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UCRL LECTURES ON NUMERICAL ANALYSIS AND APPLIED MATHEMATICS

Lecture XIX

February 24, 1953

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APPLICATIONS OF THE CALCULUS OF VARIATIONS

1. Introduction

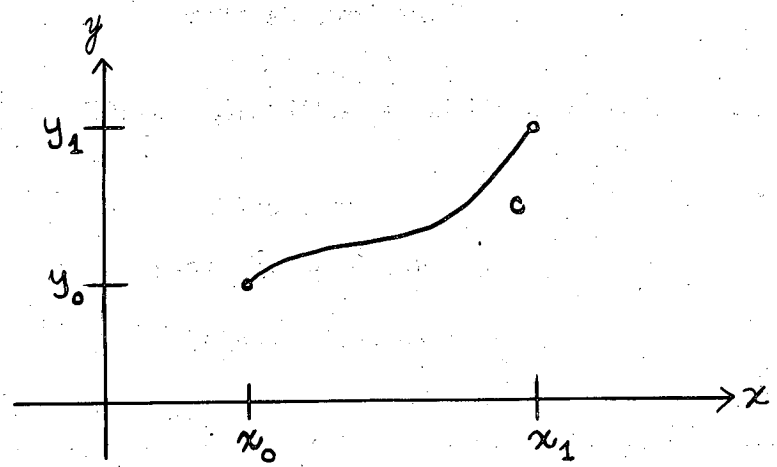
By applying the methods of the calculus of variations, we will demonstrate how to calculate the eigenvalues of certain ordinary differential equations. For example, consider the simple problem

$$y'' + \lambda y = 0$$

with boundary conditions: $y(-1) = y(+1) = 0$. The task is to determine the smallest number λ for which this problem has a non-trivial solution. We will change this problem into one involving integrals since it is so much more pleasant to deal with integrals than with derivatives in most numerical problems.

2. The Euler Equation

Suppose we are given an arbitrary curve c, with continuous derivatives, joining two given fixed points (x_0, y_0) and (x_1, y_1) in the plane



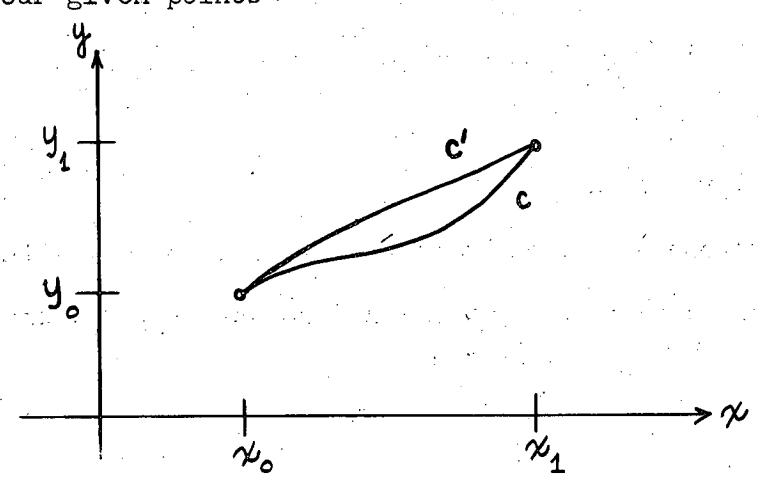
and also a given function $f(x, y, y')$. Then consider the integral

$$I = \int_{x_0}^{x_1} f(x, y, y')dx \quad .$$

For example, the arc length of the curve from $(x_0\ y_0)$ to (x_1, y_1) is given by such an integral,

$$I_{(c)} = S = \int_{x_0}^{x_1} \sqrt{1 + (y')^2}\ dx \quad , \qquad y = y_{(c)}(x) \ .$$

We then consider the value of this integral I over another curve c' passing thru our given points



$$I_{(c')} = \int_{x_0}^{x_1} f(x, y, y')\, dx \quad , \qquad\qquad y = y_{(c')}(x) \; .$$

We may consider many such curves $c, c', c'', \ldots$ and in each case we get a number $I_c, I_{c'}, I_{c''}, \ldots$. Then our problem is to <u>determine the curve $y = y(x)$ for which I is a minimum</u>. In our particular example of $f(x, y, y') = \sqrt{1 + y'^2}$ this amounts to finding the curve c joining (x_0, y_0) and (x_1, y_1) having the smallest arc length. Of course in this example the answer is known to be a straight line and in fact we shall derive this after treating the general case.

To solve our general problem, suppose the solution is already known;

$$\bar{y} = \bar{y}(x) \; .$$

We will then derive a differential equation that $\bar{y}$ must satisfy. Let $\eta(x)$ be an arbitrary function such that

$$\eta(x_0) = \eta(x_1) = 0 \; .$$

Then for any number ε

$$\bar{y}(x) + \varepsilon\eta(x) \equiv y(x)$$

is a curve which also passes thru (x_0, y_0) and (x_1, y_1). Furthermore, if $\eta(x)$ is a decently behaving function, $y(x)$ will be a curve which lies near the solution $\bar{y}(x)$ if we make ε small and which coincides with $\bar{y}(x)$ when $\varepsilon = 0$. For each of these curves corresponding to a particular ε we get a particular integral (number)

$$I(\varepsilon) = \int_{x_0}^{x_1} f(x, y, y')dx = \int_{x_0}^{x_1} f(x, \bar{y} + \varepsilon\eta, \bar{y}' + \varepsilon\eta')dx .$$

Then the numbers $I(\varepsilon)$ must have, by assumption, a minimum at $\varepsilon = 0$, i.e.

$$0 = \left.\frac{dI(\varepsilon)}{d\varepsilon}\right]_{\varepsilon = 0} = \left.\frac{d}{d\varepsilon}\int_{x_0}^{x_1} f(x, \bar{y} + \varepsilon\eta, \bar{y}' + \varepsilon\eta')dx\right]_{\varepsilon=0} .$$

But

$$\frac{d}{d\varepsilon}\int_{x_0}^{x_1} f(x, \bar{y} + \varepsilon\eta, \bar{y}' + \varepsilon\eta')dx = \int_{x_0}^{x_1} \frac{\partial}{\partial\varepsilon} f(x, \bar{y} + \varepsilon\eta, \bar{y}' + \varepsilon\eta')dx$$

$$= \left(\text{putting } \bar{y} + \varepsilon\eta = \alpha , \qquad \bar{y}' + \varepsilon\eta' = \beta\right)$$

$$= \int_{x_0}^{x_1} \left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial\varepsilon} + \frac{\partial f}{\partial\alpha}\frac{\partial\alpha}{\partial\varepsilon} + \frac{\partial f}{\partial\beta}\frac{\partial\beta}{\partial\varepsilon}\right) dx$$

$$= \int_{x_0}^{x_1} \left(0 + \frac{\partial f}{\partial\alpha}\cdot\eta + \frac{\partial f}{\partial\beta}\eta'\right) dx .$$

Thus on putting $\varepsilon = 0$ and noting that

$$\left.\frac{\partial f}{\partial \alpha}\right]_{\varepsilon=0} = \frac{\partial f}{\partial \bar{y}} \quad \text{and} \quad \left.\frac{\partial f}{\partial \beta}\right]_{\varepsilon=0} = \frac{\partial f}{\partial \bar{y}'}$$

our condition is

$$0 = \left.\frac{dI(\varepsilon)}{d\varepsilon}\right]_{\varepsilon=0} = \int_{x_0}^{x_1} \left(\frac{\partial f}{\partial \bar{y}}\,\eta + \frac{\partial f}{\partial \bar{y}'}\,\eta'\right) dx \ .$$

Now integration by parts shows that

$$\int_{x_0}^{x_1} \frac{\partial f}{\partial \bar{y}'}\,\eta'\,dx = \left.\frac{\partial f}{\partial \bar{y}'}\cdot\eta\right]_{x_0}^{x_1} - \int_{x_0}^{x_1} \eta\,\frac{d}{dx}\left(\frac{\partial f}{\partial \bar{y}'}\right) dx$$

$$= -\int_{x_0}^{x_1} \eta\,\frac{d}{dx}\left(\frac{\partial f}{\partial \bar{y}'}\right) dx$$

since $\eta(x_0) = \eta(x_1) = 0$. Hence

$$0 = \left.\frac{dI(\varepsilon)}{d\varepsilon}\right]_{\varepsilon=0} = \int_{x_0}^{x_1} \left[\frac{\partial f}{\partial \bar{y}} - \frac{d}{dx}\left(\frac{\partial f}{\partial \bar{y}'}\right)\right] \eta(x)\,dx \ .$$

Since $\eta(x)$ is arbitrary it is easy to show that the integrand must vanish, and hence we get the <u>Euler equation</u>,

$$\frac{\partial f}{\partial \bar{y}} - \frac{d}{dx}\frac{\partial f}{\partial \bar{y}'} = 0 ,$$

as the condition our solution must satisfy. (Drop bars from now on.)

In our previous example

$$f(x, y, y') = \left[1 + y'^2\right]^{\frac{1}{2}} .$$

Hence

$$\begin{cases} \dfrac{\partial f}{\partial y} = 0 \\[2ex] \dfrac{\partial f}{\partial y'} = \dfrac{y'}{\left[1 + y'^2\right]^{\frac{1}{2}}} \end{cases}$$

and the Euler equation becomes

$$-\frac{d}{dx}\left[\frac{y'}{\left[1 + y'^2\right]^{\frac{1}{2}}}\right] = 0 .$$

Thus

$$\left[\frac{y'}{\left[1 + y'^2\right]^{\frac{1}{2}}}\right] = \text{constant.}$$

Thus

$$y' = \text{constant}$$

and we get a straight line as promised. The constants are chosen so that the solution actually passes thru (x_o, y_o) and (x_1, y_1).

3. Isoperimetric Problem

Suppose now we require that

$$\begin{cases} J(y) = \int_{x_0}^{x_1} f(x, y, y')dx & \text{be a minimum} \\ K(y) = \int_{x_0}^{x_1} g(x, y, y')dx = 1 . \end{cases}$$

and

simultaneously, where g is some given function. Then it can be shown that the solution $y = y(x)$ which satisfies the above conditions is such that if we put

$$F = f - \lambda g , \qquad \lambda \text{ a parameter}$$

and solve

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} = 0$$

we get a solution $y = y(x, \lambda)$. Putting this y into

$$K = \int_{y_0}^{x_1} g\left[x, y(x, \lambda), y'(x, \lambda)\right] dx$$

we determine λ.

For example we may require $y(a) = y(b) = 0$ and

$$J(y) = \int_a^b \left[p(x)y'^2 + q(x)y^2\right] dx = \text{minimum}$$

and

$$K(y) = \int_a^b \rho(x) y^2 \, dx = 1 .$$

We then put

$$F = py'^2 + qy^2 - \lambda \rho y^2 .$$

Now a most important fact is that it can be shown that if $y = y(x)$ is the minimizing solution, then

$$J(y) = \lambda$$

and we can reverse our procedure as follows. Suppose we are given a differential equation

$$(py')' - qy - \lambda \rho y = 0 \qquad y(a) = y(b) = 0$$

where λ is an unknown constant. Now in general the above equation has a solution only for particular values of λ and it is these values of λ, especially the smallest one, which are needed. We then use the Rayleigh Ritz method.

4. Rayleigh Ritz Method for Smallest λ

Consider the following problem

$$y'' + \lambda y = 0 , \qquad y(-1) = y(+1) = 0$$

where λ is unknown. Now this is the Euler equation for the problem

$$\text{and} \quad \begin{cases} J(y) = \displaystyle\int_{-1}^{+1} y'^2\, dx = \text{minimum} \\ K(y) = \displaystyle\int_{-1}^{+1} y^2\, dx = 1 \end{cases}$$

or equivalently, the Euler equation for

$$H = \int_{-1}^{+1} \left[y'^2 - \lambda y^2 \right] dx = \text{minimum.}$$

We try as our first approximation a function $y(x)$ <u>which satisfies the boundary conditions</u>; say

$$y = a(1 - x^2).$$

Then

$$y^2 = a^2(1 - x^2)^2$$

$$y' = -2\,ax\,.$$

Therefore

$$H(a) = \int_{-1}^{+1} (y'^2 - \lambda y^2)dx = \int_{-1}^{+1} \left[4\,a^2\,x^2 - \lambda \left\{ a^2(1 - x^2)^2 \right\} \right] dx$$

$$= a^2 \int_{-1}^{+1} \left[4\,x^2 - \lambda \left\{ 1 - 2\,x^2 + x^4 \right\} \right] dx\,.$$

To minimize $H(a)$, put

$$0 = \frac{dH(a)}{da} = 2a \int_{-1}^{+1} \left[4x^2 - \lambda \left\{ 1 - 2x^2 + x^4 \right\} \right] dx .$$

Hence

$$\frac{4}{3} - \lambda \left\{ 1 - \frac{2}{3} + \frac{1}{5} \right\} = 0 .$$

Thus

$$\lambda = \frac{4/3}{8/15} = \frac{5}{2} = 2.5 .$$

As our second approximation, put

$$y = (1 - x^2) \left[a_1 + a_2 x^2 \right]$$

again noting that $y(-1) = y(+1) = 0$. Then puting this into

$$H(a_1\ a_2) = \int_{-1}^{+1} (y'^2 - \lambda y^2) dx$$

we then demand

$$0 = \frac{\partial H}{\partial a_1} = a_1 \left(1 - \frac{2\lambda}{5} \right) + a_2 \left(\frac{1}{5} - \frac{2\lambda}{35} \right) ,$$

$$0 = \frac{\partial H}{\partial a_2} = a_1 \left(1 - \frac{2\lambda}{7} \right) + a_2 \left(\frac{11}{7} - \frac{2\lambda}{21} \right) .$$

In order that these equations have a non-trivial solution for a_1 and a_2, we must have

$$0 = \Delta(\lambda) \equiv \begin{vmatrix} \left(1 - \frac{2\lambda}{5}\right) & \left(\frac{1}{5} - \frac{2\lambda}{35}\right) \\ \left(1 - \frac{2\lambda}{7}\right) & \left(\frac{11}{7} - \frac{2\lambda}{21}\right) \end{vmatrix}$$

i.e., $$\lambda^2 - 28\lambda + 63 = 0 ,$$

i.e., $$\lambda = 2.46744 \quad \text{and}$$
$$\lambda = 25.53256 ,$$

but we must choose the smallest λ , hence

$$\lambda = 2.46744 . \qquad \text{(True answer is 2.467401....)}$$

The procedure for higher approximations is evident.

5. The Adjoint Form

We now indicate a method of putting second order equations into a more pleasant form. Consider the general linear second order homogeneous equation

$$p_o(x)y'' + p_1(x)y' + p_2(x)y = 0 .$$

Then the adjoint form is (if it exists)

$$(py')' + V p_2 y = 0$$

where

$$\left\{ \begin{aligned} p &= c\, e^{\int (p_1/p_o)\, dx} \\ V &= \frac{p}{p_o} . \end{aligned} \right\}$$

First example:

$$y'' + \lambda y = 0$$

$$(y')' + \lambda y = 0 \qquad \text{is the adjoint form.}$$

Second example: Consider the (Bessel's) equation

$$x^2 y'' + xy' + (\lambda^2 x^2 - 1)y = 0 \qquad\qquad y(0) = y(1) = 0 \ .$$

Then

$$p = c\, e^{\int \frac{x}{x^2} dx} = cx$$

$$V = \frac{cx}{x^2} = \frac{c}{x} \ .$$

The adjoint form is

$$\left.\begin{aligned} &(cxy')' + \frac{c}{x}(\lambda^2 x^2 - 1)y = 0 \\ \text{or} \\ &(xy')' - \frac{1}{x} y + \lambda^2 xy = 0 \end{aligned}\right\} \quad \text{where} \quad \begin{aligned} p(x) &= x \\ q(x) &= \frac{1}{x} \\ \rho(x) &= x \ . \end{aligned}$$

Try $y = a\,x(1 - x)$. Then

$$J(y) = \int_0^1 (py'^2 + qy^2)dx = \int_0^1 \left\{ x\left[1 - 2x\right]^2 a^2 + \frac{1}{x} a^2 x^2 (1 - x)^2 \right\} dx$$

$$K(y) = \int_0^1 \rho y^2\, dx = \int_0^1 a^2 x^3 (1 - x)^2\, dx$$

$$H(a) \equiv J(y) - \lambda^2 K(y)$$

$$0 = \frac{\partial H}{\partial a} = 2a \int_0^1 \left\{ x\left[1 - 2x\right]^2 + \frac{1}{x} x^2(1-x)^2 - \lambda^2 x^3(1-x)^2 \right\} dx .$$

Therefore,

$$\lambda^2 = \frac{\int_0^1 \left\{ x\left[1 - 2x\right]^2 + \frac{1}{x} x^2(1-x)^2 \right\} dx}{\int_0^1 x^3 (1-x)^2 dx} = \frac{1/4}{1/60} = 15 .$$

6. Bibliography

6.1 Courant, R., Differential and Integral Calculus.

6.2 Courant and Hilbert, Methoden der Mathematischen Physik I.

6.3 Margenau and Murphy, The Mathematics of Physics and Chemistry.