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UCRL-2261

LECTURES ON MESON THEORY

Joseph V. Lepore

Radiation Laboratory, University of California
Berkeley, California

January 1953

Talk 1

I hope to present, in a few talks before this group, a treatment of problems in meson physics which will provide a basis for discussions among us rather than an exhaustive theoretical treatment. It is hoped that they will provide an avenue for stimulation of experimental and theoretical work along lines likely to be the most fruitful.

The pace of these seminars will, I hope, be set by you. I shall be grateful to know your opinions concerning this as well as those about the subject content of these lectures. At present I propose an exceedingly short introduction to meson theory with practically no development of its general aspects and then subsequent discussion of problems of immediate interest such as:

(a) photo meson production

(b) meson production in nucleon collisions

(c) meson scattering by nucleons

(d) nuclear forces.

To keep this program within practical limits I shall confine the treatment to charged and neutral pseudoscalar (spin 0 p.s) mesons. Moreover the nucleons will be treated, in so far as possible, as fixed sources of meson field; that is recoil effects subsequent to meson emission will be neglected.

At the end of these talks I propose to devote a single lecture to the main differences between the ideas and results of our theory and those of the

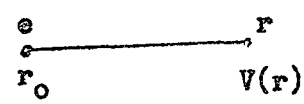
most general theory in so far as the existing very incomplete treatment of problems in this case will allow. The problem of central importance in this discussion will be the existence of nucleon pair creation, a question which will be settled by the Bevatron. These talks can then conclude with a short discussion of very high energy Bevatron experiments.

The first important fact of nuclear physics is the short range of nuclear forces. Yukawa realized that such a short range force field could be described by a modification of Poisson's equation in electrostatics. Just as

$$\nabla^2 \phi = -\rho$$

has a solution for $\rho = e\,\delta(r - r_0)$ of

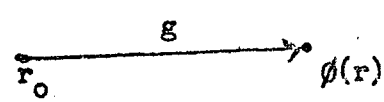
$$\phi = \frac{e}{4\pi\,|r - r_0|}$$



(1)

The equation $(\nabla^2 - \mu^2)\phi = -\rho$ has a solution for $\rho = g\,\delta(r - r_1)$

$$\phi = \frac{g\,e^{-\mu|r - r_1|}}{4\pi\,|r - r_1|}$$



(2)

The "potential" of a particle of charge "g" giving rise to a "field", ϕ, satisfying Equation (2) has the simple property of being short ranged

$$\text{range} = \frac{1}{\mu} = \lambda_0 \qquad (3)$$

In exact analogy with electrostatics one can propose that the potential energy of a nucleon in a field ϕ is

$$V = \int \phi(r)\, \rho(r)\, d^3r \quad . \tag{4}$$

Thus, the potential energy between two nucleons at r_1, r_2 is

$$V = \frac{g_1 \; g_2}{4\pi \, | \, r_1 - r_2 |} \; e^{-\mu | r_1 - r_2 |} \tag{5}$$

has the required property of cutting off sharply when

$$| r_1 - r_2 | > \frac{1}{\mu} = \lambda_0 \tag{6}$$

So far a theory of a short range field has been sketched but no reference has been made to any particle associated with this field. That there must be a particle so associated follows from the general philosophy of quantum mechanics as embodied in Bohr's Complementarity Principle. This states the complete reciprocity between the field and particle aspects of physical situations. We must now decide on a physical interpretation of our theory so as to bring it into harmony with this principle but first we must replace Equation (2) by a time dependent one which reduces to it for the case of static fields

$$(\Box^2 - \mu^2)\phi = -\rho \tag{7}$$

This combination is chosen since as we all know it is the proper relativistic generalization of the Laplacian. In a region of space remote from sources Equation (7) has the solution

$$\phi = a\, e^{i(k \cdot r - \omega t)} \quad . \tag{8}$$

Writing

$$p = \hbar k \quad ; \qquad E = \hbar \omega \tag{9}$$

we see that

$$\phi = a\, e^{\frac{i}{\hbar}(p \cdot r - Et)} \tag{10}$$

can be interpreted as the "wave function" of a particle with momentum p and energy E. The form of the wave Equation (7) implies that

$$-\frac{p^2}{\hbar^2} + \frac{E^2}{\hbar^2 c^2} - \mu^2 = 0$$

or that

$$E^2 = \mu^2 c^2 \hbar^2 + p^2 c^2 \quad . \tag{11}$$

This can be interpreted as the usual relation between energy and momentum of a particle if we write

$$E^2 = m^2 c^4 + p$$

$$m = \frac{\mu \hbar}{c} \quad ; \qquad \mu = \frac{m c}{\hbar} \quad . \tag{12}$$

Going back to the fundamental question of the short range of nuclear forces we see

$$\lambda = \frac{\hbar}{m c} \tag{13}$$

is just the Compton wave length of the particle of mass m. For $\lambda \sim 1.4 \times 10^{-13}$ cm. m comes out to be just about the mass of the π meson.

To sum up so far: A theory of a short range field has been built up and it has been shown that when interpreted in harmony with the Complementarity Principle it is associated with particles of mass m. Nevertheless nothing

about the mathematics of the theory forces us to interpret it as we have. It can be concluded, therefore, that it is incomplete and must be modified so that these conclusions will follow as mathematical consequences of our basic assumptions with no need to draw on physical principles more general than the theory itself.

The simplest way to approach this problem is to consider a general solution of $(\Box^2 - \mu^2)\phi = 0$. We may expand this in terms of plane waves

$$\phi = \frac{1}{\sqrt{V}} \sum A_k(t)\, e^{i(k\cdot r)} + \text{c.c.} \tag{14}$$

Suppose we try to interpret ϕ as a wave function according to the usual theory we would interpret

$$A_k^* A_k \propto \text{prob. that the system is found by a measurement}$$

in the state k

$$= \frac{\text{no. of systems in k}}{\text{total number of systems}} \quad . \tag{15}$$

To bring out the particle features of the theory we modify this by assuming that A_k is proportional to an operator a_k and that

$$a_k^* a_k = N_k = \text{number of systems in k.}$$
$$= 0, 1, 2, 3 \text{ only} \tag{16}$$

This can be done by assuming that a_k^*, a_k are operators, not numbers so that a_k^*, a_k is an operator with the eigenvalues 0, 1, 2, 3, ... Let us set

$$a_k \psi(N_k) = \sqrt{N_k}\, \psi(N_k - 1) \tag{17}$$

$$a_k^\dagger \psi(N_k - 1) = \sqrt{N_k}\, \psi(N_k)$$

where $\psi(N_k)$ is interpreted as a state with N_k particles in the state k. Then

$$a_k^+ \; a_k = N_k \tag{18}$$

$$\left[a_k , a_k^+ = 1 \right] \tag{19}$$

This is not a general enough prescription to allow us to calculate since we must manipulate quantities with different values of k. We may generalize (19) to

$$\left[a_k , a_k^+ \right] = \delta_{kk'} \quad . \tag{19a}$$

We now expect that the energy of our field will be (in what follows units will be chosen in which $\hbar = c = 1$)

$$E = \sum E_k N_k + \text{const.}$$

$$E = \sum \sqrt{k^2 + \mu^2} \; a_k^+ a_k + \text{const.}$$

Let us consider the quantity

$$\phi = \frac{1}{\sqrt{V}} \sum \frac{(a_k e^{i(k \cdot r)} + a_k e^{-i(k \cdot r)}}{\sqrt{2 E_k}} \tag{20}$$

and

$$\tfrac{1}{2} (\phi^2 + (\nabla \phi)^2 + \mu^2 \phi^2) = \mathcal{H} \tag{21}$$

Consider a typical term of (21)

$$\phi^2 = \frac{1}{V} \sum_{kk'} \frac{\left(a_k e^{i(k\cdot r)} + a_k e^{-i(k\cdot r)}\right)^2}{\sqrt{2 E_k}}$$

$$= \frac{1}{V} \sum_{kk'} a_k a_{k'} e^{i(k+k')}$$

$$a_k a_k^+ \quad i(k-k')\cdot r$$

$$a_k^+ a_k \quad e^{-i(k-k')\cdot r}$$

$$a_k^+ a_{k'}^+ \quad e^{-i(k+k')\cdot r} \quad .$$

Integrating over all space

$$\tfrac{1}{2} \int \phi^2 \, d^3r = \sum a_k \; a_{-k} + a_k \, a_{-k}^+ + a_k^+ a_k + a_k^+ \; a_{-k}^+$$

$$= \sum (a_k^+ \; a_k + a_k \; a_k^+) \quad .$$

Similarly we can verify that

$$\int d^3r \, \mathcal{H} = \tfrac{1}{2} \int (\phi^2 + (\nabla \phi)^2 + \mu^2 \phi_2) d^3 r$$

$$= \sum \frac{E_k}{2} (a_k^+ \; a_k + a_k \, a_k^+)$$

$$= \sum (N_k + \tfrac{1}{2}) E_k \qquad (22)$$

Thus the expression $\mathcal{H}$ represents the energy density for a system of particles.

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Talk 2

Last time I offered an intuitive discussion of the need for and the method of quantization of fields. There were various points over which I skipped in order to bring out the essential physical ideas. Now I would like to make these statements precise by discussing the problem of the scalar field.

We saw that the quantity representing the energy/unit volume of a scalar field ϕ is

$$\mathcal{H} = \tfrac{1}{2}\left(\dot{\phi}^2 + (\nabla\phi)^2 + \mu^2\phi^2\right) \quad . \tag{23}$$

The total energy or Hamiltonian of the system is

$$H = \int d^3r\,\mathcal{H} \quad .$$

Let us introduce

$$\pi = \dot{\phi} \tag{24}$$

The momentum density conjugate to the variable ϕ. This is analogous to the p, q relationship.

Note that whereas ϕ represents the amplitude of the field at a point r only the quantity

$$p(r) = \int_V \pi(r)\, d^3r \tag{25}$$

has significance. For small volume, V, about a point r this integral may be approximated by

$$p(r_k) = \pi(r_k)\Delta V_k \quad . \tag{26}$$

Now note that the Hamiltonian

$$H = \int d^3r \; \tfrac{1}{2}(\pi^2 + (\nabla\phi)^2 + \mu^2\phi^2)$$

may also be written as

$$H = \int d^3r \; \tfrac{1}{2}(\pi^2 - \phi\nabla^2\phi + \mu^2\phi^2) \tag{27}$$

since

$$\int d^3r \; (\nabla\phi)^2 = \int d^3r \left[\nabla\cdot(\phi\,\nabla\phi) - \phi\nabla^2\phi\right]$$

and

$$\int d^3r \; \nabla\cdot(\phi\,\nabla\phi) = \int_{\substack{\text{Surface}\\ \text{at } \infty}} \phi\,\nabla\phi\cdot dS = 0$$

We shall find (27) a convenient form with which to work.

For mathematical simplicity we can divide the volume containing field into small blocks of volume ΔV_k about the point r_k. Then

$$H = \sum \Delta V_k \, \tfrac{1}{2}\left[\dot{\phi}^2(r_k) + \phi(r_k)(\mu^2 - \nabla_k^2)\,\phi(r_k)\right] \quad . \tag{28}$$

Hamilton's equations

$$\dot{p}_k = -\frac{\partial H}{\partial q_k} \qquad\qquad \dot{q}_k = +\frac{\partial H}{\partial p_k} \tag{29}$$

may be applied to yield

$$\dot{\phi}(r_k) = \frac{\partial H}{\partial \pi(r_k)\Delta V_k} = \dot{\phi}(r_k) \quad ; \tag{30}$$

this just agrees with the expression we took for π. The equation for π is

more interesting

$$\dot{p}(r_k) = \dot{\pi}(r_k)\Delta V_k = -\frac{\partial H}{\partial \dot{\phi}(r_k)}$$

$$= -(\mu^2 - \nabla_k^2)\phi\,\Delta V_k \quad .$$

Cancelling out the unwanted ΔV_k one gets

$$\dot{\pi}(r_k) = -(\mu^2 - \nabla^2)\phi(r_k) \tag{31}$$

$$= \ddot{\phi}(r_k) \quad .$$

Hence,

$$(\mu^2 - \nabla^2)\phi + \ddot{\phi} = 0 \qquad \text{or}$$

$$\left[\nabla^2 - \frac{\partial^2}{\partial t^2} - \mu^2\right]\phi = (\Box^2 - \mu^2)\phi = 0 \quad . \tag{32}$$

This verifies that H is the correct Hamiltonian and that our interpretation yields correct equations of motion. We now proceed to "quantize the field" by assuming

$$\left[\phi(r_k),\, p(r_q)\right] = i\,\delta_{kq} \tag{33}$$

or

$$\left[\phi(r_k),\, \pi(r_q)\right] = i\,\frac{\delta_{kq}}{\Delta V_q} \quad . \tag{34}$$

We can compactly represent this relationship by letting $\Delta V_q \to 0$ since

$$\int d^3r\, \left[\phi(r),\, \pi(r')\right] = \sum_k \Delta V_k \left[\phi(r_k),\, \pi(r_q)\right]$$

$$= i \sum_k \frac{\Delta V_k}{\Delta V_q}\, \delta_{kq}$$

$$= i.$$

by recognizing that

$$\frac{\delta_{kq}}{\Delta V_q} = \delta(r_k - r_q) \text{ the familiar Dirac delta function.}$$

$$\int \delta(r - r')d^3r = 1$$

hence, the correct relationship between π, ϕ is compactly written

$$\left[\phi(r), \pi(r')\right] = i\,\delta(r - r') \,. \tag{35}$$

In order to approach the results of last time we again make a Fourier decomposition of the field

$$\left.\begin{aligned} \phi(r) &= \frac{1}{\sqrt{V}} \sum q_k \, e^{i\,k\cdot r} \\ \pi(r) &= \frac{1}{\sqrt{V}} \sum p_k \, e^{-i\,k\cdot r} \end{aligned}\right\} \,. \tag{36}$$

Using Eqs. (35) and (36) one finds

$$\begin{aligned} \left[\phi(r), \pi(r')\right] &= i\,\delta(r - r') \\ &= \frac{i}{V}\sum e^{i\,k\cdot(r-r')} = \frac{1}{V}\sum_{k'k''} (q_{k'}, p_k)\, e^{i\,kr - kr'} \end{aligned} \tag{37}$$

Only those terms in which the right hand side is a function of $r - r'$ can survive so we have

$$\left[q_k, p_{k'}\right] = i\,\delta_{kk'} \,. \tag{38}$$

We can now proceed directly to the results of last time by writing

$$H = \tfrac{1}{2}\sum\left(p_k^{\dagger}\, p_k + E_k^2\, q_k^{\dagger}\, q_k\right) \,. \tag{39}$$

Here $p_k^+ = p_{-k}$ and $q_k^+ = q_{-k}$.

If we now introduce new variables

$$a_k^+ = \frac{i\ p_k^+}{\sqrt{2\ E_k}} + \sqrt{\frac{E_k}{2}}\ q_k$$

$$a_k = \frac{-i\ p_k}{\sqrt{2\ E_k}} + \sqrt{\frac{E_k}{2}}\ q_k \tag{40}$$

the Hamiltonian

$$H = \sum E_k \left[\frac{p_k\ p_k}{2\ E_k} + \frac{E_k}{2}\ q_k^+\ q_k\right]$$

$$H = \sum \frac{E_k}{2}\left\{\left(\frac{i\ p_k^+}{\sqrt{2\ E_k}} + \sqrt{\frac{E_k}{2}}\ q_k\right)\left(\frac{-i\ p_k}{\sqrt{2\ E_k}} + \sqrt{\frac{E_k}{2}}\ q_k^+\right)\right.$$

$$\left. + \left(\frac{-i\ p_k}{\sqrt{2\ E_k}} + \sqrt{\frac{E_k}{2}}\ q_k^+\right)\left(\frac{i\ p_k^+}{\sqrt{2\ E_k}} + \sqrt{\frac{E_k}{2}}\ q_k\right)\right\}$$

becomes

$$= \sum E_k\ \tfrac{1}{2}\left[a_k^+\ a_k + a_k\ a_k^+\right] . \tag{41}$$

This is just the expression we found by elementary arguments last time.

Using Eq. (40) the field operator

$$\phi(r) = \frac{1}{\sqrt{V}} \sum_{k>0} \left(q_k\ e^{i\ k\cdot r} + q_k^+\ e^{-i\ k\cdot r}\right)$$

can also be expressed in terms of $a_k\ a_k^+$, so one finds

$$\phi(r) = \frac{1}{\sqrt{V}} \sum \left(\frac{a_k\ e^{i\ k\cdot r}}{\sqrt{2\ E_k}} + \frac{a_k^+\ e^{-i\ k\cdot r}}{\sqrt{2\ E_k}}\right) \tag{42}$$

since

$$-i\, p_k^{+}\, e^{-i\, k\cdot r} + i\, p_k\, e^{i\, k\cdot r} = 0$$

if we sum.

This brings us back to the point where we were last time and completes the brief discussion of the essential features in the development of a consistent quantum theory.

So far we have talked only about non-electrical properties of mesons. How can we describe charge? We have seen that the simple real scalar field is completely described by its momentum eigenstates. There is no degree of freedom associated with electrical charge which must be conserved. In order to describe charge we must introduce two different kinds of mesons. Suppose

$a_k \rightarrow$ annih op. for $\oplus$ mesons

$b_k \rightarrow$ annih op. for $\ominus$'s

then we would expect

$$H = \sum E_k (a_k^{+} a_k + b_k^{+} b_k)$$

$$Q = e \sum (a_k^{+} a_k - b_k^{+} b_k) .$$

We must consider <u>two</u> fields to describe charged mesons.

Let's now talk classically--suppose I want to write down a charge density--its volume integral must be a constant in time. Besides this the quantity ρ a charge density ought to change sign when t is changed to $-t$.

We could try

$$\phi \pi = \phi \dot{\phi} = \frac{d}{dt} \phi^2$$

but this quantity is not conserved by the Hamiltonian and so could not represent charge.

The answer to our predicament is this. There is no quantity which can be built out of our framework. It is too narrow. To get a quantity which represents charge we must allow ϕ to be complex. This can be done by introducing two real fields ϕ_1, ϕ_2 and forming

$$\phi = (\phi_1 + \phi_2)\frac{1}{\sqrt{2}}$$

$$\phi^+ = (\phi_1 - \phi_2)\frac{1}{\sqrt{2}} \quad . \tag{43}$$

We may then consider .

$$\rho = i\,e\,(\phi_1 \pi_2 - \phi_2 \pi_1) \,, \tag{44}$$

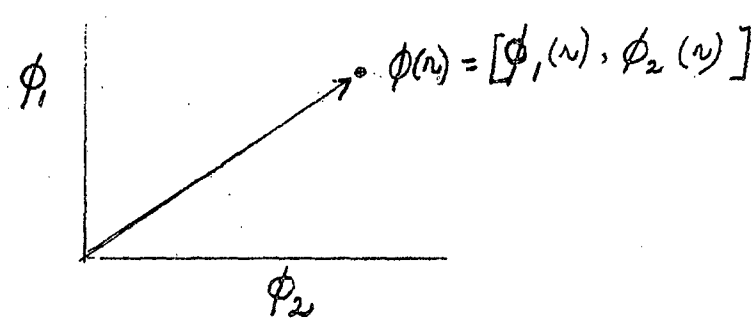
an expression which may be regarded as a charge density since its volume integral is a constant of motion.

$$Q = i\,e \int \rho \, d^3 r$$

$$\dot{Q} = i\,e \int (\dot{\phi}_1 \pi_2 + \phi_1 \dot{\pi}_2 - \dot{\phi}_2 \pi_1 - \phi_2 \dot{\pi}_1)$$

$$= i\,e \int (\pi_1 \pi_2 + \phi_1(\mu^2 - \nabla^2)\phi_2 - \pi_2 \pi_1 - \phi_2(\mu^2 - \nabla^2)\phi_1) = 0$$

We can also see this trivially since we may regard ϕ as a vector in two dimensions.



Then since H depends only on $\phi_{1,2}^2$, $\pi_{1,2}^2$ the quantity behaves like the third component $\rho = i\, e(\phi \times \pi)_3$ of an "angular momentum". Since $\mathcal{H}$ is invariant under rotations about this axis it is a constant of motion.

In term of ϕ, ϕ^+ we verify that

$$\rho = i\, e\, (\phi \dot{\phi}^+ - \phi^+ \dot{\phi}) \quad .$$

How can we interpret ϕ, ϕ^+? If we write

$$\phi = \frac{1}{\sqrt{V}} \left[\frac{a_k\, e^{i\, k\cdot r}}{\sqrt{2\, E_k}} + b_k^+\, e^{-i\, k\cdot r} \right. \tag{45}$$

$$\pi = \frac{i}{\sqrt{V}} \sum (a_k^+\, e^{-i\, k\cdot r} + b_k\, e^{i\, k\cdot r}) \sqrt{2\, E_k} \quad .$$

Then

$$H = \sum (a_k^+\, a_k + b_k^+\, b_k)\, E_k \tag{46}$$

and

$$\dot{\phi} = \left[\phi, H \right]$$

$$\dot{\phi} = q\, \frac{1}{\sqrt{V}} \sum \left(e^{i\, k\cdot r}\, a_k - b_k^+\, e^{-i\, k\cdot r} \right) \frac{E_k}{\sqrt{2\, E_k}} \quad .$$

One finds

$$\int \phi\, \dot{\phi}^+\, d^3 r = \frac{i}{2} \, (a_k\, a_k^+ - b_k^+\, b_k) + \text{other terms.}$$

Subtracting $\int \phi^+ \dot{\phi}$ one finds

$$Q = i\, e \sum (a_k^+\, a_k - b_k^+\, b_k)$$

$$= i\, e \sum (N_k^+ - N_k^-) \quad .$$

Thus the fields ϕ , ϕ^{+} represent $\oplus$, $\ominus$ mesons. We can make the following interpretation.

$\phi \longrightarrow$ creates $\ominus$ mesons

annih $\oplus$ mesons

$\phi^{+} \longrightarrow$ creates $\oplus$ mesons

annih. $\ominus$ mesons

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Talk 3

Last time I gave you a theoretically complete formulation of the problem of the neutral scalar field.

This formulation was based on the observation that the classical equation of motion of this field

$$(\Box^2 - \mu^2)\phi = 0$$

may be derived from the Hamiltonian of the system.

$$H = \int \mathcal{H}\, d^3r = \tfrac{1}{2}\int \left[\dot{\phi}^2 + (\nabla\phi)^2 + \mu^2\phi^2\right] d^3r \quad . \qquad (\dot{\phi} = \pi)$$

We proceeded to <u>quantize</u> this field according to the usual formula

$$(q_k, p_\ell) = i\,\delta_{k\ell} \qquad \begin{aligned} q_k &\longrightarrow \phi(r_k) \\ p_k &\longrightarrow \pi(r_k)\Delta V_k \end{aligned} \quad .$$

The relationship between ϕ, π necessary to accomplish the quantization was

$$\left[\phi(r', t), \pi(r, t)\right] = i\,\delta(r - r') \quad .$$

With quantization the Hamiltonian equations were to read

$$i\,\dot{\phi}(r, t) = \left[\phi(r, t), H\right]$$
$$i\,\dot{\pi}(r, t) = \left[\pi(r, t), H\right] \quad .$$

In order to diagonalize our Hamiltonian in plane waves, we introduced a fourier decomposition

$$\phi(r) = \frac{1}{\sqrt{V}} \sum q_k \, e^{i\, k\cdot r} \qquad q_k = q^{+}_{-k}$$

$$\pi(r) = \frac{1}{\sqrt{V}} \sum p_k \, e^{-i\, k\cdot r} \quad . \qquad p_k = p^{+}_{-k}$$

In terms of the variables q_k the Hamiltonian become

$$H = \sum p_k^{+} p_k + E_k^2 \; q_k^{+} q_k$$

$$(q_k, p_\ell) = i \; \delta_{k\ell}$$

In order to complete the diagonalization of H we introduced new variables

$$a_k^{+} = \frac{1}{\sqrt{2}} \left[q_k \sqrt{E_k} + \frac{i \; p_k^{+}}{\sqrt{E_k}} \right]$$

$$a_k = \frac{1}{\sqrt{2}} \left[q_k^{+} \sqrt{E_k} - \frac{i \; p_k}{\sqrt{E_k}} \right]$$

The a_k, a_k^{+} satisfy the following commutation relations

$$\left[a_k, a_{k'}^{+} \right] = \tfrac{1}{2} \left[i(q_k^{+}, p_{k'}^{+}) - i(p_k, q_{k'}) \right]$$

$$\left[a_{k'}^{+}, a_k \right] = \delta_{kk'}$$

and in terms of them

$$H = \tfrac{1}{2} \sum E_k (a_k^{+} a_k + a_k a_k^{+}) \quad .$$

To diagonalize H it was only necessary to observe that the a_k's are the variables about which we talked before. The substitution

$$a_k \psi(N_k) = \sqrt{N_k}\ \psi(N_k - 1)$$

$$a_k^\dagger \psi(N_k) = \sqrt{N_k + 1}\ \psi(N_k + 1)$$

satisfies the commutation relations and diagonalizes H since

$$N_k = a_k^\dagger \quad a_k$$

and

$$H = \sum E_k(a_k^\dagger \ a_k + \tfrac{1}{2})$$

$$= \sum E_k(N_k + \tfrac{1}{2})$$

Let's look at the quantization process from the point of view of the old quantum theory

$$\mathcal{H} = \tfrac{1}{2}\left[\dot{\phi}^2 + (\nabla\phi)^2 + \mu^2\phi^2\right]$$

$$\phi = \frac{1}{\sqrt{V}}\sum A_k\, e^{i(k\cdot r - E_k t)} + A_k^*\, e^{-i(k\cdot r - E_k t)}$$

$$\tfrac{1}{2}\int \dot{\phi}^2\, d^3r = \int \tfrac{1}{2} A_k A_{k'}\, e^{i(k\cdot r - E_k t)}\, e^{i(k'\cdot r - E_{k'} t)} = (-i\,E_k)(-i\,E_{k'})$$

$$+\int \tfrac{1}{2} A_k^* A_{k'}^*\, e^{-i(k\cdot r - E_k t)}\, e^{-i(k'\cdot r - E_{k'} t)}(-E_k E_{k'})$$

$$+ \tfrac{1}{2}\int A_{k'}^* A_k\, e^{i(k\cdot r - E_k t)}\, e^{-i(k'\cdot r - E_{k'} t)}(E_k E_{k'} + \text{c.c.})$$

$$= A_k^* A_k E_k^2 + \tfrac{1}{2}\sum A_k A_{-k}\, e^{-2i E_k t}(-E_k^2) + \tfrac{1}{2}\sum A_k^* A_{-k}^*\, e^{2i E_k t}(-E_k^2)$$

$$\tfrac{1}{2}\phi(\mu^2 - \nabla^2)\phi = \sum A_k^* A_k E_k^2 + \tfrac{1}{2}\sum A_k A_{-k} E_k^2 e^{-2i E_k t} + \tfrac{1}{2}\sum A_k A_{-k} e^{2i E_k t} E_k^2 .$$

Hence

$$H = \sum 2 E_k^2 A_k A_k^* = \sum E_k^2 (A_k^* A_k + A_k A_k^*) .$$

Applying the old quantum conditions

$$H = \sum H_k$$

$$H_k = N_k E_k = N_k E_k$$

one gets

$$E_k^2 A_k^* A_k = N_k E_k$$

or

$$A_k^* A_k = \frac{N_k}{\sqrt{2 E_k}}$$

so with

$$A_k^* \sim \frac{\sqrt{N_k}}{\sqrt{2 E_k}}$$

$$A_k \sim \frac{N_k}{2 E_k}$$

one can accomplish the quantization. Hence if

$$A_k = \frac{a_k}{\sqrt{2 E_k}} \qquad \text{and} \qquad A_k^\dagger = \frac{a_k^\dagger}{\sqrt{2 E_k}}$$

we get the results we have developed before and

$$\phi(r, t) = \frac{1}{\sqrt{V}} \sum \frac{a_k e^{i(k\cdot r - E_k t)}}{\sqrt{2 E_k}} + \frac{a_k^\dagger e^{i(k\cdot r - E_k t)}}{\sqrt{2 E_k}} .$$

The matrix element of ϕ is

$$(N_k \mid \phi(r) \mid N_{k-1}) = \frac{e^{i(k\cdot r - E_k t)}}{\sqrt{2\,E_k}} \qquad \text{(absorption)}$$

$$(N_{k+1} \mid \phi(r) \mid N_k) = \frac{e^{-i(k\cdot r - E_k t)}}{\sqrt{2\,E_k}} \qquad \text{(emission)} \quad .$$

Last time I remarked that there was no variable in the theory to describe the existence of charged meson since the Hamiltonian was completely diagonalized in plane wave states. Thus in order to meet this possibility it is necessary to broaden our framework by introducing another field. We can expect that if we write

$$H = \sum E_k\, a_k^{+} a_k = \sum 2\, E_k^2\, A_k^{*} A_k$$

(quantum mechanics) (classical mechanics)

for mesons of type one and a similar expression for mesons of type two then

$$H = \sum 2\, E_k^2\, (A_k^{*} A_k + B_k^{*} B_k) \quad .$$

If the type A's $\longrightarrow \oplus$ mesons and B's $\longrightarrow \ominus$ mesons we have

$$Q = \sum (A_k^{*} A_k - B_k^{*} B_k) \quad .$$

Let us look at this problem from another point of view. We want to describe charge. Thus an expression is needed which behaves like a charge density. If $t \longrightarrow -t$ it ought to change sign. If we consider <u>two</u> real fields ϕ_1, ϕ_2 we can write

$$\phi_1\, \pi_2 \qquad \text{or} \qquad \pi_1\, \phi_2$$

$$\phi_1\, \dot{\phi}_2 \qquad \text{or} \qquad \dot{\phi}_1\, \phi_2$$

but you can easily verify that although each reverses sign as $t \longrightarrow -t$

only the combination

$$(\phi_1 \pi_2 - \phi_2 \pi_1)$$

is a constant of the motion. We are led to write

$$\rho = i\, e(\phi_1 \pi_2 - \phi_2 \pi_1) \quad .$$

This can be re-written

$$\rho = i\, e \left[\frac{(\phi_1 + i\,\phi_2)}{\sqrt{2}} \frac{(\pi_1 - i\,\pi_2)}{\sqrt{2}} - \frac{(\phi_1 - i\,\phi_2)}{\sqrt{2}} \frac{(\pi_1 + i\,\pi_2)}{\sqrt{2}} \right]$$

$$= i\, e \left[\phi\, \pi - \phi^* \pi^* \right]$$

if we define ϕ, π suitably. Introducing

$$\phi = \frac{1}{\sqrt{V}} \sum A_k\, e^{i(k\cdot r - E_k t)} + B_k^*\, e^{-i(k\cdot r - E_k t)}$$

$$\phi^* = \frac{1}{\sqrt{V}} \sum A_k^*\, e^{-i(k\cdot r - E_k t)} + B_k\, e^{i(k\cdot r - E_k t)}$$

$$A_{-k} = A_k^* \quad ; \quad B_{-k} = B_k^* \quad .$$

One finds

$$\int \rho\, d^3r = i\, e \sum A_k A_k^* (+ i\, E_k)$$

$$- B_k^* B_k^* (i\, E_k)$$

$$- A_k B_{-k} (+ i\, E_k)\, e^{-2i\, E_k t} + A_{-k}^* B_k^* (+ i\, E_k)\, e^{2\, i\, E_k t}$$

$$- \text{complex conjugate.}$$

Hence

$$Q = \int \rho\, d^3r = - i\, e \sum (A_k^* A_k - B_k^* B_k)\, 2\, i\, E_k \quad .$$

Similarly

$$H = H_1 + H_2 = \int \left[\dot{\phi}^* \dot{\phi} + \phi^* (\mu^2 - \nabla^2) \phi \right] d^3 r$$

$$= \sum A_k A_k^* E_k^2 + B_k B_k^* E_k^2 + A_k B_k (-E_k^2) e^{-2i E_k t} + A_k^* B_k^* (-E_k^2) e^{2i E_k t}$$

$$+ \sum A_k A_k^* E_k^2 + B_k B_k^* E_k^2$$

$$+ \sum A_k B_k (E_k^2) e^{-2i E_k t} + \sum A_k^* B_k^* (E_k^2) e^{2i E_k t} .$$

Thus,

$$Q = \sum (A_k^* A_k - B_k^* B_k)(+2 e E_k)$$

$$H = \sum (A_k^* A_k + B_k^* B_k)(2 E_k^2) .$$

Quantizing one has

$$H_k = N_k E_k = N_k^+ E_k + N_k E_k$$

so

$$A_k^* A_k (2 E_k^2) = N_k^+ E_k^+$$

$$A_k^* A_k = \frac{N_k^+}{\sqrt{2 E_k}}$$

so one finds the same result for $B_k^* B_k$ and

$$Q = \sum (N_k^+ - N_k^-)(+e) .$$

Now let us look back at what we have done. We introduced charged mesons by introducing two real fields, ϕ_1, $)\phi_2$.

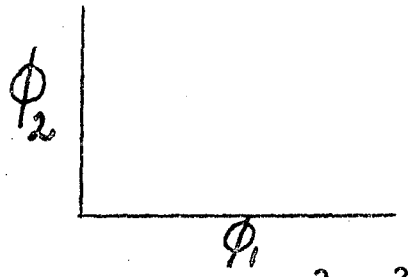
$$\phi = \frac{1}{\sqrt{2}} (\phi_1 + i \phi_2)$$ emission of $\ominus$ mesons abs. of $\oplus$ mesons.

$$\phi^* = \frac{1}{\sqrt{2}} (\phi_1 - i \phi_2)$$ emission of $\oplus$ mesons abs. of $\ominus$ mesons.

The charge

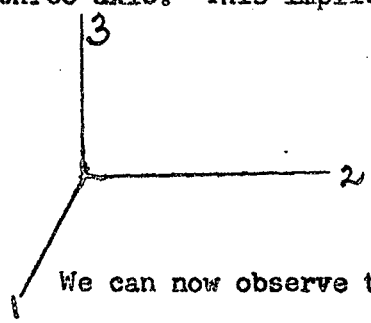
$$\rho = -i\, e(\phi_1 \pi_2 - \phi_2 \pi_1)$$

may be looked upon as the 3-component of angular momentum if ϕ_1, ϕ_2 are regarded as components of a vector ϕ in "isotopic spin space" since



$$L_3 = (\phi \times \pi)_3 .$$

H depends only on ϕ_1^2, ϕ_2^2 and is therefore invariant under rotations about the three axis. This implies L_3 is conserved.



We can now observe that to describe <u>charged</u> and neutral mesons simultaneously we can introduce three fields, ϕ_1, ϕ_2, ϕ_3, and associate.

$\phi = (\phi_1 + i\,\phi_2)\frac{1}{\sqrt{2}}$ emission $\ominus$ mesons

$\phi^* = (\phi_1 - i\,\phi_2)\frac{1}{\sqrt{2}}$ emission $\oplus$ mesons

$\phi_3 = \phi_3$ emission neutral mesons

If ϕ_1, ϕ_2, ϕ_3 are regarded as the components of a vector in "isotopic spin space" we obtain a compact and symmetrical treatment of the two kinds of mesons. Since H is, as before, only a function of ϕ_k^2 it is now invariant under an arbitrary rotation in isotopic spin space. We can introduce

$\vec{\rho} = -i\, e(\vec{\phi} \times \vec{\pi})$, the isotopic angular momentum,

and can specify ρ^2 and ρ_3 independently. ρ_3 is as usual the electric

charge density and ρ^2 is the square of the total isotopic angular momentum.

The problem of non-interacting particles has been thoroughly discussed and we have seen how to conveniently describe charged and neutral mesons. Nucleons must now be introduced into the picture. This can be done by regarding neutron and proton as two states of the same particle. The charge density of such a particle is

$$\rho = e\,\delta(r - r_1). \qquad \begin{matrix} 1 & \text{if a proton} \\ 0 & \text{if a neutron.} \end{matrix}$$

ρ has two eigenvalues. It may be represented by a matrix ρ_{ij} which, barring degeneracy, must therefore be two by two

$$\begin{vmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{vmatrix} .$$

It must be hermitian so $\rho^*_{12} = \rho_{21}$. Its eigenvalues given by

$$\det.(\rho - \lambda I) = 0 \text{ or}$$

$$\begin{vmatrix} \rho_{11} - \lambda & \rho_{12} \\ \rho^*_{12} & \rho_{22} - \lambda \end{vmatrix} = 0 \qquad \text{must be 0, 1.}$$

A possible choice is

$$\begin{vmatrix} 1 - \lambda & 0 \\ 0 & \lambda \end{vmatrix} = 0 \qquad \lambda(1 - \lambda) = 0$$

$$\lambda = 0, 1$$

Hence

$$\rho = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix}$$

$$= \tfrac{1}{2} \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + \tfrac{1}{2} \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}$$

$$= \tfrac{1}{2}(1 + \tau_3) \quad .$$

If the eigenstates of ρ (or τ_3) are

$$\chi_n = \begin{vmatrix} 1 \\ 0 \end{vmatrix} \qquad \chi_p = \begin{vmatrix} 0 \\ 1 \end{vmatrix}$$

then

$$\begin{vmatrix} 0 \\ 1 \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} \begin{vmatrix} 1 \\ 0 \end{vmatrix}$$

$$\chi_p = \tau_- \chi_n$$

also

$$\begin{vmatrix} 1 \\ 0 \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} \begin{vmatrix} 0 \\ 1 \end{vmatrix}$$

$$\chi_n = \tau_+ \chi_p \quad .$$

Hence

$$\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = \rho = \tfrac{1}{2}\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + \tfrac{1}{2}\begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} = \tfrac{1}{2}(1 + \tau_3)$$

$$\begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} = \tau_- = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} - i\begin{vmatrix} 0 & -i \\ i & 0 \end{vmatrix} = (\tau_1 - i\tau_2)$$

$$\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = \tau_+ = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + i\begin{vmatrix} 0 & -i \\ i & 0 \end{vmatrix} = (\tau_1 + i\tau_2) \quad .$$

The state of the nucleon is completely described by τ_1, τ_2, τ_3 where τ_1, τ_2, τ_3 are just the Pauli matrices.

Suppose now we consider interaction between a charged meson field and a nucleon. This interaction must be of such analytical form that it conserves charge;

$$Q = \text{charge of nucleon} + \text{charge of field}$$

$$Q = \tfrac{1}{2}(1 + \tau_3) + (-i\,e) \int d^3r\,(\phi_1 \pi_2 - \phi_2 \pi_1) \; .$$

If the interaction $H_{int.}$ contains a term in ϕ^* (emission of a $+$ charge) it must also contain a term in τ_+ (change a proton to a neutron) and vice versa. Analytically

$\tau_+ \phi^*$ must be in combination

or

$\tau_- \phi$ must be in combination.

If one has a linear coupling of the type found in electromagnetic theory

$$j \cdot A$$

then

$$\mathcal{H}_{int} \sim (f^{+} \tau_{+} \phi^{*} + f^{-} \tau_{-} \phi) \sqrt{2} \quad .$$

If f is a real number then in order for H to be hermitian $f^{+} = f^{-}$ and

$$\mathcal{H}_{int} = \sqrt{2}\, f(\tau_{+} \phi^{*} + \tau_{-} \phi) \qquad \text{or}$$

$$\mathcal{H}_{int} = f\, \tau_1 \phi_1 + \tau_2 \phi_2 = \quad \text{scalar function of } \tau, \phi \, .$$

If one in addition considers neutral mesons one can coule them in two ways:

$$\mathcal{H}_{int} = g\, \phi_3 \qquad \text{to neutrons}$$

$$g\, \phi_3 \qquad \text{to protons}$$

or

$$g\, \phi_3 \qquad \text{to neutrons}$$

$$-g\, \phi_3 \qquad \text{to protons} \quad .$$

The special choice

$$\mathcal{H}_{int} = f(\tau_1 \phi_1 + \tau_2 \phi_2 + \tau_3 \phi_3)$$

is called the symmetrical coupling and has interesting properties since it is invariant under rotations in isotopic spin space and so conserves total isotopic angular momentum as well as charge. It leads also to <u>charge independent</u> nuclear forces. Thus the nucleon can be described by its isotopic spin vector τ in isotopic spin space.

LECTURES ON MESON THEORY

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Talk 4

A theoretical basis has now been laid for the treatment of problems involving meson-nucleon interactions. The two simplest such problems are the problem of a single nucleon and the problem of the scattering of mesons by a single nucleon; moreover, these problems are intimately related and are especially interesting at this time since an understanding of them leads to a simple interpretation of many recent experimental results.

So far we have talked only about scalar mesons but as you know there are many reasons which lead one to the conclusion that the π meson is pseudoscalar and that nuclear forces are charge independent. Accordingly I shall base these discussions on the symmetrical pseudoscalar meson theory, since it leads to charge independent nuclear forces.

A possible hamiltonian for a single nucleon interacting with the meson field will now be written down in an approximation in which the nucleon is supposed not to recoil. In other words we shall think of it as sitting, fixed, at the origin of the coordinate system. Moreover it will be assumed that it is not a point but that it has a finite extension described by a density function $U(r)$.

$$\int U(r)d^3r = 1 \qquad (1)$$

This is a necessary requirement in order to give meaning to some of our mathematical results. To a limited extent such a view would also result from an exact relativistic theory.

It is convenient to introduce a parameter, a, which roughly describes the size of the nucleon.

$$\frac{1}{a} = \int U(r) \frac{1}{|r - r'|} U(r') d^3r \, d^3r' \tag{2}$$

The hamiltonian describing the meson field alone is, as in the scalar theory,

$$H = \int \tfrac{1}{2} \left[\pi_\alpha^2 + \phi_\alpha (\mu^2 - \nabla^2) \phi_\alpha \right] d^3r \quad . \tag{3}$$

There are an infinite number of possible choices for the terms describing the interaction of pseudoscalar mesons with nuclear matter. The simplest choices however are

$$H_{int}^{(1)} = \frac{f}{\mu} \int \sigma \cdot \nabla \phi_\alpha (r) \tau_\alpha \; U(r) d^3r \tag{4}$$

and

$$H_{int}^{(2)} = \frac{g^2}{2M} \int \phi_\alpha^2 (r) \, U(r) d^3r \quad . \tag{5}$$

In these formulae, σ , τ are the nucleon spin and isotopic spin operators, μ , M are the meson and nucleon masses, respectively and f, g are dimensionless coupling constants. The particular combinations of these latter quantities chosen will have some significance when we try to relate the present simplified theory to the exact relativistic one.

The first type of coupling, one which increases in strength linearly with meson momentum, is the so-called pseudovector (or gradient) coupling of the pseudoscalar field. The second corresponds to a scalar pair coupling of the pseudoscalar field. It has many interesting properties since it leads to nuclear forces with range $\hbar/2\mu c$, it dominates the scattering of very low energy mesons by nucleons, and it leads to many body forces which may be used in understanding the saturation properties of nuclei [illegible]

To spin and isotopic spin variables σ, τ satisfy the following commutation relations in a quantum theory

$$\left[\sigma_i, \sigma_j\right] = 2i\,\varepsilon_{ijk}\,\sigma_k \qquad (i, j, k = 1, 2, 3)$$

$$\left[\tau_\alpha, \tau_\beta\right] = 2i\,\varepsilon_{\alpha\beta\gamma}\,\tau_\beta \qquad (\alpha, \beta, \gamma = 1, 2, 3) \tag{6}$$

ε_{ijk} is an alternating symbol which takes the values 1, -1, according to whether i, j, k are an even or odd permutation of 1, 2, 3. It is zero if any two are the same. In a classical theory these commutation relations are replaced by symbolic classical poisson brackets

$$\left[\sigma_i, \sigma_j\right]_{CPB} = 2\,\varepsilon_{ijk}\,\sigma_k$$

$$\left[\tau_\alpha, \tau_\beta\right]_{CPB} = 2\,\varepsilon_{\alpha\beta\gamma}\,\tau_\gamma \tag{7}$$

I mention these facts because it will be convenient in the course of the discussion to distinguish those features of our problems which are purely of classical origin from those which are present only in a quantum theory.

The total hamiltonian, H , from which the equation of motion for $\phi_\alpha, \sigma, \tau$ may be derived is the sum

$$H = H^0 + H^1 + H^2 \quad . \tag{8}$$

The equations of motion for σ, τ are

$$i\,\dot{\sigma}_i = \left[\sigma_i, H\right]; \qquad i\,\dot{\tau}_\alpha = \left[\tau_\alpha, H\right] \tag{9}$$

in a quantum theory and

$$\dot{\sigma}_i = \left[\sigma_i, H\right]_{CPB} \quad ; \quad \dot{\tau}_\alpha = \left[\tau_\alpha, H\right]_{CPB} \tag{10}$$

in a classical theory.

The field equations may be obtained from the hamiltonian equations of motion

$$\dot{\phi}_\alpha(r) = \frac{\partial H}{\partial\left[\pi_\alpha(r)d\tau\right]} \quad ; \quad \dot{\pi}_\alpha(r) = -\frac{\partial H}{\partial \phi_\alpha(r)} \quad . \tag{11}$$

These yield

$$\dot{\phi}_\alpha(r) = \pi_\alpha(r) \quad ; \quad \dot{\pi}_\alpha(r) = \ddot{\phi}_\alpha(r) = -(\mu^2 - \nabla^2)\phi_\alpha + \frac{f}{\mu}\sigma_i \cdot \partial_i U(r)\,\tau_\alpha + \frac{g^2}{2M}\,U(r)\phi_\alpha(r) \tag{12}$$

or

$$(\Box^2 - \mu^2)\phi_\alpha = -\frac{f}{\mu}\sigma_i\,\partial_i\,U(r)\tau_\alpha + \frac{g^2}{2M}\,U(r)\phi_\alpha \quad . \tag{13}$$

It is interesting to note that the last term, arising from the scalar pair coupling, implies an increase in mass of the meson in the vicinity of the source of amount

$$\delta\mu = \left[\frac{g^2}{2M}\,U(r)\right]^{\frac{1}{2}} \quad . \tag{14}$$

If equations (9) are now applied they yield the quantum equation for the motion of spin and isotopic spin

$$\dot{\sigma}_i = 2i\,\frac{f}{\mu}\,\varepsilon_{ijk}\int\sigma_j\,\partial_k\,U(r)\,\tau_\alpha\,\phi_\alpha(r)d^3r \tag{15}$$

$$\dot{\tau}_\alpha = 2i\,\frac{f}{\mu}\,\varepsilon_{\alpha\beta\gamma}\int\tau_\beta\,\phi_\gamma(r)\,\sigma_i\,\partial_i\,U(r)d^3r$$

These equations are of the form one meets in the study of spinning bodies

$$\frac{d\vec{L}}{dt} = \vec{L} \times \vec{T}$$

The torque in this case is that due to the field at the presence of the heavy particle. We shall see that motion of the spin and isctopic spin is responsible for the radiation and scattering of mesons just as the actual physical motion of charge is responsible for radiation in electrodynamics.

We can now ask about the nature of the solutions $\phi_\alpha(r, t)$ to our system of equations. These equations describe all phenomena associated with a single nucleon. Different physical situations are differentiated and described by various types of boundary conditions; for example, we may ask if there any solutions corresponding to bound, i.e. non-radiating, states of the meson nucleon system or what is the solution corresponding to the scattering of mesons. The former situation is distinguished by the boundary condition that $\phi_\alpha(r)$ vanish at remote distances from the nucleon; the latter by finding that solution which corresponds, asymptotically, to an incoming plane wave and an outgoing spherical one.

The most fundamental problem is the first of these although they are intimately connected. Let us try to find a time independent solution to the equations of motion.

$$(\Box^2 - \mu^2)\phi_\alpha(r) = -\frac{f}{\mu}\sigma_i\,\partial_i\,U(r)\,\tau_\alpha + \frac{g^2}{2M}\,U(r)\,\phi_\alpha(r) \tag{16}$$

If the discussion is confined to low lying states of the system it is convenient to set

$$\begin{aligned}
\phi_\alpha &= \phi_\alpha(0) + A_\alpha(E_0)\,e^{iE_0t} + A_\alpha^+(E_0)\,e^{-iE_0t} \\
\sigma_i &= \sigma_i(0) + \sigma_i(E_0)\,e^{iE_0t} + \sigma_i(E_0)\,e^{-iE_0t} \\
\tau_\alpha &= \tau_\alpha(0) + \tau_\alpha(E_0)\,e^{iE_0t} + \tau_\alpha(E_0)\,e^{-iE_0t}
\end{aligned} \tag{17}$$

These lead to the following equations for $\phi_\alpha(0)$, $\phi_\alpha(E_0)$:

$$\left[\nabla^2 - \mu^2 - \frac{g^2}{2M} U(r)\right] \phi_\alpha(0) = -\frac{f}{\mu} \partial_i U(r) \left[\sigma_i(0) \tau_\alpha(0) + \sigma_i(E_0) \tau_\alpha^+(E_0) + \sigma_i^+(E_0) \tau_\alpha(E_0)\right]$$

$$= -\frac{f}{\mu} \partial_i U(r) \Lambda_{i\alpha}(0)$$

$$\left[\nabla^2 - \mu^2 + E_0 - \frac{g^2}{2M} U(r)\right] A_\alpha(E_0) = -\frac{f}{\mu} \partial_i U(r) \left[\sigma_i(0) \tau_\alpha(E_0) + \sigma_i(E_0) \tau_\alpha(0)\right]$$

$$= -\frac{f}{\mu} \partial_i U(r) \Lambda_{i\alpha}(E_0) \quad . \tag{18}$$

To be sure terms involving higher frequencies should have been included but our aim here is to discuss only the low lying states of the system. We shall see that for the pseudoscalar theory the first excited state can be expected to be unbound, i.e. at least of frequency $E_0 > \mu$.

The important point which we learn from Eq. (18) is that in zero'th approximation

$$\phi_\alpha(0) = -\frac{f}{\mu} \frac{\partial_i U(r) \Lambda_{i\alpha}(0)}{\left[\nabla^2 - \mu^2 - \frac{g^2}{2M} U(r)\right]} \quad . \tag{19}$$

If the source function $U(r)$ is replaced by its spectrum

$$U(r) = \frac{1}{(2\pi)^{3/2}} \int U(k) e^{i k \cdot r} d^3 k \tag{20}$$

one finds (setting $\mu_1^2(r) = g^2 U(r)/2M$)

$$\phi_\alpha(0) = \frac{f}{\mu} \frac{\Lambda_1^{(0)} \partial_i}{(2\pi)^{3/2}} \int \frac{U(k) e^{i k \cdot r} d^3 k}{\left[k^2 + \mu^2 + \mu_1^2(r)\right]} \quad . \tag{21}$$

This integral may be evaluated approximately by replacing $U(k)$ by its value for a point source

$$U(k)\Big|_{a=0} = \frac{1}{(2\pi)^{3/2}} \quad . \tag{22}$$

At distances large compared to a, the source size,

$$\phi_\alpha(0) = \frac{f}{\mu} \Lambda_{i\alpha}(0)\, \partial_i \frac{e^{-\mu r}}{4\pi r} \quad . \tag{23}$$

This is the Yukawa potential in the gradient coupling theory. The high singularity of the potential is a consequence of this type of coupling. A rough solution valid in the interior of the source is

$$\phi_\alpha(0) \sim \frac{f}{\mu} \Lambda^{(0)}_{i\alpha}\, \partial_i \frac{e^{-\sqrt{\mu^2+\mu_1^2(r)}\; r}}{4\pi r} \quad . \tag{24}$$

The pair coupling thus causes a dimunition of the field in the interior region. Mesons probing this region behave as if they are very heavy and they are repelled from it.

The solution for $\phi_\alpha(E_0)$ is more interesting. For r a one finds

$$A\ (E_0) = \frac{f}{\mu} \Lambda_{i\alpha}(E_0)\, \partial_i \frac{e^{-\sqrt{\mu^2 - E_0^2}\; r}}{4\pi r} \quad . \tag{25}$$

This solution corresponds to an excited state of the meson nucleon system of larger range than the conventional Yukawa potential. It should be emphasized however that E_0^2 has been assumed smaller than μ^2. We shall soon learn that this is not to be expected for pseudoscalar mesons. This state, although unstable, does however, as we shall see, play a dominant role in the interpretation of high energy meson nucleon scattering since when excited by an external meson beam it leads to an unusually large cross section because of the large spatial

volume it occupies.

An expression for E_o in terms of the coupling constant, f, and the source size, a, can be found by substituting the expressions for the field $\phi_\alpha(r)$ into the equations for the spin motion and demanding that they be consistent with these forms. In carrying out this task one meets a formidable analytical difficulty; the equations for the spin motion are non-linear and are not understood from a mathematical point of view. In order to deal with them at all it is necessary to make a linear approximation, that is, only terms linear in $\sigma_i(E_o)$, $\tau_\alpha(E_o)$ will be retained.

The general solution for $\phi_\alpha(r)$ is

$$\phi_\alpha(r) = \phi_\alpha(0) + A_\alpha(E_o)\, e^{i E_o t} + A^+_\alpha(E_o)\, e^{-i E_o t} + \phi^H_\alpha(r) . \tag{26}$$

$\phi^H_\alpha(r)$ is a solution of the homogeneous equation

$$\left[\Box^2 - \mu^2 - \mu_1^2(r)\right] \phi_\alpha(r) = 0 \tag{27}$$

which for $r > a$ is just a sum of plane waves

$$\phi^H_\alpha(r) = \frac{1}{\sqrt{V}} \sum_k \frac{1}{\sqrt{2 E_k}} \left[a_k\, e^{i k\cdot r} + a^+_k\, e^{-i k\cdot r}\right] . \tag{28}$$

Upon combining equations 15, 18, 19, and making the above mentioned approximations one finds

$$\dot{\sigma}_i(0) = 0$$

$$E_o \sigma_i(E_o) = -2i \frac{f^2}{\mu^2} \varepsilon_{ijk} \int \left[\frac{\sigma_j(0)\, \partial_k U(r)\, \tau_\alpha(0) \sigma_\ell(E_o) \tau_\alpha(0) \partial_\ell U(r)}{\mu^2 - \nabla^2 - E_o^2} + \frac{\sigma_j(E_o)\, \partial_k U(r)\, \tau_\alpha(0) \sigma_\ell(0) \tau_\alpha(0) \partial_\ell U(r)}{\mu^2 - \nabla^2 - E_o^2} \right] d^3$$

Since the integrands are spherically symmetric only terms with $k = \ell$ give any contribution. We may also introduce vector notation at this point since there is no danger of confusing vectors in space with those in isotopic spin space.

$$E_o \sigma(E_o) = -\frac{2i f^2}{3\mu^2} \gamma^2(0) \sigma(0) \times \sigma(E_o) \left[I(E_o) - I(0) \right] \tag{30}$$

where

$$I(E_o) = \int \frac{\left[\partial_k U(r)\right]^2}{\mu^2 - \nabla^2 - E_o^2} d^3r$$

$$I(0) = \int \frac{\left[\partial_k U(r)\right]^2}{\mu^2 - \nabla^2} d^3r \quad . \tag{31}$$

The analogous equation for $\tau_\alpha(E_o)$ yields

$$E_o \tau_\alpha(E_o) = \frac{-2 i f^2}{3\mu^2} \sigma^2(0) \tau(0) \times \tau(E_o) \left[I(E_o) - I(0) \right] \tag{32}$$

so that the motions of spin and isotopic spin are identical.

If one now takes the vector product of $\sigma(0)$ and equation 30

$$E_o \sigma(0) \times \sigma(E_o) = \frac{-2i f^2}{3\mu^2} \tau_\alpha^2(0) \left[I(E_o) - I(0) \right] \sigma(0) \times \left[\sigma(0) \times \sigma(E_o) \right]$$

and employs the relation

$$\sigma(0) \times \left[\sigma(0) \times \sigma(E_o) \right] = -\sigma^2(0) \sigma(E_o) \tag{33}$$

(since $\sigma(0) \cdot \sigma(E_o) = 0$) one finds

$$\sigma(0) \times \left[\sigma^R(E_o) + i\sigma^I(E_o)\right] = -\frac{2f^2}{3\mu^2}\frac{\gamma^2(0)\,\sigma^2(0)}{E_o}\left[I(E_o) - I(0)\right]$$

$$\times \left[i\sigma^R(E_o) - \sigma^I(E_o)\right]$$

or

$$\sigma(0) \times \sigma^R(E_o) = +C\,\sigma^I(E_o)$$

$$\sigma(0) \times \sigma^I(E_o) = -C\,\sigma^R(E_o) \tag{34}$$

where σ^R, σ^I are the real and imaginary parts of $\sigma(E_o)$ and

$$C = \frac{2f^2}{3\mu^2}\frac{\gamma^2(0)\,\sigma^2(0)}{E_o}\left[I(E_o) - I(0)\right] \quad . \tag{35}$$

These show that $\sigma(0)$, $\sigma^R(E_o)$, $\sigma^I(E_o)$ form a system of vectors at right angles to each other. Also from (33) and (34),

$$\sigma(0) \times \left[\sigma(0) \times \sigma^R(E_o)\right] = C\,\sigma(0) \times \sigma^I(E_o)$$

$$-\sigma^2(0)\,\sigma^R(E_o) = -C^2\,\sigma_R(E_o)$$

or

$$C = \sqrt{\sigma^2(0)} \quad . \tag{36}$$

This is the relation which determines the energy, E_o, of a possible excited state of the meson-nucleon system. The integrals $I(0)$, $I(E_o)$ may be evaluated by introducing the Fourier transform of the source function $U(r)$.

$$U(r) = \frac{1}{(2\pi)^{3/2}}\int U(k)\,e^{i\,k\cdot r}\,d^3k$$

$$I(0) = \frac{-1}{(2\pi)^3} \int \frac{U(k)\, U(-k')\, e^{i(k-k')\cdot r}\, (k\cdot k')\; d^3k\, d^3k'\, d^3r}{k^2+\mu^2} \tag{37}$$

$$= \int \frac{|U(k)|^2\, k^2\, d^3k}{k^2+\mu^2} = 4\pi \int_0^\infty \frac{|U(k)|^2\, k^4\, dk}{k^2+\mu^2} \quad .$$

It is now convenient to split the integral into those parts which would be infinite for a point source, $1/a = \infty$, and those which would be finite. Write

$$\frac{k^4}{k^2+\mu^2} = (k^2-\mu^2) + \frac{\mu^4}{k^2+\mu^2} \quad ;$$

a similar reduction for $I(E_o)$ involves

$$\frac{k^4}{k^2+\mu^2-E_o^2} = k^2-\mu^2+E_o^2+\frac{(\mu^2-E_o^2)^2}{k^2+\mu^2-E_o^2}$$

so that the difference $I(E_o) - I(0)$ is just

$$I(E_o) - I(0) = 4\pi \int_0^\infty |U(k)|^2\, dk \left\{ E_o^2 + \frac{(\mu^2-E_o^2)^2}{k^2+\mu^2-E_o^2} - \frac{\mu^4}{k^2+\mu^2} \right\}. \tag{38}$$

It may be readily verified that the first term depends on the source size defined by equation (2) which when expressed in terms of $U(k)$ is

$$\int |U(k)|^2\, dk = \frac{1}{(4\pi)^2\, a} \quad . \tag{39}$$

The remaining integrals in (38), finite for a point source may be approximated by writing $U(k) = 1/(2\pi)^{3/2}$, the value of $U(k)$ appropriate to such a source. Then provided $E_o \ll \mu$,

$$\frac{1}{(2\pi)^3}\int_0^\infty \frac{dk}{k^2+\mu^2-E_o^2} = \frac{1}{(2\pi)^3}\cdot\frac{\pi}{2}\cdot\frac{1}{\sqrt{\mu^2-E_o^2}} .$$

Upon combining all these results

$$I(E_o) - I(0) = \frac{1}{4\pi}\left[\frac{E_o^2}{a} + (\mu^2 - E_o^2)^{3/2} - \mu^3\right] . \tag{40}$$

For $E_o \ll \mu$ this simplifies to

$$I(E_o) - I(0) = \frac{E_o^2}{4\pi}\mu\left[\frac{1}{a\mu} - \frac{3}{2}\right] . \tag{41}$$

If (41) is combined with (35) and (36), one finds

$$\left(\frac{E_o}{\mu}\right) = \frac{1}{\left(\frac{f^2}{4\pi}\right)\gamma^2(0)\sqrt{\sigma^2(0)}\left[\frac{2}{3\,a\mu} - 1\right]} . \tag{42}$$

Using (42) we can now conclude that <u>no stable isobars are possible in the pseudoscalar theory with gradient coupling</u>. Estimates of the coupling constant $f^2/4\pi$ always yield a value

$$0 \leq f^2/4\pi \leq 0.3$$

and we may expect that

$$a \geq \frac{1}{M} .$$

the nucleon compton wave length; furthermore $\gamma^2(0)$, $\sigma^2(0)$ are of order of magnitude unity so one finds

$$E_o/\mu > 1 \tag{43}$$

contrary to the supposition $E_o < \mu$ which has been used in deriving this result. One can verify that this same result holds provided one employs the

more accurate expression (40) to determine E_o.

It has been remarked before that although no stable excited state of the meson nucleon system may be expected to exist it does however play an important role in high energy meson processes. The relation (42) always defines an energy of a meson mode which is easily excited by an external stimulus, for example a beam of mesons, a beam of photons or a nucleon impact. The situation is analogous to a circuit containing resistance inductance and capacitance. In the absence of resistance the resonant frequency (stationary state) of the system is $\omega = (LC)^{-\frac{1}{2}}$; even in the presence of resistance the same formula serves to approximately define the frequency of that mode of oscillation of the system most easily excited by a generator placed in the circuit. Because of these features it is of importance to examine all properties of an isobaric state of the nucleon meson system. Of particular interest are the spin and isotopic spin values associated with this state.

Before discussing the above features it is necessary to examine the equation of motion of the isotopic spin. It may be treated in precisely the same manner as the equation for the spin to yield

$$C = \sqrt{\gamma^2(0)} \quad . \tag{44}$$

This result implies that

$$\gamma^2(0) = \sigma^2(0) \quad . \tag{45}$$

The expression for the angular momentum of the meson-nucleon system is

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[1] G. Wentzel, "Quantum Theory of Fields", page 10.

$$J = \frac{\sigma}{2} = \int d^3r \left[r \times \nabla \phi_\alpha \right] \dot{\phi}_\alpha \quad . \tag{46}$$

This formula shows that no angular momentum is present in the field due to its static component which varies as $e^{-\mu r}$ since the canonical momentum $\pi_\alpha^{(0)} = \dot{\phi}_\alpha(0)$ associated with it vanishes. On the other hand

$$\pi_\alpha(E_o) = i E_o \left[A_\alpha(E_o) e^{i E_o t} - A_\alpha^+(E_o) e^{-i E_o t} \right] . \tag{47}$$

One finds

$$J = \frac{\sigma}{2} - i \int d^3r \; r \times \left[\nabla A_\alpha(E_o) A_\alpha^+(E_o) - \nabla A_\alpha^+(E_o) A_\alpha(E_o) \right] E_o$$
$$- i \int d^3r \; r \times \nabla \phi_\alpha(0) \left[A_\alpha(E_o) e^{i E_o t} - A_\alpha^+(E_o) e^{-i E_o t} \right] E_o \; . \tag{48}$$

It is easy to verify that the time dependent terms are cancelled, in the expression for the total angular momentum, J, by the corresponding non-secular terms in σ. Thus

$$J = \frac{\sigma}{2}(0) - i \int d^3r \; r \times \left[\nabla A_\alpha(E_o) A_\alpha^+(E_o) - \nabla A_\alpha^+(E_o) A_\alpha(E_o) \right] E_o \; . \tag{49}$$

Upon using the expressions for $A_\alpha(E_o)$, $A^+(E_o)$ one finds

$$J = \frac{\sigma(0)}{2} + \frac{i f^2}{\mu^2} \Big\{ \left[\sigma_\ell(0) \sigma_m(0) \right] \left[\tau_\alpha(E_o) \tau_\alpha^+(E_o) + \tau_\alpha^+(E_o) \tau_\alpha(E_o) \right]$$
$$+ \tau_\alpha^2(0) \left[\sigma_\ell(E) \sigma_m^+(E) + \sigma_m^+(E) \sigma_\ell(E) \right] \Big\} F_{\ell m} E_o \tag{50}$$

where

$$F_{\ell m} = \int d^3r \; \frac{\left[r \times \nabla \partial_\ell U \right]}{(\mu^2 - \nabla^2 - E_o^2)^2} \partial_m U \; . \tag{51}$$

This integral may be evaluated to yield

$$\frac{\epsilon_{i\ell m}}{3} \int \frac{(\nabla U)^2 \, d^3}{(\mu^2 - \nabla^2 - E_o^{\,2})^2} \qquad \text{or}$$

$$\frac{\epsilon_{i\ell m}}{12\pi} \left[\frac{1}{a} - \frac{3}{2} (\mu^2 - E_o^{\,2})^{\frac{1}{2}} \right] . \tag{52}$$

Combining this with (50) one obtains

$$J = \frac{\sigma(0)}{2} + 2\,i\left(\frac{f^2}{4\pi}\right) \frac{\tau_\alpha^2(0)}{3\mu^2} \left[\sigma(E) \times \sigma^+(E)\right] \cdot \left[\frac{1}{a} - \frac{3}{2}(\mu^2 - E_o^{\,2})^{\frac{1}{2}}\right] E_o \tag{53}$$

with

$$\sigma(E_o) \times \sigma^+(E_o) = -2\,i\,\sigma_R(E_o) \times \sigma_I(E_o)$$

$$= -2\,i\,\sigma_R^2(E_o) \frac{\sigma(0)}{\sqrt{\sigma^2(0)}}$$

equation (53) becomes

$$J = \frac{\sigma(0)}{2} + \left(\frac{f^2}{4\pi}\right) \frac{\tau^2(0)}{3\mu^2} (3 - \sigma^2(0)) \left[\frac{1}{a} - \frac{3}{2}(\mu^2 - E_o^{\,2})^{\frac{1}{2}}\right] E_o \frac{\sigma(0)}{\sqrt{\sigma^2(0)}} .$$

This shows that the angular momentum associated with the field is in the direction $\sigma(0)$. By using equation (42) this may be simplified to

$$J = \frac{\sigma(0)}{2} + \frac{\sigma(0)}{\sqrt{\sigma^2(0)}} . \tag{54}$$

At this point it is useful to note that the requirements of the equations of motion for σ_i, τ_α may be met by the following substitutions.

$$\sigma_1(0) = 0 \quad ; \quad \sigma_1(E_o) = \sigma_+ \quad ; \quad \sigma_1^+(E_o) = \sigma_-$$

$$\sigma_2(0) = 0 \quad ; \quad \sigma_2(E_o) = -i\,\sigma_+ \quad ; \quad \sigma_2^+(E_o) = i\,\sigma_-$$

$$\sigma_3(0) = \sigma_3 \quad ; \quad \sigma_3(E_o) = 0 \quad ; \quad \sigma_3^+(E_o) = 0 \tag{55}$$

Where $\sigma_{\pm} = \frac{1}{\sqrt{2}} (\sigma_1 \pm i \sigma_2)$, etc. are defined in terms of the Pauli matrices $\sigma_1, \sigma_2, \sigma_3$ introduced earlier.

This implies that the angular momentum associated with the isobaric state of the nucleon, E_o , is

$$J = \frac{3}{2} \sigma(0) \quad . \tag{56}$$

In other words the angular momentum associated with this state is 3/2. It should be noted that J^2 cannot be computed using (56) since products of terms in $e^{i E_o t}$, $e^{-i E_o t}$ which we have neglected contribute secular terms to J^2.

We now may determine in precisely the same fashion the value of the isotopic spin associated with this state.

$$T_\alpha = -\frac{\tau_\alpha}{2} + \int d^3r \, \epsilon_{\alpha\beta\gamma} \phi_\beta \pi_\gamma \, . \tag{57}$$

By using (47) and (17) one may verify that the time dependent terms in (57) cancel as they did in equation (48) leaving only

$$T_\alpha = -\frac{\tau_\alpha(0)}{2} + i \int d^3r \, \epsilon_{\alpha\beta\gamma} \left[A_\beta(E_o) A^+_\gamma(E_o) - A^+_\beta(E_o) A_\gamma(E_o) \right] E_o$$

$$T_\alpha = -\frac{\tau_\alpha(0)}{2} - \left(\frac{e^2}{4\pi}\right) \frac{\sigma^2(0)}{3\mu^2} \left(3 - \tau_\alpha^2(0)\right) \frac{\tau_\alpha(0)}{\sqrt{\tau_\alpha^2(0)}} \left[\frac{1}{a} - \frac{3}{2} (\mu^2 - E_o^2)^{\frac{1}{2}} \right] E_o$$

If we use equation (42) this may be reduced to

$$-T_\alpha = \frac{3}{2} \tau_\alpha(0) \quad . \tag{58}$$

This relation shows that the <u>isotopic spin</u> associated <u>with the isobaric state</u> <u>E_o</u> , has the value 3/2 .

The results of the foregoing section show that there are no stable isobaric states of a nucleon in the symmetrical pseudoscalar theory because of the smallness of the coupling constant. The properties of such states have been discussed in detail however since one expects that although they are unstable they may be excited by an external meson beam and thus influence the scattering of mesons. It has been shown that the first excited state of the meson nucleon system has spin and isotopic spin 3/2. One may therefore expect strong resonance scattering of mesons when this state is excited. That this is indeed the case will be shown in future talks.