_{31} & z_{23} - z_{31}w_{12} & (z_{33} - z_{31}w_{13} - w_{32}w_{23}) & (z_{34} - z_{31}w_{14} - w_{32}w_{24})\dfrac{\bar{w}_{33}}{|w_{33}|^2} & (z_{35} - z_{31}w_{15} - w_{32}w_{25})\dfrac{\bar{w}_{33}}{|w_{33}|^2} \\
 & = w_{32} & = w_{33} & = w_{34} & = w_{35} \\
z_{41} & z_{42} - z_{41} & z_{43} - z_{41}w_{13} - w_{42}w_{23} & z_{44} - z_{41}w_{14} - w_{42}w_{24} - w_{43}w_{34} & (z_{45} - z_{41}w_{15} - w_{42}w_{25} - w_{43}w_{35})\dfrac{\bar{w}_{44}}{|w_{44}|^2} \\
 & = w_{42} & = w_{43} & = w_{44} & = w_{45}
\end{vmatrix}
$$

$$f_4(w) = w_{45}$$

$$f_3(w) = w_{35} - w_{34} f_4$$

$$f_2(w) = w_{25} - w_{24} f_4 - w_{23} f_3$$

$$f_1(w) = w_{15} - w_{14} f_4 - w_{13} f_3 - w_{12} f_2 \quad .$$

4. Errors

Consider the following transcendental equation

$$\tan \theta = x \quad .$$

Let us assume that this equation has been derived theoretically and that an experiment has been performed in order to verify the theory. Let us suppose that the angle θ is very nearly equal to $\pi/2 = 1.57079633$.

Now note the different values of x when the measured value of θ is slightly less than $\pi/2$ and when θ is slightly greater than $\pi/2$.

For example, if

$$\theta = 1.5707 \qquad \tan \theta = +10381.32742$$

$$\theta = 1.5709 \qquad \tan \theta = -9645.69385 \quad .$$

Thus, we see that errors independent of mathematical manipulation may be encountered. However, after considering the validity of improving the results of a determinantal computation, i.e., extending the range of significant figures, Crout offers an easy method for achieving this goal. The method consists of substituting the computed values into the original equations and subtracting the results thereby obtained from the given result. These differences are then added to the original matrix as an extra column. This added column is operated on in the same manner as the original columns. The corrections to the original answers are obtained in the same manner as were the original answers. Add the corrections to the original answers to obtain the first corrected answers. This method may be repeated until the results converge to the desired number of significant figures.

5. Conclusions

Crout's method may be considered superior to that of Cramer for the following reasons:

(a) Minimum number of computations required.

(b) Continuous check on the computations.

(c) Number of significant figures may be readily extended.

6. Bibliography

6.1 Milne, "Numberical Calculus".

6.2 Crout, Prescott, "Transactions of the American Institute for Electrical Engineers, Vol. 60, 1941.