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UCRL LECTURES ON NUMERICAL ANALYSIS AND APPLIED MATHEMATICS

Lectures XVI and XVII

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INTEGRAL EQUATIONS AND TRANSFORMS

1. Introduction

An integral equation can often be solved by the use of transform methods. In particular, the application of the Laplace, Hankel and Mellin transforms to integral equations will be discussed in this paper. Also an illustration of the Wiener-Hopf technique will be given.

2. The Laplace Transform

An integral equation

$$h(x)\, f(x) = g(x) + \int k(x, y)\, f(y)dy$$

where the kernel k is of the form $k(x - y)$, $k(x + y)$, $k(xy)$, or $k(y/x)$ can often be solved by Fourier integral methods.

The basic theorem is:

Theorem 1 If f is a function of bounded variation over $(-\infty, \infty)$ and

$$\int_{-\infty}^{\infty} e^{-sx} f(x)dx = F(s)$$

converges absolutely for all complex s such that $a < \mathrm{Re}\, s < b$, then

$$\frac{f(x+0) + f(x-0)}{2} = \lim_{T \to \infty} \frac{1}{2\pi i} \int_{c-Ti}^{c+Ti} e^{xs} F(s)ds ,$$

where $a < c < b$. (Refer to G. Doetsch, Handbuch der Laplace-Transformation, p.210.) F is the (two-sided) Laplace Transform of f and we use the notation:

$$Lf = F \quad \text{and} \quad f = L^{-1}F .$$

We shall systematically reserve a small letter for the original function and the corresponding capital letter for its transform. $a < \mathrm{Re}\, s < b$ is the strip of convergence of f in the complex plane. This strip arises from a consideration of the functions

$$f_+(x) = \begin{cases} f(x) & x > 0 \\ f(x)/2 & x = 0 \\ 0 & x < 0 \end{cases}$$

$$f_-(x) = \begin{cases} 0 & x > 0 \\ f(x)/2 & x = 0 \\ f(x) & x < 0 \end{cases} .$$

We have

$$F_+(s) = \int_0^\infty e^{-sx} f(x)dx$$

$$F_-(s) = \int_{-\infty}^0 e^{-sx} f(x)dx .$$

If $e^{-sx} f(x)$ is absolutely integrable over $(0, \infty)$ for $\operatorname{Re} s > a$, then $F_+(s)$ is analytic for $\operatorname{Re} s > a$. Similarly, if $e^{-sx} f(x)$ is absolutely integrable over $(-\infty, 0)$ then $F_-(s)$ is analytic for $\operatorname{Re} s < b$, Then $F(s) = F_+(s) + F_-(s)$ is analytic in its strip of convergence $a < \operatorname{Re} s < b$.

The <u>unit step</u> u

$$u(x - h) = \begin{cases} 0 & x < 0 \\ \frac{1}{2} & x = h \\ 1 & x > h \end{cases}$$

has the transform e^{-hs}/s and is analytic in its strip of convergence $\operatorname{Re} s > 0$.

Consider $f(x) = ce^{ax} u(x) + de^{bx} u(-x)$.

Hence

$$F(s) = c \int_0^\infty e^{(a-s)x} dx + d \int_{-\infty}^0 e^{(b-s)x} dx .$$

The first integral converges for $\operatorname{Re} s > a$. The second integral converges for $\operatorname{Re} s < b$. Hence, the Laplace transform exists only if $a < b$ and then

$$F(s) = \frac{c}{s - a} - \frac{d}{s - b}, \quad a < \operatorname{Re} s < b .$$

3. Application of the Laplace Transform.

The shift rule

$$L\, f(x + h) = e^{hs}\, L\, f(x)$$

is basic for the formal solution of integral equations by these methods. Thus, if one has

$$h(x) = \int_{-\infty}^{\infty} f(x - y)\, g(y)dy ,$$

upon formal application of the shift rule

$$H(s) = \int_{-\infty}^{\infty} e^{-ys} F(s)\, g(y)dy$$

$$= F(s)\, G(s)$$

h is the <u>faltung</u> or <u>convolution</u> of f and g. Deutsch on page 121 shows that if f and g have a common strip of convergence, then H(s) exists on this common strip and

$$Lh = H .$$

<u>Example 1</u>: Solve for f in

$$\int_{-\infty}^{\infty} f(x - y)\, f(y)dy = \sin x\, u(x);$$

but the left hand side is the faltung of f with itself

$$F(s) \cdot F(s) = \frac{1}{s^2 + 1} \qquad \text{Re } s > 0$$

$$F(s) = \frac{1}{\sqrt{s^2 + 1}}$$

$$f(x) = J_o(x)\, u(x)$$

where $J_o(x)$ is the ordinary Bessel function.

Hence,

$$\int_{-\infty}^{\infty} J_o(x - y)\, u(x - y)\, J_o(y)\, u(y)dy = \sin x\, u(x)$$

and

$$\int_{0}^{x} J_o(x - y)\, J_o(y)dy = \sin x \qquad x \geq 0 \ .$$

<u>Example 2:</u>

Solve for f in the integral equation of the first kind:

$$h(x) = \int_{0}^{\infty} \cos y\, f(x - y)dy \qquad -\infty < x < \infty$$

$$h(x) = \int_{-\infty}^{\infty} \cos y\, u(y)\, f(x - y)dy \ .$$

$$H(s) = \frac{s}{s^2 + 1} F(s)$$

$$F(s) = \frac{s^2 + 1}{s} H(s) = (s + \frac{1}{s}) H(s)$$

$$f(x) = h'(x) + \int_{-\infty}^{x} h(y)dy \ .$$

Example 3:

Solve for f in the integral equation of the second kind

$$f(x) = g(x) + \int_{-\infty}^{\infty} k(x - y)\, f(y)dy$$

$$F(s) = G(s) + K(s)\, F(s)$$

$$F(s) = \frac{G(s)}{1 - K(s)} .$$

We define (if it exists) the reciprocal kernel q of k so that

$$Q(s) = \frac{K(s)}{1 - K(s)}$$

$$F(s) - G(s) = G(s) \left(\frac{1}{1 - K(s)} - 1 \right)$$

$$= G(s) \frac{K(s)}{1 - K(s)}$$

$$= G(s)\, Q(s) .$$

So

$$f(x) = g(x) + \int_{-\infty}^{\infty} q(y)\, g(x - y)dy$$

is the solution .

Example 4:

$$f(x) = g(x) + \int_0^x e^{-y} f(x-y)dy \qquad x \geq 0$$

$$f(x)u(x) = g(x)\,u(x) + \int_{-\infty}^{\infty} e^{-y}\,u(y)\,f(x-y)\,u(x-y)dy\ .$$

$$F(s) = G(s) + \frac{1}{s+1}\,F(s)$$

$$Q(s) = \frac{\frac{1}{s+1}}{1 - \frac{1}{s+1}} = \frac{1}{s}\ .$$

Hence the reciprocal kernel

$$q(x) = u(x)$$

so

$$f(x) = g(x) + \int_0^x g(y)dy\ .$$

Note that we have only used the one sided Laplace transform throughout, that is, our functions considered were all of the form

$$f(x)\,u(x)\ .$$

For this case we can use an inversion formula due to Widder to obtain $f(x)$ from $F(s)$. $F_+(s) = F(s)$ is analytic in its strip of convergence $a < \mathrm{Re}\ s < \infty$ so $\frac{d^N}{ds^N}F(s)$ is defined for $a < \mathrm{Re}\ s$.

If $L(f(x)u(x)) = F(s)$, then under the conditions as stated in Theorem 1

$$\frac{f(x+0)+f(x-0)}{2} = \lim_{N\to\infty}\left(\frac{s^{N+1}}{N!}\left(-\frac{d}{ds}\right)^{N}F(s)\right)_{s=\frac{N}{t}} . \qquad t > 0$$

This form has an advantage in that $F(s)$ is required to be known only for real values of s, so analytic continuation from the real axis is not needed.

Example 5:

Consider

$$f(x) = g(x) + \lambda \int_{-\infty}^{\infty} k(x-y)f(y)dy$$

$$F(s) = G(s) + \lambda K(s)F(s)$$

$$F(s) = \frac{G(s)}{1-\lambda K(s)} .$$

Formally we have

$$F(s) = G(s)\sum_{n=0}^{\infty}\lambda^{n}\left(K(s)\right)^{n} \quad \text{for } \lambda \text{ sufficiently small .}$$

$$F(s) = \sum_{n=0}^{\infty}\lambda^{n} Fn(s)$$

$$f(x) = \sum_{n=0}^{\infty}\lambda^{n} fn(x)$$

where

$$f_o(x) = g(x)$$

$$f_1(x) = \int_{-\infty}^{\infty} g(x - y)\, k(y)dy$$

$$f_2(x) = \int_{-\infty}^{\infty} f_1(x - y)\, k(y)dy$$

$$f_{n+1}(x) = \int_{-\infty}^{\infty} f_n(x - y)\, k(y)dy$$

which is the formal Liouville-Neumann series for the integral equation.

Example 6:

$$f(x) = g(x) + \int_{-\infty}^{\infty} k(y)\, f(x + y)dy$$

$$F(s) = G(s) + \int_{-\infty}^{\infty} k(y)\, e^{ys}\, F(s)dy$$

$$= G(s) + K(-s)\, F(s)$$

$$F(s) = \frac{G(s)}{1 - K(-s)}$$

$$f(x) = L^{-1}\left(\frac{G(s)}{1 - K(-s)}\right) .$$

Example 7:

$$f(x) = g(x) + \int_{-\infty}^{\infty} k(x + y)\, f(y)dy$$

$$F(s) = G(s) + \int_{-\infty}^{\infty} e^{ys}\, K(s)\, f(y)dy$$

$$= G(s) + K(s)\, F(-s) .$$

Hence,

$$F(-s) = G(-s) + K(-s)\,F(s)$$

$$F(s) = G(s) + K(s)\big(G(-s) + K(-s)\,F(s)\big)$$

$$F(s) = \frac{G(s) + G(-s)\,K(s)}{1 - K(-s)\,K(s)} .$$

<u>Example 8:</u> <u>Wiener-Hopf Technique.</u>

$$f(x) = c \int_0^\infty e^{-|x-y|} f(y)dy , \qquad -\infty < x < \infty$$

$$f(x) = c \int_{-\infty}^\infty e^{-|x-y|} f(y)\,u(y)dy .$$

Note

$$\int_{-\infty}^\infty e^{-|y|}\, e^{-sy}\, dx = -\frac{2}{s^2 - 1} \qquad \text{for} \quad -1 < \operatorname{Re} s < 1 .$$

Thus, we seek a solution f such that $F(s)$ has a strip of convergence

$$-1 < a < \operatorname{Re} s < b < 1 , \quad \text{so}$$

$$F(s) = F_+(s) + F_-(s) = \frac{-2c}{s^2 - 1}\, F_+(s)$$

$$F_+(s)\left(1 + \frac{2c}{s^2 - 1}\right) = -F_-(s)$$

$$F_+(s)\,\frac{s^2 + 2c - 1}{s^2 - 1} = -F_-(s)$$

$$F_+(s)\,\frac{s^2 + 2c - 1}{s + 1} = F_-(s)(1 - s) = P(s) .$$

The left side is analytic for $\operatorname{Re} s > a$ and the right side is analytic for $\operatorname{Re} s < b$. Hence both sides are analytic in the whole plane and hence is an integral function $P(s)$. Paley and Wiener in Fourier Integrals in the Complex Domain prove in fact $P(s)$ is a polynomial, and in our case $P(s) = p$, a constant.

So

$$F_+(s) = \frac{p(s+1)}{s^2 + 2c - 1}$$

$$f_+(s) = \frac{p}{2(1-2c)^{\frac{1}{2}}} \left[\left((1-2c)^{\frac{1}{2}} + 1 \right) e^{(1-2c)^{\frac{1}{2}} x} + \left((1-2c)^{\frac{1}{2}} - 1 \right) e^{-(1-2c)^{\frac{1}{2}} x} \right]$$

for $0 < c < \frac{1}{2}$ and $x \geq 0$.

$$F_-(s) = \frac{p}{1-s}$$

$$f_-(s) = -p\, e^{x} \qquad \text{for} \qquad x \leq 0 .$$

4. The Mellin and Hankel Transforms.

We shall now introduce the Mellin transform by allowing

$$x = -\ln t \qquad\qquad g(t) = f(-\ln t)$$

in Theorem 1 and thus we obtain:

$$G(P) = \int_0^\infty t^{P-1} g(t)dt \qquad\qquad a < \operatorname{Re} \; ;$$

and its inversion

$$\frac{g(t+0) + g(t-0)}{2} = \frac{1}{2\pi i} \int_{c-\infty i}^{c+\infty i} t^{-P} G(P)dP .$$

G is the Mellin transform of g,

$$G = M g , \qquad g = M^{-1} G ,$$

and we will systematically use P for the transform parameter and capital letters for the transformed function.

Example 9:

$$g(x) = \int_0^\infty k(yx)\, f(y)\, y^r\, dy .$$

$$G(P) = \int_0^\infty f(y)\, y^r\, dy \int_0^\infty k(yx)\, x^{P-1}\, dx$$

$$= \int_0^\infty f(y)\, y^{r-P}\, dy \int_0^\infty k(yx)(xy)^{P-1}\, d(yx)$$

$$= F(r - P + 1)\, K(P) .$$

Example 10:

Invert

$$g(x) = \int_0^\infty J_\gamma(yx)\, f(y)\, y\, dy , \qquad \gamma \geq 0 .$$

Let $k(x) = J_\gamma(yx)$.

From Titchmarch, Theory of Fourier Integrals, p.196,

$$K(P) = \frac{1}{K(2 - P)} .$$

Now,

$$G(P) = F(2 - p)\, K(P) .$$

$$F(2 - P) = \frac{G(P)}{K(P)} .$$

$$F(P) = \frac{G(2 - P)}{K(2 - P)} = G(2 - P)\, K(P) .$$

$$f(x) = \int_0^\infty J_\gamma(yx)\, g(y)\, dy .$$

Hence, we have inverted our integral equation. The transform pair gives the basic definition of the Hankel Transform.

Example 11: Use of the Hankel Transform.

Solve

$$\frac{d^2 f}{dx^2} + \frac{1}{x}\frac{df}{dx} - k^2 f = -\frac{\delta(r)}{r}$$

where δ is the Dirac delta function. We have

$$-P^2 F(P) - k^2 F(P) = -1 .$$

$$F(P) = \frac{1}{P^2 + k^2} .$$

$$f(x) = K_o(kr) ,$$

where K_o is the modified Bessel function. Hence this transform is useful whenever one has need of solving Laplace's equation in cylindrical coordinates.

Example 12:

$$f(x) = g(x) + \int_0^\infty k(ux)\, f(u)\, du .$$

$$F(P) = G(P) + F(1 - P)\, K(P) .$$

$$F(1 - P) = G(1 - P) + F(P)\,K(1 - P) .$$

Substitute for $F(1 - P)$ to obtain

$$F(P) = \frac{G(P) + G(1 - P)\,K(P)}{1 - K(P)\,K(1 - P)} .$$

5. Bibliography

5.1 Titchmarch, Theory of Fourier Integrals.

5.2 Van der Pol and Bremmer, Two Sided LaPlace Transform.

5.3 Doetsch, Laplace Transformation.

5.4 Doetsch, Tafeln Laplace Transformation.

5.5 MacLachlan et Humpert, Formulaire pour le Calcul Symbolique.