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Contract No. W-7405-eng-48

ON THE ANALYSIS OF NUCLEON-NUCLEON SCATTERING EXPERIMENTS

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January 15, 1949

Berkeley, California

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Abstract

A method of perturbation calculation, especially adapted to nucleon-nucleon scattering problems, is described. Any contribution to the energy of the system which is relatively small where the nuclear potential is large may be treated as the perturbation. Two principal examples are discussed. (1) Energy as the perturbation: An expansion of the phase shifts in powers of the energy is written down which extends earlier results of Schwinger, Blatt and Jackson. (2) The coulomb field as the perturbation in the proton-proton problem: Expansions are given which relate the nuclear phase shifts in a combined nuclear and coulomb field to the corresponding phase shifts for a purely nuclear problem. Attention is confined to central forces throughout.

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I. Introduction

The study of nucleon-nucleon scattering has long been recognized as a source of direct information about the nature of nuclear forces. In the absence of a detailed theory of nuclear forces, the procedure commonly adopted in the past has been to describe the interaction phenomenologically by a short range potential of some particular shape. Calculations of scattering cross-sections were carried out and comparisons made with experimental data, with varying degrees of success depending on the energy region in question.

In the low energy region ($<$ 10 Mev) it was found that no potential was obviously superior to any other. The underlying reason for this lack of discrimination was first pointed out qualitatively by Smorodinsky[1] and more recently quantitatively by Schwinger[2]. The reason is that the scattering characteristics of a given potential shape may be described, in this limited energy region, by only two parameters. These may be chosen as the scattering length at zero energy[3] and the so-called effective range[4]. Since there are only two parameters to be fitted, it is evident that any shape potential is suitable provided the proper choice of range and depth is made.

Schwinger's analysis, which has been confined thus far to central forces, is based on an expansion of the quantity, k cot δ, in powers of k^2. k is the wave number and δ is the S phase shift for neutron-proton scattering.

$$k \cot \delta = A_0 + A_1 k^2 + A_2 k^4 + \ldots \qquad (A)$$

For <u>any</u> positive (or negative) definite short range potential, A_2 turns out to be so small that the third term cannot be noticed experimentally at low energies. This, of course, is the underlying reason for the well shape independence of low energy cross sections. It is not clear from Schwinger's procedure, however, why the quantity $k \cot \delta$ should be representable by a rapidly convergent power series in the energy. In fact, the appropriate quantity to expand in the case of proton-proton scattering is considerably more complicated. In this paper we derive a general expression for that quantity which in a particular problem can be expected to have a convergent power series representation. The actual determination of any number of the terms in the expansion can be carried out in a straightforward way, and the form of these terms is, in general, simpler in appearance than that given by Schwinger's variational method. The higher terms in the energy expansions are of physical interest because of the information they give about the shape of the nuclear potential, but we shall not discuss this aspect of the problem. Blatt and Jackson[5] have made a detailed investigation of the coefficient A_2, using the variational principle. They discuss the physical implications in detail.

The expansion (A) will also be generalized to an arbitrary angular momentum. The possibility of making analogous expansions in parameters other than the energy is pointed out, and in particular the case of the coulomb field as a perturbation is discussed in considerable detail. The usefulness of the various expansions will be discussed in Section V .

We shall confine ourselves, for simplicity, to central forces. The generalization to tensor forces has been made by R. S. Christian and will be published by him.

II The General Method

The problem to be solved is the determination of the asymptotic form of the regular solution of the following differential equation:

$$\left\{L + f\right\} u(x) = - \varepsilon g(x)\ u(x) \tag{1}$$

where

$$L = \frac{d^2}{dx^2} + q(x)\ , \tag{2}$$

f is a short range self adjoint operator, ε is a parameter of smallness, q(x) is an arbitrary function no more singular at the origin than $1/x^2$, and g(x) another function no more singular than $1/x$.

Equation (1) is therefore a general form for the radial Schrodinger equation for an arbitrary angular momentum. The choice of q(x), ε, and g(x) will vary from problem to problem. The asymptotic form of u (x) shall be called X(x); i.e., X(x) is an irregular solution of the equation

$$L\ X(x) = - \varepsilon g\ (x)\, X(x) \tag{3}$$

Since an expansion in powers of ε is the object here, we define explicitly the limiting forms of u(x) and X(x) as ε approaches zero. Thus, $v_R(x)$ is the regular solution of the equation

$$\left\{L + f\right\} v_R\ (x) = 0 \tag{4}$$

and $Y_I(x)$ is the asymptotic form of $v_R(x)$, being in turn an irregular solution of the equation

$$L\ Y_I\ (x) = 0$$

It is convenient also to define at this time an irregular solution of equation (4) and the regular solution of equation (5). These will be designated as $v_I(x)$ and $Y_R(x)$ respectively, and chosen so that

$$\left.\begin{aligned} W_x(\mathcal{V}_I, \mathcal{V}_R) &= 1 \\ W_x(Y_I, Y_R) &= 1 \end{aligned}\right\} \quad \text{for all } x \tag{6}$$

and so that asymptotically $\mathcal{V}_I(x)$ approaches $-Y_R(x)$. The notation $W_x(f,g)$ means $\left[f(x)\frac{dg(x)}{dx} - g(x)\frac{df(x)}{dx}\right]$. These conditions can be achieved by a proper choice of the constant in the following representation of $\mathcal{V}_I(x)$:

$$\mathcal{V}_I(x) = \mathcal{V}_R(x)\left\{-\int^x \frac{dx'}{\mathcal{V}_R^2(x')} + \text{constant}\right\} \tag{7}$$

We now derive the fundamental identity from which all subsequent results can be obtained. From equations (1) and (4), one sees that

$$W_{x_0}(\mathcal{V}_R, u) = W_{x_1}(\mathcal{V}_R, u) + \varepsilon \int_{x_0}^{x_1} g(x)\, \mathcal{V}_R(x) u(x)\, dx \tag{8}$$

Similarly from equations (3) and (5)

$$W_{x_0}(Y_I, X) = W_{x_1}(Y_I, X) + \varepsilon \int_{x_0}^{x_1} g(x)\, Y_I(x)\, X(x)\, dx \tag{9}$$

The difference of these equations gives

$$W_{x_0}(Y_I, X) - W_{x_0}(\mathcal{V}_R, u) = W_{x_1}(Y_I, X) - W_{x_1}(\mathcal{V}_R, U) + \varepsilon \int_{x_0}^{x_1} g(x)\left[Y_I(x)\, X(x) - \mathcal{V}_R(x) u(x)\right] dx$$

Now pass to the limit $x_1 \longrightarrow \infty$, $x_0 \longrightarrow 0$. We obtain

$$\lim_{x_0 \to 0} W_{x_0}(Y_I X) = \lim_{x_0 \to 0} \varepsilon \int_{x_0}^{\infty} g(x)\left[Y_I(x)\, X(x) - \mathcal{V}_R(x)\, u(x)\right] dx \tag{10}$$

since the Wronskian of the two regular functions $u(x)$ and $\mathcal{V}_R(x)$ must vanish at the origin. Henceforth the limiting process, $x_0 \longrightarrow 0$, will be implied wherever x_0 is written.

Later examples will show that $W_{x_0}(Y_I, X)$ is a measure of the influence of the term $\epsilon\, g(x)$ on the scattering characteristics of the system. It is evident that $W_{x_0}(Y_I, X)$ can be represented by a rapidly convergent power series in ϵ whenever the functions, $u(x)$ and $X(x)$, can be so represented <u>in the region where the nuclear potential is important</u>. This, in turn, will be the case if $\epsilon\, g(x)$ is relatively small in the region where f is effective.

It should be noted that equation (10) is homogeneous, i.e., it is independent of the normalization of $v_R(x)$ and $u(x)$. This follows from the fact that if v_R or u is multiplied by a constant factor, the corresponding asymptotic form must be multiplied by the same factor. In subsequent examples the normalization will be chosen for convenience.

A procedure is now set up by which any number of terms in the required expansion can be obtained. The expansions of $u(x)$ and $X(x)$, if they exist, can be obtained by the iteration method which follows[6].

The differential equation (1), together with its boundary conditions, may be replaced by the following integral equation,

$$u(x) = W_{x_1}(v_I, u)\, v_R(x) + \epsilon \left\{ v_I(x) \int_0^x g(x')\, v_R(x')\, u(x')\, dx' + v_R(x) \int_x^{x_1} g(x')\, v_I(x')\, u(x')\, dx' \right\} \tag{11}$$

If we choose x_1 sufficiently large to be outside the range of f, then

$$\begin{aligned} W_{x_1}(v_I, u) &= W_{x_1}(-Y_R, X) \\ &= -W_0(Y_R, X) + \epsilon \int_0^{x_1} g(x')\, Y_R(x')\, X(x') \end{aligned} \tag{12}$$

Thus

$$u(x) = -W_0(Y_R, X) + \epsilon \left\{ \begin{aligned} & v_I(x) \int_0^x g(x')\, v_R(x')\, u(x')\, dx' - v_R(x) \int_0^x g(x')\, v_I(x')\, u(x')\, dx' \\ & + v_R(x) \int_0^{x_1} g(x') \left[Y_R(x')\, X(x') + v_I(x')\, u(x') \right] dx' \end{aligned} \right\}$$

It is obvious now that x_1 may just as well be taken as $+\infty$, and one can always adjust the relative normalizations so that

$$W_0(Y_R, X) = -1$$

Assuming this to be done and defining

$$H_{12}(x) = Y_R(x)X(x) + v_I(x)u(x) \tag{13}$$

we arrive at the final integral equation

$$u(x) = v_R(x) + \epsilon \left\{ v_I(x)\int_0^X g(x')v_R(x')u(x')dx' - v_R(x)\int_0^X g(x')v_I(x')u(x')dx' + v_R(x)\int_0^\alpha g(x')H_{12}(x')dx' \right\} \tag{14}$$

The corresponding equation for X(x) is

$$X(x) = Y_I(x) + W_{x_0}(Y_I,X)Y_R(x) + \epsilon \left\{ Y_I(x)\int_0^X g(x')Y_R(x')X(x')dx' - Y_R(x)\int_{X_0}^X g(x')Y_I(x')X(x')dx' \right\} \tag{15}$$

The three equations (10), (14) and (15) may now be used to find the desired expansion of $W_{x_0}(Y_I X)$:

$$W_{x_0}(Y_I,X) = \epsilon \sum_{n=0}^{\infty} W_n \epsilon^n \tag{16}$$

Designate the corresponding expansions of u(x) and X(x) as follows:

$$u(x) = \sum_{n=0}^{\infty} u_n(x)\epsilon^n \tag{17}$$

$$X(x) = \sum_{n=0}^{\infty} X_n(x)\epsilon^n \tag{18}$$

It is convenient to define $h_{11}(x) = Y_I^2(x) - v_R^2(x)$, $h_{22}(x) = Y_R^2(x) - v_I^2(x)$, $h_{12}(x) = Y_R(x)\, Y_I(x) + v_R(x)v_I(x)$.

The zeroth order terms are written down at once.

$$\begin{aligned} u_0(x) &= v_R(x) \\ X_0(x) &= Y_I(x) \\ W_0 &= \int_{X_0}^{\alpha} g(x)h_{11}(x)dx \end{aligned} \tag{19}$$

To obtain W_1, we must substitute in (10) $u_0 + \epsilon u_1$ for u and $X_0 + \epsilon X_1$ for X. X_1 is written down from (15) ,

$$X_1(x) = W_0 Y_R(x) + Y_I(x)\int_0^x g(x')\, Y_R(x')Y_I(x')dx' - Y_R(x)\int_{x_0}^x g(x')Y_I^2(x')dx'$$

and u_1 from (14)

$$u_1(x) = v_I(x)\int_0^x g(x')v_R^2(x')dx' - v_R(x)\int_0^x v_I(x')v_R(x') + v_R(x)\int_0^\infty g(x')h_{12}(x')dx'$$

This leads to

$$W_1 = 2\int_{x_0}^\infty g(x)h_{11}(x)\int_0^x g(x')Y_R(x')Y_I(x')dx' - 2\int_{x_0}^\infty g(x)h_{12}(x)\int_0^x g(x')v_R^2(x')dx' \qquad (20)$$

This procedure may obviously be continued to give any number of terms which may be desired. We write down only one more, since the complication is rapidly increasing and there is as yet not sufficient experimental information to permit conclusions about the size of higher terms, in the energy expansion at least.

$$\begin{aligned} W_2 = \int_{x_0}^\infty dx\, g(x)\Big\{ & 2h_{11}(x)\Big[\int_0^x g(x')Y_I(x')Y_R(x')dx'\Big]^2 - 2v_R^2(x)\Big[\int_0^x g(x')Y_I(x')Y_R(x')dx'\Big]\Big[\int_x^\infty g(x'')h_{12}(x'')dx''\Big] \\ & + h_{22}(x)\Big[\int_0^x g(x')v_R^2(x')dx'\Big]^2 - 2h_{12}(x)\Big[\int_0^x g(x')Y_I(x')dx'\Big]\Big[\int_0^x g(x'')v_R^2(x'')dx''\Big] \\ & + Y_R^2(x)\Big[\int_x^\infty g(x')h_{11}(x')dx'\Big]^2 - 2Y_R^2(x)\Big[\int_0^x g(x')v_R^2(x')dx'\Big]\Big[\int_x^\infty g(x'')h_{11}(x'')dx''\Big] \\ & - v_R^2(x)\Big[\int_x^\infty g(x')h_{12}(x')dx'\Big]^2 - 2v_I(x)v_R(x)\Big[\int_0^x g(x')v_R^2(x')dx'\Big]\Big[\int_x^\infty g(x'')h_{12}(x'')dx''\Big]\Big\} \end{aligned} \qquad (21)$$

Applications of the expansion (16) will now be given.

III. Neutron-Proton Scattering

(a) Low Energies

The chief motivation for this investigation was to find a systematic scheme for determining the coefficients in the expansion (A). The differential equation in question is that for an S state of the n-p system:

$$\left\{\frac{d^2}{dx^2} + f\right\}\overset{o}{u}(x) = -K^2\overset{o}{u}(x) \qquad (22)$$

where the unit of length has been taken to be the range of the nuclear potential, f, and K is the relative wave number of the incident proton in these units. As usual,

$u(x)$ is r times the radial wave function. We wish to treat the energy as a perturbation, so that in terms of our general notation, $q(x) = 0$, $g(x) = 1$, and $\epsilon = \kappa^2$.

Let us first determine the meaning of $W_{x_0}(Y_I, X)$. Write

$$\overset{\circ}{X}(x) = \kappa \cot \overset{\circ}{\delta} \frac{\sin \kappa x}{\kappa} + \cos \kappa x \tag{23}$$

where $\overset{\circ}{\delta}$ is the S phase shift. A possible way of writing $Y_I(x)$ is

$$\overset{\circ}{Y}_I(x) = 1 + \frac{x}{\overset{\circ}{d}} \tag{24}$$

where $\overset{\circ}{d}$ is the scattering length. It follows immediately that

$$W_{x_0}(Y_I, X) = \kappa \cot \overset{\circ}{\delta} - \frac{1}{\overset{\circ}{d}} \tag{25}$$

and we have from (16)

$$\kappa \cot \overset{\circ}{\delta} - \frac{1}{\overset{\circ}{d}} = \kappa^2 \overset{\circ}{W}_0 + \kappa^4 \overset{\circ}{W}_1 + \kappa^6 \overset{\circ}{W}_2 + \ldots \tag{26}$$

where $\overset{\circ}{W}_0, \overset{\circ}{W}_1, \overset{\circ}{W}_2, \ldots$ are given by the expressions (19), (20), (21). . . in which $v_R(x)$ is taken as the zero energy solution of equation (22), normalized to (24) at infinity, and $v_I(x)$ is the irregular solution, normalized to $-\overset{\circ}{Y}_R(x) = -x$.

In (26), the coefficient $\overset{\circ}{W}_0$ is half the effective range, as defined by Schwinger[2,4] and the coefficient $\overset{\circ}{W}_1$ has been obtained by Blatt and Jackson[5] by a variational method. Their method, however, does not introduce explicitly the irregular zero energy solution $v_I(x)$, and consequently leads to a more complicated expression. $\overset{\circ}{W}_2$ has not previously been given. From a practical point of view, the introduction of $\overset{\circ}{v}_I(x)$ is justified because it can always be constructed from its representation (7) in terms of $\overset{\circ}{v}_R(x)$, once the latter is known.

The expansion (26) is correct for both singlet and triplet S states of the n-p system. In the triplet case, however, it is desirable to relate the scattering length, $\overset{\circ}{d}$, to the binding energy of the deuteron. The relation may be obtained at once from the above formalism.

The radial equation for the bound state of the deuteron is

$$\left\{\frac{d^2}{dx^2}+f\right\} u_D(x)=\alpha^2 u_D(x) \qquad \left(\alpha r_0=\sqrt{\frac{MB}{\hbar^2}}\right. \tag{27}$$

with r_0 the range of the force, B the binding energy of the deuteron.)

The boundary condition is that $U_D(x)$ approach zero for large x. We must set $\epsilon=-\alpha^2$, $q(x)=0$, and $g(x)=1$. Then $v_R(x)$, $v_I(x)$, $Y_R(x)$, $Y_I(x)$ will be the same functions as above. If we choose

$$X_0(x)=e^{-\alpha x}, \text{ then}$$

$$W_0(Y_I,X) = W_0\left(1+\frac{x}{\mathring{d}}, e^{-\alpha x}\right) = -\alpha-\frac{1}{\mathring{d}},$$

so that

$$\frac{1}{\mathring{d}}=-\alpha+\alpha^2\mathring{W}_0-\alpha^4\mathring{W}_1+\alpha^6\mathring{W}_2+\ldots \tag{28}$$

The coefficients in (28) are the same as in (26), so that we may rewrite the latter for the triplet case as

$$\kappa\cot\mathring{\delta}=-\alpha+(\kappa^2+\alpha^2)\mathring{W}_0+(\kappa^4-\alpha^4)\mathring{W}_1+(\kappa^6+\alpha^6)\mathring{W}_2+ \tag{29}$$

The coefficients here are defined in terms of zero energy <u>scattering</u> functions, not the deuteron functions, as is sometimes done.

(b) High Energies

If the value of $\kappa\cot\delta$ is known at some energy other than zero, say at κ'^2, then the energy dependence in the neighborhood of this value is given by the identity (10). The generalization of the procedure in (a) is obvious. The result is

$$\kappa\cot\mathring{\delta}=\kappa'\cot\mathring{\delta}_0'+(\kappa^2-\kappa'^2)\mathring{W}_0+(\kappa^2-\kappa'^2)\mathring{W}_1'+\ldots \tag{30}$$

where the coefficients $\mathring{W}_n'$ are defined in terms of the regular and irregular solutions at the energy κ'^2 and $\mathring{\delta}'$ is the corresponding phase shift.

(c) Higher angular momenta

We next generalize equation (22) to

$$\left[\frac{d^2}{dx^2} - \frac{\ell(\ell+1)}{x^2} + f\right] \overset{\circ}{U}_\ell(x) = -\kappa^2 \overset{\circ}{U}_\ell(x) \tag{31}$$

still treating κ^2 as the perturbation. Choose

$$\overset{\circ}{X}(x) = \kappa^{2\ell+1} \cot \overset{\circ}{\delta}_\ell \frac{j_\ell(x)}{\kappa^{\ell+1}} + \kappa^\ell n_\ell(x)$$

and

$$\overset{\circ}{Y}_I(x) = \frac{(2\ell)!}{2^\ell \ell!} x^{-\ell} + \frac{1}{\overset{\circ}{a}_\ell} \frac{2^\ell \ell!}{(2\ell+1)!} x^{\ell+1}$$

where $j_\ell(x)$ and $n_\ell(x)$ are respectively $\left(\frac{\pi x}{2}\right)^{\frac{1}{2}} J_{\ell+\frac{1}{2}}(\kappa x)$ and $(-1)^\ell \left(\frac{\pi x}{2}\right)^{\frac{1}{2}} J_{-\ell-\frac{1}{2}}(\kappa x)$.

Asymptotically, $j_\ell(x) \longrightarrow \sin\left(\kappa x - \frac{\ell\pi}{2}\right)$, $n_\ell(x) \longrightarrow \cos\left(\kappa x - \frac{\ell\pi}{2}\right)$

Then

$$W_{x_0}(Y_I, X) = \kappa^{2\ell+1} \cot \overset{\circ}{\delta}_\ell - \frac{1}{\overset{\circ}{a}_\ell} - \frac{(2\ell)!}{2^\ell \ell!} W_{x_0}\left(x^{-\ell}, \kappa^\ell n_\ell\right) \tag{32}$$

The last term on the right hand side will in general become infinite as x_0 approaches zero, but reference to (10) shows that this will be exactly compensated by terms on the right hand side of that equation. This fact is guaranteed by the identity (9).

Therefore let us define a series

$$-\frac{(2\ell)!}{2^\ell \ell!} W_{x_0}\left(x^{-\ell}, \kappa^\ell n_\ell\right) = \kappa^2 \sum_{n=0}^{\ell} D_{\ell,n} (\kappa^2)^n$$

and write out the expansion (16) as

$$\kappa^{2\ell+1} \cot \overset{\circ}{\delta}_\ell - \frac{1}{\overset{\circ}{a}_\ell} = \left(\overset{\circ}{W}_{\ell,0} - D_{\ell,0}\right)\kappa + \left(\overset{\circ}{W}_{\ell,1} - D_{\ell,1}\right)\kappa^4 + \ldots \tag{33}$$

where the $\overset{\circ}{W}_{\ell,n}$ are defined in terms of the zero energy solutions of equation (31).

For potentials without "a tail", such as the square well, the expansion (33) converges equally well for all ℓ. The criterion for convergence stated in II shows us, however, that if a small tail is important in producing the phase shift in question the perturbation approach fails as soon as the energy is comparable to the strength of the tail. Since the phase shifts for higher ℓ have a strong tail dependence, the convergence of (33) will become rapidly poorer as ℓ increases.

It is quite feasible, however, to use the analogue of (30) to investigate the effect on any phase shift of small fractional changes in the energy.

IV. Proton-proton scattering

(a) Low energies

Proton-proton scattering may be treated in much the same way as above. The radial equation for the S state of this case is

$$\left[\frac{d^2}{dx^2} - \beta/x + f\right] u(x) = -\kappa^2 u(x) \tag{34}$$

where $\beta = r_0/(\hbar/Me^2)$, with r_0 the range of the nuclear potential and M the proton mass. For low energies, we again identify κ^2 with ϵ to obtain the expansion analogous to (26). Choose

$$X(x) = C_0^2 \kappa \cot \delta \frac{F_0(x)}{C_0 \kappa} + C_0 G_0(x) \tag{35}$$

where $F_0(x)$ and $G_0(x)$ are the regular and irregular solutions in a repulsive coulomb field defined by Breit, Wheeler and Yost[6].

$$F_0 = C_0 \kappa x \left\{1 + \frac{\beta x}{2} - \frac{1}{6}\kappa^2 x^2 + \frac{\beta^2 x^2}{12} + \cdots\right\}$$

$$G_0 = \frac{1}{C_0}\left[1 - \frac{\kappa^2 x^2}{2} + \frac{3}{4}\beta^2 x^2 + \beta\alpha\left\{\ln \gamma^2 \beta x - 1 + Q\right\} + \cdots\right] \tag{36}$$

$$F_0 \xrightarrow[a.s.]{} \sin(\kappa x + \delta_c)$$

$$G_0 \xrightarrow[a.s.]{} \cos(\kappa x + \delta_c)$$

where

$$C_0^2 = 2\pi\alpha/(e^{2\pi\alpha} - 1)$$

$$\delta_c = \arg \Gamma(1 + i\alpha) - \alpha \ln 2\kappa x$$

$$\alpha = \beta/2\kappa$$

$$Q = \left\{\text{Real Part } \frac{\Gamma'(-i\alpha)}{\Gamma(-i\alpha)}\right\} - \ln \alpha$$

The zero energy asymptotic form is taken as

$$Y_I = K(x) + I(x)/a \tag{37}$$

where

$$I(x) = 1/\beta\sqrt{\beta x}\ I_1\ (2\sqrt{\beta x})$$
$$K(x) = 2\sqrt{\beta x}\ K_1\ (2\sqrt{\beta x}) \tag{38}$$

and a might be called a dimensionless "coulomb scattering length".

I (x) and $K(x)$ are the modified Bessel functions of the second kind of pure imaginary argument as defined by Watson[7]. The behavior of I(x) and K(x) near the origin is

$$I(x) \sim x + \beta x^2/2$$
$$K(x) \sim 1 + \beta x \left\{ \ln \gamma\beta x - 1 \right\}$$

allowing us to calculate

$$W_o(Y_I, X) = C_o^2 \kappa \cot \delta - \frac{1}{a} + \beta Q \tag{39}$$

The required expansion is, therefore

$$C_o^2 \kappa \cot \delta + \beta Q = 1/a + \kappa^2 W_o + \kappa^4 W_1 + \kappa^6 W_2 \tag{40}$$

where the W_n are defined in terms of the appropriately normalized zero energy solutions of the equation (34). Here again the coefficients W_o and W_1 have been obtained and discussed by Schwinger and others. We believe that the method of approach given here is simpler and makes the relation between the n-p and p-p expansions clearer.

It is obvious that the results of III (b) and III (c) can be generalized to the p-p case. The formulae become quite complicated, so we shall write down only a single special case, one that might be of interest in the near future. This case is the p phase shift for triplet proton-proton scattering. The appropriate expansion turns out to be

$$(1+\alpha^2) C_o^2 \kappa^3 \cot \delta_1 + \beta\kappa^2 (Q_1 - 2) + \beta^3 Q_1/4 - 1/a_1 \tag{41}$$
$$= \kappa^2 \left(W_o - \beta \ln \gamma^2 \beta x_o - 1/x_o \right) + \kappa^4 W_1 + \kappa^6 W_2 + \cdots$$

where

$Q_1 = R_o P_o \dfrac{\Gamma'(2-i\alpha)}{\Gamma(2-i\alpha)} - \ln \alpha$ and the W_n are defined in terms of the solutions

of the equation

$$\left\{ \frac{d^2}{dx^2} - \beta/x - 2/x^2 + f \right\} \begin{Bmatrix} v_R \\ v_I \end{Bmatrix} = 0$$

normalized at infinity to

$$Y_I = \beta\sqrt{\beta x}\, K_3\left(2\sqrt{\beta x}\right) + \frac{1}{a_1}\,\frac{2}{\beta}\sqrt{\frac{x}{\beta}}\, I_3\left(2\sqrt{\beta x}\right)$$

and

$$Y_R = -\frac{2}{\beta}\sqrt{\frac{x}{\beta}}\, I_3\left(2\sqrt{\beta x}\right)$$

(b) A treatment of the coulomb field as a perturbation

Up to this point we have consistently had $g(x) = 1$ in all applications. As was pointed out in the introduction, there is no reason why this must be done. One might, for instance, investigate the effect of small changes in the nuclear potential itself. We describe here an application, in which the coulomb term, β/x, is considered as small. This is possible even though outside the range of the nuclear force, the coulomb term may represent the main contribution to the total energy. The point is that the integrand on the right hand side of equation (10) fails to vanish only within the range of the nuclear force. What the perturbation may do to the function u(x) outside the range, therefore, does not affect the value of the integral. The only requirement is that its relative effect <u>inside</u> the range is small. This condition is clearly satisfied by the coulomb field, even when the kinetic energy of the incident proton is small.

Let us consider equation (34) again, therefore, this time regarding $-\beta/x$ as the perturbation, i.e., $\epsilon = -\beta$, $g = 1/x$ X is again given by (35) but Y_I is now to be

$$Y_I = \kappa \cot \mathring{\delta}\, \frac{\sin \kappa x}{\kappa} + \cos \kappa x$$

where $\mathring{\delta}$ is the phase shift produced by the nuclear field alone. We calculate

$$W_{x_0}(Y_I, X) = C_0^{\,2}\kappa \cot \delta - \kappa \cot \mathring{\delta} + \beta \ln \gamma^2 \beta x_0 + \beta Q \qquad (42)$$

The expansion (16) becomes

$$C_0^2 \varkappa \cot \delta + \beta Q = \varkappa \cot \overset{\circ}{\delta} - \beta (W_0 + \ln \gamma^2 \beta x_0) + \beta^2 W_1 - \beta^3 W_2 \tag{43}$$

where the coefficients W_n are defined in terms of the <u>purely</u> <u>nuclear</u> wave functions at the energy in question. Since β is always of the order 1/10 or less, the series converges rapidly and we have a method for computing coulomb phase shifts without actual recourse to the complicated and only incompletely tabulated hypergeometric functions. The corresponding relation for the p phase shift is

$$\begin{aligned}(1+\alpha^2)\, C_0^2 \varkappa^3 \cot \delta_1 + \beta \varkappa^2 (Q_1 - 2) + \beta^3 Q_1/4 - \varkappa^3 \cot \overset{\circ}{\delta}_1 = \\ = -\beta \left[W_0 + \ln \gamma^2 \beta x_0 - 1/2\, x_0^2\right] + \beta^2 \left[W_1 - 1/2\, x_0\right] \\ - \beta^3 \left[W_2 + \tfrac{1}{4} \ln \gamma^2 \beta x_0 + 3/8\right] + \beta^4 W_3 + \cdots\end{aligned} \tag{44}$$

where the W_n are defined in terms of the purely nuclear p-wave functions at the energy $\varkappa^2$ normalized as in III (c) .

(c) Coulomb field plus energy as perturbation

The last application of the identity (10) to be given here is provided by the case in which the energy term and the coulomb term are simultaneously considered as perturbations. We restrict our attention to S scattering. In equation (34), therefore, we regard $\varkappa^2 - \beta/x$ as $\epsilon\, g(x)$. Here we use (35) for $X(x)$ and (24) for $Y_I(x)$. This leads to

$$W_{x_0}(Y_I, X) = C_0^2 \varkappa \cot \delta + \beta Q + \beta \ln \gamma^2 \beta x_0 - \frac{1}{a}$$

and thus to

$$\begin{aligned} C_0^2 \varkappa \cot \delta + \beta Q = \frac{1}{a} + \int_{x_0}^{\infty} (\varkappa^2 - \beta/x)(\overset{\circ}{Y}{}_I^2 - v_R^2)\, dx - \beta \ln \gamma^2 \beta x_0 \\ + 2 \int_{x_0}^{\infty} (\varkappa^2 - \beta/x) \Big[\overset{\circ}{h}_{11}(x) \int_0^x \overset{\circ}{Y}_I(x') \overset{\circ}{Y}_R(x') \\ (\varkappa^2 - \beta/x')\, dx' - \overset{\circ}{h}_{12}(x) \int_0^x \overset{\circ}{v}{}_R^2(x')(\varkappa^2 - \beta/x')\, dx' \Big] \end{aligned} \tag{40'}$$

A comparison of (40') with (40) allows us to express the coulomb scattering length and effective range in terms of the corresponding purely nuclear quantities.

Thus

$$1/a = 1/a_0 - \beta\left[\int_{x_0}^{\infty}\frac{dx}{x}\left(\mathring{Y}_I^2(x) - \mathring{v}_R^2(x)\right)dx - \ln \gamma^2\beta x_0\right] \tag{45}$$

$$+ 2\beta^2\left[\int_{x_0}^{\infty}\frac{dx}{x}\mathring{h}_{11}(x)\int_0^x\frac{dx'}{x'}\mathring{Y}_I(x')\mathring{Y}_R(x') - \int_{x_0}^{\infty}\frac{dx}{x}\mathring{h}_{12}(x)\int_0^x\frac{dx'}{x'}\mathring{v}_R^2(x')\right] + O(\beta^3)$$

$$W_0 = \mathring{W}_0 - 2\beta\int_{x_0}^{\infty}dx\left[\begin{array}{l}\mathring{h}_{11}(x)\int_0^x\frac{dx'}{x'}\mathring{Y}_I(x')\mathring{Y}_R(x') + \frac{\mathring{h}_{11}(x)}{x}\int_0^x dx'\,\mathring{Y}_I(x')\mathring{Y}_R(x') \\ -\mathring{h}_{12}(x)\int_0^x\frac{dx'}{x'}\mathring{v}_R^2(x') - \frac{\mathring{h}_{12}(x)}{x}\int_0^x\mathring{v}_R^2(x')\,dx'\end{array}\right] + O(\beta^3) \tag{46}$$

These last two formulae are of considerable practical use, because the best way of analyzing low energy p-p data at present is to plot the left hand side of (40) versus energy (or κ^2) and then read off (in dimensional units of course) the intercept and slope of the resulting straight line. The problem is then to find out what these two numbers, the coulomb scattering length and the effective range, imply about the nuclear potential which is acting. A definition of 1/a and W_0 in terms of a purely nuclear zero energy wave function is much easier to interpret than one which involves the combined nuclear and coulomb field. The terms actually written above give sufficient accuracy for the available data, the residual error in each expansion being less than 1% .

Discussion

The advantages of analyzing nucleon-nucleon scattering data via expansions in powers of the energy have been mentioned in the introduction and are discussed in great detail by Blatt and Jackson[5] . Unfortunately the data available at present make this a completely practical procedure only for singlet-S proton-proton scattering. Higher angular momenta and n-p scattering in general still require the direct computation of phase shifts from an assumed potential for at least part of the analysis. We believe that in many cases expansions such as presented in this paper afford the most convenient means of making such computations.

It is necessary to determine a wave function, in general by numerical integration, and then to evaluate various definite integrals involving this function. Once these integrals are known, the behavior of the phase shifts over a considerable region of variation of the parameter in question is also known. We give a few typical examples.

(1) Energy dependence of phase shifts:

(a) A knowledge of $\overset{o}{W}_o$ and $\overset{o}{W}_1$ in the expansion (26) will yield the n-p singlet-S phase shift with an accuracy of 1% up to an energy of 40 Mev. A similar situation exists in expansion (40) for the p-p singlet S phase shift. In the expansions (33) and (41) a knowledge of the coefficient of $\varkappa^2$ yields δ_1 to 2% up to the energies of 30 Mev for wells without tails.

Triplet phase shifts are given with a comparable accuracy. The triplet S is an especially favorable case because of the nearness to 90° during much of the interesting range.

(2) Modification of the nuclear phase shift due to the coulomb field: Expansion (43), keeping only terms proportional to β, may be used at all energies with roughly the same accuracy. At 32 Mev the error in the phase shift is about 0.2^o. The accuracy of the formulae in Section IV (c) has already been discussed.

We have been purposely vague about the nature of the potential, f. The only requirements that need be made are that it be a short range, self adjoint operator. Professor Breit has pointed out in this connection that an operator of the form

$$f\,\Psi(r) = \int dr'\, V(r,r')\, \Psi(r') \quad ,$$

with $V(r,r') = V(r', r)$ may be treated by this method.

Acknowledgements:

We are indebted to Professor Serber who first pointed out to us the possibility of obtaining the expansions by a straightforward non-variational procedure and who contributed helpful advice at many points. Thanks are also due R. S. Christian for the scheme of including boundary conditions in the integral equation (11) and for many enlightening discussions on the general scattering problem.

Work described in this report was performed under the auspices of the Atomic Energy Commission.

RLID/MR
2-16-49

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