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Seminar in Meson Physics — September 15, 1952

Notes on Isotopic Spin — Wilson J. Frank

References:

Cassen and Condon - - On Nuclear Forces PR 50 (846) 1936
Serber - - Meson Theory Brookhaven Publication BNL 153 (T-29)
Watson - - Hypothesis of Charge Symmetry PR 85 (852) 1952
Brueckner - - Meson-Nucleon Scattering PR 86 (106) 1952

Shortly after the discovery of the neutron, Heisenberg introduced the idea of isotopic spin, mainly as a mathematical convenience in dealing with different types of nuclear forces. From this viewpoint, a neutron and a proton are considered as different states of the same particle, called the nucleon. A negative proton is not considered as a state of the nucleon because (according to Fermi's "low-brow" explanation) it is a hole, a missing positive proton, in the negative energy proton sea. As far as charge goes, then, the nucleon has a doublet structure, and the mathematical formalism is set up by the analogy to the spin matrices and wave functions of a particle of ordinary spin $\frac{1}{2}$.

The operator $\vec{\sigma}$ becomes $\vec{\tau}$, and the matrices $\sigma_x, \sigma_y, \sigma_z$ become τ_1, τ_2, τ_3 (the axes are numbered 1, 2, 3 to emphasize that the isotopic spin operators and wave functions are in a special space, called charge space). The matrix τ_3 is chosen to be diagonal, and is related to the charge of the particle. The eigenfunctions of the isotopic spin operators $\vec{\tau}^2$ and τ_3 are the one-column matrices

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

and will be represented by $N_\tau^{\tau_3}$. The operators and wave functions are summarized below.

$$\tau_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \tau_2 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \tau_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \vec{\tau}^2 = \frac{3}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

the proton eigenfunction is $N_{\frac{1}{2}}^{\frac{1}{2}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$,

the neutron eigenfunction is $N_{\frac{1}{2}}^{-\frac{1}{2}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\tau_+ = \tau_1 + i\tau_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \tau_- = \tau_1 - i\tau_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

the charge is given by $q_\tau = e(\tau_3 + \frac{1}{2})$

The [illegible] operators change the τ_3 by [illegible], and thus connect [illegible]

These operators have the following properties (as they were designed to have):

$$\tau_3 N^{\frac{1}{2}}_{\frac{1}{2}} = \frac{1}{2} N^{\frac{1}{2}}_{\frac{1}{2}} \qquad \tau_3 N^{-\frac{1}{2}}_{\frac{1}{2}} = -\frac{1}{2} N^{-\frac{1}{2}}_{\frac{1}{2}}$$

$$\vec{\tau}^2 N^{\frac{1}{2}}_{\frac{1}{2}} = \frac{3}{4} N^{\frac{1}{2}}_{\frac{1}{2}} \qquad \vec{\tau}^2 N^{-\frac{1}{2}}_{\frac{1}{2}} = \frac{3}{4} N^{-\frac{1}{2}}_{\frac{1}{2}}$$

$$\tau_+ N^{\frac{1}{2}}_{\frac{1}{2}} = 0 \qquad \tau_+ N^{-\frac{1}{2}}_{\frac{1}{2}} = N^{\frac{1}{2}}_{\frac{1}{2}}$$

$$\tau_- N^{\frac{1}{2}}_{\frac{1}{2}} = N^{-\frac{1}{2}}_{\frac{1}{2}} \qquad \tau_- N^{-\frac{1}{2}}_{\frac{1}{2}} = 0$$

$$q_{proton} = e\left(\tfrac{1}{2} + \tfrac{1}{2}\right) = +e \qquad q_{neutron} = e\left(-\tfrac{1}{2} + \tfrac{1}{2}\right) = 0$$

(Historical note on the name "isotopic spin:" the "spin" comes from the analogy to ordinary spin; the "isotopic" comes from the isotopic number of a nucleus, defined as the number of neutrons minus the number of protons. This number was probably defined in this way to give positive integers for most nuclei, but it should be multiplied by one-half to correspond to a $\tau = 1/2$ for nucleons. This makes the proton have $\tau_3 = -1/2$. Some authors have changed the signs, so the plus τ_3 value goes with the plus charge. Both conventions are being used currently.)

The isotopic spin formalism can be extended to a group of nucleons. The total isotopic spin is $\vec{T} = \vec{\tau}(1) + \vec{\tau}(2) + \cdots + \vec{\tau}(n)$, and the third axis component is $T_3 = \tau_3(1) + \tau_3(2) + \cdots + \tau_3(n)$. The operators that raise or lower the T_3 value by 1 are $T_+ = \tau_+(1) + \cdots + \tau_+(n)$, $T_- = \tau_-(1) + \cdots + \tau_-(n)$. The spin functions, of which there are 2^n, are denoted by $N^{\tau_1}_{\tau}(1) N^{\tau_2}_{\tau}(2) \cdots N^{\tau_n}_{\tau}(n)$, while an eigenfunction of the operators $\vec{T}^2$ and T_3 is denoted by $I^{T_3}_{T}$, and is a linear combination of the spin functions. The convention will be used that when the nucleon number is omitted, it is understood to be indicated by the position of the function or the operator; for example, in the spin function $N^{\tau_1}_{\tau} N^{\tau_2}_{\tau} N^{\tau_3}_{\tau} \cdots$, the second $N^{\tau_2}_{\tau}$ refers to the second nucleon, and so forth.

The Pauli principle for neutrons and protons can be extended to nucleons by requiring that the total wave function (including space, spin, and isotopic spin) be anti-symmetric.

As the two-nucleon isotopic spin eigenfunctions will be needed later, they are derived here to illustrate the method. There are two possibilities for T: either $\frac{1}{2} + \frac{1}{2} = 1$, or $\frac{1}{2} - \frac{1}{2} = 0$. The $T = 1$ state is a triplet and has $T_3 = 1, 0, -1$, while the $T = 0$ state is a singlet and has $T_3 = 0$ only. If only the S space state (which is symmetric) is considered, the spin states can be determined from the Pauli principle: [illegible]

The four possible spin functions are $N_{\frac{1}{2}}^{\frac{1}{2}}N_{\frac{1}{2}}^{\frac{1}{2}}$, $N_{\frac{1}{2}}^{\frac{1}{2}}N_{\frac{1}{2}}^{-\frac{1}{2}}$, $N_{\frac{1}{2}}^{-\frac{1}{2}}N_{\frac{1}{2}}^{\frac{1}{2}}$, $N_{\frac{1}{2}}^{-\frac{1}{2}}N_{\frac{1}{2}}^{-\frac{1}{2}}$.
The two-nucleon operators are:

$$\vec{T}^2 = \left[\vec{\tau}(1) + \vec{\tau}(2)\right]^2 = \vec{\tau}^2(1) + \vec{\tau}^2(2) + 2\vec{\tau}(1)\cdot\vec{\tau}(2)$$

$$\vec{\tau}(1)\cdot\vec{\tau}(2) = \tau_1\tau_1 + \tau_2\tau_2 + \tau_3\tau_3$$

$$\tau_+\tau_- = \tau_1\tau_1 - i\tau_1\tau_2 + i\tau_2\tau_1 + \tau_2\tau_2$$

$$\tau_-\tau_+ = \tau_1\tau_1 + i\tau_1\tau_2 - i\tau_2\tau_1 + \tau_2\tau_2$$

$$\therefore \vec{\tau}(1)\cdot\vec{\tau}(2) = \tfrac{1}{2}\left[\tau_+\tau_- + \tau_-\tau_+\right] + \tau_3\tau_3$$

$$\vec{T}^2 = \vec{\tau}^2(1) + \vec{\tau}^2(2) + \tau_+\tau_- + \tau_-\tau_+ + 2\tau_3\tau_3$$

$$T_3 = \tau_3(1) + \tau_3(2)$$

$$T_\pm = \tau_\pm(1) + \tau_\pm(2)$$

The four eigenfunctions are I_1^1, I_1^0, I_1^{-1}, I_0^0; since they are linear combinations of the spin functions, they could be found by the straight forward way of evaluating the coefficients. (For example, consider $I_1^1 = a_1 N_{\frac{1}{2}}^{\frac{1}{2}}N_{\frac{1}{2}}^{\frac{1}{2}} + a_2 N_{\frac{1}{2}}^{\frac{1}{2}}N_{\frac{1}{2}}^{-\frac{1}{2}} + a_3 N_{\frac{1}{2}}^{-\frac{1}{2}}N_{\frac{1}{2}}^{\frac{1}{2}} + a_4 N_{\frac{1}{2}}^{-\frac{1}{2}}N_{\frac{1}{2}}^{-\frac{1}{2}}$.
The equation $\vec{T}^2 I_1^1 = 2I_1^1$ leads to the result that $a_1 = a_1;\ a_2 = 2a_3;\ a_3 = 2a_2;\ a_4 = a_4$.
The equation $T_3 I_1^1 = 1I_1^1$ leads to the result that $a_1 = a_1;\ a_2 = 0;\ a_3 = 0;\ a_4 = -a_4$.
Normalizing imposes the condition that $\Sigma a_i^2 = 1$. Thus, $I_1^1 = N_{\frac{1}{2}}^{\frac{1}{2}}N_{\frac{1}{2}}^{\frac{1}{2}}$.
This perhaps was obvious from the beginning since $N_{\frac{1}{2}}^{\frac{1}{2}}N_{\frac{1}{2}}^{\frac{1}{2}}$ is the only spin function having $T_3 = 1$. Similarly, $I_1^{-1} = N_{\frac{1}{2}}^{-\frac{1}{2}}N_{\frac{1}{2}}^{-\frac{1}{2}}$.

Once the "lead" eigenfunction of a multiplet is known, the rest can be gotten by using the T_- operator, since $T_- I_T^{T_3} = \sqrt{(T+T_3)(T-T_3+1)}\ I_T^{T_3-1}$.

$$T_- I_1^1 = \sqrt{2}\, I_1^0 = N_{\frac{1}{2}}^{-\frac{1}{2}}N_{\frac{1}{2}}^{\frac{1}{2}} + N_{\frac{1}{2}}^{\frac{1}{2}}N_{\frac{1}{2}}^{-\frac{1}{2}} \qquad \therefore I_1^0 = \tfrac{1}{\sqrt{2}}\left[N_{\frac{1}{2}}^{\frac{1}{2}}N_{\frac{1}{2}}^{-\frac{1}{2}} + N_{\frac{1}{2}}^{-\frac{1}{2}}N_{\frac{1}{2}}^{\frac{1}{2}}\right]$$

$$T_- I_1^0 = \sqrt{2}\, I_1^{-1} = \tfrac{1}{\sqrt{2}}\left[0 + N_{\frac{1}{2}}^{-\frac{1}{2}}N_{\frac{1}{2}}^{-\frac{1}{2}} + N_{\frac{1}{2}}^{-\frac{1}{2}}N_{\frac{1}{2}}^{-\frac{1}{2}} + 0\right] \qquad \therefore I_1^{-1} = N_{\frac{1}{2}}^{-\frac{1}{2}}N_{\frac{1}{2}}^{-\frac{1}{2}}$$

The I_0^0 eigenfunction can be found by noting that it must involve the same two spin functions as I_1^0, and also be orthonormal to I_1^0.

$$\therefore I_0^0 = \tfrac{1}{\sqrt{2}}\left[N_{\frac{1}{2}}^{\frac{1}{2}}N_{\frac{1}{2}}^{-\frac{1}{2}} - N_{\frac{1}{2}}^{-\frac{1}{2}}N_{\frac{1}{2}}^{\frac{1}{2}}\right]$$

The two-nucleon system is summarized in Table I.

Table I

Space and Spin	T	T_3		Eigenfunction
$^1S_0$	1	1	(p,p)	$I_1^1 = N_{\frac{1}{2}}^{\frac{1}{2}} N_{\frac{1}{2}}^{\frac{1}{2}}$
		0	(p,n)	$I_1^0 = \frac{1}{\sqrt{2}}[N_{\frac{1}{2}}^{\frac{1}{2}} N_{\frac{1}{2}}^{-\frac{1}{2}} + N_{\frac{1}{2}}^{-\frac{1}{2}} N_{\frac{1}{2}}^{\frac{1}{2}}]$
		-1	(n,n)	$I_1^{-1} = N_{\frac{1}{2}}^{-\frac{1}{2}} N_{\frac{1}{2}}^{-\frac{1}{2}}$
$^3S_1$	0	0	(p,n)	$I_0^0 = \frac{1}{\sqrt{2}}[N_{\frac{1}{2}}^{\frac{1}{2}} N_{\frac{1}{2}}^{-\frac{1}{2}} - N_{\frac{1}{2}}^{-\frac{1}{2}} N_{\frac{1}{2}}^{\frac{1}{2}}]$

Finding the eigenfunctions for three-nucleon groups is somewhat more complicated (see Schiff (229) for the analogous three-electron ordinary spin case, or Messiah PR 86 (430) 1952 for the three-nucleon case). Instead, for these and heavier nuclei, the nucleus will be taken as a single particle with a certain isotopic spin. The two nuclei, He^3 and H^3, will be considered to have the same space and spin configurations, the only difference between them lying in the isotopic spin part of the wave function. It is assumed that they form a doublet ($T = \frac{1}{2}$), which seems reasonable if He^3 is looked at as a deuteron plus a proton, and H^3 as a deuteron plus a neutron. Other examples, such as the $T = 1$ triplet C^{14}, N^{14} (excited state), O^{14} are discussed by Adair PR 87 (1041) 1952.

Before extending the isotopic spin formalism to pions, one reason for the current interest in isotopic spin might be mentioned. Charge symmetry of nuclear forces assumes that the p-p and n-n forces are equal (except for the Coulomb forces). Charge independence extends this equality to include n-p forces, and says all three are the same for the same space and spin state. This last hypothesis places stringent conditions on certain reactions, and predicts, with the use of the isotopic spin formalism, ratios of cross-sections of related processes. Many of these experiments are now being done, and their results will help decide the validity of the charge independence hypothesis.

The three π-mesons suggest that the pion has a triplet structure (with $\tau = 1$, and $\tau_3 = +1, 0, -1$ for positive, neutral, and negative pions. The isotopic spin notation to be used follows by analogy to a particle with ordinary spin of one. The τ^2 and τ_3 matrices are chosen to be diagonal; the eigenfunctions of these two operators are designated by $\pi_\tau^{\tau_3}$ and are three-row, one-column matrices.

The positive pion eigenfunction is $\pi^1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$,

the neutral pion eigenfunction is $\pi^0 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$,

[illegible]

The pion isotopic spin operators are:

$$\tau_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \tau_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \qquad \tau_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\vec{\tau}^2 = 2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\tau_+ = \tau_1 + i\tau_2 = \sqrt{2}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad \tau_- = \tau_1 - i\tau_2 = \sqrt{2}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

the pion charge is given by $q = e\tau_3$.

These operators and eigenfunctions have the following properties.

$\tau_3 \Pi_1^1 = 1\,\Pi_1^1$	$\tau_3 \Pi_1^0 = 0$	$\tau_3 \Pi_1^{-1} = -1\,\Pi_1^{-1}$
$\vec{\tau}^2 \Pi_1^1 = 2\,\Pi_1^1$	$\vec{\tau}^2 \Pi_1^0 = 2\,\Pi_1^0$	$\vec{\tau}^2 \Pi_1^{-1} = 2\,\Pi_1^{-1}$
$\tau_+ \Pi_1^1 = 0$	$\tau_+ \Pi_1^0 = \sqrt{2}\,\Pi_1^1$	$\tau_+ \Pi_1^{-1} = \sqrt{2}\,\Pi_1^0$
$\tau_- \Pi_1^1 = \sqrt{2}\,\Pi_1^0$	$\tau_- \Pi_1^0 = \sqrt{2}\,\Pi_1^{-1}$	$\tau_- \Pi_1^{-1} = 0$
$q_{\pi^+} = +e$	$q_{\pi^0} = 0$	$q_{\pi^-} = -e$

Combinations of pions and nucleons are treated in the same manner as groups of nucleons. The case of one nucleon and one pion will be the example, since their eigenfunctions are needed. The possibilities for T are $1+\frac{1}{2} = \frac{3}{2}$ (a quadruplet) and $1-\frac{1}{2} = \frac{1}{2}$ (a doublet). The operators are the same as the two-nucleon ones on page 3, except that the second nucleon is replaced by a pion. The possible spinfunctions are: $N_{\frac{1}{2}}^{\frac{1}{2}}\Pi_1^1$, $N_{\frac{1}{2}}^{-\frac{1}{2}}\Pi_1^1$, $N_{\frac{1}{2}}^{\frac{1}{2}}\Pi_1^0$, $N_{\frac{1}{2}}^{-\frac{1}{2}}\Pi_1^0$, $N_{\frac{1}{2}}^{\frac{1}{2}}\Pi_1^{-1}$, $N_{\frac{1}{2}}^{-\frac{1}{2}}\Pi_1^{-1}$. The "lead" function of the quadruplet is $\Pi_{\frac{3}{2}}^{\frac{3}{2}}$ and must be equal to $N_{\frac{1}{2}}^{\frac{1}{2}}\Pi_1^1$ since that is the only spinfunction with $m_s = \frac{1}{2}$. The others follow from the relation $\tau_- \Pi_{\frac{3}{2}}^{m_s} = \sqrt{(\frac{3}{2}-m_s)(\frac{3}{2}+m_s+1)}\,\Pi_{\frac{3}{2}}^{m_s-1}$

$$\tau_- \Pi_{\frac{3}{2}}^{\frac{3}{2}} = \sqrt{3}\,\Pi_{\frac{3}{2}}^{\frac{1}{2}} = N_{\frac{1}{2}}^{-\frac{1}{2}}\Pi_1^1 + \sqrt{2}\,N_{\frac{1}{2}}^{\frac{1}{2}}\Pi_1^0 \qquad \Pi_{\frac{3}{2}}^{\frac{1}{2}} = \frac{1}{\sqrt{3}}\left[N_{\frac{1}{2}}^{-\frac{1}{2}}\Pi_1^1 + \sqrt{2}\,N_{\frac{1}{2}}^{\frac{1}{2}}\Pi_1^0\right]$$

$$\tau_- \Pi_{\frac{3}{2}}^{\frac{1}{2}} = \sqrt{2}\,\Pi_{\frac{3}{2}}^{-\frac{1}{2}} = \frac{1}{\sqrt{3}}\left[\sqrt{2}\,N_{\frac{1}{2}}^{-\frac{1}{2}}\Pi_1^0 + \sqrt{2}\,N_{\frac{1}{2}}^{\frac{1}{2}}\Pi_1^{-1} + 2N_{\frac{1}{2}}^{-\frac{1}{2}}\Pi_1^1\right]$$

The doublet eigenfunctions are found in this way: for $I^{1/2}_{1/2}$, we have the same two spin functions as for $I^{1/2}_{3/2}$; these two eigenfunctions must be orthogonal, so $I^{1/2}_{1/2} = \frac{1}{\sqrt{3}}\left[\sqrt{2}\, N^{-1/2}_{1/2}\Pi^{1}_{1} - N^{1/2}_{1/2}\Pi^{0}_{1}\right]$.
Similarly, $I^{-1/2}_{1/2} = \frac{1}{\sqrt{3}}\left[N^{-1/2}_{1/2}\Pi^{0}_{1} - \sqrt{2}\, N^{1/2}_{1/2}\Pi^{-1}_{1}\right]$
These functions are summarized for convenience in table II.

Table II

	$T = 3/2$	$T = 1/2$	
$T_3 = 3/2$	$I^{3/2}_{3/2} = N^{1/2}_{1/2}\Pi^{1}_{1}$ (p,π^+)		
$T_3 = 1/2$	$I^{1/2}_{3/2} = \frac{1}{\sqrt{3}}\left[N^{-1/2}_{1/2}\Pi^{1}_{1} + \sqrt{2}\, N^{1/2}_{1/2}\Pi^{0}_{1}\right]$	$I^{1/2}_{1/2} = \frac{1}{\sqrt{3}}\left[\sqrt{2}\, N^{-1/2}_{1/2}\Pi^{1}_{1} - N^{1/2}_{1/2}\Pi^{0}_{1}\right]$	$(n,\pi^+ + p,\pi^0)$
$T_3 = -1/2$	$I^{-1/2}_{3/2} = \frac{1}{\sqrt{3}}\left[\sqrt{2}\, N^{-1/2}_{1/2}\Pi^{0}_{1} + N^{1/2}_{1/2}\Pi^{-1}_{1}\right]$	$I^{-1/2}_{1/2} = \frac{1}{\sqrt{3}}\left[N^{-1/2}_{1/2}\Pi^{0}_{1} - \sqrt{2}\, N^{1/2}_{1/2}\Pi^{-1}_{1}\right]$	$(n,\pi^0 + p,\pi^-)$
$T_3 = -3/2$	$I^{-3/2}_{3/2} = N^{-1/2}_{1/2}\Pi^{-1}_{1}$ (n,π^-)		

Later the spin functions will be needed expressed in terms of the eigenfunctions. These are gotten by solving the expressions of tables I and II backwards, and table III below gives the results.

Table III

(p,p) $N^{1/2}_{1/2}N^{1/2}_{1/2} = I^{1}_{1}$

(p,n) $N^{1/2}_{1/2}N^{-1/2}_{1/2} = \frac{1}{\sqrt{2}}\left[I^{0}_{1} + I^{0}_{0}\right]$

(n,n) $N^{-1/2}_{1/2}N^{-1/2}_{1/2} = I^{-1}_{1}$

(p,π^+) $N^{1/2}_{1/2}\Pi^{1}_{1} = I^{3/2}_{3/2}$

(p,π^0) $N^{1/2}_{1/2}\Pi^{0}_{1} = \frac{1}{\sqrt{3}}\left[\sqrt{2}\, I^{1/2}_{3/2} - I^{1/2}_{1/2}\right]$

(p,π^-) $N^{1/2}_{1/2}\Pi^{-1}_{1} = \frac{1}{\sqrt{3}}\left[I^{-1/2}_{3/2} - \sqrt{2}\, I^{-1/2}_{1/2}\right.$

(n,π^+) $N^{-1/2}_{1/2}\Pi^{1}_{1} = \frac{1}{\sqrt{3}}\left[I^{1/2}_{3/2} + \sqrt{2}\, I^{1/2}_{1/2}\right]$

(n,π^0) $N^{-1/2}_{1/2}\Pi^{0}_{1} = \frac{1}{\sqrt{3}}\left[\sqrt{2}\, I^{-1/2}_{3/2} + I^{-1/2}_{1/2}\right]$

(n,π^-) $N^{-1/2}_{1/2}\Pi^{-1}_{1} = I^{-3/2}_{3/2}$

If the Hamiltonian is unchanged by a rotation in coordinate space, it commutes with the angular momentum, and the angular momentum is a constant of the motion. Similarly, if the Hamiltonian is unchanged by a rotation in charge space, it will commute with the total isotopic spin and the total isotopic spin will be a constant of the motion. (T_3 is a constant of the motion, since it is related to the charge and charge is conserved.) If the Hamiltonian is to be unchanged by a rotation in charge space, it cannot contain any isotopic spin operators and thus cannot distinguish between protons and neutrons, or between different types of pions. Assuming that protons and neutrons have the same properties is, of course, the hypothesis of charge independence of nuclear forces. Therefore, the consequences of charge independence can be investigated by assuming that the total isotopic spin is a constant of the motion. The method of investigation usually involves the comparison of cross-sections for several similar processes.

For example, consider the related processes:

$$p + p \rightarrow d + \pi^+$$

$$n + p \rightarrow d + \pi^0$$

$$n + n \rightarrow d + \pi^-$$

By comparing the cross-sections for the same incident energies, the phase space factors are eliminated. (Since the space and spin parts of the wave functions involved are the same, the differential cross-sections can also be compared at the same laboratory angles.) Using Fermi's "Golden Rule #2"

$$\frac{\sigma_{pp}}{\sigma_{np}} = \frac{|H|^2_{pp}}{|H|^2_{np}} \quad \text{and} \quad \frac{\sigma_{nn}}{\sigma_{np}} = \frac{|H|^2_{nn}}{|H|^2_{np}} \quad \text{where} \quad H = \int \psi_f^* \mathcal{H} \psi_i \, d\tau .$$

The only difference between the ψ for the three reactions lies in the isotopic spin part, and the interaction matrix H reduces to

$$H_{pp} = A\left[pp|d\pi^+\right], \quad H_{np} = A\left[np|d\pi^0\right], \quad \text{and} \quad H_{nn} = A\left[nn|d\pi^-\right].$$

The isotopic spin of the right half of the reactions is the same as that for a pion, since the deuteron has zero isotopic spin. Using table III,

$$\left[pp|d\pi^+\right] = \left[I_1'|I_1'\right] = \left[I_1|I_1\right]$$

$$\left[pn|d\pi^0\right] = \left[\tfrac{1}{\sqrt{2}}(I_1^0 + I_0^0)|I_1^0\right] = \tfrac{1}{\sqrt{2}}\left[I_1|I_1\right]$$

$$\left[nn|d\pi^-\right] = \left[I_1^-|I_1^-\right] = \left[I_1|I_1\right]$$

The factor $\left[I_0^0|I_1^0\right]$ is zero since the eigenfunctions are orthogonal; the T_3 notation is dropped from the final expressions because the result is independent of the charge of the state.

The relative cross-sections are: $\sigma_{pp} = 2\,\sigma_{np} = \sigma_{nn}$.

Another group of related processes is pion scattering from hydrogen:

$$\pi^+ + p \rightarrow \pi^+ + p$$

$$\pi^- + p \rightarrow \pi^0 + n$$

$$\pi^- + p \rightarrow \pi^- + p$$

Using table III again, the matrix elements in this case are:

$$[p\pi^+ | p\pi^+] = [I^{1/2}_{3/2} | I^{1/2}_{3/2}] = [I_{3/2} | I_{3/2}]$$

$$[p\pi^- | n\pi^0] = \left[\tfrac{1}{\sqrt{3}}\left(I^{-1/2}_{3/2} - \sqrt{2}\, I^{-1/2}_{1/2}\right) \middle| \tfrac{1}{\sqrt{3}}\left(\sqrt{2}\, I^{-1/2}_{3/2} + I^{-1/2}_{1/2}\right)\right]$$

$$= \tfrac{\sqrt{2}}{3}[I_{3/2} | I_{3/2}] - \tfrac{\sqrt{2}}{3}[I_{1/2} | I_{1/2}]$$

$$[p\pi^- | p\pi^-] = \left[\tfrac{1}{\sqrt{3}}\left(I^{-1/2}_{3/2} - \sqrt{2}\, I^{-1/2}_{1/2}\right) \middle| \tfrac{1}{\sqrt{3}}\left(I^{-1/2}_{3/2} - \sqrt{2}\, I^{-1/2}_{1/2}\right)\right]$$

$$= \tfrac{1}{3}[I_{3/2} | I_{3/2}] + \tfrac{2}{3}[I_{1/2} | I_{1/2}]$$

If we now assume that the scattering is mostly from the $T = 3/2$ state and set $[I_{1/2} | I_{1/2}] = 0$, the relative cross-sections are:

$$\sigma_{p\pi^+} : \sigma_{n\pi^0} : \sigma_{p\pi^-} :: 1 : \tfrac{2}{9} : \tfrac{1}{9}$$

The fact that the Chicago results from pion-proton scattering seemed to agree with these numbers was responsible to a large degree for the present interest in isotopic spin.

Another set of reactions is:

$$p + d \rightarrow t + \pi^+$$
$$p + d \rightarrow He^3 + \pi^0$$

The H^3 and He^3 are assumed to behave isotopic-spin wise like nucleons, so that the results of table III can be used. The left side of the reactions has the isotopic spin of the proton, which is $I^{1/2}_{1/2}$.
The matrix elements are:

$$[pd | t\pi^+] = [I^{1/2}_{1/2} | \tfrac{1}{\sqrt{3}}(I^{1/2}_{3/2} + \sqrt{2}\, I^{1/2}_{1/2})] = \sqrt{\tfrac{1}{3}}\,[I_{1/2} | I_{1/2}]$$

$$[pd | He^3\pi^0] = [I^{1/2}_{1/2} | \tfrac{1}{\sqrt{3}}(\sqrt{2}\, I^{1/2}_{3/2} - I^{1/2}_{1/2})] = \sqrt{\tfrac{1}{3}}\,[I_{1/2} | I_{1/2}]$$

The relative cross-sections are $\sigma_{t\pi^+} = 2\,\sigma_{He^3\pi^0}$.

This result can also be seen in the eigenfunction $I^{1/2}_{1/2} = \frac{1}{\sqrt{3}}[\sqrt{2}\, N^{-1/2}_{1/2}\Pi^+_1 - N^{1/2}_{1/2}\Pi^0_1]$, which indicates that H^3, π^+ is formed twice as often as He^3, π^0.