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ON THE ANALYSIS OF NEUTRON-PROTON SCATTERING WITH TENSOR FORCES

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A perturbation method for the analysis of nucleon-nucleon-scattering by central forces has recently been proposed by Chew and Goldberger[1] among others.[2,3,4]

[1] G. F. Chew and M. L. Goldberger, Phys. Rev. in press.

[2] J. M. Blatt and J. D. Jackson, Phys. Rev. in press.

[3] J. Schwinger, Hectographed Notes on Nuclear Physics, Harvard, 1947.

[4] Peirels and Barker, Phys. Rev. 75, 312, (1945).

It is the purpose of the present note to extend the work of Chew and Goldberger to include tensor force scattering. We will, therefore, follow their notation as closely as practicable and their paper should be consulted for a discussion of the general problem.

The second order differential system which arise in the consideration of the $({}^3S_1 + {}^3D_1)$ state with tensor forces, we write in the matrix form

$$(L(x) + F(x))U(x) = -K^2U(x), \qquad (1)$$

where $L = (L_{ij})$ and $F = (F_{ij})$ for $i, j = 1, 2$. That is, explicitly, $L_{11} = \frac{d^2}{dx^2}$, $L_{22} = \frac{d^2}{dx^2} - \frac{6}{x^2}$, $L_{12} = L_{21} = 0$ and $F_{11} = f(x)$, $F_{22} = f(x) - 2g(x)$ and $F_{12} = F_{21} = 2^{3/2}g(x)$, where $f(x)$ and $g(x)$ are self adjoint short range operators. $U(x)$ is a two component vector, with components $u(x)$ and $w(x)$. There are in general four such independent vector solutions of equation (1). The wronskian of any two vector functions is defined by

$$W_x(U_i, U_j) = u_i(x)u_j'(x) - u_i'(x)u_j(x) + w_i(x)w_j'(x) - w_i'(x)w_j(x). \qquad (2)$$

It is a consequence of this definition and the self adjoint nature of equation (1) that the wronskian of any two solutions of (1) is a constant. We adopt the convention that the subscripts 1 and 3 shall indicate solutions which are zero at the origin, the so-called regular solutions, and the subscripts 2 and 4 those which are irregular at the origin. For convenience we specify that the solutions are chosen such that

$$W_x(U_1, U_2) = W_x(U_3, U_4) = 1, \tag{3}$$

and all other wronskian combinations vanish identically.

The asymptotic forms of U_1 and U_3 are defined as the two component vectors X_2 and X_4 respectively, which are themselves solutions of the system

$$LX_i = -\kappa^2 X_i. \tag{4}$$

The vectors U_2 and U_4 are defined to be such as to have the asymptotic forms $-X_1$ and $-X_3$ respectively.

The solutions of equations (1) and (3) contain only three independent constants, these being sufficient to characterize scattering completely. We may conveniently choose the regular solutions such that both components of U_i have the same phase shift, δ_i. The remaining constant will then express the asymptotic ratio between the two components of a solution. This completes the definition of a complete set of U vectors which are explicitly tabulated below in terms of their asymptotic forms, the X vectors:

$$U_2 \longrightarrow -X_1 = (1/1 + R\kappa^4)\begin{pmatrix} (\sin \kappa x/\kappa) \\ R\,\kappa^4(j_2(\kappa x)/\kappa^3) \end{pmatrix} \tag{5a}$$

$$U_1 \longrightarrow X_2 = \begin{pmatrix} \kappa \operatorname{ctg} \delta_1\ (\sin \kappa x/\kappa) + (\cos \kappa x) \\ R\left[\kappa^5 \operatorname{ctg} \delta_1\ (j_2(\kappa x)/\kappa^3) + (\kappa^2 n_2(\kappa x))\right] \end{pmatrix} \tag{5b}$$

$$U_4 \longrightarrow -X_3 = (1/1 + R\kappa^4)\begin{pmatrix} -R\ (\sin \kappa x/\kappa) \\ (j_2(\kappa x)/\kappa^3) \end{pmatrix} \tag{5c}$$

$$U_3 \longrightarrow X_2 = \begin{pmatrix} -R\left[\kappa^5 \operatorname{ctg}\delta_3 \,(\sin \kappa x/\kappa\,) + \kappa^4(\cos\kappa x)\right] \\ \kappa^5 \operatorname{ctg}\delta_3 \,(j_2(\kappa x)/\kappa^3) + (\kappa^2 n_2(\kappa x)) \end{pmatrix} . \tag{5d}$$

In general, $\kappa \operatorname{ctg}\delta_1$, $\kappa^5 \operatorname{ctg}\delta_3$, and the ratio R are functions of κ^2. In the limit of $\kappa^2 \to 0$ the forms $\kappa \operatorname{ctg}\delta_1$ and $\kappa^5 \operatorname{ctg}\delta_1$ are finite and are the reciprocals of the scattering lengths a_1 and a_3 respectively. The contribution to the total cross section from this state is given by

$$\sigma = (4\pi/k^2)(\sin^2\delta_1 + \sin^2\delta_3). \tag{6}$$

The U vectors in the limit of $\kappa^2 \to 0$, we call the V vectors. They obviously satisfy the relation

$$(L + F)V_i = 0. \tag{7}$$

The limiting X vectors we call the Y vectors. They satisfy the relation

$$(L)Y_i = 0. \tag{8}$$

From equations (1) and (7) we derive

$$W_{x_0}(V_i, U_j) = W_{x_1}(V_i, U_j) + \kappa^2 \int_{x_0}^{x_1} V_i U_j , \tag{9}$$

where scalar multiplication is implied by the juxtaposition of the two vectors as in the integrand above.

Similarly from equations (4) and (8) we get

$$W_{x_0}(Y_i, X_j) = W_{x_1}(Y_i, X_j) + \kappa^2 \int_{x_0}^{x_1} Y_i X_j . \tag{10}$$

To eliminate the singular terms occurring in this last equation in the limit $x_0 \to 0$ it is convenient to define the new vectors $X_2^{(\ell)}$ and $X_4^{(\ell)}$ which are formed by subtracting from the second component of X_2 and X_4, R and 1 (one) times the first ℓ terms in the expansion of $(\kappa^2 n_2(\kappa x))$ in powers of $(\kappa x)^2$, respectively. These satisfy the relations

$$(L)X_i^{(\ell)} = -\kappa^2 X_i^{(\ell-1)}. \tag{11}$$

Hence from equations (8) and (11) we get

$$W_0(Y_i, X_j^{(\ell)}) = W_{x_1}(Y_i, X_j^{(\ell)}) + \kappa^2 \int_0^{x_1} Y_i X_j^{(\ell-1)} \tag{12}$$

From equations (9) and (12) and the definitions of the asymptotic forms we can derive several relations analogous to the central force fundamental identity, equation (10) of Chew and Goldberger. We consider only the combination formed from the first two solutions

$$W_0(Y_2, X_2^{(3)}) = \kappa^2 \int_0^\infty (Y_2 X_2^{(2)} - V_1 U_1). \tag{13}$$

Evaluating the left hand side we find

$$W_0(Y_2, X_2^{(3)}) = (\kappa \operatorname{ctg} \delta_1 - 1/a_1) + R_0 R_k \kappa^5 \operatorname{ctg} \delta_1 . \tag{14}$$

The right hand side of equation (13) can be expanded in powers of κ^2 by an iteration procedure based on the integral equation satisfied by U_1, which is shown in the appendix to be

$$U_1 = W_x(V_2, U_1)V_1 + W_x(V_4, U_1)V_3$$
$$\kappa^2 \left[V_1 \int_x^{x_1} (V_2 U_1) + V_3 \int_x^{x_1} (V_4 U_1) + V_2 \int_0^x (V_1 V_1) + V_4 \int_0^x (V_3 V_1) \right], \tag{15}$$

which by use of equations (9) and (10) can be written in the form

$$U_1 = V_1 + \kappa^2 \left[-V_1 \int_0^x (V_2 U_1) - V_3 \int_0^x (V_4 U_1) + V_2 \int_0^x (V_1 V_1) + V_4 \int_0^x (V_3 V_1) \right. \tag{16}$$
$$\left. V_1 \int_0^\infty (V_2 U_1 - X_1 Y_2) + V_3 \int_0^\infty (V_4 U_1 - X_3 Y_2) \right] .$$

Similarly we find for the irregular solution $X_2^{(2)}$ of equation (11) the integral equation

$$X_2^{(2)} = Y_2^{(1)} + \kappa^2 \left[-Y_1 \int_0^x (Y_2 X_2^{(1)}) - Y_3 \int_0^x (Y_4 X_2^{(1)}) + Y_2^{(1)} \int_0^x (Y_1 X_2) + Y_4^{(1)} \int_0^x (Y_3 X_2) \right.$$
$$\left. + Y_1 \int_0^\infty (Y_2 X_2^{(1)} - V_1 U_1) + Y_3 \int_0^\infty (Y_4 X_2^{(1)} - V_3 U_1) \right] . \tag{17}$$

Using these last two equations and equations (13) and (14) we can write

$$\kappa \operatorname{ctg} \delta_1 = 1/a_1 + W_0 \kappa^2 + W_1 \kappa^4 + \ldots \tag{18}$$

where

$$W_o = \int_0^\infty \left[(x/a_1 + 1)^2 - V_1 V_1 \right] , \tag{19}$$

and

$$W_1 = -R_o^2/a_1 + 2\int_0^\infty \left[(x/a_1 + 1)x - V_1 V_2 \right] \int_0^x (V_1 V_2) - 2 \int_0^\infty \left[(x/a_1 + 1)^2 - V_1 V_1 \right] \int_0^x (x/a_1 + 1)x$$

$$2\int_0^\infty \left[(x/a_1 + 1) R_o x - R_o x/5 - V_4 V_1 \right] \int_0^x (V_1 V_3) \tag{20}$$

$$-2\int_0^\infty \left[(x/a_1 + 1)\ (-R_o x/5a_3) + R_o x/5a_3 - V_1 V_3 \right] \int_0^x \left[R_o x(x/a_1 + 1) - R_o x/5 \right] .$$

It can be noted here that the integral of $V_1 V_1$ is larger than that of the corresponding central force function, hence the effective range will be reduced by the addition of tensor forces. Also that the exact expression for W_1 is complicated by the presence of all four V solutions, however that it can be approximated by neglecting terms proportional to R^2. The result is then identical with the central force case. (The neglected terms are smaller by a factor of the order of the fraction of $^3D_1$ admixture in the ground state of the deuteron.)

The preceding analysis was carried out by expanding about the zero energy state, however it is clear that this may be generalized to an expansion about any energy, even to negative energies corresponding to a bound state. In particular since the deuteron bound state wave function is generally available from calculations of the binding energy, quadrupole moment, etc., it is desirable to calculate the expansion coefficients using the bound state wave function, $V_1(\alpha)$. We will then be able to calculate an approximate third coefficient simply from the change in the second coefficient. The expansion about the bound state is

$$\kappa \operatorname{ctg} \delta_1 = -\alpha + W_o' (\kappa^2 + \alpha^2) + W_1' (\kappa^2 + \alpha^2)^2 + \dots , \tag{21}$$

where $-\alpha^2$ gives the binding energy on the bound level, and

$$W_o' = \int_0^\infty \left[e^{-2\alpha x} - V_1 (\alpha)\ V_1 (\alpha) \right] , \tag{22}$$

with an expansion for W_1' similar to the one for W_1.

We can further relate the scattering length to the binding energy of the bound level, that is, we expand the bound level wave functions in the zero energy wave functions as in equation (15). The result is evidently

$$1/a_1 = -\alpha + W_0\alpha^2 - W_1\alpha^4 + \ldots \quad . \tag{23}$$

Combining this result with equations (18) and (21) we get the approximate expression for W_1

$$W_1 \cong (1/2\alpha^2)\,(W_0' - W_0) \quad . \tag{24}$$

(The coefficient W_0' has been given by Schwinger[3] using a variational procedure and is one half the effective range in the current terminology.)

Calculations with these formulas have been made for the meson potentials by E. W. Hart and the present author[5] in connection with the analysis of the high energy scattering. The modifications in the effective range due to the addition of tensor forces is at most a few percent for effective ranges less than 2×10^{-13}cm and with tensor strengths sufficient to account for the quadrupole moment. For example, the results with a meson potential of range 1.35×10^{-13}cm can be approximately expressed in dimensional units by

$$2W_0 = 1.62\,(1 - .0178\,P_D) \times 10^{-13}\text{cm}. \tag{25}$$

where P_D is the percentage $^3D_1$ state in the deuteron. The modification of W_1 is also small.

In particular it is possible to characterize scattering below 10 Mev by means of only two parameters scattering length and effective range, even in the presence of tensor forces.

The author wishes to thank Professor Serber and Drs. Chew and Goldberger for helpful discussions and advice.

[5] R. Christian and E. W. Hart, Bull. Am. Phys. Soc. 24, No. 2, F6 (1949).

APPENDIX

The coupled second order differential system, equation (1) in the text, can be replaced by a system of four first order differential equations by means of the

substitutions $y_{1i} = u_i$, $y_{2i} = u_i'$, $y_{3i} = w_i$, and $y_{4i} = w_i'$. We can therefore write equation (1) as the matrix equation[6]

[6] G. D. Birkhoff and R. E. Langer, Proc. of the Am. Acad. of Arts and Sciences, 58, No. 2, (1923).

$$Y'(x) = A(x)Y(x) \tag{1A}$$

where

$$A(x) = -\begin{pmatrix} 0 & -1 & 0 & 0 \\ F_{11}(x) & 0 & F_{12}(x) & 0 \\ 0 & 0 & 0 & -1 \\ F_{21}(x) & 0 & F_{22}(x)-\frac{6}{x^2} & 0 \end{pmatrix} \tag{2A}$$

and $Y = (y_{ij})$. That is, each column represents an independent solution. It is readily verified by direct application of (1A) that $\det Y(x)$ is a constant and further that if C is a constant matrix, $Y(x)C$ is the most general solution of (1A).

We also consider the related system

$$Z'(x) = -Z(x)A(x) \tag{3A}$$

which is the so-called adjoint system. By use of (1A) and (3A) it can be shown that the product $Z(x)Y(x)$ is a constant matrix. We can therefore choose an adjoint solution, such that $Z(x)Y(x) = I$, i.e., $Z(X) = Y^{-1}(x)$.

A general adjoint solution of (3A) is given by

$$Z(x) = \begin{pmatrix} u_2(x) & -u_2(x) & w_2(x) & -w_2(x) \\ -u_1(x) & u_1(x) & -w_1(x) & w_1(x) \\ u_4(x) & -u_4(x) & w_4(x) & -w_4(x) \\ -u_3(x) & u_3(x) & -w_3(x) & w_3(x) \end{pmatrix} \tag{4A}$$

It is further equal to $Y^{-1}(x)$ if the conditions of equation (3) are imposed upon the otherwise arbitrary solutions. The relation $Y(x)Z(x) = I$, yields the following identies

$$u_1(x)u_2'(x) - u_1'(x)u_2(x) + u_3(x)u_4'(x) - u_3'(x)u_4(x) = 1 \tag{5A}$$

$$w_1(x)w_2'(x) - w_1'(x)w_2(x) + w_3(x)w_4'(x) - w_3'(x)w_4(x) = 1 \tag{6A}$$

$$w_1(x)u_2(x) - w_2(x)u_1(x) + w_3(x)u_4(x) - w_4(x)u_3(x) = 0. \tag{7A}$$

which are just the relation necessary for establishing the validity of the integral equation (14) of the text.

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