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THE DISTRIBUTIONS IN ENERGY AND IN NUMBER OF ELECTRONS AND PHOTONS IN CASCADE

Derived with Energy-Dependent Bremsstrahlung, Materialization, Collision Loss and Multiple Scattering

Bayard Rankin

September 17, 1953

Berkeley, California

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ABSTRACT

Diffusion equations for a cascade shower which include (1) an energy cutoff (below which the electrons and photons are no longer observed) and (2) energy parameters for each particle have not previously been studied. In this paper it is found that an inclusion of both permits collision loss to be represented as a discrete process by modifying the differential radiation cross section. It is assumed that (in addition to the continuous spectrum of radiation losses) energy is lost through collision in amounts equal to the cutoff energy. The foreshortening of the shower penetration by multiple scattering is accomplished by increasing the radiation and collision loss cross sections by the ratio of the integrated to the projected electron path length. The correction ratio can be made a function of energy, as can the cross sections. Full screening approximations are unnecessary. After integrating over the energy variables and resorting to the matrix notation of Messel and Potts, one writes and solves the equations as a double recurrence relation. The distribution functions themselves are obtained, rather than their moments. They are expressed in terms of products and sums of matrices and one-dimensional integrations. The mathematical approximations necessary to bring a solution appear in the energy integrations and become better when the energy cutoff is small compared to the energy of the shower primary. With better approximation, however, more addition and multiplication of matrices are required in order to achieve a numerical solution.

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INTRODUCTION

The cascade phenomenon that occurs when high-energy electrons and photons penetrate matter is difficult to investigate because of the complex branching probabilities and the interdependence in energy among the particles. In the first place, the cross sections for materialization and radiation loss become inhomogeneous functions* of energy when the screening of the nucleus by the outer electrons is not complete. The behavior of the cross sections complicates the usual continuity equations. In the second place, the conservation of energy leaves the particles genetically dependent after each branching. For that reason more parameters enter the equations. Beyond the two chief difficulties it is impractical to incorporate in the symbolism the three-dimensional behavior that develops at lower energies. It is hard to describe mathematically the multiple energy degradations that occur with increasing importance along an electron's path, the Compton collisions of the photon, and the final removal of the particles from the shower altogether. Writers do not even consider processes such as trident production by electrons and positron annihilation.

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* A cross section is a homogeneous function of the primary energy, if its product with the primary energy is a function of a single variable only, the variable being the ratio of one of the secondary energies to the primary energy. This definition holds only when the total energy of the primary is shared between the two secondaries.

Some idealization of the physical phenomenon is imperative in order to bring a solution. The common idealization is to imagine showers initiated by primaries of infinitely high energy. In this case, often approached in cosmic ray studies, one-dimensional development is predominant; the number of electrons and photons with unspecified energies changes only as a function of absorber thickness and primary energy. The initiating particle is little deflected by the coulomb fields of the nuclei, multiple scattering is slight, and its momentum is transferred through its secondaries largely into the forward direction. The same is true of each member in the shower over the greater part of the shower's development, because energy is uniformly split, approximately, at the branch points. Hence, asymptotic cross sections, correct for the high-energy particles that establish the shower characteristics, can be used throughout. Two processes only are assumed: bremsstrahlung and pair production. An energy cutoff, below which particles are no longer counted, usually serves to diminish the cascade population after the maximum in particle number has been reached. When chosen wisely (to be about half the energy at which collision losses equal radiation losses) it also approximates the effect of collision loss.

Any further refinement of the above idealization, such as inclusion of exact collision loss or multiple scattering, seems to change the basic structure of the diffusion equations, so far as they have been investigated up to the present. Consequently, few mathematical techniques are available for moderate-energy showers. It is partly for this reason that a different approach to the cascade problem is attempted here. An equation that includes an energy cutoff and also energy parameters for each particle has not previously been studied. Jánossy included an energy cutoff in his equations without requiring energy parameters, while others have included energy parameters without a cutoff. In this paper it is found that an inclusion of both permits collision loss to be represented as a discrete process by modifying the differential radiation cross section. It is assumed that (in addition to continuous spectrum of radiation losses) energy is lost through collision in amounts equal to the cutoff energy. The foreshortening of the shower penetration by multiple scattering is accomplished by increasing the radiation and collision loss cross sections by the

ratio of the integrated to the projected electron path length. The correction ratio can be made a function of energy just as the cross sections are. Similiar modifications to account for Compton effect and angular emission at branching are also feasible, though they are not tried here. There is a further advantage: The distribution functions themselves can be solved for, approximately, rather than their moments. This will be made clear after reviewing briefly the work of other authors.

Nothing will be said about the valuable work on the first moment of the electron distribution beyond a remark: The complexity of the results made it look hopeless[1] even to write down the equations for the distribution itself. Bhabha, Heitler,[2] Furry,[3] and Arley,[4] who were among the first to investigate the one-dimensional number distribution of electrons beyond its first moment, constructed simplified models. With models some qualitative idea could be gotten about the shape of the distribution, but without a quantum mechanical calculation for the second moment even the half-width was in doubt. Collision loss and removal of the particles from the shower were not adequately represented in the models, if at all; hence, the structure of the distribution as a function of thickness was virtually unknown. Arley[4] did simulate the effects of collision loss by assuming that each particle had a probability proportional to the thickness of "dying" in a differential increment of path. Later Messel,[5] following up the calculations of Jánossy and Messel,[6] did insert a reliable second moment into a Pólya distribution for the number of electrons. There was reason to believe[4] that the Pólya distribution, lying between the Poisson and the Furry distributions, was somewhere near the truth.

Scott and Uhlenbeck[7] in 1942 published the first in a series of more or less independent papers in which they, and later Bhabha,[8] Bhabha and Ramakrishnan,[9] and Ramakrishnan[10,11] all contributed to the development of a powerful technique for finding the number distribution of electrons and photons. The essential entity is a function of the energy parameters (variously called product density, correlation function, and moment density) which is related to the moments through summation and energy integration and which is expressible in terms of more elementary functions. Practically, the technique

has yielded good expressions for the second moments.[7,9] Theoretically, it has yielded the distribution,[12] though the answer is not explicit and is far from a workable form. A cutoff calculation, excluding the possibility of particles below a given energy, has been used by most of these writers to account for collision loss.

Simultaneously with the above studies, Jánossy[13,14,15] investigated the moment-generating function for the distribution in number of electrons and photons above a given energy. His unique contributions are his incorporation of an energy cutoff in the distribution function and his use of the multiplicative property of the stochastic process in the equations for the distribution. (The electron-photon phenomenon is multiplicative in the sense that each particle generates an independent subpopulation.) Ideally, his equations, too, will yield the desired distribution after tedious iteration and use of numerical methods, but only the first step - the first two moments and the correlation coefficient[6] between electrons and photons - has been achieved.

Still a fourth technique, exploited fully for the first time by Messel and Potts[16,17], has characteristics in common with the two preceding methods: It uses a moment density function but includes a cutoff in the energy integration, so that the moments of Jánossy's distribution are obtained.* It is a natural outgrowth of studies on nucleon cascades[18,19] though the essential logic, barring the energy cutoff and the solution, appears in Scott's[20] independent and suggestive article. The equations for the density function are linear. After carrying out a single Laplace and n + m fold Mellin transform and resorting to matrix notation, one can write down a solution in the transform space. Of course, a numerical answer for the moments can be recovered after one performs an n + m + 1 fold inverse integral transformation numerically. When collision loss is included as a constant energy degradation, an approximation must be used by way of an expansion.

None of the above methods is adequate when multiple scattering or Compton effect is important. The lack of mathematical techniques for moderate-energy showers is made more obvious by R. R. Wilson's[21] introduction of a

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* Messel's original proof[18] of the relation between moment densities and moments used an incorrect equation similar to Jánossy's original G-equation.[14] No errors seem to have resulted.

Monte Carlo method for that region. He includes Compton scattering as well as bremsstrahlung and materialization, for particles are observed to zero velocity in his game. Collision loss appears as a constant energy degradation and multiple scattering enters as an independent correction. The valuable results obtained shed light on the mathematical problem by emphasizing the need for collision loss and multiple scattering.

Aside from R. R. Wilson, the model builders, together with the three groups represented by Uhlenbeck, Jánossy, and Messel, have been the prominent investigators into the sixteen-year-old problem.* They have with rare exceptions approached the distribution of the number of electrons or the number of electrons and photons through its moments. One exception in Scott and one in Messel occurs when a formula is given for the probability of n particles of specified energies. In each case the formulas are derived with asymptotic cross sections and without collision loss or cutoff. They will not be physically meaningful, as each author remarks, until an infinite number of particle energies is integrated out below some energy ϵ. In view of this difficulty and the above mentioned complexities in the moments, it is generally felt that an explicit expression for physically meaningful probabilities would be prohibitive. On the contrary, it is one purpose of this paper to show that a set of equations can be written down whose solution is the probability distribution in energy and number of electrons and photons above a cutoff ϵ.

The basic equations presented here are first simplified by integrating over energy increments equal to the cutoff energy. Slight approximations are used in the process. After resorting to the matrix notation of Messel and Potts, one writes the equations as a double recurrence relation. The first of the recurrence variables represents the total number of observable particles in the shower and the second their total energy. The recurrence relation is then solved in analogy to a shower's development: low number and high energy first, with high number and low energy depending on previous calculations. Instead of a Laplace transform on the penetration variable, as used by Messel and

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* The extensive researches into the first moment can be examined independently of the fluctuation problem.

Potts, an exponential matrix method is used. In other words, the solution is essentially in series form. These and many other cascade shower equations are derivable from the Kolmogoroff-Chapman equations. The present equations and Messel's are both "forward" diffusion equations in Kolmogoroff's[22] terminology, while Jánossy's come from a "backwards" equation. On the other hand, the present equations and Jánossy's are both satisfied by distributions that include cutoffs. In effect, the advantages of Jánossy's method can be gained without dependence on the multiplicative property which brings non-linearity.

It should be pointed out that the present paper is in keeping with the trend in recent years to be unbaffled by the complicated form of a complete solution. The important thing is to see that a solution is well suited for electronic machines, for they provide us with numerical techniques that were unheard of by the original investigators into the cascade problem. In the present case, the basic operations in obtaining a numerical solution are multiplication and addition of matrices and one-dimensional integration. There are no multiple integrals to compute (except for an especially accurate computation of a few coefficients), since the integrations over the energy variables are already carried out in increment steps in this paper. It will be clear that the solution developed here for moderate-energy showers (primary energies well above the critical energy) will take on a simpler form in the asymptotic case of infinite-energy primaries. The numerical solution for the ultra-high-energy case will be practical, however, only if the cutoff energy is increased above the critical energy.

THE EQUATIONS*

For the purpose of this paper, the following assumptions are made:

(1) the cascade shower is one dimensional in the sense defined in the introduction and shower penetration is measured in radiation lengths;

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* The notation used by Messel and Potts[16,17] will be adhered to as closely as possible.

(2) only particles of energy greater than or equal to ϵ $(0 \leqslant \epsilon < 1)$ are under observation, the primary having unit energy;

(3) three processes are operative: energy loss by electron collisions, radiation loss by electrons, and pair production by photons;

(4) the differential cross sections for radiation loss and pair production are given by

$$w^{(1)}(u-\eta, \eta)\, d\eta = d\eta/\eta\left[g^{(1)}(u, \eta/u)\right] \text{and}$$
$$w^{(2)}(u-\eta, \eta)\, d\eta = d\eta/u\left[g^{(2)}(u, \eta/u)\right],$$

where u represents the primary energy and η, in the first case, represents the photon energy and, in the second case, the electron energy; the g's, as functions of η/u, are slowly varying over most of the range and $g^{(2)}$ is symmetric about the point $\eta/u = 1/2$, as it is, for instance, when computed under the Born approximation;

(5) an electron of energy η can lose energy (in addition to radiation losses) only in amounts ϵ and the cross section per radiation length for this loss is $e(\eta)/\epsilon$, where $e(\eta)$ is the true collision loss per radiation length for an electron of energy η;

(6) the effect of multiple scattering by an electron of energy η is to increase the cross sections for radiation and collision loss by the factor $s(\eta)$, where $bs^{-1}(\eta)$ is the average distance that an electron of energy η would penetrate into the scattering medium after entering normally to the surface and traveling b radiation lengths (the primary electron is identified with its secondary if radiation occurs); b is some constant which for simplicity can be chosen as 1;

(7) $\epsilon \gg \mu c^2/E_0$, where E_0 is the energy of the shower primary and μc^2 is the rest energy of the electron (the mathematical approximations used to solve the equations are better if it is also true that $\epsilon \ll 1$).

It is convenient to say that the system under observation is the set of particles whose energies are greater than or equal to ϵ. The state of the system S_a at a fixed thickness a is a random variable whose value in a particular shower is a point $\lambda_{n,m} = (n, m; \eta_1, \ldots, \eta_n; \eta_{n+1}, \ldots, \eta_{n+m})$, $n = 1, 2, \ldots,$ $m = 1, 2, \ldots, \epsilon \leq \eta_1 \leq \ldots \leq \eta_n \leq 1, \epsilon \leq \eta_{n+1} \leq \ldots \leq \eta_{n+m} \leq 1, \sum_{i=1}^{n+m} \eta_i \leq 1,$ representing n electrons of energies $\eta_1, \ldots, \eta_n$ and m photons of energies $\eta_{n+1}, \ldots, \eta_{n+m}$, so long as there is at least one electron and one photon; otherwise its value is one of the points $\lambda_{n,0} = (n, 0; \eta_1, \ldots, \eta_n; 0)$, $\lambda_{0,m} = (0, m; 0; \eta'_1, \ldots, \eta'_m)$, or $\lambda_{0,0} = (0, 0; 0, 0)$, $n \geq 1$, $m \geq 1$, $\epsilon \leq \eta_1 \leq \ldots \leq \eta_n \leq 1$, $\epsilon \leq \eta'_1 \leq \ldots \leq \eta'_m \leq 1$, $\sum_{i=1}^{n} \eta_i \leq 1$, $\sum_{i=1}^{m} \eta'_i \leq 1$, with obvious interpretation.

For mathematical clarity let $\Omega_{n,m}$ be the set of all points of the above form with fixed n, m, $\Omega = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \Omega_{n,m}$, β a σ-field of sets in Ω. The state of the system

$$S = \{S_a, 0 \leq a < \infty\}$$

is a random function which takes its value in the space Ω. It corresponds to a Markoff (or stochastically definite) process in the sense that the probability distribution of S_t, given $\{S_y; y \leq \tau\}$, $\tau < t$, and that of S_t, given S_τ, are the same. In other words, the future state of the system is independent of the history prior to τ, so long as S_τ is known. This point has been under question by Bartlett and Kendall,[23] but is seems that the whole difficulty lies in whether the system is defined as all particles or just those above an energy ϵ. In the former case, $\lambda_{n,m}$ would be an incomplete description of a Markovian system, and would in general lead to a non-Markovian process. Owing to the conservation of energy, however, the set of particles that are left out of the description, those below the limit of observation ϵ, have no influence on the future state of the system of particles above ϵ. It should also be noticed here and in Appendix III that the Markoff property has nothing to do with the multiplicative property necessary to derive Jánossy's equations.

Letting λ be any point in Ω and Λ any set $\Lambda \subset \Omega$, $F(\tau;\ \lambda:\ t:\Lambda)$ is defined as the conditional probability of S_t being contained in Λ, given that at a fixed depth $\tau < t$, S_τ coincided with the point λ of Ω. It is clear that F will be a nonnegative, completely additive function of sets on $\mathcal{B}$ and will satisfy the continuity and measurability conditions stated by Feller[24] for a Markoff process. Moreover, it can be seen that F satisfies all the conditions of a purely discontinuous Markoff process. Along with the normalizing and boundary conditions, F satisfies the Kolmogoroff-Chapman equation:

$$F(y;\ \lambda:\ x;\Lambda) = \int_\Omega F(y;\ \lambda:\ z:\ d\Omega_\mu)\, F(z;\ \mu:\ x;\Lambda) \tag{1}$$

In Appendix III the following equations for the symmetric densities $f^{(j)}_{n,m}(\eta_1, \ldots, \eta_n;\ \eta_{n+1}, \ldots, \eta_{n+m};\ x)$ are derived rigorously from (1) and the symmetric densities are defined in terms of $F(\tau;\ \lambda:\ t;\Lambda)$. Here it is sufficient to describe $f^{(j)}_{n,m}(\eta_1, \ldots, \eta_n;\ \eta_{n+1}, \ldots, \eta_{n+m};\ x)$, $n = 0, 1, 2, \ldots$, $m = 0, 1, 2, \ldots$, $\epsilon \leqslant \eta_k \leqslant 1$, intuitively as the probability density for finding n electrons of energies $\eta_1, \ldots, \eta_n$ and m photons of energies $\eta_{n+1}, \ldots, \eta_{n+m}$ at thickness x under the condition that they were initiated by an electron (in the case $j = 1$) or a photon (in the case $j = 2$) of unit energy. Since they are symmetric in the electron energies and in the photon energies, though the electrons or photons are indistinguishable among themselves, it is clear that a normalization factor, $(n!\ m!)^{-1}$, is necessary when integrating. The divergence in the second term on the left-hand side of (2e) and the third term on the right-hand side will be found in Appendix I to cancel. The positive number c that appears, for instance, in the sixth term on the right hand side of (2e) prevents divergence of the integral. It is chosen from the interval $0 < c < \epsilon$.

$$f^{(j)}_{0,0}(0;\ 0;\ 0) = 0, \tag{2a}$$

$$f^{(j)}_{n,m}(\eta_1, \ldots, \eta_n;\ \eta_{n+1}, \ldots, \eta_{n+m};\ 0) = \delta(n+j-2)\,\delta(m+1-j)\,\delta(1-\eta_1),\quad n+m>0,$$

$$f^{(j)}_{n,m}(\eta_1, \ldots, \eta_n;\ \eta_{n+1}, \ldots, \eta_{n+m};\ x) = 0 \quad \text{If}\ \Big(\sum_{i=1}^{n+m} \eta_i > 1\Big)\ \text{or}\ (\eta_k < \epsilon,\ k = 1, 2, \ldots, n+m),$$

$$x > 0,\quad n+m > 0.$$

$$\frac{\partial}{\partial x} f_{0,0}^{(j)}(0;0;x) = \int_{\epsilon}^{2\epsilon} du\, f_{1,0}^{(j)}(u;0;x) \int_{\mathrm{Max}[c,\,u-\epsilon]}^{\epsilon} dv\, s(u)\, w^{(1)}(u-v,\, v)$$

$$+ \int_{\epsilon}^{2\epsilon} du\, f_{0,1}^{(j)}(0;u;x) \int_{u-\epsilon}^{\epsilon} dv\, w^{(2)}(u-v,\, v) \tag{2b}$$

$$+ \int_{\epsilon}^{2\epsilon} du\, f_{1,0}^{(j)}(u;0;x)\, s(u)\, \frac{e(u)}{\epsilon}.$$

$$\left(\frac{\partial}{\partial x} + \int^{\eta_1} dv\, s(\eta_1)\, w^{(1)}(\eta_1 - v,\, v) + s(\eta_1)\, \frac{e(\eta_1)}{\epsilon}\right) f_{1,0}^{(j)}(\eta_1;0;x)$$

$$= \int_{\eta_1}^{\eta_1+\epsilon^0} du\, f_{1,0}^{(j)}(u;0;x)\, s(u)\, w^{(1)}(\eta_1,\, u-\eta_1)$$

$$+ \int_{\eta_1}^{\eta_1+\epsilon} du\, f_{0,1}^{(j)}(0;u;x)\, 2\, w^{(2)}(u-\eta_1,\, \eta_1)$$

$$+ f_{1,0}^{(j)}(\eta_1+\epsilon;0;x)\, s(\eta_1+\epsilon)\, \frac{e(\eta_1+\epsilon)}{\epsilon}$$

$$+ \int_{\epsilon}^{2\epsilon} du\, f_{2,0}^{(j)}(\eta_1,\, u;0;x) \int_{\mathrm{Max}[c,\,u-\epsilon]}^{\epsilon} dv\, s(u)\, w^{(1)}(u-v,\, v) \tag{2c}$$

$$+ \int_{\epsilon}^{2\epsilon} du\, f_{1,1}^{(j)}(\eta_1;u;x) \int_{u-\epsilon}^{\epsilon} dv\, w^{(2)}(u-v,\, v)$$

$$+ \int_{\epsilon}^{2\epsilon} du\, f_{2,0}^{(j)}(\eta_1,\, u;0;x)\, s(u)\, \frac{e(u)}{\epsilon}.$$

$$\left(\frac{\partial}{\partial x} + \int_0^{\eta_1} dv\, w^{(2)}(\eta_1 - v,\ v)\right) f_{0,1}^{(j)}(0;\ \eta_1;\ x)$$

$$= \int_{\eta_1}^{\eta_1+\epsilon} du\, f_{1,0}^{(j)}(u;\ 0;\ x)\, s(u)\, w^{(1)}(u - \eta_1,\ \eta_1)$$

$$+ \int_{\epsilon}^{2\epsilon} du\, f_{1,1}^{(j)}(u;\ \eta_1;\ x) \int_{\mathrm{Max}[c,\ u-\epsilon]}^{\epsilon} dv\, s(u)\, w^{(1)}(u - v,\ v) \tag{2d}$$

$$+ \int_{\epsilon}^{2\epsilon} du\, f_{0,2}^{(j)}(0;\ \eta_1,\ u;\ x) \int_{u-\epsilon}^{\epsilon} dv\, w^{(2)}(u - v,\ v)$$

$$+ \int_{\epsilon}^{2\epsilon} du\, f_{1,1}^{(j)}(u;\ \eta_1;\ x)\, s(u) \frac{e(u)}{\epsilon}\ .$$

$$\left(\frac{\partial}{\partial x}+\sum_{C_1^n}\int_0^{\eta_1'} dv\, s(\eta_n')\, w^{(1)}(\eta_n'-v,\, v)+\sum_{C_1^n} s(\eta_n')\frac{e(\eta_n')}{\epsilon}+\sum_{C_1^m}\int_0^{\eta_{n+m}'} dv\, w^{(2)}(\eta_{n+m}'-v,\, v)\right)\cdot$$

$$\cdot\, f^{(j)}_{n,m}(\eta_1,\ldots,\eta_n;\,\eta_{n+1},\ldots,\eta_{n+m};\,x)$$

$$=\sum_{C_1^n}\sum_{C_1^m} f^{(j)}_{n,m-1}(\eta_1',\ldots,\eta_{n-1}',\,\eta_n'+\eta_{n+m}';\,\eta_{n+1}',\ldots,\eta_{n+m-1}';\,x)\, s(\eta_n'+\eta_{n+m}')\, w^{(1)}(\eta_n',\,\eta_{n+m}')$$

$$+\sum_{C_2^n} f^{(j)}_{n-2,\,m+1}(\eta_1',\ldots,\eta_{n-2}';\,\eta_{n+1},\ldots,\eta_{n+m},\,\eta_{n-1}'+\eta_n';\,x)\, w^{(2)}(\eta_{n-1}',\,\eta_n')$$

$$+\sum_{C_1^n}\int_{\eta_n'}^{\eta_n'+\epsilon} du\, f^{(j)}_{n,m}(\eta_1',\ldots,\eta_{n-1}'\,u;\,\eta_{n+1},\ldots,\eta_{n+m};\,x)\, s(u)\, w^{(1)}(\eta_n',\,u-\eta_n')$$

$$+\sum_{C_1^m}\int_{\eta_{n+m}'}^{\eta_{n+m}'+\epsilon} du\, f^{(j)}_{n+1,\,m-1}(\eta_1,\ldots,\eta_n,\,u;\,\eta_{n+1}',\ldots,\eta_{n+m-1}';\,x)\, s(u)\, w^{(1)}(u-\eta_{n+m}',\,\eta_{n+m}')$$

$$+\sum_{C_1^n}\int_{\eta_n'}^{\eta_{n+\epsilon}'} du\, f^{(j)}_{n-1,\,m+1}(\eta_1',\ldots,\eta_{n-1}';\,\eta_{n+1},\ldots,\eta_{n+m},\,u;\,x)\, 2\, w^{(2)}(u-\eta_n',\,\eta_n') \tag{2e}$$

$$+\int_{\epsilon}^{2\epsilon} du\, f^{(j)}_{n+1,\,m}(\eta_1,\ldots,\eta_n,\,u;\,\eta_{n+1},\ldots,\eta_{n+m};\,x)\int_{\mathrm{Max}[c,\,u-\epsilon]}^{\epsilon} dv\, s(u)\, w^{(1)}(u-v,\,v)$$

$$+\int_{\epsilon}^{2\epsilon} du\, f^{(j)}_{n,\,m+1}(\eta_1,\ldots,\eta_n;\,\eta_{n+1},\ldots,\eta_{n+m},\,u;\,x)\int_{u-\epsilon}^{\epsilon} dv\, w^{(2)}(u-v,\,v)$$

$$+\sum_{C_1^n} f^{(j)}_{n,m}(\eta_1',\ldots,\eta_{n-1}',\,\eta_n'+\epsilon;\,\eta_{n+1},\ldots,\eta_{n+m};\,x)\, s(\eta_n'+\epsilon)\,\frac{e(\eta_n'+\epsilon)}{\epsilon}$$

$$+\int_{\epsilon}^{2\epsilon} du\, f^{(j)}_{n+1,\,m}(\eta_1,\ldots,\eta_n,\,u;\,\eta_{n+1},\ldots,\eta_{n+m};\,x)\, s(u)\,\frac{e(u)}{\epsilon},\quad n+m>1.$$

In (2e), the sums C_1^n and C_2^n signify summations over all possible choices of η_n' and η_{n-1}', η_n' respectively from the η_k, $k = 1, \ldots, n$ and the sum over C_1^m signifies summation over all possible choices of η_{n+m}' from the $\eta_{n+\ell}$, $\ell = 1, \ldots, m$.

The differential operator in (2e) acting on the density function represents the change within dx in the probability density that is associated with the state $\lambda_{n,m} = (n, m; \eta_1, \ldots, \eta_n; \eta_{n+1}, \ldots, \eta_{n+m})$. This change is equal to the flux of probability into the state minus the flux away from the state, of which the first is represented by the sum of the right-hand terms and the second by the sum of the three remaining left-hand terms. Each of the first seven terms on the right corresponds to a bremsstrahlung or materialization process which during the increment of path dx brings the system into the state $\lambda_{n,m}$. The last two terms correspond to collision losses which do the same thing. In the first two terms on the right, both secondaries remain in the observable energy range. In the third and fifth terms only the secondary electron remains observable, while in the fourth only the photon does. Both secondaries fall below the observable energy limit ϵ in the sixth and seventh terms.

The first two terms on the right of (2e) appear in Messel's[16] equations for the corresponding densities when $\epsilon = 0$ and $s(\eta) = 1$. They remain when $\epsilon = 0$ in (2e), which is as it should be. The next three terms are similar, except for the limits, to the corresponding terms in Messel's moment density equations. When $1 - \sum_{i=1}^{n+m} \eta_i < \epsilon$, $n + m > 0$, $s(\eta) = e(\eta)/\epsilon = 1$, and the cross sections are specialized to full-screening approximations, the densities in (2) become identical to Messel's moment densities[17] and the equations, except for the collision loss terms, are the same, accordingly. In that case, the sixth, seventh, and ninth terms drop out and the remaining integrands go to zero before the upper limits of the integrations are reached. The remaining left- and right-hand collision loss terms combine here to give a difference instead of a differential coefficient.

THE SOLUTION

The first step toward the solution of (2) is to convert the equations from integro-differential-difference form to differential-difference form. This is done by applying one of the operators,

$$\int_{a_1}^{a_1+\epsilon} d\eta_1 \cdots \int_{a_{n+m}}^{a_{n+m}+\epsilon} d\eta_{n+m}, \qquad a_k = \mu_k \epsilon,\ \mu_k = 1, 2, \ldots$$

$$\sum_{k=1}^{n+m} a_k = \left(\left[\frac{1}{\epsilon}\right] - \nu\right)\epsilon,\ \nu = 0, 1, 2, \ldots, \left[\frac{1}{\epsilon}\right] - (n+m),$$

to both sides, where $[1/\epsilon]$ is the greater integer in $1/\epsilon$. The solution of the converted equations then becomes the integral function,

$$F_{n,m,\nu}(a_1, \ldots, a_n; a_{n+1}, \ldots, a_{n+m}; x; \epsilon) = \int_{a_1}^{a_1+\epsilon} du_1 \cdots \int_{a_{n+m}}^{a_{n+m}+\epsilon} du_{n+m}\ f^{(j)}_{n,m}(u_1, \ldots, u_n; u_{n+1}, \ldots, u_{n+m}; x),$$

$n + m > 0$, which conveniently gives the number distribution, $P^{(j)}_{n,m}(x)$, of electrons and photons at thickness x through a finite summation:

$$P^{(j)}_{0,0}(x) = f^{(j)}_{0,0}(0; 0; x)$$

$$P^{(j)}_{n,m}(x) = \frac{1}{n!\,m!} \sum_{\nu=0}^{[1/\epsilon]-(n+m)} \sum F^{(j)}_{n,m,\nu}(a_1, \ldots a_n; a_{n+1}, \ldots, a_{n+m}; x; \epsilon),\ n+m > 0, \tag{3}$$

where the second summation is extended over all ways of selecting the $a_k = \mu_k \epsilon$, $\mu_k = 1, 2, \ldots$, so that $\sum_{k=1}^{n+m} a_k = ([1/\epsilon] - \nu)\epsilon$. The converted equations which follow are exactly equivalent to (2), if the coefficients G_k, $B^{\epsilon}_{k,\ell}$, $B_{k,\ell}$, and R_k are understood to be weakly dependent on the unknown probability densities. Otherwise, they are close approximations to (2), becoming more exact as ϵ becomes small compared to 1. Practically, the weak dependence of the

coefficients on the densities will be ignored, so that they can be computed from the known cross sections and the multiple scattering factor. Appendix I should be consulted for their explicit form with and without approximation. The necessary transformations and integrations by parts which yield the following are also evident there.

$$f^{(j)}_{0,0}(0;\,0;\,0) = 0, \qquad J = \begin{cases} 1 & \text{If } 1/\epsilon \text{ is an integer.} \\ 0 & \text{Otherwise.} \end{cases} \tag{4a}$$

$$F^{(j)}_{n,m,\nu}(a_1, \ldots, a_n; a_{n+1}, \ldots, a_{n+m}; 0;\,\epsilon) = \delta(n+j-2)\,\delta(m+1-j)\,\delta(\nu - J),\ n+m>0,$$

$$F^{(j)}_{n,m,\nu}(a_1, \ldots, a_n; a_{n+1}, \ldots, a_{n+m}; x;\,\epsilon) = 0 \qquad \text{If } (\nu<0) \text{ or } (a_k<\epsilon,\ k=1,2,\ldots,n+m),$$

$$x>0,\ n+m>0.$$

$$\begin{aligned} \frac{\partial}{\partial x} f^{(j)}_{0,0}(0;\,0;\,x) &= F^{(j)}_{1,0,[1/\epsilon]-1}(a_1;\,0;\,x;\,\epsilon)\,R_1(\epsilon;\,\epsilon) \\ &\quad + F^{(j)}_{0,1,[1/\epsilon]-1}(0;\,a_1;\,x;\,\epsilon)\,R_2(\epsilon;\,\epsilon). \end{aligned} \tag{4b}$$

$$\begin{aligned} &\left(\frac{\partial}{\partial x} + B_{1,1}(a_1;\,a_1;\,\epsilon)\right) F^{(j)}_{1,0,\nu}(a_1;\,0;\,x;\,\epsilon) \\ &= -F^{(j)}_{0,1,\nu}(0;\,a_1;\,x;\,\epsilon)\,B_{1,2}(a_1;\,a_1;\,\epsilon) \\ &\quad + F^{(j)}_{0,1,\nu-1}(0;\,a_1+\epsilon;\,x;\,\epsilon)\,B^{\epsilon}_{1,2}(a_1;\,a_1;\,\epsilon) \\ &\quad + F^{(j)}_{1,0,\nu-1}(a_1+\epsilon;\,0;\,x;\,\epsilon)\,B^{\epsilon}_{1,1}(a_1;\,a_1;\,\epsilon) \\ &\quad + F^{(j)}_{2,0,\nu-1}(a_1,\,\epsilon;\,0;\,x;\,\epsilon)\,R_1(a_1;\,\epsilon) \\ &\quad + F^{(j)}_{1,1\nu-1}(a_1;\,\epsilon;\,x;\,\epsilon)\,R_2(a_1;\,\epsilon). \end{aligned} \tag{4c}$$

$$\left(\frac{\partial}{\partial x} + B_{2,2}(a_1; a_1; \epsilon)\right) F^{(j)}_{0,1,\nu}(0; a_1; x; \epsilon)$$

$$= - F^{(j)}_{1,0,\nu}(a_1; 0; x; \epsilon)\, B_{2,1}(a_1; a_1; \epsilon)$$

$$+ F^{(j)}_{1,0,\nu-1}(a_1 + \epsilon; 0; x; \epsilon)\, B^{\epsilon}_{2,1}(a_1; a_1; \epsilon) \tag{4d}$$

$$+ F^{(j)}_{1,1,\nu-1}(\epsilon; a_1; x; \epsilon)\, R_1(a_1; \epsilon)$$

$$+ F^{(j)}_{0,2,\nu-1}(0; a_1, \epsilon; x; \epsilon)\, R_2(a_1; \epsilon).$$

$$\left(\frac{\partial}{\partial x} + \sum_{C_1^n} B_{1,1}(a_1, \ldots, a_{n+m}; a_n'; \epsilon) + \sum_{C_1^m} B_{2,2}(a_1, \ldots, a_{n+m}; a_{n+m}'; \epsilon)\right)$$

$$\cdot F^{(j)}_{n,m,\nu}(a_1, \ldots, a_n; a_{n+1}, \ldots, a_{n+m}; x; \epsilon)$$

$$= \sum_{C_1^n} \sum_{C_1^m} F_{n,m-1,\nu}(a_1', \ldots, a_{n-1}', a_n' + a_{n+m}'; a_{n+1}', \ldots, a_{n+m-1}'; x; \epsilon)\, G_1(a_1, \ldots, a_{n+m}; a_n', a_{n+m}'; \epsilon)$$

$$+ \sum_{C_2^n} F^{(j)}_{n-2,m+1,\nu}(a_1', \ldots, a_{n-2}'; a_{n+1}, \ldots, a_{n+m}, a_{n-1}' + a_n'; x; \epsilon)\, G_2(a_1, \ldots, a_{n+m}; a_{n-1}', a_n'; \epsilon)$$

$$- \sum_{C_1^m} F^{(j)}_{n+1,m-1,\nu}(a_1, \ldots, a_n, a_{n+m}'; a_{n+1}', \ldots, a_{n+m-1}'; x; \epsilon)\, B_{2,1}(a_1, \ldots, a_{n+m}; a_{n+m}'; \epsilon)$$

$$- \sum_{C_1^n} F^{(j)}_{n-1,m+1,\nu}(a_1', \ldots, a_{n-1}'; a_{n+1}, \ldots, a_{n+m}, a_n'; x; \epsilon)\, B_{1,2}(a_1, \ldots, a_{n+m}; a_n'; \epsilon)$$

$$+ \sum_{C_1^n} F^{(j)}_{n,m,\nu-1}(a_1', \ldots, a_{n-1}', a_n' + \epsilon; a_{n+1}, \ldots, a_{n+m}; x; \epsilon)\, B^{\epsilon}_{1,1}(a_1, \ldots, a_{n+m}; a_n'; \epsilon) \tag{4e}$$

$$+ \sum_{C_1^m} F^{(j)}_{n+1,m-1,\nu-1}(a_1, \ldots, a_n, a_{n+m}' + \epsilon; a_{n+1}', \ldots, a_{n+m-1}'; x; \epsilon)\, B^{\epsilon}_{2,1}(a_1, \ldots, a_{n+m}; a_{n+m}' \epsilon)$$

$$+ \sum_{C_1^n} F^{(j)}_{n-1,m+1,\nu-1}(a_1', \ldots, a_{n-1}'; a_{n+1}, \ldots, a_{n+m}, a_n' + \epsilon; x; \epsilon)\, B^{\epsilon}_{1,2}(a_1, \ldots, a_{n+m}; a_n'; \epsilon)$$

$$+ \quad F^{(j)}_{n+1,m,\nu-1}(a_1, \ldots, a_n, \epsilon; a_{n+1}, \ldots, a_{n+m}; x; \epsilon)\, R_1(a_1, \ldots, a_{n+m}; \epsilon)$$

$$+ \quad F^{(j)}_{n,m+1,\nu-1}(a_1, \ldots, a_n; a_{n+1}, \ldots, a_{n+m}, \epsilon; x; \epsilon)\, R_2(a_1, \ldots, a_{n+m}; \epsilon), \; n+m > 1.$$

Using the notation of Messel and Potts,[16,17] (4) can be written as a matrix equation:

$$\mathbb{F}_0(0;\epsilon) = 0, \qquad J = \begin{cases} 1 & \text{If } 1/\epsilon \text{ is an integer} \\ 0 & \text{Otherwise.} \end{cases}$$

$$\mathbb{F}_{N,\nu}(a_1, \ldots, a_N; 0; \epsilon) = \mathbb{E}_1\, \delta(N-1)\, \delta(\nu - J), \quad N > 0, \tag{5a}$$

$$\mathbb{F}_{N,\nu}(a_1, \ldots, a_N; x; \epsilon) = 0 \quad \text{If } (\nu<0) \text{ or } (a_k < \epsilon,\ k = 1, 2, \ldots, N),\ x>0,\ N>0.$$

$$\frac{\partial}{\partial x}\mathbb{F}_0(x;\epsilon) = \sum_{r=1} R_r(\epsilon;\epsilon)\, \mathbb{F}^{(r)}_{1,[1/\epsilon]-1}(a_1; x; \epsilon). \tag{5b}$$

$$\left(\frac{\partial}{\partial x} + \mathbb{B}_1(a_1; a_1; \epsilon)\right)\mathbb{F}_{1,\nu}(a_1; x; \epsilon)$$

$$= \mathbb{B}^{\epsilon}_1(a_1; a_1; \epsilon)\, \mathbb{F}_{1,\nu-1}(a_1 + \epsilon; x; \epsilon) \tag{5c}$$

$$+ \sum_{r=1}^{2} R_r(a_1; \epsilon)\, \mathbb{F}^{(r)}_{2,\nu-1}(\epsilon, a_1; x; \epsilon).$$

$$\left(\frac{\partial}{\partial x} + \sum_{k=1}^{N} \mathbb{B}_N(a_1, \ldots, a_N; a_k; \epsilon)\right)\mathbb{F}_{N,\nu}(a_1, \ldots, a_N; x; \epsilon)$$

$$= \sum_{C^n_2} \mathbb{G}_{N-1}(a_1, \ldots, a_N; a_{N-1}, a_N; \epsilon)\, \mathbb{F}_{N-1,\nu}(a_1, \ldots, a_{N-2}, a_{N-1} + a_N; x; \epsilon)$$

$$+ \sum_{k=1}^{N} \mathbb{B}^{\epsilon}_N(a_1, \ldots, a_N; a_k; \epsilon)\, \mathbb{F}_{N,\nu-1}(a_1, \ldots, a_k + \epsilon, \ldots, a_N; x; \epsilon) \tag{5d}$$

$$+ \sum_{r=1}^{2} R_r(a_1, \ldots, a_N; \epsilon)\, \mathbb{F}^{(r)}_{N+1,\nu-1}(\epsilon, a_1, \ldots, a_N; x; \epsilon), \quad N>1.$$

Most of the matrices of equations (5), as defined in the following paragraphs, admit duplicate rows when some of the a_k are equal. A modified set of definitions that deleted all but one of a set of duplicate rows and introduced appropriate weighting factors, could be used to advantage during actual calculation but at present would encumber the notation.

$\mathbb{E}_1$ is the unit matrix of order 2.

$\mathbb{F}_0(x;\epsilon) = \left[f^{(1)}_{0,0}(0;0;x)\ f^{(2)}_{0,0}(0;0;x) \right]$. $\mathbb{F}_{N,\nu}$ is a $2^N \times 2$ matrix the columns of which correspond to $F^{(1)}$ and $F^{(2)}$. The rows are ordered by writing $a_1, \ldots, a_N$ as a binary number with digits 1 and 2 standing for an electron and photon respectively. For instance, in $\mathbb{F}_{3,\nu}(a_1, a_2, a_3; x; \epsilon)$, $F^{(j)}_{2,1,\nu}(a_1, a_3; a_2; x; \epsilon)$ appears in the row above $F^{(j)}_{1,2,\nu}(a_1; a_2, a_3; x; \epsilon)$, because $a_1a_2a_3 = 121 < a_1a_2a_3 = 122$. $\mathbb{F}^{(r)}_{N,\nu}$, $r = 1, 2$, are submatrices of $\mathbb{F}_{N,\nu}$ each of which have 2^{N-1} rows:

$$\mathbb{F}_{N,\nu} = \begin{bmatrix} \mathbb{F}^{(1)}_{N,\nu} \\ \mathbb{F}^{(2)}_{N,\nu} \end{bmatrix}.$$

The $\mathbb{B}_N(a_1, \ldots, a_N; a_k; \epsilon)$ and $\mathbb{B}^{\epsilon}_N(a_1, \ldots, a_N; a_k; \epsilon)$ are defined as the $2^N \times 2^N$ - order matrices,

$$\mathbb{B}_N(a_1, \ldots, a_N; a_k; \epsilon) = \Big[\!\Big[\gamma_{\alpha_1\alpha_2\ldots\alpha_N, \beta_1\beta_2\ldots\beta_N} = \delta(\alpha_1-\beta_1)\,\delta(\alpha_2-\beta_2)\ldots B_{\alpha_k\beta_k}(a_1, \ldots, a_N; a_k; \epsilon)\ldots\delta(\alpha_N-\beta_N) \Big]\!\Big]$$

$$\mathbb{B}^{\epsilon}_N(a_1, \ldots, a_N; a_k; \epsilon) = \Big[\!\Big[\gamma_{\alpha_1\alpha_2\ldots\alpha_N, \beta_1\beta_2\ldots\beta_N} = \delta(\alpha_1-\beta_1)\,\delta(\alpha_2-\beta_2)\ldots B^{\epsilon}_{\alpha_k\beta_k}(a_1, \ldots, a_N; a_k; \epsilon)\ldots\delta(\alpha_N-\beta_N) \Big]\!\Big],$$

in which the subscripts of the general element, $\gamma_{\alpha_1\alpha_2\cdots\alpha_N,\beta_1\beta_2\cdots\beta_N}$ are written as binary numbers: The α_k and β_k can take on the values 1 or 2. $B^{\epsilon}_{2,2}(a_1, \ldots, a_N; a_k; \epsilon) = 0$.

$\mathbb{G}_{N-1}(a_1, \ldots, a_N; a_{N-1}, a_N; \epsilon)$ is a $2^N \times 2^{N-1}$ matrix in which the non zero elements $G_r(a_1, \ldots, a_N; a_{N-1}, a_N; \epsilon)$, $r = 1, 2$, are ordered according to the following rules:

(1) If in the binary number $a_1 \ldots a_N$, $a_{N-1} = 1$ and $a_N = 1$, then all the elements of the row corresponding to this number are zero except for the term $G_2(a_1, \ldots, a_N; a_{N-1}, a_N; \epsilon)$, which is placed in the first even-numbered column in which this term has not already appeared.

(2) If $a_{N-1} = 1$ and $a_N = 2$, then all the elements of the row are zero except for the term $G_1(a_1, \ldots, a_N; a_{N-1}, a_N; \epsilon)$, which is placed in the first odd-numbered column in which this term has not already appeared.

(3) If $a_{N-1} = 2$ and $a_N = 1$, then all the elements of the row are zero except for the term $G_1(a_1, \ldots, a_N; a_N, a_{N-1}; \epsilon)$, which is placed in the first odd-numbered column in which this term has not already appeared.

(4) If $a_{N-1} = 2$ and $a_N = 2$, all the elements of the row are zero.

The matrix $(\partial/\partial x)\mathbb{F}_{N,\nu}$ is obtained from $\mathbb{F}_{N,\nu}$ by replacing all the elements of the latter by their differential coefficients with respect to x. Similarly, in what follows, the integral of a matrix will be constructed by

replacing all the elements of that matrix by their integrals. The exponential of a matrix $\mathbb{A}_N$ of order 2^N is defined as

$$\exp\left\{\mathbb{A}_N\right\} = \mathbb{E}_N + \mathbb{A}_N + (1/2!)\mathbb{A}_N^2 + (1/3!)\mathbb{A}_N^3 + \ldots,$$

where $\mathbb{E}_N$ is the unit matrix of order 2^N. For examples of the matrix notation used here, see Appendix II.

Equation (5) can now be converted from differential-difference form to difference form by treating its x-dependence as in a simple differential equation. The formal step is justified by putting the succeeding solution back into (5).

$$\mathbb{F}_0(x;\epsilon) = \int_0^x d\xi \sum_{r=1}^{2} R_r(\epsilon;\epsilon)\, \mathbb{F}^{(r)}_{1,[1/\epsilon]-1}(a_1;\xi;\epsilon). \tag{6a}$$

$$\mathbb{F}_{1,\nu}(a_1;x;\epsilon)$$

$$= \exp\left\{-x\mathbb{B}_1(a_1;a_1;\epsilon)\right\}\int_0^x d\xi \exp\left\{\xi\mathbb{B}_1(a_1;a_1;\epsilon)\right\}\cdot \tag{6b}$$

$$\cdot\left[\mathbb{B}_1^{\epsilon}(a_1;a_1;\epsilon)\mathbb{F}_{1,\nu-1}(a_1+\epsilon;\xi;\epsilon) + \sum_{r=1}^{2} R_r(a_1;\epsilon)\mathbb{F}^{(r)}_{2,\nu-1}(\epsilon,a_1;\xi;\epsilon)\right]$$

$$+\mathbb{E}_1 \exp\left\{-x\mathbb{B}_1(a_1,a_1;\epsilon)\ \delta(\nu-J)\right\}.$$

$$\mathbb{F}_{N,\nu}(a_1, \ldots, a_N; x; \epsilon) \tag{6c}$$

$$= \exp\left\{-x \sum_{k=1}^{N} {}_N(a_1, \ldots, a_N; a_k; \epsilon)\right\} \int_0^x d\xi \exp\left\{\xi \sum_{k=1}^{N} \mathbb{B}_N(a_1, \ldots, a_N; a_k; \epsilon)\right\} \cdot$$

$$\cdot \Bigg[\sum_{C_2^N} \mathbb{G}_{N-1}(a_1, \ldots, a_N; a_{N-1}, a_N; \epsilon) \mathbb{F}_{N-1,\nu}(a_1, \ldots, a_{N-2}, a_{N-1} + a_N; \xi; \epsilon)$$

$$+ \sum_{k=1}^{N} \mathbb{B}_N^{\epsilon}(a_1, \ldots, a_N; a_k; \epsilon) \mathbb{F}_{N,\nu-1}(a_1, \ldots, a_k + \epsilon, \ldots, a_N; \xi; \epsilon)$$

$$+ \sum_{r=1}^{2} R_r(a_1, \ldots, a_N; \epsilon) \mathbb{F}^{(r)}_{N+1,\nu-1}(\epsilon, a_1, \ldots, a_N; \xi; \epsilon) \Bigg], \quad N > 1.$$

The apparent complexity of the recurrence in the two variables N and ν is illusory. By starting with $\mathbb{F}_{1,0}$ and solving for the other matrices in the order prescribed by Table 1, one eventually arrives at $\mathbb{F}_{N,\nu}$ in terms of previous calculations whatever N and ν, $N > 0$, $\nu > 0$. $\mathbb{F}_0$ will be found in terms of $\mathbb{F}_{1,[1/\epsilon]-1}$. It is unnecessary to write down the general term, $\mathbb{F}_{N,\nu}$, explicitly, because actual calculations will proceed according to Table 1. Each matrix will be physically meaningful, as a part of the energy-number distribution, and, together with others, will yield the number distribution through (3). It should be noticed that the series form of the solution is not a consequence of the inclusion of collision loss, as it is in Messel's solution. A Laplace transform could easily be used on the penetration variable to avoid the series, but inverted matrices would result.

$N+\nu \backslash \nu$	0	1	2	3	...	$[1/\epsilon]-1$
1	1					
2	2	3				
3	4	5	6			
⋮				⋱		
$[1/\epsilon]$						$\frac{[1/\epsilon]([1/\epsilon]+1)}{2}$

Table I

As illustration of the use of Table 1 to solve (6), the first three matrices are given explicitly in the case when $1/\epsilon$ is not an integer:

$$\mathbb{F}_{1,0}(a_1; x; \epsilon) = \mathbb{E}_1 \exp\left\{-x\,\mathbb{B}_1(a_1; a_1; \epsilon)\right\}.$$

$$\mathbb{F}_{2,0}(a_1, a_2; x; \epsilon)$$

$$= \exp\left\{-x\sum_{k=1}^{2}\mathbb{B}_2(a_1, a_2; a_k; \epsilon)\right\}\int_0^x d\xi\, \exp\left\{\xi\sum_{k=1}^{2}\mathbb{B}_2(a_1, a_2; a_k; \epsilon)\right\}\cdot$$

$$\cdot\,\mathbb{G}_1(a_1, a_2; a_1, a_2; \epsilon)\,\mathbb{E}_1 \exp\left\{-\xi\,\mathbb{B}_1(a_1+a_2; a_1+a_2; \epsilon)\right\}.$$

$$\mathbb{F}_{1,1}(a_1;\, x;\, \epsilon)$$

$$= \exp\left\{-x\,\mathbb{B}_1(a_1;\, a_1;\, \epsilon)\right\}\int_0^x d\xi\, \exp\left\{\xi\,\mathbb{B}_1(a_1;\, a_1;\, \epsilon)\right\}\cdot$$

$$\cdot\Bigg[\mathbb{B}_1^{\epsilon}(a_1;\, a_1;\, \epsilon)\,\mathbb{E}_1 \exp\left\{-\xi\,\mathbb{B}_1(a_1+\epsilon;\, a_1+\epsilon;\, \epsilon)\right\}$$

$$+\sum_{r=1}^{2} R_r(a_1;\, \epsilon)\Bigg(\exp\left\{-\xi\sum_{k=1}^{2}\mathbb{B}_2(a_1',\, a_2';\, a_k';\, \epsilon)\right\}\cdot$$

$$\cdot\int_0^{\xi} d\zeta\, \exp\left\{\zeta\sum_{k=1}^{2}\mathbb{B}_2(a_1',\, a_2';\, a_k';\, \epsilon)\right\}\cdot$$

$$\cdot\,\mathbb{G}_1(a_1',\, a_2';\, a_1',\, a_2';\, \epsilon)\,\mathbb{E}_1 \exp\left\{-\zeta\,\mathbb{B}_1(a_1'+a_2';\, a_1'+a_2';\, \epsilon)\right\}\Bigg)^{(r)}\Bigg],$$

$a_1' = \epsilon,\ a_2' = a_1.$

DISCUSSION

The cutoff energy ϵ plays a dominant role throughout this paper. Its presence permits the inclusion of collision loss as a natural modification of the radiation process, for an electron that emits a photon behaves identically (except for angular deflection) to one that collides with another electron, provided that the energy carried of by the photon or δ-ray is just under the observable limit ϵ. Its presence also permits the inclusion of generalized cross sections and a multiple-scattering factor which are energy-dependent. This is true because the solution can be developed over energy increments equal to ϵ; within each increment the energy-dependent functions do not vary appreciably. The number ϵ is everywhere treated as though it were large compared to $\mu c^2/E_0$ and yet small compared to 1. It is also treated as though it excludes from observation the effects of Compton scattering. It is well interpreted as the critical energy.

Interpreting ϵ as the critical energy, one sees, first, that Compton scattering that decreases rapidly above ϵ has a cross section there one-fifth[25] as great in air as the cross section for pair production. The ratio is about one-third in lead. Second, the criterion $\epsilon \ll 1$ will be satisfied if the shower primary has energy 7×10^7 ev or greater in lead and 10^9 ev or greater in air. At those minimum limits for primary energy, $\epsilon = 1/10$. This means that in an actual calculation Table 1 contains about 55 steps. (The order of the matrices is not an indication of the amount of numerical work required to obtain as answer, because the matrices of higher order will have duplicate rows.)

The multiple-scattering factor, as defined, will actually be slightly increased by angular emission during radiation. However, as pointed out by Rossi and Greisen,[25] the average angle of emission per radiation length for primary electrons and primary photons is roughly one-fortieth the root mean square angle of scattering of electrons in one radiation length. It is clear that angular emission has a negligible effect on the accuracy of the present solution, and to the extent that it is important, it is already included in the multiple-scattering factor. The biggest objection to the scattering factor is that it does not have a "memory", in the sense that the cumulative effect of scattering is not present beyond a radiation length. The dependence of the factor on energy, however, serves to make it important at the end of the shower, which is as it should be.

It is important to notice that at the beginning of the shower, the total energy in the observable particles can be decreased by two modes, either by collision or by a branching process in which one of the secondaries takes a small portion of the primary energy, specifically, less than ϵ. The second mode of energy decrease will be strongly dependent on the shape of the function $g^{(1)}(u, v)$ near $v = 1$ and the function $g^{(2)}(u, v)$ near $v = 1$ and $v = 0$. Unfortunately, these functions have been computed[26] for high primary energies under the Born approximation, which is not valid near the end points cited above. More recent calculations,[27] while not limited by the Born approximation, are still not valid when, for instance, the electron radiates most of its energy. Experiments[28, 29] that have been performed on the radiation and pair-production spectrums shed no light on this point. The resolution in determining the secondary energies is too broad to resolve a rapid change in curvature. The experiments are not inconsistent with relatively uniform spectrums and nonzero cutoffs. In view of our knowledge of nonrelativistic radiation spectrums, it would seem wisest to modify the Bethe-Heitler calculations so that $g^{(1)}(u, v)$ and $g^{(2)}(u, v)$ are weak functions of v when the Born approximation is no longer valid. The Bethe-Heitler values should be further normalized in accordance with attenuation experiments[30-35] and more accurate theory.[36]

As mentioned previously, the method of solution developed here is easily adaptable to ultra-high-energy showers in which collision loss and multiple scattering can be neglected. The present form of the solution is well adapted for the same energy range that was investigated with Monte Carlo techniques[21] and is derived from the same basic elements, except for Compton effect, that were used here. The increment steps in the penetration variable, which are taken with slight approximation by Wilson, may be contrasted with the increment energy integrations and approximations used here.

Before attempting a large scale computing project to obtain meaningful curves, the author has completed some trial calculations for the case in which $\epsilon = 1/4$. Though the criterion for accuracy stated in the text is not satisfied in this case, most of the qualitative features of the solution should be present. It was found, in fact, that the exponential matrices converged extremely rapidly for small penetration. These exponentials could quickly be raised to successive powers in order to obtain solutions for increasing penetration. The numerical work was carried out on a desk calculator. It is hoped that comprehensive results can be presented soon.

APPENDIX I

The coefficients of equations (4) are most conveniently expressed in terms of the following functions and operators:

$$I^{\epsilon}(u_1, a_1; \ldots, ; u_N, a_N) = \begin{cases} 1 & \text{If } a_1 \le u_1 \le a_1 + \epsilon, \ \ldots, \ a_N \le u_N \le a_N + \epsilon \\ & \text{and } \sum_{i=1}^{N} u_i \le 1. \\ 0 & \text{Otherwise.} \end{cases}$$

$$I^{2\epsilon}(u_1, a_1; \ldots ; u_N, a_N) = \begin{cases} 1 & \text{If } a_1 \le u_1 \le a_1 + \epsilon, \ \ldots, \ a_{N-1} \le u_k \le a_{N-1} + \epsilon \\ & a_N \le u_N \le a_N + 2\epsilon, \\ & \text{and } \sum_{i=1}^{N} u_i \le 1. \\ 0 & \text{Otherwise.} \end{cases}$$

$$\mathcal{E}\left\{g(u_1, \ldots, u_N)\, I^{r\epsilon}(u_1, a_1; \ldots; u_N, a_N)\, h(u_k)\right\}$$

$$= \frac{\int_0^1 du_1 \ldots \int_0^1 du_N\, g(u_1, \ldots, u_N)\, I^{r\epsilon}(u_1, a_1; \ldots; u_N, a_N)\, h(u_k)}{\int_0^1 du_1 \ldots \int_0^1 du_N\, g(u_1, \ldots, u_N)\, I^{r\epsilon}(u_1, a_1; \ldots, u_N, a_N)}, \quad r = 1, 2.$$

$g(u_1, \ldots, u_N)$ and $h(u_k)$ are any continuous functions.

$B_{k,\ell}$, $B^{\epsilon}_{k,\ell}$, G_r, and R_r will be given below in an approximate form which is obtained from the exact form of the coefficients of equations (4) by the following substitutions:

(1) The exact coefficient of $F^{(j)}_{n-1,m+1,\nu}(a'_1, \ldots, a'_{n-1}; a_{n+1}, \ldots, a_{n+m}, a'_n; x; \epsilon)$ which has the form

$$\mathcal{E}\left\{I^{\epsilon}(u_1, a_1; \ldots; u_{n+m}, a_{n+m})\, f^{(j)}_{n-1,m+1}(u'_1, \ldots, u'_{n-1}; u_{n+1}, \ldots, u_{n+m}, u'_n; x; \epsilon)\, h(u'_n)\right\}$$

is replaced by

$$\mathcal{E}\left\{I^{\epsilon}(u_1, a_1; \ldots; u_{n+m}, a_{n+m})\, h(u'_n)\right\}$$

to obtain $B_{1,2}(a_1, \ldots, a_{n+m}; a'_n; \epsilon)$. Similar substitutions are made in the other coefficients.

(2) $f^{(j)}_{n,m}\left(u'_1, \ldots, u'_{n-1}, \frac{u'_n}{1-v}; u_{n+1}, \ldots, u_{n+m}; x\right)$, which appears in one of the exact coefficients of

$F^{(j)}_{n,m,\nu}(a_1, \ldots, a_n; a_{n+1}, \ldots, a_{n+m}; x; \epsilon)$ is replaced by

$f^{(j)}_{n,m}(u_1, \ldots, u_n; u_{n+1}, \ldots, u_{n+m}; x)$ for $0 \leq v \leq 1 - \dfrac{u'_n}{u'_n + \epsilon}$

to obtain $B_{1,1}(a_1, \ldots, a_{n+m}; a'_n; \epsilon)$.

Substitutions (1) and (2) are good if $f^{(j)}_{n,m}$, $f^{(j)}_{n-1,m+1}$, etc. are not strong functions of their energy arguments or if $\epsilon \ll 1$. (2) is also good if $a'_n \gg \epsilon$.

Though it will appear that all the coefficients are given in terms of multiple integrals, it is true for about half of them only, for whenever $\sum_{k=1}^{N} a_k + N\epsilon + (r-1)\,\epsilon \leq 1$, then

$$\mathcal{E}\left\{ I^{r\epsilon}\,(u_1,\ a_1;\ \ldots,\ u_N,\ a_N)\, h\,(u_k)\right\} = 1/r\epsilon \int_{a_k}^{a_k+r\epsilon} du_k\, h(u_k), \begin{cases} r=1,\ k=1,2,\ \ldots,\ N \\ r=2,\ k=N. \end{cases}$$

Even in the cases where true multiple integrals are involved, it is possible to get a good answer by writing approximately:

$$\mathcal{E}\left\{ I^{r\epsilon}\,(u_1,\ a_1;\ \ldots,\ u_N,\ a_N)\, h\,(u_k)\right\} \cong h\,(\bar{u}_k),$$

$$\bar{u}_k = \mathcal{E}\left\{ I^{r\epsilon}\,(u_1,\ a_1;\ \ldots;\ u_N,\ a_N)\, u_k\right\},\ r = 1, 2.$$

Integrations by parts are necessary to compute $B_{1,2}$, $B_{2,1}$, $B^{\epsilon}_{1,2}$ and $B^{\epsilon}_{2,1}$ and some obvious transformations are necessary for G_1 and G_2. It should be noticed that the divergence of the radiation spectrum cancels out in $B_{1,1}$.

$$B_{1,1}\,(a_1,\ \ldots,\ a_N;\ a_k;\ \epsilon)$$

$$= \mathcal{E}\left\{ I^{\epsilon}\,(u_1,\ a_1;\ \ldots;\ u_N,\ a_N)\left[\int_{1-u_k/(u_k+\epsilon)}^{1} dv/v\Big(s(u_k)\, g^{(1)}\,(u_k,\ v)\Big) \right.\right.$$

$$\left.\left. + \int_{0}^{1-u_k/(u_k+\epsilon)} dv/v\left(s(u_k)\, g^{(1)}\,(u_k,\ v) - s\left(\frac{u_k}{1-v}\right)\frac{g^{(1)}\left(\frac{u_k}{1-v},\ v\right)}{1-v}\right) + s(u_k)\,\frac{e(u_k)}{\epsilon}\right]\right\}.$$

$$B_{2,2}\,(a_1,\ \ldots,\ a_N;\ a_k;\ \epsilon)$$

$$= \mathcal{E}\left\{ I^{\epsilon}\,(u_1,\ a_1;\ \ldots;\ u_N,\ a_N)\int_0^1 dv\, g^{(2)}\,(u_k,\ v)\right\}.$$

$$B^{\epsilon}_{1,1}(a_1, \ldots, a_N; a_k; \epsilon)$$

$$= \mathcal{E}\left\{I^{\epsilon}(u_1, a_1; \ldots; u_k, a_k + \epsilon; \ldots; u_N, a_N)\, s(u_k)\, \frac{e(u_k)}{\epsilon}\right\}.$$

$$B^{p\epsilon}_{1,2}(a_1, \ldots, a_N; a_k; \epsilon) = \mathcal{E}\left\{I^{\epsilon}(u_1, a_1; \ldots; u_k, a_k + p\epsilon; \ldots; u_N, a_N)\, 2s(u_k)\left[\frac{v}{u_k}\, g^{(2)}(u_k, \frac{v}{u_k}) + \sum_{\ell=1}^{\infty} \frac{(-1)^{\ell}}{(\ell+1)!}\, \frac{v^{\ell+1}}{u_k}\, \frac{\partial^{\ell}}{\partial v^{\ell}} g^{(2)}(u_k, \frac{v}{u_k})\right]_{v = u_k - p\epsilon}^{v = a_k + p\epsilon}\right\},$$

$$B^{p\epsilon}_{2,1}(a_1, \ldots, a_N; a_k; \epsilon) = \mathcal{E}\left\{I^{\epsilon}(u_1, a_1; \ldots; u_k, a_k + p\epsilon; \ldots; u_N, a_N)\, s(u_k)\left[(\log v)\, g^{(1)}(u_k, \frac{v}{u_k}) + \sum_{\ell=1}^{\infty} (-1)^{\ell}\, v^{\ell}\left(\frac{\log v}{\ell} - \sum_{q=1}^{\ell} \frac{q!}{\ell!\, q^2}\right) \frac{\partial^{\ell}}{\partial v^{\ell}} g^{(1)}(u_k, \frac{v}{u_k})\right]_{v = u_k - p\epsilon}^{v = a_k + p\epsilon}\right\},$$

$p = 0, 1.$

$$G_1(a_1, \ldots, a_N; a_{N-1}, a_N; \epsilon)$$

$$= \mathcal{E}\left\{I^{2\epsilon}(u_1, a_1; \ldots; u_{N-2}, a_{N-2}; u, a_{N-1} + a_N)\, s(u) \int_{\mathrm{Max}[\frac{a_N}{u}, 1 - \frac{a_{N-1} + \epsilon}{u}]}^{\mathrm{Min}[\frac{a_N + \epsilon}{u}, 1 - \frac{a_{N-1}}{u}]} \frac{dv}{v}\, g^{(1)}(u, v)\right\}.$$

$G_2\,(a_1, \ldots, a_N; a_{N-1}, a_N; \epsilon)$

$$= \mathcal{E}\left\{ I^{2\epsilon}\,(u_1, a_1; \ldots; u_{N-2}, a_{N-2}; u, a_{N-1} + a_N)\; 2\,s(u) \int_{\mathrm{Max}[\frac{a_N}{u},\, 1-\frac{a_{N-1}+\epsilon}{u}]}^{\mathrm{Min}[\frac{a_N+\epsilon}{u},\, 1-\frac{a_{N-1}}{u}]} dv\; g^{(2)}\,(u, v) \right\}.$$

$R_1\,(a_1, \ldots, a_N; \epsilon)$

$$= \mathcal{E}\left\{ I^{\epsilon}\,(u_1, a_1; \ldots; u_N, a_N; u, \epsilon)\; s(u) \left[\int_{\mathrm{Max}[c,\, 1-\frac{\epsilon}{u}]}^{\frac{\epsilon}{u}} \frac{dv}{v}\, g^{(1)}\,(u, v) + \frac{e(u)}{\epsilon} \right] \right\}.$$

$R_2\,(a_1, \ldots, a_N; \epsilon)$

$$= \mathcal{E}\left\{ I^{\epsilon}\,(u_1, a_1; \ldots; u_N, a_N; u, \epsilon) \int_{1-\frac{\epsilon}{u}}^{\frac{\epsilon}{u}} dv\; g^{(2)}\,(u, v) \right\}.$$

APPENDIX II

Some examples of the matrices appearing in (5) and (6) are given here,

$$\mathbb{F}_{2,\nu}(a_1, a_2; x; \epsilon) = \begin{bmatrix} F^{(1)}_{2,0,\nu}(a_1, a_2; 0; x; \epsilon) & F^{(2)}_{2,0,\nu}(a_1, a_2; 0; x; \epsilon) \\ F^{(1)}_{1,1,\nu}(a_1; a_2; x; \epsilon) & F^{(2)}_{1,1,\nu}(a_1; a_2; x; \epsilon) \\ F^{(1)}_{1,1,\nu}(a_2; a_1; x; \epsilon) & F^{(2)}_{1,1,\nu}(a_2; a_1; x; \epsilon) \\ F^{(1)}_{0,2,\nu}(0; a_1, a_2; x; \epsilon) & F^{(2)}_{0,2,\nu}(0; a_1\ a_2; x; \epsilon) \end{bmatrix}$$

$$\mathbb{F}^{(1)}_{3,\nu}(\epsilon, a_1, a_2; x; \epsilon) = \begin{bmatrix} F^{(1)}_{3,0,\nu}(\epsilon, a_1, a_2; 0; x; \epsilon) & F^{(2)}_{3,0,\nu}(\epsilon, a_1, a_2; 0; x; \epsilon) \\ F^{(1)}_{2,1,\nu}(\epsilon, a_1; a_2; x; \epsilon) & F^{(2)}_{2,1,\nu}(\epsilon, a_1; a_2; x; \epsilon) \\ F^{(1)}_{2,1,\nu}(\epsilon, a_2; a_1; x; \epsilon) & F^{(2)}_{2,1,\nu}(\epsilon, a_2; a_1; x; \epsilon) \\ F^{(1)}_{1,2,\nu}(\epsilon; a_1, a_2; x; \epsilon) & F^{(2)}_{1,2,\nu}(\epsilon; a_1, a_2; x; \epsilon) \end{bmatrix}$$

$$\mathbb{F}^{(2)}_{3,\nu}(\epsilon, a_1, a_2; x; \epsilon) = \begin{bmatrix} F^{(1)}_{2,1,\nu}(a_1, a_2; \epsilon; x; \epsilon) & F^{(2)}_{2,1,\nu}(a_1, a_2; \epsilon; x; \epsilon) \\ F^{(1)}_{1,2,\nu}(a_1; \epsilon, a_2; x; \epsilon) & F^{(2)}_{1,2,\nu}(a_1; \epsilon, a_2; x; \epsilon) \\ F^{(1)}_{1,2,\nu}(a_2; \epsilon, a_1; x; \epsilon) & F^{(2)}_{1,2,\nu}(a_2; \epsilon, a_1; x; \epsilon) \\ F_{0,3,\nu}(0; \epsilon, a_1, a_2; x; \epsilon) & F_{0,3,\nu}(0; \epsilon, a_1, a_2; x; \epsilon) \end{bmatrix}$$

$$\mathbb{B}_2(a_1, a_2; a_1; \epsilon) = \begin{bmatrix} B_{1,1}(a_1, a_2; a_1; \epsilon) & 0 & B_{1,2}(a_1, a_2; a_1; \epsilon) & 0 \\ 0 & B_{1,1}(a_1, a_2; a_1; \epsilon) & 0 & B_{1,2}(a_1, a_2; a_1; \epsilon) \\ B_{2,1}(a_1, a_2; a_1; \epsilon) & 0 & B_{2,2}(a_1, a_2; a_1; \epsilon) & 0 \\ 0 & B_{2,1}(a_1 a_2; a_1; \epsilon) & 0 & B_{2,2}(a_1, a_2; a_1; \epsilon) \end{bmatrix}$$

$$\mathbb{B}_2(a_1, a_2; a_2; \epsilon) = \begin{bmatrix} B_{1,1}(a_1, a_2; a_2; \epsilon) & B_{1,2}(a_1, a_2; a_2; \epsilon) & 0 & 0 \\ B_{2,1}(a_1, a_2; a_2; \epsilon) & B_{2,2}(a_1, a_2; a_2; \epsilon) & 0 & 0 \\ 0 & 0 & B_{1,1}(a_1, a_2; a_2; \epsilon) & B_{1,2}(a_1, a_2; a_2; \epsilon) \\ 0 & 0 & B_{2,1}(a_1, a_2; a_2; \epsilon) & B_{2,2}(a_1, a_2; a_2; \epsilon) \end{bmatrix}$$

$$\mathbb{G}_1(a_1, a_2; a_1, a_2; \epsilon) = \begin{bmatrix} 0 & G_2(a_1, a_2; a_1, a_2; \epsilon) \\ G_1(a_1, a_2; a_1, a_2; \epsilon) & 0 \\ G_1(a_1, a_2; a_2, a_1; \epsilon) & 0 \\ 0 & 0 \end{bmatrix}$$

APPENDIX III

In this section the symmetric densities are defined in terms of $F(\tau;\ \lambda:\ t;\Lambda)$ and it is shown how equations (2) as well as Janossy's equations can be derived from the Kolmogoroff-Chapman equations.

Let $\Lambda_{n,m}$ be any set in Ω of the form $\Lambda_{n,m} = \bigcup_{\eta_1 \in E_1} \cdots \bigcup_{\eta_{n+m} \in E_{n+m}}$ $(n, m;\ \eta_1, \ldots, \eta_n;\ \eta_{n+1}, \ldots, \eta_{n+m})$, $n + m > 0$. The E_i are open sets on the real line. Let Ω^* be any set containing Ω. The function $\bar{f}(\tau;\ \lambda'_{n',m'}:\ t;\ \lambda_{n,m})$, $\lambda'_{n',m'} \in \Omega^*$, $\lambda_{n,m} \in \Omega^*$ is defined by the following:

(1) $\bar{f}(\tau;\ \lambda'_{n',m'}:\ t;\ \lambda_{n,m}) = 0$ If $\lambda'_{n',m'} \notin \Omega$ or $\lambda_{n,m} \notin \Omega$, $n'+m' > 0$, $n+m > 0$.

(2) $$\int_{E_1} d\eta_1 \int_{E_{n+m}} d\eta_{n+m}\ \bar{f}(\tau;\ \lambda'_{n',m'}:\ t;\ n, m;\ \eta_1, \ldots, \eta_n;\ \eta_{n+1}, \ldots, \eta_{n+m}) =$$

$$F(\tau;\ \lambda'_{n',m'}:\ t;\Lambda_{n,m}),\ \lambda'_{n',m'} \in \Omega,\ n'+m' > 0,\ n+m > 0. \tag{7}$$

(3) $\bar{f}(\tau;\ \lambda_{0,0}:\ t;\ \lambda_{0,0}) = F(\tau;\ \lambda_{0,0}:\ t;\ \lambda_{0,0})$.

(4) $\bar{f}(\tau;\ \lambda_{0,0}:\ t;\ \lambda_{n,m}) = 0$, $\lambda_{n,m} \in \Omega$, $n+m > 0$.

(5) $\bar{f}(\tau;\ \lambda'_{n',m'}:\ t;\ \lambda_{0,0}) = F(\tau;\ \lambda'_{n',m'}:\ t;\ \lambda_{0,0})$, $\lambda'_{n',m'} \in \Omega$, $n'+m' \geq 0$.

For greater simplicity Messel's notation will be followed:

$$\bar{f}^{(1)}_{n,m}(\eta_1, \ldots, \eta_n;\ \eta_{n+1}, \ldots, \eta_{n+m};\ t) = \bar{f}(0;\ 1,\ 0;1;\ 0:\ t;\ n,m;\ \eta_1, \ldots, \eta_n;\ \eta_{n+1} \ldots, \eta_{n+m}),$$

$$\bar{f}^{(2)}_{n,m}(\eta_1, \ldots, \eta_n;\ \eta_{n+1}, \ldots, \eta_{n+m};\ t) = \bar{f}(0;\ 0,\ 1;\ 0;1:\ t;\ n,\ m;\ \eta_1, \ldots, \eta_n;\ \eta_{n+1}, \ldots, \eta_{n+m}).$$

The symmetrized densities defined on Ω^* are, for fixed n, m, symmetric in the first n variable and in the last m and coincide with $\bar{f}^{(j)}_{n,m}$ on Ω:

$$f^{(j)}_{n,m}(\eta_{i_1}, \ldots, \eta_{i_n}; \eta_{n+1}, \ldots, \eta_{n+m}; t) = f^{(j)}_{n,m}(\eta_{k_1}, \ldots, \eta_{k_n}; \eta_{n+1}, \ldots, \eta_{n+m}; t), \tag{8a}$$

$$j = 1, 2;\ n + m > 1.$$

$$f^{(j)}_{n,m}(\eta_1, \ldots, \eta_n; \eta_{n+1}, \ldots, \eta_{n+m}; t) = \bar{f}^{(j)}_{n,m}(\eta_1, \ldots, \eta_n; \eta_{n+1}, \ldots, \eta_{n+m}; t), \tag{8b}$$

$$(n, m; \eta_1, \ldots, \eta_n; \eta_{n+1}, \ldots, \eta_{n+m}) \in \Omega_{n,m},$$

$$j = 1, 2;\ n + m > 0.$$

$$f^{(j)}_{n,m}(\eta_1, \ldots, \eta_n; \eta_{n+1}, \ldots, \eta_{n+m}; t) = 0, \quad \text{If } \left(\sum_{i=1}^{n+m} \eta_1 > 1\right) \text{ or } (\eta_k < \epsilon,\ k = 1, 2, \ldots, n+m),$$

$$j = 1, 2;\ n + m > 0. \tag{8c}$$

A definition similar to (8a) holds for the symmetry in the last m variables. (7) and (8) do not extend the definitions of the densities to all the boundaries of $\Omega_{n,m}$. Hence, for fixed $\lambda'_{n',m'}$, t, and τ, the values of the $\bar{f}^{(j)}_{n,m}$ and $f^{(j)}_{n,m}$ at any point on the boundary of $\Omega_{n,m}$, $n + m > 1$ is taken as the limit from within $\Omega_{n,m}$ at that point. In particular, this defines the symmetric densities on the hyperplanes $\sum_{i=1}^{n+m} \eta_i = 1$.

For the sets $\Delta\Lambda_{n,m}$,

$$\Delta\Lambda_{n,m} = \left\{ \bigcup_{\eta_1 < u_1 < \eta_1 + \Delta\eta_1} \cdots \bigcup_{\eta_{n+m} < u_{n+m} < \eta_{n+m} + \Delta\eta_{n+m}} (n, m; u_1, \ldots, u_n; u_{n+1}, \ldots, u_{n+m}) \right.,$$

$$\Delta\Lambda_{n,m} \subset \Omega ,$$

$n \geqslant 0$, $m \geqslant 0$, and for small $(t - \tau)$, $\Delta\eta_1, \ldots, \Delta\eta_{n+m}$, it is possible to write down $F(\tau; \lambda'_{n',m'}: t; \Delta\Lambda_{n,m})$ immediately up to terms of the type $o(\Delta\eta_k)$ and $o(t - \tau)o(\Delta\eta)$. $o(\Delta\eta)$ represents higher order terms in one or more of the energy variables, while the other symbols have well known meanings. By inserting $F(\tau; \lambda'_{n',m'}: t; \Delta\Lambda_{n,m})$ into (1) and using definition (7), Janossy's equations can be obtained. To do this, set $\lambda = (1, 0; 1, 0)$ or $\lambda = (0, 1; 0, 1)$ in (1) and take limits in $(z-y)$ and the energy increments $\Delta\eta_i$. (It is necessary also to set $s(\eta) = 1$, $e(\eta) = 0$, and to specialize the cross sections to full-screening approximations, if the simplest form of the equations is desired.) Make use of the fact that the process is temporally homogeneous by substituting $x - y$ for x and 0 for y, integrate out the energy variables and apply the multiplicative property. For instance, set

$$F(0;2,0;\ \eta_1,\ \eta_2;0:x-y;\Omega_{n,m}) =$$

$$= \sum_{\substack{n'+n''=n \\ m'+m''=m}} F(0;1,0;\ \eta_1;0:x-y,\Omega_{n',m'})\, F(0;1,0;\ \eta_2, 0:x-y, \Omega_{n'',m''}) .$$

Finally, use the definition of the moment generating function.

The basic equations of this paper are obtained by inserting $F(\tau; \lambda'_{n',m'}: t; \Delta\Lambda_{n,m})$ into (1) and using definitions (7) and (8). It is necessary to set $y = 0$, $\lambda = (1, 0; 1, 0)$ or $\lambda = (0, 1; 0, 1)$ in (1) and then take limits in $(x-z)$ and the energy increments.

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