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"THE THEORY AND INTERPRETATION OF POLARIZATION PHENOMENA IN NUCLEAR SCATTERING"

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THE THEORY AND INTERPRETATION OF POLARIZATION PHENOMENA IN NUCLEAR SCATTERING

Henry Pierce Stapp

(Thesis)

August, 1955

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ABSTRACT

This thesis deals with a theoretical investigation of polarization phenomena in nuclear scattering. In Part I, the expressions needed for a phase-shift analysis of polarization, triple scattering, and correlation experiments in nucleon-nucleon scattering are derived, and the results of a phase-shift analysis are given for proton-proton (P-P) scattering at 310 Mev. The theory of the correlation experiments is then developed and an explicit expression for the scattering matrix at 90° as a function of these correlation experiments together with the triple scattering experiments is obtained. The symmetry effects in P-P scattering and the formalism relating the N-P to P-P experiments is developed, and the problem of separating the nuclear phase shifts from the coulomb parts is discussed. In Part II, a covariant treatment of polarization phenomena in double and triple scattering of Dirac particles from spin-zero targets and from Dirac particles is developed and the relativistic triple scattering and correlation expressions are obtained. In Part III, the nonrelativistic scattering matrix of spin-one particles by spin-zero targets is developed. The available data on deuteron polarization are analyzed in terms of first and second Born approximations. The effect of the D-state of the deuteron upon the polarization phenomena is also considered.

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INTRODUCTION

During recent months a considerable number of polarization experiments have been performed both here[1] and abroad.[2] In these experiments particles emerging from nuclear collisions are found to have their spin orientations partially aligned. This alignment, or polarization as it is called, may be studied by means of a further scattering process. The degree to which a particular type of nuclear scattering polarizes the particles will depend upon the spin dependence of the forces which produce this scattering, and the importance of polarization experiments lies in this information about the spin dependence of nuclear forces which they provide. A basic purpose of this dissertation is to develop the theory needed to extract from the experimental data the basic theoretical parameters of the problem and to apply this theory to the analysis of the data which are now available.

The presence of spin dependence in nuclear forces, which is evidenced at low energies by the deuteron quadrupole moment and by the success of the shell model, is shown at cyclotron energies by the large polarizations which are obtained. Forces of this type were used several years ago in the phenomenological models of Jastrow[3], Christian et al[4,5], and Case and Pais,[6] and were designed to explain the differential cross sections in nucleon-nucleon scattering. More recently Goldfarb and

Feldman,[7] and Swanson[8] have investigated the polarization effects predicted by these models. The hard core model of Jastrow was found to give polarizations much smaller than the experimental values, and only the L·S model of Case and Pais was found to give the very large polarizations which are observed. The qualitative fit of the Case and Pais differential cross section with the experimental results is, however, almost wholly destroyed in the more exact variational treatment of Goldfarb and Feldman.

This apparent failure of the potential models suggests an alternative approach in which the experimental data are taken as the starting point and an effort is made to extract the information which they contain. A first step along this direction would be a phase-shift analysis of the available data. Until recently, when the polarization data became available, such an analysis was in principle impossible unless only S-waves were considered. For triplet states each value of L, the orbital angular momentum, which is higher than one has four associated phase shifts corresponding to the three possible values of J and to the admixture parameter. The number of fourier coefficients (independent pieces of information) in the differential cross section is $2L + 1$ for the N - P system and $L + 1$ in the P - P system where L is the highest orbital angular momentum which contributes. Thus the number of phase shifts to be determined increases with L much faster than the number of independent pieces of information in the differential cross section. For example, if the partial waves higher than f-waves are neglected in the analysis of the proton-proton experiments there are nine independent phase shifts. However, the corresponding differential cross section contains only four fourier coefficients and the four equations on the nine phase shifts are indeterminate. Even when the

ordinary polarization data, which are described by three fourier coefficients, are added the equations do not become determinate. However, besides the ordinary polarization experiment, which involves two scatterings, the first to polarize a beam and the second to analyze this beam, there are a number of independent experiments involving three or more scatterings. In the basic triple scattering experiments a first scattering produces a polarized beam. This beam is then scattered by the interaction being studied and the emerging beam is analyzed by means of a third scattering. There are a variety of experiments of this type corresponding both to the different orientations of the three scattering planes and to the various possible values of external parameters such as magnetic fields. These experiments, which give new information about the system, are described by several new and independent parameters, two of which have been measured for the P - P system in recent experiments at Berkeley.[9] With this additional information the equations on the P - P phase shifts become determinate and a phase shift analysis becomes possible. Such an analysis has been carried out and is discussed in the first two sections of Part One.

Part One is devoted, in general, to the treatment of nucleon-nucleon polarization phenomena. In its first section the phase-shift expressions for the various quantities measured in the P - P polarization and triple-scattering experiments are derived. The treatment of these experiments is based upon the theory developed by Wolfenstein[10] and by Wolfenstein and Ashkin,[11] while the treatment of the P - P phase shifts is similar to that of Blatt and Biedenharn.[12] The results obtained in section one are recorded in Tables A, B, C, and D, and they are the basis of the phase-shift analysis of the P - P experiments discussed in section two.

In the analysis of the Berkeley experiments, the incident beam energy of which is about 300 Mev, it was assumed that partial waves higher than f-waves could be neglected. However, the validity of this assumption is uncertain and alternatives to the phase shift-method are desirable. In the third section of Part One the theory of another type of triple-scattering experiments, called correlation experiments, is developed and it is shown how, with the aid of these experiments, the scattering matrix at particular angles may be determined without the use of a phase-shift analysis. In particular the explicit form of the scattering matrix at 90° is given in terms of the triple scattering and correlation parameters measured at this angle. One of the correlation parameters (i.e., C_{NN}) is found to have a particularly simple significance at 90° where it gives a direct measure of the singlet part of the scattering.

Polarization experiments have also been carried out on the N - P system. The fourth section is devoted first to the discussion of the symmetry properties of the P - P scattering matrix and then to the relationships between the N - P and P - P scattering matrices required by the hypothesis of charge independence. Here it is assumed that for large angles the coulomb contributions to the P - P scattering matrix are negligible. Several direct relationships among the observed quantities in the N - P and P - P experiments are then obtained. The experiments which are involved, however, have not yet been performed.

The analysis of the nucleon-nucleon system outlined above and developed in Part One is nonrelativistic. However, the incident beam energy of the Berkeley and other cyclotron experiments together with the still higher energies which are becoming available indicate the desirability of a completely relativistic treatment of nucleon

polarization phenomena. Such a treatment is carried out in Part Two. The first three sections are devoted to the development of covariant forms for the S-matrix, the density matrix, and the polarization formalism for the case of the scattering of a Dirac particle by a series of spin zero, finite-mass particles. In the fourth section the scattering of a Dirac particle by a Dirac particle is treated. It is shown that with certain interpretations and modifications the nonrelativistic formalism is applicable in the relativistic region. The relativistic formulas for Wolfenstein's triple scattering parameter $R^{(10)}$ and for the correlation parameter C_{KP} differ from the nonrelativistic formulas, and their relativistic forms are given. It is found that the relativistic corrections for the Berkeley experiments are of order 10%.

A different type of polarization experiment, which has been carried out at Berkeley,[13] is the polarization of deuterons. Because the deuteron spin is one rather than one-half, its state of polarization is not completely specified by the orientation of the spin axis. There are also orientational features which may be described in terms of tensors, as opposed to the vectors which specify the spin orientation. The first section of the Third Part contains a general development of the theory of the scattering of a spin-one particle from a spin-zero target. The treatment is a generalization of the M-matrix formalism used in the first two parts for the polarization theory of Dirac particles. Using a different approach the general problem of the polarization of the deuteron has been studied by Lakin.[14] Insofar as they overlap the results of the two treatments are in agreement. The results of section one are used in the next three sections in which calculations based upon various models and methods of approximations are performed and

the results compared to the experiments. In section two a Thomas[15] type of spin-orbit force used in nucleon scattering problems by Fermi[16] and others[17] is assumed and the Born approximation to the solution is used to investigate the strength of the spin-orbit coupling. This gives a measure of the spin-orbit interaction for 82 Mev nucleons. Subject to the validity of certain assumptions a spin-orbit term about twenty times the Thomas term seems indicated both for this energy and for 300 Mev nucleons. This value is also consistent with the strength of the spin orbit term needed by the shell model.[18] However, the Born approximation does not give particularly good overall agreement with the experiments. Since, moreover, the scattering matrix is restricted to a very special form by the Born approximation, with two of the four parameters vanishing, a qualitative estimate of the effects of the higher-order corrections seems desirable. In the third section the second order Born approximation is carried out for the case of a Gaussian potential. Features of the deuteron-carbon experimental data which are not contained in the first Born approximation are given by the second Born approximation. Another source of higher order effects is the contribution to the scattering matrix from the D-state part of the deuteron wave function. Since the interference effects between the S and D states vary as the product of the amplitudes, and since the D-state amplitude is about 20%, these contributions might be appreciable. In the last section the D-state contributions are evaluated in an approximation in which the center of mass coordinates of the deuteron are treated in the Born approximation but in which the deuteron wave functions are used for the relative coordinate part of the problem. The D-state contributions are then found to be much smaller than would have been expected.

PART I

Section 1. Basic Equations for the P - P System.

In the analysis of polarization experiments it is convenient to use the M-matrix introduced by Wolfenstein and Ashkin. This is a matrix in the composite spin space of the two particles in a collision process and is defined by

$$\sum_{\mu} f_{\mu}(\theta\,\phi)\,\chi_{\mu} \equiv f(\theta\,\phi) = M(\theta\,\phi)\,\chi_{inc} \,. \tag{1}$$

Here the χ_{μ} are basis vectors in the composite spin space of the two particles, χ_{inc} is the spin part of the state vector in an incident plane wave state, and the scattering amplitude $f(\theta\,\phi)$ is a vector in spin space which is defined by

$$\Psi_{scat}(r,\,\theta\,\phi) = f(\theta\,\phi)\,e^{iKr}/r \;, \tag{2}$$

where the left-hand side is the asymptotic form of the scattered wave in the relative coordinate system. Following Wolfenstein and Ashkin the spin-space density matrices ρ_{inc} and $\rho(\theta\,\phi)$ related by

$$\rho(\theta\,\phi) = M(\theta\,\phi)\;\rho_{inc}\,\overline{M}(\theta\,\phi) \tag{3}$$

will be introduced, where $\overline{M}(\theta\,\phi)$ is the hermitian conjugate of $M(\theta\,\phi)$. The average values of the quantity related to any spin-space operator A, when the measurements are made on particles in the incident plane wave or, alternatively, on particles in the beam corresponding to the scattering angles $\theta\,\phi$, are[10] respectively

$$\begin{aligned} \langle A\rangle_{inc} &= \mathrm{Tr}\,\rho_{inc}\,A/\mathrm{Tr}\,\rho_{inc} \\ \langle A\rangle_{\theta\,\phi} &= \mathrm{Tr}\,\rho(\theta\,\phi)\,A/\mathrm{Tr}\,\rho(\theta\,\phi) \;, \end{aligned} \tag{4}$$

while the differential cross section is given by

$$I(\theta\,\phi) = \mathrm{Tr}\,\rho(\theta\,\phi)/\mathrm{Tr}\,\rho_{inc}\,. \tag{5}$$

If the two protons are treated as Pauli particles then the composite spin space is four dimensional and the M matrix and density matrices are four by four. They may therefore be written as a linear combination of the sixteen linearly independent matrices $(\sigma_{1i}\,\sigma_{2j})$ where σ_{1i} and σ_{2i} $(i = 0, 1, 2, 3)$ are the unit matrix and the three Pauli matrices for the first and second particles, respectively. Calling these sixteen matrices the S_n and noticing that

$$\tfrac{1}{4}\,\mathrm{Tr}\,S_n\,S_m = \delta_{nm} \tag{6}$$

the density matrices may be written, with the help of Eqs. (4), in the form

$$\begin{aligned} \rho_{inc} &= (\tfrac{1}{4}\,\mathrm{Tr}\,\rho_{inc})\,\sum_n \langle S_n\rangle_{inc}\,S_n \\ \rho(\theta\,\phi) &= (\tfrac{1}{4}\,\mathrm{Tr}\,\rho(\theta\,\phi))\sum_n \langle S_n\rangle_{\theta\,\phi}\,S_n \quad . \end{aligned} \tag{7}$$

The $\langle S_n\rangle$ are the quantities which determine the state of polarization of the beam. In particular, the $\langle\vec{\sigma}_1\rangle$ and $\langle\vec{\sigma}_2\rangle$ give the expectation value of the spin of the first and second particles, respectively, and will be called the polarization of these particles. The quantities $\langle\sigma_{1i}\,\sigma_{2j}\rangle$, $(i, j = 1, 2, 3)$, called the correlation parameters, which are also needed to specify completely the state of polarization, will be discussed in section three.

In the analysis of the P - P scattering it is possible to treat the two particles as if they were distinguishable, provided that the

M matrix is appropriately symmetrized. This point is discussed in section four. Thus, the first particle will be taken as the incident particle, the second as the target.

The quantities measured in the recent Berkeley experiments are called $P(\theta)$, $D(\theta)$ and $R_K(\theta)$. $P(\theta)$ and $D(\theta)$ are the polarization and depolarization functions and may be measured by experiments in which the incident particles are polarized along the direction N, the normal to the plane of scattering. If the magnitude of the incident polarization is P_{inc} and the polarization of the particles scattered in the direction $\theta\phi$ is $\vec{P}(\theta\phi)$, then the $P(\theta)$ and $D(\theta)$ are defined by

$$\vec{P}(\theta\phi)\cdot\vec{N}(\theta\phi) = \frac{I_0}{I(\theta\phi)} \left(P(\theta) \quad D(\theta)P_{inc}\right) , \tag{8}$$

where I_0 is the cross section when $P_{inc} = 0$. If, on the other hand, the incident polarization is in the plane of the scattering and along $\vec{N} \times \vec{K}_{in}$, then the part of the polarization vector of the scattered beam which lies in the plane of scattering has a magnitude proportional to P_{inc} and is denoted by $(P_{inc}\,\vec{R}(\theta\phi))$. This defines the vector $\vec{R}(\theta\phi)$. Now, as has been shown by Wolfenstein[10], the asymmetry in the differential cross section after a scattering gives a measure of the components of polarization which are perpendicular to the (laboratory) velocity of the incident particle. The component of $\vec{R}(\theta\phi)$ which is perpendicular to the laboratory velocity is called $R_K(\theta)$ and this is therefore the measured quantity.* With the help of Eqs. (3), (4) and (5) it is easily seen that these quantities may be written in the forms stated and their

---

* With the use of magnetic fields to cause the spin to precess relative to the direction of motion, other components may be measured.[10]

formal expressions in terms of the M-matrix may be obtained. Using the definitions

$$\sigma_{in} \equiv \vec{\sigma}_1 \cdot \vec{N} \quad , \text{etc}$$

$$\vec{N} = (\vec{K}_{in} \times \vec{K}_{out}) / |\vec{K}_{in} \times \vec{K}_{out}|$$

$$\vec{P} = (\vec{K}_{out} + \vec{K}_{in}) / |\vec{K}_{out} + \vec{K}_{in}|$$

$$\vec{K} = (\vec{K}_{out} - \vec{K}_{in}) / |\vec{K}_{out} - \vec{K}_{in}|$$

$$\vec{S} = (N \times K_{in}) / |\vec{N} \times \vec{K}_{in}|$$

$$h\vec{K}_{out} = \text{final relative momentum}$$

$$h\vec{K}_{in} = \text{incident relative momentum} \quad , \tag{9}$$

the observables may be expressed by

$$I_0(\theta) = \tfrac{1}{4} \operatorname{Tr} M(\theta\phi)\, \bar{M}(\theta\phi)$$

$$I_0 P(\theta) = \tfrac{1}{4} \operatorname{Tr} M(\theta\phi)\, \bar{M}(\theta\phi)\, \sigma_{1N}$$

$$I_0 D(\theta) = \tfrac{1}{4} \operatorname{Tr} M(\theta\phi)\, \sigma_{1N}\, M(\theta\phi)\, \sigma_{1N}$$

$$I_0 R_K(\theta) = \tfrac{1}{4} \operatorname{Tr} M(\theta\phi)\, \sigma_{1S}\, M(\theta\phi)\, \sigma_{1K}$$

where $\bar{M}$ is the Hermitian conjugate of M. By the use of symmetry and time reversal arguments Wolfenstein and Ashkin have shown that the M-matrix for the P - P system may be written in the form

$$M(\theta\phi) = a(\theta) + c(\theta)(\sigma_{1N} + \sigma_{2N}) + m(\theta)\sigma_{1N}\sigma_{2N}$$
$$+ g(\theta)(\sigma_{1P}\sigma_{2P} + \sigma_{1K}\sigma_{2K}) + h(\theta)(\sigma_{1P}\sigma_{2P} - \sigma_{1K}\sigma_{2K}) \tag{10}$$

By substituting this form of $M(\theta\phi)$ into the expressions for $I_0(\theta)$, $P(\theta)$, $D(\theta)$ and $R_K(\theta)$ and evaluating the traces, one obtains expressions for these observables which are quadratic functions of the M-matrix coefficients $a(\theta)$, $c(\theta)$, $m(\theta)$, $g(\theta)$ and $h(\theta)$. The results of these calculations are recorded in Table A. Equivalent formulas have now become available in the literature and the reader is referred to these papers[19] for a more detailed discussion of them.

The expressions for the observables in terms of the phase shifts will be obtained by first expressing the M-matrix coefficients, $a(\theta)$, etc., in terms of the matrix elements of M and then obtaining these matrix elements as functions of the phase shifts. In order to obtain expressions for the coefficients $a(\theta),\ldots h(\theta)$, the orthogonality property of the S_n expressed in Eq. (6) is used. By multiplying both sides of Eq. (10) by the various S_n and taking one fourth of the trace one obtains

$$
\begin{aligned}
a(\theta) &= \tfrac{1}{4}\,\mathrm{Tr}\,M \\
c(\theta) &= \tfrac{1}{4}\,\mathrm{Tr}\,M\,\sigma_{1N} = \tfrac{1}{4}\,\mathrm{Tr}\,M\,\sigma_{2N} \\
m(\theta) &= \tfrac{1}{4}\,\mathrm{Tr}\,M\,\sigma_{1N}\,\sigma_{2N} \\
g(\theta) &= \tfrac{1}{8}\,\mathrm{Tr}\,M\,\sigma_{1P}\,\sigma_{2P} + \tfrac{1}{8}\,\mathrm{Tr}\,M\,\sigma_{1K}\,\sigma_{2K} \\
h(\theta) &= \tfrac{1}{8}\,\mathrm{Tr}\,M\,\sigma_{1P}\,\sigma_{2P} - \tfrac{1}{8}\,\mathrm{Tr}\,M\,\sigma_{1K}\,\sigma_{2K}\ .
\end{aligned}
\tag{11}
$$

In order to compute the traces, specific representations of the matrices will be introduced. The S_n take their most simple form in the single particle representation where the basis vectors are

$$
\begin{aligned}
\alpha(1)\,\alpha(2) &= \phi(\tfrac{1}{2},\tfrac{1}{2}) \\
\alpha(1)\,\beta(2) &= \phi(\tfrac{1}{2},-\tfrac{1}{2}) \\
\beta(1)\,\alpha(2) &= \phi(-\tfrac{1}{2},\tfrac{1}{2}) \\
\beta(1)\,\beta(2) &= \phi(-\tfrac{1}{2},-\tfrac{1}{2})\ .
\end{aligned}
\tag{12}
$$

The $\alpha(N)$ and $\beta(N)$ are the spin up and spin down state of the Nth particle. In this representation, where vectors are specified by a "couple" (a, b), the matrix elements will be specified by a pair of couples; the first index in each couple referring to the first particle, the second index to the second particle. The matrix elements of the S_n are therefore

$$(\sigma_{1i}\,\sigma_{2j})_{(a,b)(c,d)} = (\sigma_i)_{a,c}\,(\sigma_j)_{b,d} \tag{13}$$

where a,b,c and d may take on the values $+\frac{1}{2}$ and $-\frac{1}{2}$ corresponding to the first and second rows and columns. Taking the usual representations of the Pauli matrices, letting the 3-axis be directed along the incident beam, letting $\theta\phi$ be the usual polar angles describing the scattered beam direction, and taking the order of the four states to be the one used in Eq. (12), the pertinent S_n are:

$$I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\sigma_{1N} = \begin{pmatrix} 0 & 0 & -ie^{-i\phi} & 0 \\ 0 & 0 & 0 & -ie^{-i\phi} \\ ie^{i\phi} & 0 & 0 & 0 \\ 0 & ie^{i\phi} & 0 & 0 \end{pmatrix}$$

$$\sigma_{2N} = \begin{pmatrix} 0 & -ie^{-i\phi} & 0 & 0 \\ ie^{i\phi} & 0 & 0 & 0 \\ 0 & 0 & 0 & -ie^{-i\phi} \\ 0 & 0 & ie^{i\phi} & 0 \end{pmatrix}$$

$$\sigma_{1N}\sigma_{2N} = \begin{pmatrix} 0 & 0 & 0 & -e^{-2i\phi} \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -e^{2i\phi} & 0 & 0 & 0 \end{pmatrix}$$

$$(\sigma_{1P}\sigma_{2P} + \sigma_{1K}\sigma_{2K}) = \begin{pmatrix} 1 & 0 & 0 & e^{-2i\phi} \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ e^{2i\phi} & 0 & 0 & 1 \end{pmatrix}$$

$$(\sigma_{1P}\sigma_{2P} - \sigma_{1K}\sigma_{2K}) = \begin{pmatrix} \cos\theta & \sin\theta\, e^{-i\phi} & \sin\theta\, e^{-i\phi} & -\cos\theta e^{-2i\phi} \\ \sin\theta\, e^{i\phi} & -\cos\theta & -\cos\theta & -\sin\theta e^{i\phi} \\ \sin\theta\, e^{i\phi} & -\cos\theta & -\cos\theta & -\sin\theta e^{i\phi} \\ -\cos\theta\, e^{2i\phi} & -\sin\theta\, e^{i\phi} & -\sin\theta\, e^{i\phi} & \cos\theta \end{pmatrix}$$

(14)

The M matrix elements which are most easily expressed in terms of the phase shifts are, on the other hand, those in singlet-triplet representation. In this representation the M-matrix may be written in the form*

$$\begin{pmatrix} M_{11} & M_{10}\, e^{-i\phi} & M_{1-1}\, e^{-2i\phi} & 0 \\ M_{01}\, e^{i\phi} & M_{00} & M_{0-1}\, e^{-i\phi} & 0 \\ M_{-11}\, e^{2i\phi} & M_{-10}\, e^{i\phi} & M_{-1-1} & 0 \\ 0 & 0 & 0 & M_{SS} \end{pmatrix}$$

(15)

where the $M_{\mu\nu}$ are functions only of θ. The indices $\mu = 1, 0, -1$, S refer to the basis vectors

---

* S is a constant of the motion due to the antisymmetry of the wave function and the conservation of parity. The ϕ dependence follows from the conservation of the Z component of angular momentum.

$$\chi_1 = \phi(\tfrac{1}{2}, \tfrac{1}{2})$$

$$\chi_0 = \frac{1}{\sqrt{2}} (\phi(\tfrac{1}{2}, -\tfrac{1}{2}) + \phi(-\tfrac{1}{2}, \tfrac{1}{2}))$$

$$\chi_{-1} = \phi(-\tfrac{1}{2}, -\tfrac{1}{2})$$

$$\chi_s = \frac{1}{\sqrt{2}} (\phi(\tfrac{1}{2}, -\tfrac{1}{2}) - \phi(-\tfrac{1}{2}, \tfrac{1}{2})) \tag{16}$$

where χ_1, χ_0 and χ_{-1} are the three triplet states and χ_s is the singlet state. To obtain the traces needed in Eq. (11) the M-matrix may be transformed to the single particle representation by means of the equation

$$\begin{aligned} M(a, b)(c, d) &= (a, b \mid M \mid c, d) \\ &= (a, b \mid \mu)(\mu \mid M \mid \nu)(\nu \mid c, d) . \end{aligned} \tag{17}$$

The $(a, b \mid \mu)$ and $(\nu \mid c, d)$ are Clebsch-Gordon coefficients which are easily obtained from Eq. (16). Performing the matrix multiplication one obtains the M matrix in the single particle representation in terms of the $M_{\mu\nu}$. It is

$$\begin{pmatrix} M_{11} & \frac{1}{\sqrt{2}} M_{10}\, e^{-i\phi} & \frac{1}{\sqrt{2}} M_{10}\, e^{-i\phi} & M_{1-1}\, e^{-2i\phi} \\ \frac{1}{\sqrt{2}} M_{01}\, e^{i\phi} & \frac{1}{2}(M_{00} + M_{SS}) & \frac{1}{2}(M_{00} - M_{SS}) & \frac{1}{\sqrt{2}} M_{0-1}\, e^{-i\phi} \\ \frac{1}{\sqrt{2}} M_{01}\, e^{i\phi} & \frac{1}{2}(M_{00} - M_{SS}) & \frac{1}{2}(M_{00} + M_{SS}) & \frac{1}{\sqrt{2}} M_{0-1}\, e^{-i\phi} \\ M_{-11}\, e^{2i\phi} & \frac{1}{\sqrt{2}} M_{-10}\, e^{i\phi} & \frac{1}{\sqrt{2}} M_{-10}\, e^{i\phi} & M_{-1-1} \end{pmatrix} . \tag{18}$$

Comparing this matrix with the matrices in Eq. (14), out of which it must be built, one observes that

$$
\begin{aligned}
M_{11} &= M_{-1-1} \\
M_{-11} &= M_{1-1} \\
M_{01} &= -M_{0-1} \\
M_{10} &= -M_{-10} \quad .
\end{aligned}
\tag{19}
$$

The four $M_{\mu\nu}$ together with M_{00} and M_{SS} give six variables. Expressing them in terms of the five parameters. $a(\theta),\ldots, h(\theta)$ and eliminating the latter one finds the additional relation*

$$
\frac{\sqrt{2}}{\sin\theta}\,(M_{10} + M_{01}) = \frac{1}{\cos\theta}\,(M_{11} - M_{1-1} - M_{00}) \quad . \tag{20}
$$

Calculating now the traces in Eqs. (11) one obtains the $a(\theta),\ldots,h(\theta)$ as linear combinations of the $M_{\mu\nu}$. The results are given in Table B. Combining Tables A and B one obtains the observables as functions of the $M_{\mu\nu}$, and these are given in Table C.

The expressions in Table C will, when the $M_{\mu\nu}$ are expressed in terms of the phase shifts, give the observables in terms of the phase shifts. In the derivation of the phase shift expressions for the $M_{\mu\nu}$ it is convenient to use a bracket notation. The vector $|L\ L_Z)$ will represent the spherical harmonic, the $|S\ S_Z)$ will denote the spin vector previously denoted by χ_μ and $|L\ S\ L_Z\ S_Z)$ is the product of these vectors. In this notation Eqs. (1) and (2) become

---

* This relation is obtained in a somewhat similar manner in Ref. (11).

$$(\theta\phi \mid f) = (\theta\phi \mid M\,\chi_{inc}) = M(\theta\phi) \mid \chi_{inc}) \tag{21}$$

and

$$\Psi_{scat}(r, \theta, \phi) = (\theta\phi \mid f)e^{iKr}/r \tag{22}$$

where $\mid f)$ is a vector in spin-angle space and $M \equiv \mid M)$ is a vector in angle space and an operator in spin space.

The phase shifts are directly related to an operator R which may be defined by

$$\mid f) = R \mid f_0) \tag{23}$$

where $(\theta\phi \mid f_0)e^{iKr}/r$ is the outgoing part of the incident plane wave. (The operator $S = R + 1$ is an operator in the spin-angle space which transforms the spin angle vector $\mid f_0)$, which describes the unperturbed outgoing wave, into the spin angle vector for the actual outgoing wave. The connection between this definition and certain other definitions is discussed in the appendix.) The analysis of the incident plane wave in spherical harmonics gives

$$\begin{aligned} \mid f_0) &= \mid L\,L_Z)(L\,L_Z \mid f_0) \\ &= \sum_{L\,L_Z} \mid L\,L_Z)\ \delta_{L_Z 0}\left[\pi(2L+1)\right]^{\frac{1}{2}}(-i/k)\ \chi_{inc} \quad , \\ &= \sum_L \mid L\,0\,\chi_{inc})\ b_L \end{aligned} \tag{24}$$

where

$$b_L = \left[\pi\,(2L+1)\right]^{\frac{1}{2}}(-i/k) \quad . \tag{25}$$

The substitution of Eq. (24) into Eq. (23) gives

$$|f) = \sum_L R\,|L\,0\,\chi_{inc})\,b_L \tag{26}$$

and a comparison to Eq. (21) shows that

$$M = \sum_L R\,|L\,0)b_L\,. \tag{27}$$

The matrix elements of $M(\theta\phi)$ are therefore

$$(S'\,S'_Z\,|\,M(\theta\phi)\,|\,S\,S_Z) = \sum_L (\theta\phi\;S'\,S'_Z\,|\,R\,|\,L\,S\,0\,S_Z)b_L$$

$$= \sum_L (\theta\phi\,|\,L'\,L'_Z)(L'\,S'\,L'_Z\,S'_Z,\,|\,R\,|\,L\,S\,0\,S_Z)b_L$$

$$\equiv \sum_{L'L'_Z} Y_{L'L'_Z}(\theta\phi)\sum_L b_L(L'\,S'\,L'_Z\,S'_Z\,|R|\,L\,S\,0\,S_Z)\,. \tag{28}$$

Since the matrix elements of R are known in the L, S, J, J_Z representation, one may write

$$(L'\,S'\,L'_Z\,S'_Z\,|\,R\,|\,L\,S\,0\,S_Z) =$$

$$(L'\,S'\,L'_Z\,S'_Z\,|\,L'\,S'\,J'\,J'_Z)(L'\,S'\,J'\,J'_Z\,|R|\,L\,S\,J\,J_Z)$$

$$\times(L\,S\,J\,J_Z\,|\,L\,S\,0\,S_Z) \tag{29}$$

where indices appearing three times are not summed.* Since J, J_Z and S are constants of the motion, this may be simplified to

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* The fact that

$$(L'\,S'\,J\,J_Z\,|\,L\,S\,L_Z\,S_Z) = (L\,S\,L_Z\,S_Z\,|\,L'\,S'\,J\,J_Z)$$

is a multiple of $\delta_{LL'}\,\delta_{SS'}$ is used here.

$$(L' \; S' \; L'_Z \; S'_Z \mid R \mid L \; S \; 0 \; S_Z) =$$

$$\delta_{SS'}(L' \; S \; L'_Z \; S'_Z \mid L' \; S \; J \; J_Z)(L' \mid R^{JJ_Z}_{,S} \mid L)(L \; S \; J \; J_Z \mid L \; S \; 0 \; S_Z) \tag{30}$$

where $R^{JJ_Z}_{,S}$ is an operator only in the L part of space. Because of spherical symmetry the R matrix is independent of J_Z and this index may be dropped. The diagonal elements of $R^J_{,S}$ will be called $R^J_{L,S}$; thus

$$(L \mid R^J_{,S} \mid L) \equiv R^J_{L,S} \; . \tag{31}$$

For the case $S = 0$, the vector addition law gives $L = J = L'$ and there are no off diagonal elements. The antisymmetry of the wave function requires L to be even when $S = 0$. Thus for the singlet state the only contributions are from

$$R^J_{L,0} = R^L_{L,0} \qquad L \text{ even.} \tag{32}$$

For $S = 1$, the values of L and L' must be $J + 1$, J, $J - 1$, and odd. Thus, the only off diagonal elements are from J even and $L = J \pm 1$ $L' = J \mp 1$. These will be defined

$$\begin{aligned} (L' = J + 1 \mid R^J_{,1} \mid L = J - 1) &= R^J \\ (L' = J - 1 \mid R^J_{,1} \mid L = J + 1) &= R^J \; . \end{aligned} \tag{33}$$

The equality of these two matrix elements is a consequence of time reversal.[12]

Inserting these expressions for the $(L' \mid R^J_{,S} \mid L)$ into Eq. (30) and making use of the properties of the Clebsch-Gordon coefficients $C_{LS}(J;\, L_Z\, S_Z)$ defined by

$$(L',\, S'\, L'_Z S'_Z | L\, S\, J\, J_Z) = \delta_{LL'}\, \delta_{SS'}\, \delta_{L'_Z,\, J_Z - S'_Z}\, C_{L'S'}(J;\, L'_Z,\, S'_Z)\ .$$

$$= (L\, S\, J\, J_Z \mid L'\, S'\, L'_Z\, S'_Z)\ , \tag{34}$$

the expressions for the matrix elements $M_{\mu'\mu}(\theta\phi)$ may be simplified to

$$M_{\mu'\mu}(\theta\phi) = \sum_{\text{odd } L} Y_{L,S_Z - S'_Z}(\theta\phi)\, b_L\, N^L_{S'_Z S_Z}\, \delta_{S,1}$$

$$+ \sum_{\text{even } L} Y_{L,0}(\theta\phi)\, b_L\, R^L_{L0}\, \delta_{S,0} \tag{35}$$

where

$$N^L_{S'_Z S_Z} = \sum_J C_{L1}(J;\, S_Z - S'_Z,\, S'_Z)\, C_{L1}(J;\, 0,\, S_Z)\, R^J_{L1}$$

$$+ \sum_{\pm} C_{L1}(L \pm 1;\, S_Z - S'_Z,\, S'_Z)\, C_{L\pm 2,1}(L \pm 1,\, 0\, S_Z)\, R^{L\pm 1}\, \frac{b_{L\pm 2}}{b_L}\ . \tag{36}$$

The $M_{\mu'\mu}$ of Eq. (15) are obtained by evaluating the $M_{\mu'\mu}(\theta\phi)$ at $\phi = 0$ and their expressions in terms of the $R^J_{L,S}$ and R^J are given in Table D for the case that partial waves with $L \geqslant 4$ do not contribute.

The $R^J_{L,S}$ and R^J are closely related to the usual phase shifts. The equations

$$R^L_{L,0} = \exp 2i\, \delta_L - 1 \tag{37}$$

define the singlet phase shifts δ_L, which will be real. Similarly the equations

$$R^J_{L,1} = \exp\left[2i\ \delta^J_L\right] - 1 \qquad \begin{cases} J = 1, 3, 5, \ldots \\ J = 0 \end{cases} \tag{38}$$

define triplet phase shifts for the values of J which are indicated. For even values of J which are greater than zero there are off-diagonal elements and the above definition of $R^J_{L,1}$ would lead, in general, to complex δ^J_L. Following Blatt and Biedenharn[12] one defines in this case real phase shifts $\delta^J_{J\pm 1}$ and a real admixture parameter ε^J such that

$$R^J_{J\pm 1,1} = (\cos^2 \varepsilon^J \exp 2i\ \delta^J_{J\pm 1} + \sin^2 \varepsilon^J \exp 2i\ \delta^J_{J\mp 1}) - 1$$

$$R^J = \tfrac{1}{2} \sin 2\ \varepsilon^J (\exp 2i\ \delta^J_{J+1} - \exp 2i\ \delta^J_{J-1}) . \tag{39}$$

An alternative method of defining the real phase shifts, which seems convenient when coulomb effects are considered is discussed in the appendix.

Eqs. (37), (38), and (39), together with Tables C and D, give expressions for the observed quantities in terms of the phase shifts. These were used in a phase shift analysis of the P - P experiments. This analysis is discussed in the next section.

Section 2. Phase-Shift Analysis of the Berkeley 310-Mev P-P Data.

The data obtained in the recent polarization and triple-scattering experiments at Berkeley, together with earlier data on the total and differential cross sections, have been used as the basis of a phase-shift analysis of the P-P system. This work was carried out in close collaboration with Dr. T. J. Ypsilantis and with the invaluable assistance of other members of the experimental group, in particular Dr. Owen Chamberlain and Dr. Emilio Segre. In this section a general discussion of the numerical computations that have been made is given together with a summary of the preliminary results obtained. To begin these calculations a preliminary run was carried out on the Univac at Livermore, and the body of the computing was then done by the Maniac at Los Alamos.

The input data consisted of twenty-nine pieces of experimental data. There were six measurements of R_K at angles ranging from 22° to 80° (center of mass) and six D measurements in the range 23° to 80°. The polarization parameter was given at six points between 21° and 76°. The cross-section data were introduced in the following way: absolute magnitudes for the total cross section and for the 90° differential cross section were given, and then at nine angles the ratios of the differential cross section to that at 90° were used. The actual values used are summarized in Table I.* These data were kindly supplied by the members of the experimental group, much of them prior to publication.

In the analysis of these data the general method was the same as that used by Fermi and Metropolis and others[20] in the analysis

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* For a detailed discussion of the experimental data see T. J. Ypsilantis, Reference 30.

of the π meson-proton system. The procedure is to first express the various quantities as functions of the phase shift. These relations were developed in section one, and the results have been tabulated in Tables C and D. Let these functions be denoted by $O_i(\delta_K)$ where i runs over the number of experimental observables and K runs over the number of phase shifts. Denoting the measured values by O_i and the corresponding experimental errors by ε_i the quantity

$$\sum \equiv \sum_i \left(\frac{O_i(\delta_K) - O_i}{\varepsilon_i} \right)^2$$

is then formed. $\sum$ is a function of the phase shifts and it is a measure of the fit of the phase shifts to the experimental data. A trial set of phase shifts is introduced as a starting point and the $\sum$ is computed. By slight variations of the phases the fit is gradually improved until no more improvement is possible within the framework of the particular method of searching being used. Three methods of search were used. The gradient method is one in which the gradient of $\sum$, considered as a function in the space whose coordinates are the phase shifts, is computed at the trial point, and then the $\sum$ is evaluated at a succession of points along the gradient line until the fit starts to get worse. A new gradient is then computed and the process repeated. This method leads to paths in the phase-shifts space which seem to oscillate from one side to another of narrow channels and make only gradual improvement. A second method is the grid method in which only one phase shift is varied at a time. It was found that when one of these methods reached a point of no improvement, the other method could many times give further improvement, and when both methods were stopped a random step method would usually give further progress. The

time required to compute Σ on the Maniac was very close to one second and the time required to pursue a given run from the arbitrary starting point to a relative minimum, using the grid method was between twenty minutes and one hour. During this time the step sizes in S were progressively diminished by factors of two from 1° to $1/64^{\circ}$. The last few steps usually produced little improvement in Σ and only small over-all changes in the phase shifts.

Ninety-six initial points have been used to date in the runs at Los Alamos and 56 of these were random points; the remaining points were solutions obtained from a preceding run in which fewer pieces of experimental data were used. From these 96 starting points 28 relative minima were obtained, and of these 28 solutions six were obtained only a single time, indicating that a further search would probably uncover additional solutions. The values of Σ for the various solutions range between 20 and 180 with the exception of one solution for which $\Sigma = 1131$. These values may be compared to the expected value of Σ at the relative minimum which lies in the neighborhood of the true solution. This expected value is equal to number of observables minus the number of variables (phase shifts) and is therefore 20. The probability that the value lies between 16 and 24 is $\sim 50\%$ and the probability that it is larger than 50 is less than 0.1%. These statistical results are based upon the assumptions that the true values of the measured quantities can be exactly represented by the nine phase shifts and that the errors are all of a statistical nature (as opposed to errors of a systematic kind).

There are 18 solutions for which Σ is less than 50 and these are given in Table J. Some choice may be made among these solutions by

the use of the Coulomb interference effects in the small-angle differential cross section. The experiments of Chamberlain, Pettengill, Segre and Wiegand[21] and those of Fischer and Goldhaber[22] indicate a rather large destructive interference in the region where the Coulomb and nuclear scattering amplitudes are of equal magnitude. Since the Coulomb amplitude is predominantly negative imaginary in this region the nuclear part of the scattering amplitude is required to have its imaginary part positive. Nine of the eighteen solutions satisfy this condition, but for three of these nine the real part of the amplitude is smaller by an order of magnitude than that which is needed to account for the interference observed.*

Recently another parameter of the P-P system has been measured by James E. Simmons together with Jack Baldwin, Dave Fischer and other members of the experimental group mentioned above. This parameter, called A by Wolfenstein, is measured by passing the polarized beam through a magnetic field which rotates the direction of incident polarization, giving it a component along the incident direction. This parameter has been measured at three angles of scattering. At each of these angles separately the best fit from among the remaining six solutions is given by the second of the solutions listed in Table J (the solution with $\sum = 27.2$). This solution lies within the experimental error at the two large angle points. Only one other solution lies within the experimental error at either of these points and this solution gives an extremely poor fit at the other

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* Ypsilantis[30] calculates the real part of the amplitude in the interference region to be of the order of 0.19×10^{-13} cm.

large angle point. At the small angle point, which lies at 25.4°, none of the solutions attains the large negative value that is measured, and even this best solution is too small by 50%, which is three times the experimental error. In spite of this poor small-angle fit, this solution is by far the best of the remaining six solutions and appears to be the only solution found so far that gives even a fair fit to all the experimental data. The values of the other observables that are predicted by this solution listed under the heading θ(Theo) in Table I for comparison with the experimental values. One will notice that there is good agreement with all except the R data, and here again it is at the small angle points that the disagreement becomes large. These discrepancies at small angles suggest that the higher order phases shifts, though perhaps small, are playing a significant role in this small-angle region, where their effects would be expected to become most pronounced. Nevertheless, the policy of neglecting the higher order phases shifts gains some general support in the smallness shown in this best solution of the d and f phase shifts relative to those for the s and p waves. It should be pointed out that if it is admitted that the higher partial waves play a significant role in the small-angle region that the validity of the arguments concerning the Coulomb interference is placed in doubt. However, the A parameter has also been calculated for the solutions which have negative imaginary amplitudes and whose $\sum$'s are less than 40. All these solutions give strong disagreement with the large angle experimental data.

Section 3. Theory of Correlation Experiments.

In this section the information which may be obtained from correlation experiments is discussed. In these experiments an unpolarized proton beam is scattered by protons and components of the polarization of both the recoil and scattered protons are measured in coincidence. Thus the correlation between the spin directions of the two particles is determined.

Since the center-of-mass momentum is not a constant throughout this process the wave function will be expressed again in terms of the individual coordinates r_1 and r_2. The part of the wave function $\Psi(r_1, r_2)$ after the first scattering which will contribute to the correlation measurements will be a product of plane waves in the r_1 and r_2 spaces and its spin state will be described by $\rho\ (\theta\phi)$. This will be the incident beam for the second process which involves a scattering of both the first and second particle. This scattering of two particles can be represented by a generalization of the Wolfenstein-Ashkin M-matrix. The generalized matrix will be a function of two sets of angles $\theta_1\phi_1$ and $\theta_2\phi_2$ and for the simple case in which the second scatterers are spin zero it will take the form

$$M(\theta_1\phi_1, \theta_2\phi_2) = (f_1(\theta_1) + g_1(\theta_1)\vec{\sigma}_1\cdot\vec{N}_1)(f_2(\theta_2) + g_2(\theta_2)\vec{\sigma}_2\cdot\vec{N}_2) . \tag{40}$$

In general the matrix $M(\theta_1\phi_1; \theta_2\phi_2)$ is just the direct product of the M matrices for the individual scatterings. The density matrix which represents the spin state after the second scatterings is

$$\rho(\theta_1\phi_1, \theta_2\phi_2, \theta\phi) = M(\theta_1\phi_1, \theta_2\phi_2)\,\rho(\theta\phi)\,\overline{M}(\theta_1\phi_1, \theta_2\phi_2) . \tag{41}$$

The $\rho(\theta\phi)$ is the density matrix for the incident state of the second scatterings. Suppressing the $\theta\phi$ dependence, this may be written

$$\rho = \tfrac{1}{4}(\mathrm{Tr}\,\rho)\,(1 + \vec{P}_1\cdot\vec{\sigma}_1 + \vec{P}_2\cdot\vec{\sigma}_2 + C_{ij}\,\sigma_{1i}\,\sigma_{2j}) \tag{42}$$

where $\vec{P}_1$ and $\vec{P}_2$ are the polarization vectors discussed in section one and C_{ij} is the correlation parameter which, according to Eq. (4), is $\langle \sigma_{1i}\,\sigma_{2j} \rangle$. The analog of Eq. (5) is

$$I(\theta_1\phi_1, \theta_2\phi_2) = \mathrm{Tr}\,\rho\,(\theta_1\phi_1, \theta_2\phi_2)/\mathrm{Tr}\,\rho \tag{43}$$

where $I(\theta_1\phi_1, \theta_2\phi_2)d\Omega_1\,d\Omega_2$ is the coincidence cross section. If now Eqs. (40) and (42) are substituted into Eq. (41) and this in turn is inserted in Eq. (43), the resulting expression becomes after some simplification

$$I(\theta_1\phi_1, \theta_2\phi_2)$$
$$= I_0(\theta_1)\,I_0(\theta_2)\left\{1 + \vec{P}_1\cdot\vec{P}(\theta_1\phi_1) + \vec{P}_2\cdot\vec{P}(\theta_2\phi_2) + C_{ij}\,P_i(\theta_1\phi_1)P_j(\theta_2\phi_2)\right\} \tag{44}$$

where

$$I_0(\theta_1) \equiv |f_1(\theta_1)|^2 + |g_1(\theta_1)|^2$$

$$I_0(\theta_2) \equiv |f_2(\theta_2)|^2 + |g_2(\theta_2)|^2$$

$$I_0(\theta_1)\vec{P}(\theta_1\phi_1) \equiv (f_1(\theta_1)\,g_1^*(\theta_1) + f_1^*(\theta_1)\,g_1(\theta_1))\,\vec{N}_1(\theta_1\phi_1)$$

$$I_0(\theta_2)\vec{P}(\theta_2\phi_2) \equiv (f_2(\theta_2)\,g_2^*(\theta_2) + f_2^*(\theta_2)\,g_2(\theta_2))\,\vec{N}_2(\theta_2\phi_2)\ . \tag{45}$$

Defining
$$I(\theta_1\, 0,\ \theta_2\, 0) = LL$$
$$I(\theta_1\, \pi,\ \theta_2\, 0) = RL$$
$$I(\theta_1\, 0,\ \theta_2\, \pi) = LR$$
$$I(\theta_1\, \pi,\ \theta_2\, \pi) = RR \tag{46}$$

and letting $\vec{e}_1$ and $\vec{e}_2$ be the normal vectors $\vec{N}_1$ and $\vec{N}_2$ when $\phi_1 = 0$ and $\phi_2 = 0$, respectively, one finds that

$$C_{e_1e_2} \equiv C_{ij}\, e_{1i}e_{2j} = \frac{1}{P(\theta_1)\, P(\theta_2)}\ \frac{LL + RR - LR - RL}{LL + RR + LR + RL} \tag{47}$$

where $P(\theta_1) = |\vec{P}(\theta_1\phi_1)|$ and $P(\theta_2) = |\vec{P}(\theta_2\phi_2)|$. Eq. (47) provides the relationship between the quantities LL, etc. which are measured and the quantity

$$C_{e_1e_2}(\theta\phi) = \left\langle \vec{\sigma}_1\cdot\vec{e}_1\ \ \vec{\sigma}_2\cdot\vec{e}_2 \right\rangle_{\theta\phi}$$
$$= \tfrac{1}{4}\, \mathrm{Tr}(M(\theta\phi)\, \overline{M}(\theta\phi)\ \vec{\sigma}_1\cdot\vec{e}_1\ \ \vec{\sigma}_2\cdot\vec{e}_2\ ) \tag{48}$$

where the $\theta\phi$ is now no longer suppressed. Eq. (48) allows $C_{e_1e_2}(\theta\phi)$ to be expressed in terms of M-matrix coefficients $a(\theta)$ etc.

There are various possible $C_{e_1e_2}$ according to the choice made for the directions of $\vec{e}_1$ and $\vec{e}_2$ in different experiments. In one type of experiment the $\vec{e}_1$ and $\vec{e}_2$ are taken along $\vec{N}$, the normal to the original scattering plane. Then one measures $C_{NN}(\theta)$ where

$$I_0\, C_{NN}(\theta) = \tfrac{1}{4}\, \mathrm{Tr}\, M(\theta\phi)\, \overline{M}(\theta\phi)\, \sigma_{1N}\, \sigma_{2N}$$
$$= 2\, \mathrm{Re}\, am^* + 2\,|c|^2 - 2\,|g|^2 + 2\,|h|^2\ . \tag{49}$$

In other experiments $\vec{e}_1$ or $\vec{e}_2$ or both may lie in the plane of the original scattering. If $\vec{M}$ and $\vec{M}'$ are unit vectors in this plane

and $\vec{N}$ is still the normal vector, then one finds

$$I_0\, C_{NM'} = I_0\, C_{MN} = 0 \tag{50}$$

and

$$I_0\, C_{MM'} = \cos(\beta - \beta')\, 2\, \mathrm{Re}(a - m)g^* - \cos(\beta + \beta')\, 2\, \mathrm{Re}(a + m)h^* + 2 \sin(\beta + \beta')\, 2\, \mathrm{Re}(ich^*)$$

where β is the angle between $\vec{M}$ and $\vec{K}$, the vector along the momentum transfer, and β' is the angle between $\vec{M}'$ and $\vec{K}$. The sign of these angles is such that β and β' equal $\pi/2$, when the $\vec{M}$ and $\vec{M}'$ lie along $\vec{P}$. Now in the laboratory frame in which the measurements are made the particles emerge, neglecting a relativistic correction, in the directions of $\vec{P}$ and $-\vec{K}$, and the components in the plane which are measured are along the respective perpendiculars $\vec{K}$ and $\vec{P}$. Thus $\beta = 0$ and $\beta' = \pi/2$ and

$$I_0\, C_{KP}(\theta) = 4\, \mathrm{Re}\, ich^* \quad . \tag{51}$$

The experimental determination of the correlation functions is made difficult by the fact that the particle scattered into the backward direction has a small energy in the laboratory frame and the analyzing power $P(\theta)$ is correspondingly small. The experiments are the easiest when $\theta = \pi/2$. The interpretation of the experiments at this angle is also considerably simplified by the vanishing of some of the $M_{\mu\nu}$. This may be seen by first noting that L_Z of the incident plane wave is zero and since J_Z is a constant of the motion, the L_Z' of the final state must be the difference of the incident and final S_Z's. However, the parity of the wave function in the triplet states is odd and thus whenever the ϕ dependence is even the θ dependence is odd.

Thus

$$M_{11}(\pi/2) = M_{00}(\pi/2) = M_{1-1}(\pi/2) = 0 \ . \tag{52}$$

At $\theta = \pi/2$ Table B then gives

$$\begin{aligned} a &= \tfrac{1}{4} M_{SS} = -m = -g \\ c &= i(8)^{-\frac{1}{2}} \left[M_{10} - M_{01} \right] \\ h &= (8)^{-\frac{1}{2}} \left[M_{10} + M_{01} \right] . \end{aligned} \tag{53}$$

Inserting these relations in Table A, it is then found that:

$$\begin{aligned} I_0(1 - C_{NN}) &= \tfrac{1}{2} \left| M_{SS} \right|^2 \\ I_0(1 + C_{NN}) &= \left| M_{01} \right|^2 + \left| M_{10} \right|^2 \\ I_0 \, C_{KP} &= \tfrac{1}{2} \left[\left| M_{01} \right|^2 - \left| M_{10} \right|^2 \right] \\ I_0 \, R_K &= \tfrac{1}{2} \operatorname{Re} M_{01} M_{SS}^* \\ &= \tfrac{1}{2} \left| M_{01} \right| \cdot \left| M_{SS} \right| \cos \theta_{01,SS} \\ I_0 \, D &= - \operatorname{Re} M_{10} M_{01}^* \\ &= - \left| M_{10} \right| \cdot \left| M_{01} \right| \cos \theta_{10,01} \ . \end{aligned} \tag{54}$$

Combining the second and third equations

$$\begin{aligned} I_0(1 + C_{NN} + 2 \, C_{KP}) &= 2 \left| M_{01} \right|^2 \\ I_0(1 + C_{NN} - 2 \, C_{KP}) &= 2 \left| M_{10} \right|^2 . \end{aligned} \tag{55}$$

Thus at this angle the absolute values of all three of the nonvanishing matrix elements are determined by the correlation experiments. Furthermore, the R and D measurements determine the relative phases of

these matrix elements, up to a four fold ambiguity arising from the double-valuedness of the arc-cosines. Except for this ambiguity and the ambiguity of the overall phase the M-matrix can be completely determined at this angle by these four experiments, and the differential cross section. In the regions of higher energy where a phase shift analysis becomes impractical this method of determining the M-matrix will take on increased importance. Even when the phase shift analysis is used it can provide a rather stringent condition on the phase shifts.

By the use of more complicated types of experiments this general method can be applied to angles other than $\pi/2$. These generalizations are straightforward but will not be discussed here, since the experiments are much more difficult than the ones used above and even these have not been satisfactorily performed as yet.

Section 4. Symmetry Considerations and the N - P System.

In the first part of this section the consequences of the indistinguishability of the two protons are considered. In the second part the relationships between experiments on the N-P and P-P systems are discussed and some consequences of charge independence derived.

Consider first a system which consists of a single particle. The probability $\omega(R)$ that this particle will be found in a region R may be expressed as

$$\omega(R) = \langle P(R) \rangle \equiv \int d\vec{r}\, \psi^*(\vec{r})\, P(R)\, \psi(\vec{r})$$

where P(R) is the operator which projects onto the region R and is defined by

$$P(R)\psi(\vec{r}) = \psi(\vec{r}) \qquad \text{for } \vec{r} \text{ in } R$$
$$P(R)\psi(\vec{r}) = 0 \qquad \text{otherwise .}$$

Furthermore, the average over particles found in the region R of the quantity which corresponds to the spin space operator A is

$$\overline{A}(R) = \langle A\, P(R) \rangle / \langle P(R) \rangle \quad .$$

For the system in which there are several distinguishable particles, let A_n denote the operators in the spin spaces of the various particles which correspond to the same type of physical measurement A. The expectation value of A for a measurement in the region R upon the nth particle is

$$\overline{A}_n(R) = \langle A_n\, P_n(R) \rangle / \langle P_n(R) \rangle$$

where

$$P_n(R)\,\Psi(r_1, r_2, \ldots) = \Psi(r_1, r_2, \ldots) \quad \text{for} \quad r_n \text{ in } R$$

$$= 0 \qquad \text{otherwise.}$$

If the measurement does not distinguish between the different particles (though they may be distinguishable) then the expectation value is

$$\overline{A}(R) = \left\langle \sum_n A_n P_n(R) \right\rangle \Big/ \left\langle \sum_n P_n(R) \right\rangle \quad . \tag{56}$$

The denominator $\left\langle \sum_n P_n(R) \right\rangle \equiv \omega(R)$ is the probability of finding <u>some</u> particle in the region R. In the case of indistinguishable particles the operators corresponding to various measurements should be of this form. In this case there is, in addition, a condition on the symmetry properties of the wave function. For two protons the wave function may be written

$$\Psi(r_1, r_2) = (2)^{-\frac{1}{2}}(1 - T\,S)\Psi^0(r_1, r_2) \tag{57}$$

where $\Psi^0(r_1, r_2)$ is the unsymmetrized wave function and T and S are the spin and space exchange operators defined by

$$T = \tfrac{1}{2}(1 + \vec{\sigma}_1 \cdot \vec{\sigma}_2) \quad , \tag{58}$$

$$S\Psi^0(r_1, r_2) = \Psi^0(r_2, r_1) \tag{59}$$

where only the space coordinates of Ψ^0 are interchanged in the last equation. They satisfy the equations

$$T\,A_1\,T = A_2 \tag{60}$$

$$S\,P_1(R)\,S = P_2(R) \; . \tag{61}$$

Using the equation

$$T\,S\;\psi(r_1, r_2) = T\,S\,(2)^{-\frac{1}{2}}\,(1 - T\,S)\,\psi^0(r_1, r_2)$$

$$= -\,\psi(r_1, r_2)\;, \tag{62}$$

one finds that

$$\langle A_2\,P_2(R)\rangle = \langle T\,S\,A_2\,P_2(R)\,T\,S\rangle$$

$$= \langle A_1\,P_1(R)\rangle \tag{63}$$

and

$$\langle P_2(R)\rangle = \langle P_1(R)\rangle\;. \tag{64}$$

Therefore

$$\overline{A(R)} = \frac{\langle A_1\,P_1(R)\rangle}{\langle P_1(R)\rangle} = \frac{\langle A_2\,P_2(R)\rangle}{\langle P_2(R)\rangle} \tag{65}$$

$$w(R) = 2\,\langle P_1(R)\rangle = 2\,\langle P_2(R)\rangle \tag{66}$$

and the nonsymmetrized operators may be used to calculate expectation values and probabilities, so long as the symmetrized wave functions are used.

In polarization experiments the initial conditions are specified by giving the expectation value of the spin for the regions R and R' corresponding to the locations of the two particles before the scattering. The operators whose expectation values are fixed are of the symmetrized forms given in Eq. (56) since the two particles cannot be distinguished. The density matrix ρ_{inc}, as it is used here, corresponding to such expectation values in the incident state cannot be constructed, in general, since ρ_{inc} is a function only of spin and does not possess the complexity required to describe the relationship between spin and position which characterizes the incident state. However one may use instead the density matrix ρ^0_{inc} which describes the polarization of

the unsymmetrized state $\Psi^{0}(r_1, r_2)$. Letting $M^{0}(\theta\phi)$ be the M-matrix which propagates the $\Psi^{0}(r_1\ r_2)$, one has

$$\rho^{0}(\theta\phi) = M^{0}(\theta\phi)\, \rho^{0}_{inc}\, M^{0}(\theta\phi) \ . \tag{67}$$

According to Eq. (57) and the definition of the density matrix, the actual density matrix is then

$$\rho\,(\theta\phi) = (2)^{-\frac{1}{2}}\,(1 - T\,S)\,\rho^{0}(\theta\phi)\,(1 - T\,S)(2)^{-\frac{1}{2}}. \tag{68}$$

Defining $M(\theta\phi) = (2)^{-\frac{1}{2}}\,(1 - T\,S)\,M^{0}(\theta\phi)$ this may be written

$$\rho\,(\theta\phi) = M(\theta\phi)\,\rho^{0}_{inc}\,M(\theta\phi) \tag{69}$$

which gives a relationship between the properly symmetrized density matrix after scattering and the unsymmetrized one used for the incident particles. If this symmetrized form of the M-matrix is used then in specifying the state of polarization of the incident beam one may neglect the indistinguishability of the two protons and consider one to be the incident particle and the other to be the target. The operators corresponding to expectation values in the final state may according to Eq. (65) be taken as operators referring to the first or second particle if the corresponding projection operators are used. It should be noticed that in relative coordinate space the projections on the first and second particle coordinates become respectively $P(R)$ and $R(\tilde{R})$ where $\tilde{R}$ is the inversion trhough the origin of R. Thus the expectation value of A measured in the beam traveling in the direction $\theta\phi$ may be expressed as $\langle A_1\, P(\theta\phi)\rangle / \langle P(\theta\phi)\rangle$ or as $\langle A_2\, P(\theta'\phi')\rangle / \langle P(\theta'\phi')\rangle$ where $\theta' = \pi - \theta$, $\phi' = \pi + \phi$. In order to automatically include the effect of the factor 2 in Eq. (66) it is convenient to multiply the

M matrix by $(2)^{\frac{1}{2}}$. The $\rho(\theta\phi)$ is then doubled and the expression for $I(\theta\phi)$ remains

$$I(\theta\phi) = \mathrm{Tr}\,\rho(\theta\phi)/\mathrm{Tr}\,\rho_{\mathrm{inc}} .$$

The above remarks provide the justification of the treatment of the P - P system given in section one* and are the basis of the following remarks on the relationship between the experiments on the P - P and N - P systems.

The content of the hypothesis of charge independence is that the M-matrix for the N-P system is just the $M^0(\theta\phi)$ discussed above; that is, aside from the requirements of antisymmetrization of the P-P wave function the N-P and P-P systems are identical.** It is useful, therefore, to obtain the relations between the coefficients $a^0(\theta)$, $c^0(\theta)$, ... $h^0(\theta)$ defined by

$$M^0(\theta, \phi) = a^0(\theta) + c^0(\theta)(\sigma_{1N} + \sigma_{2N}) + m^0(\theta)\,\sigma_{1N}\,\sigma_{2N}$$
$$+ g^0(\theta)(\sigma_{1P}\,\sigma_{2P} + \sigma_{1K}\,\sigma_{2K}) + h^0(\theta)(\sigma_{1P}\,\sigma_{2P} - \sigma_{1K}\,\sigma_{2K})$$

and the corresponding coefficients of the $M(\theta\phi)$. According to its definition,

$$\begin{aligned} M(\theta\phi) &= (1 - T\,S)\,M^0(\theta\phi) \\ &= M^0(\theta\phi) - T\,M^0(\theta'\phi') . \end{aligned} \tag{70}$$

(This matrix $M(\theta\phi)$ is the M-matrix for the P-P system which has been used in the earlier sections.) The $M^0(\theta'\phi')$ is obtained by replacing

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* See Breit, Ehrman and Hull, Reference (31), for another discussion of these points.

** The coulomb effects in the P-P system are neglected here.

$\vec{K}_{out}$ by $-\vec{K}_{out}$ in $M^0(\theta\phi)$. In terms of the vectors $\vec{N}$, $\vec{P}$ and $\vec{K}$ associated with the angles $\theta\phi$, $M^0(\theta'\phi')$ is

$$M^0(\theta'\phi') = a^0(\theta') - c^0(\theta')(\sigma_{1N} + \sigma_{2N}) + m^0(\theta')(\sigma_{1N}\,\sigma_{2N})$$

$$+ g^0(\theta')(\sigma_{1P}\,\sigma_{2P} + \sigma_{1K}\,\sigma_{2K}) - h^0(\theta')(\sigma_{1P}\,\sigma_{2P} - \sigma_{1K}\,\sigma_{2K}). \tag{71}$$

Using the definition of T in Eq. (58)

$$\begin{aligned} M(\theta, \phi) &\equiv M^0(\theta\phi) - T\,M^0(\theta'\phi') \\ &= M^0(\theta\phi) - M^0(\theta'\phi') + \tfrac{1}{2}(1 - \vec{\sigma}_1\cdot\vec{\sigma}_2)\,M^0(\theta'\,\phi') \\ &= a^0(\theta) - a^0(\theta') + (c^0(\theta) + c^0(\theta'))(\sigma_{1N} + \sigma_{2N}) \\ &\quad + (m^0(\theta) - m^0(\theta'))\,\sigma_{1N}\,\sigma_{2N} + (g^0(\theta) - g^0(\theta')(\sigma_{1P}\,\sigma_{2P} + \sigma_{1K}\,\sigma_{2K}) \\ &\quad + (h^0(\theta) + h^0(\theta'))(\sigma_{1P}\,\sigma_{2P} - \sigma_{1K}\,\sigma_{2K}) \\ &\quad + \tfrac{1}{2}(1 - \sigma_1\cdot\sigma_2)(a^0(\theta') - m^0(\theta') - 2\,g(\theta')) \quad . \end{aligned}$$

Collecting terms and comparing them to the terms of $M(\theta\phi)$ defined in Eq. (10) one obtains, with subscripts s and a denoting symmetric and antisymmetric parts with respect to $\theta = \pi/2$,

$$\begin{aligned} a(\theta) &= 2\,a_a^0(\theta) - t_a^0(\theta) + t_s^0(\theta) \\ m(\theta) &= 2\,m_a^0(\theta) + t_a^0(\theta) - t_s^0(\theta) \\ g(\theta) &= 2\,g_a^0(\theta) + t_a^0(\theta) - t_s^0(\theta) \\ h(\theta) &= 2\,h_s^0(\theta) \\ c(\theta) &= 2\,c_s^0(\theta) \end{aligned} \tag{72}$$

where $t^0(\theta) = \tfrac{1}{2}(a^0(\theta) - m^0(\theta) - 2\,g^0(\theta)$.

The symmetry properties of the coefficients $a(\theta)$ ..., $h(\theta)$ are apparent from this equation and one may also notice that

$$a_a(\theta) - m_a(\theta) - 2\, g_a(\theta) = 0 . \tag{73}$$

If the hypothesis of charge independence is not valid then the $M^0(\theta\phi)$ will not describe the N-P scattering. This process will be described rather by an M-matrix $M^{NP}(\theta\phi)$ whose coefficients will be denoted by $a^{NP}(\theta),\ldots, h^{NP}(\theta)$. If, moreover, charge symmetry is not valid there will be an extra term

$$b^{NP}(\theta)(\sigma_{1N} - \sigma_{2N}) .$$

In N-P scattering the polarization of the proton scattered at $\theta\phi$ (the proton will be considered the first particle and the neutron the second, where $\hat{r} = (\vec{r}_1 - \vec{r}_2)$) will be denoted by $\vec{P}(P, \theta\phi)$ and its magnitude is

$$P(P, \theta) = \frac{\mathrm{Tr}\, M^{NP}(\theta\phi)\, \bar{M}^{NP}(\theta\phi)\, \sigma_{1N}}{\mathrm{Tr}\, M^{NP}(\theta\phi)\, \bar{M}^{NP}(\theta\phi)}$$

$$= \Big\{ 2\, \mathrm{Re}\, c^{NP}(\theta)(a^{NP}(\theta) + m^{NP}(\theta))^* \\ 2\, \mathrm{Re}\, b^{NP}(\theta)(a^{NP}(\theta) - m^{NP}(\theta))^* \Big\} I_0^{-1} .$$

The polarization of the neutron scattered at $\theta'\phi'$ is denoted by $\vec{P}(N, \theta'\phi')$ and its magnitude is

$$P(N, \theta') = (\mathrm{Tr}\, M^{NP}(\theta\phi)\, \bar{M}^{NP}(\theta\phi)\, \sigma_{2N})\, (I_0)^{-1}$$

$$= \Big\{ 2\, \mathrm{Re}\, c^{NP}(\theta)\, (a^{NP}(\theta) + m^{NP}(\theta))^* \\ - 2\, \mathrm{Re}\, b^{NP}(\theta)(a^{NP}(\theta) - m^{NP}(\theta))^* \Big\}\, (I_0)^{-1}$$

A difference between $P(N, \theta)$ and $P(P, \theta)$ would indicate the lack of validity of the hypothesis of charge symmetry. The quantities $P(N\ \theta'\phi')$ and $P(P\ \theta\ \phi)$ can be measured either by measuring the polarization of the neutron or the proton, respectively, after an N-P collision, or by measuring the asymmetry in an N-P collision when the incident neutron or proton is polarized.[11]

In the depolarization and rotation experiments one may polarize either the incident neutron or the incident proton, and then measure the polarization of either particle after the scattering. To denote the depolarization function when the neutron is polarized and the proton emerging at θ is analyzed, the symbol $D(N, P, \theta)$ will be used. If the proton is polarized and the neutron emerging at θ' is measured the symbol will be $D(P, N, \theta')$. The expressions for the various quantities measured in terms of the $(a^{NP}(\theta),\ldots, h^{NP}(\theta), b^{NP}(\theta)$ are given in Table E where, however, $a^{NP}(\theta)$ is abbreviated by a and similarly for the other coefficients. The consequences of charge independence are obtained by identifying these coefficients with the $a^{0}(\theta),\ldots, h^{0}(\theta)$ of Eqs. (72), the coulomb effects being neglected here. Since the P-P coefficients are, according to Eq. (72), functions of the N-P coefficients for both θ and θ', the relationships between the N-P and P-P experiments will involve measurements at both angles. However, at $\pi/2$ where $\theta = \theta'$ the relationships will be relatively simple. Since the antisymmetric parts are zero here one sees immediately that

$$(I_0\ C_{KP}(\pi/2))_{PP} = 4(I_0\ C_{KP}(\pi/2))_{NP} \tag{74}$$

and a little manipulation shows that

$$\left[I_0(1 + C_{NN}(\pi/2) - 2D(\pi/2) \right]_{PP} = 4 \left[I_0(1 + C_{NN}(\pi/2) - D(NP\ \pi/2) - D(PP\ \pi/2) \right]_{NP} \tag{75}$$

and

$$\left[I_0\ R_K(\pi/2) \right]_{PP} = \left[I_0(R_K(PP\ \pi/2) + R_P(PN\ \pi/2) - R_K(NP\ \pi/2) - R_P(NN\ \pi/2)) \right]_{NP} . \tag{76}$$

These relationships would provide some direct tests of the hypothesis of charge independence for the two nucleon system. The necessary experiments are, however, considered quite difficult at the present time.

PART II

Section 1. Covariant S-Matrix.

In the preceding part the polarization phenomena is treated using the simplifying assumption that the nucleons are Pauli particles. In view of the 300 Mev incident beam energy of the Berkeley experiments and the still larger energies now available a relativistic treatment is desirable. In this part a covariant treatment of the problem is carried out.

In this first section the covariant form of the S-matrix for the collision of a Dirac particle with a spin zero particle is developed. Relativistic invariance requires that the element of the S-matrix which transforms the spinor in the initial state into the spinor in the final state be of the form*

$$S_P(\underline{k}', \underline{t}, \underline{k}) = A + B_\mu \gamma_\mu + \tfrac{1}{2} C_{\mu\nu} \sigma_{\mu\nu} + D_\mu (i \gamma_5 \gamma_\mu) + E \gamma_5 \tag{1}$$

where A, B_μ, $C_{\mu\nu}$, D_μ and E are respectively scalar, vector, antisymmetric tensor, pseudovector and pseudoscalar functions of the three independent four momenta $\underline{k}$, $\underline{k}'$ and $\underline{t}$. The $\underline{k}$ and $\underline{k}'$ denote the relative four-momenta in the initial and final states respectively, while $\underline{t}$ is the total four-momentum of the system, the sum of the initial or the final four-momenta of the two particles. The general matrix of this form is, however, not consistent with the requirements of hole theory. This interpretation of the Dirac equation requires that a Dirac particle which is in a plane wave state at both $t = +\infty$ and $t = -\infty$ must have the sign of its energy the same at these two times.

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* $S_p(\underline{k}', \underline{t}, \underline{k})$ is a matrix <u>element</u> in momentum space and a matrix in spinor space. The subscript P distinguishes it from a symbol to be defined later.

Stated in physical terms the Dirac particle cannot be changed from an ordinary particle at $t = -\infty$ to an antiparticle at $t = +\infty$, or vice versa.* Before expressing this condition in mathematical form some notation must be introduced.

If the incident Dirac particle is in a positive energy state then its wave function may be expressed as**

$$\psi_{inc} = e^{i\underline{f}\cdot\underline{x}} (a_1 u_1 (\underline{f}) + a_2 u_2 (\underline{f})) ,$$

while for a negative energy state

$$\psi_{inc} = e^{-i\underline{f}\cdot\underline{x}} (a_3 u_3 (\underline{f}) + a_4 u_4 (\underline{f})).$$

Here $\underline{f}$ is the four-momentum representing the physically measured energy and momentum of the Dirac particle. Thus $f_0 > 0$; and the space part of $\underline{f}$ has the same direction and sense as the incident velocity. Notice that $\underline{f}$ is not the relative momentum, like $\underline{k}$, but the momentum in the basic reference frame. The four spinors $u_i(\underline{f})$ each have four components $u_{si}(\underline{f})$ which are given by***

$$u_{si}(\underline{f}) = (\mp i\, \underline{f}\cdot \underline{\gamma}_{si} + M) \left[2\, M(f_0 + M)\right]^{-\frac{1}{2}} .$$

Here, and in what follows, the upper sign refers to indices $i = 1, 2$ (positive energy states) and the lower sign refers to $i = 3,4$ (negative energy states). The covariant normalization condition

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* Cases in which <u>real</u> particles are created during a collision may be treated by an extension of the S-matrix formalism, but will not be considered here.

** $\underline{f}\cdot\underline{x} \equiv f_\mu x_\mu$ where $f_4 = i f_0$, etc.

*** $\hbar = c = 1$; M = proton rest mass.

$$\bar{u}_i(\underline{f})\, u_j(\underline{f}) = u_i^*(\underline{f})\, \beta\, u_j(\underline{f}) = \pm \delta_{ij}$$

is satisfied by these spinors. In this relativistic treatment a star is used to denote complex conjugate transpose and $\bar{u}$ denotes $u^*\beta$, the adjoint of u. The $u_i(\underline{f})$ introduced above are easily seen to be solutions of the Dirac equation

$$(\pm i\, \underline{f}\cdot\underline{\gamma} + M)\, u_i(\underline{f}) = 0 \quad .$$

It is now convenient to introduce for any four vector $\underline{v}$ the symbol

$$\gamma(\underline{v}) \equiv (\underline{\gamma}\cdot\underline{v})/(\underline{v}\cdot\underline{v})^{\frac{1}{2}} \quad ,$$

where the square root in the denominator is to be taken as positive or positive imaginary. The Dirac equation then becomes

$$\gamma(\underline{f})\, u_i(\underline{f}) = \pm\, u_i(\underline{f}) \quad . \tag{3}$$

Using this relation the hole theory condition may be expressed by the equation

$$S(\underline{f}', \underline{t}, \underline{f}) = \gamma(\underline{f}')\, S(\underline{f}', \underline{t}, \underline{f})\, \gamma(\underline{f}) \tag{4}$$

where $S(\underline{f}', \underline{t}, \underline{f})$ denotes the S matrix element between states in which the Dirac particle has the physical momenta $\underline{f}$ and $\underline{f}'$ in the initial and final states respectively. It will prove convenient, however, to cast the condition expressed by Eq. (4) into the form of a commutation relation. This may be done with the help of the operator

$$\gamma(\underline{u}, \underline{w}) \equiv \gamma(\underline{u}\, |\underline{u}\cdot\underline{u}|^{-\frac{1}{2}} + \underline{w}\, |\underline{w}\cdot\underline{w}|^{-\frac{1}{2}})$$
$$\propto \left[\gamma(\underline{u}) + \gamma(\underline{w})\right] \quad .$$

Using the equations

$$\gamma(\underline{u})\,\gamma(\underline{u}) = 1 = \gamma(w)\,\gamma(w)\,,$$

one finds that

$$\gamma(\underline{u})\,\gamma(\underline{u},\underline{w}) = \gamma(\underline{u},\underline{w})\,\gamma(\underline{w})\,. \tag{5}$$

With the aid of this equation and $S_q(\underline{k}', \underline{t}, \underline{k})$ defined by

$$S(\underline{f}', \underline{t}, \underline{f}) = \gamma(\underline{f}', \underline{t})\, S_q(\underline{k}', \underline{t}, \underline{k})\, \gamma(\underline{t}, \underline{f})\,, \tag{6}$$

the hole theory condition may be expressed as

$$S_q(\underline{k}', \underline{t}, \underline{k}) = \gamma(\underline{t})\, S_q(\underline{k}', \underline{t}, \underline{k})\, \gamma(\underline{t})\,. \tag{7}$$

Since the S-matrix and the $\gamma(\underline{u}, \underline{w})$ have convariant forms the $S_q(\underline{k}', \underline{t}, \underline{k})$ must also be covariant and it may be written in the form given by Eq. (1) with the subscript P replaced now by q. The commutation relation Eq. (7) may be used to restrict the coefficients in this expression for S_q to the forms

$$B_\mu = N_b(b\, t_\mu) \tag{8}$$

$$C_{\mu\nu} = N_c\, C\left\{k_\mu k'_\nu - k_\nu k'_\mu - \frac{(M^2 - m^2)}{|\underline{t}\cdot\underline{t}|}\left[t_\mu(k'_\nu - k_\nu) - t_\nu(k'_\mu - k_\mu)\right]\right\}$$

$$D_\mu = N_d\, d(-i)\, k_\lambda k'_\rho\, t_\sigma\, \varepsilon_{\mu\lambda\rho\sigma} \equiv d\, n_\mu\,.$$

Here the coefficients b, c and d are scalar functions, m is the rest mass of the second particle and the normalization factors N_b, N_c, and N_d are choosen so that

$$B_\mu B_\mu = b^2\,, \qquad C_{\mu\nu} C_{\mu\nu} = 2\,c^2\,, \qquad D_\mu D_\mu = d^2\,.$$

The $\varepsilon_{\mu\lambda\rho\sigma}$ is the antisymmetric symbol and $\underline{n}$ is a unit pseudovector which satisfies

$$(\underline{k}\cdot\underline{n}) = (\underline{k}'\cdot\underline{n}) = (\underline{t}\cdot\underline{n}) = (1 - \underline{n}\cdot\underline{n}) = 0 .$$

This pseudovector $\underline{n}$ is the four dimensional generalization of $\vec{N}$, the three dimensional vector normal to the plane of scattering. The auxiliary operator $S_q(\underline{k}', \underline{t}, \underline{k})$ which has just been introduced has a rather simple interpretation. To see this let Eq. (7) be substituted into Eq. (6) to give

$$S(\underline{f}', \underline{t}, \underline{f}) = (\gamma(\underline{f}', \underline{t})\, \gamma(\underline{t}))\, S_q(\underline{k}', \underline{t}, \underline{k})(\gamma(\underline{t})\, \gamma(\underline{t}, \underline{f})) . \tag{6'}$$

The operator $(\gamma(\underline{t})\, \gamma(\underline{t}, \underline{f}))$ is closely related to the Lorentz transformation between the center of mass frame and the rest frame of the incident Dirac particle, and the operator $(\gamma(\underline{f}', \underline{t})\, \gamma(\underline{t}))$ is similarily related to the rest frame of the scattered particle. This may be seen by reducing the Lorentz transformation

$$L(f) \equiv \exp\left[-\tfrac{1}{2}\, \theta(\vec{\alpha}\cdot\vec{f})\, |\vec{f}|^{-1}\right]$$

to the form

$$L(\underline{f}) = \beta\,(-i\, \underline{\gamma}\cdot\underline{f} + M\beta)\left[2M(f_0 + M)\right]^{-\frac{1}{2}} . \tag{11}$$

In the center of mass frame in which $\gamma(\underline{t}) = \beta$, one may immediately identify terms to obtain

$$\gamma(\underline{t}_1)\, \gamma(\underline{t}_1\, \underline{f}_1) = L(\underline{f}_1)$$

$$\gamma(\underline{f}_1', \underline{t}_1)\, \gamma(\underline{t}_1) = L^{-1}(\underline{f}_1') \tag{12}$$

where the subscript one indicates the center of mass value. Thus

$$S(\underline{f}_1', \underline{t}_1, \underline{f}_1) = L^{-1}(\underline{f}_1')\, S_q(\underline{k}_1', \underline{t}_1, \underline{k}_1)\, L(\underline{f}_1) .$$ This

equation has the following interpretation: the S-matrix in the center of mass frame may be decomposed into a product of two Lorentz transformations and a scattering matrix S_q. The first factor is a Lorentz transformation which converts the spinors of the incident wave function from their values in the center of mass frame to their values in a rest frame of the incident Dirac particle. It converts the spinors to their "proper" values, one might say. Then the unitary operator S_q gives the effect of the scattering upon the "proper" spinors and finally a Lorentz transformation converts the "proper" spinors of the scattered particle back to their value as seen in the center-of-mass frame.

The form of S_q in the center-of-mass frame is particularly simple. The Eqs. (8) give in this case

$$B_\mu \gamma_\mu = b\beta$$

$$\tfrac{1}{2} C_{\mu\nu} \sigma_{\mu\nu} = c\, \sigma_i N_i \qquad (i = 1, 2, 3)$$

$$D_\mu\, i\gamma_5 \gamma_\mu = d\, \beta \sigma_i N_i \,. \tag{13}$$

Here the σ_i are the usual four by four Dirac matrices

$$\sigma_i = \begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix}$$

and $\vec{N}$ is the three vector which is normal to the scattering plane in the center of mass frame. Combining these one obtains

$$S_q(\underline{k}_1', \underline{t}_1, \underline{k}_1) = \begin{pmatrix} (f^+ + g^+\sigma_N) & 0 \\ 0 & (f^- - g^-\sigma_N) \end{pmatrix} \tag{14}$$

where σ_N is the Pauli $\sigma_i N_i$ and

$$f^{\pm} = a \pm b \qquad\qquad g^{\pm} = d \pm c \quad . \tag{15}$$

The f's and g's are scalar functions which completely describe the scattering. The upper two by two matrix operates only on the positive energy "proper" spinors and the lower matrix operatoes only on negative energy parts.

In the general frame, also, the S_q may be put into a form which clearly separates the parts referring to positive and negative energy states. The desired form is obtained by first writing

$$\begin{aligned} \tfrac{1}{2} C_{\mu\nu} \sigma_{\mu\nu} &\equiv \tfrac{1}{2} C_{\mu\nu} (-i/2)(\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu) \\ &= -\tfrac{i}{2} C_{\mu\nu} \gamma_\mu \gamma_\nu \\ &= -\tfrac{i}{4} C_{\mu\nu} \gamma_\sigma \gamma_\rho \varepsilon_{\rho\sigma\mu\nu} \gamma_5 \quad . \end{aligned} \tag{16}$$

The condition that $\frac{1}{2} C_{\mu\nu} \sigma_{\mu\nu}$ commutes with $\gamma(\underline{t})$ requires that

$$t_\mu C_{\mu\nu} = - C_{\nu\mu} t_\mu = 0 \quad . \tag{17}$$

Using this relation, Eq. (16) may be written

$$\tfrac{1}{2} C_{\mu\nu} \sigma_{\mu\nu} = i \gamma(\underline{t}) \gamma_5 \underline{\gamma} \cdot \underline{c} \tag{18}$$

where

$$C_\rho = \tfrac{1}{2}(-i\, t_\sigma) \varepsilon_{\rho\sigma\mu\nu} C_{\mu\nu} \, | t \cdot t |^{-\frac{1}{2}} \quad . \tag{19}$$

If the expression for $C_{\mu\nu}$ from Eq. (8) is put into Eq. (19) and the definition of $\underline{n}$ from Eq. (8) is used, one obtains

$$\underline{c} = c\, \underline{n} \quad . \tag{20}$$

Eqs. (1), (8) and (18) now combine to give

$$S_q(\underline{k}', \underline{t}, \underline{k}) = a + b\,\gamma(\underline{t}) + d(i\,\gamma_5\,\underline{\gamma}\cdot\underline{n}) + c\,\gamma(\underline{t})(i\,\gamma_5\,\underline{\gamma}\cdot\underline{n}) \quad . \tag{21}$$

With the introduction of the covariant projection operators

$$\Lambda^{\pm}(\underline{t}) = \tfrac{1}{2}(1 \pm \gamma(\underline{t})) \,, \tag{22}$$

this reduces to

$$S_q(\underline{k}', \underline{t}, \underline{k}) = \sum_{\pm} \Lambda^{\pm}(\underline{t}) \left[f^{\pm} + g^{\pm}\, i\,\gamma_5\,\underline{\gamma}\cdot\underline{n} \right] \quad . \tag{23}$$

In this form of S_q the $\sigma_{\mu\nu}$ type of term has been eliminated in favor of projection operators and terms of the $i\,\gamma_5\,\gamma_\rho$ type. Alternatively the $i\,\gamma_5\,\gamma_\rho$ may be eliminated in favor of projection operators and $\sigma_{\mu\nu}$'s. The form of the S-matrix obtained by substituting Eq. (23) into Eq. (6') is covariant and clearly separates the parts referring to the positive and negative energy states. This form will be used in the analysis of the polarization experiments in the third section. In the next section the covariant form of the density matrix will be introduced and reduced in a manner quite similar to the reduction of the S-matrix in this section.

Section 2. Covariant Density Matrix.

In the treatment of polarization phenomena it is necessary to consider mixtures of states and a density matrix formulation is convenient. The expectation value of an operator A in the incident beam is expressed in terms of the density matrix $\rho(\underline{f})$ by the equation[10]

$$\langle A \rangle_f = \mathrm{Tr}\, \rho(\underline{f})\, A / \mathrm{Tr}\, \rho(\underline{f}) \quad . \tag{24}$$

For the scattered beam the corresponding equation is

$$\langle A \rangle_{f'} = \mathrm{Tr}\, \rho'(\underline{f}')\, A / \mathrm{Tr}\, \rho'(\underline{f}') \ . \tag{25}$$

The differential cross section is

$$I = \mathrm{Tr}\, \rho'(\underline{f}') / \mathrm{Tr}\, \rho(f) \tag{26}$$

where the density matrices before and after the scattering are related by*

$$\rho'(\underline{f}') = S(\underline{f}', \underline{t}, \underline{f})\, \rho(\underline{f})\, \bar{S}(\underline{f}', \underline{t}, \underline{f}) \quad . \tag{27}$$

The adjoint $\bar{A}$ of an operator A is defined by the equation

$$\overline{Au} = \bar{u}\, \bar{A}$$

and thus

$$\bar{S} = \beta S^* \beta$$

where the star denotes complex conjugate transpose.

The covariant density matrices $\rho(\underline{f})$ and $\rho'(\underline{f}')$ may be expressed in the forms

$$\rho(\underline{f}) = (\tfrac{1}{4}\, \mathrm{Tr}\, \rho(\underline{f}))\left\{ 1 + \lambda_\mu \gamma_\mu + \tfrac{1}{2} s_{\mu\nu} \sigma_{\mu\nu} + i\, \gamma_5 \gamma_\mu p_\mu + e\, \gamma_5 \right\}$$

$$\rho'(f') = (\tfrac{1}{4}\, \mathrm{Tr}\, \rho'(f'))\left\{ 1 + \lambda'_\mu \gamma_\mu + \tfrac{1}{2} s_{\mu\nu} \sigma_{\mu\nu} + i\, \gamma_5 \gamma_\mu p'_\mu + e'\, \gamma_5 \right\} \tag{28}$$

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* See Appendix for a discussion of the covariant density matrix used here.

where λ_μ, $s_{\mu\nu}$, p_μ and e are respectively vector, antisymmetric tensor, pseudovector and pseudoscalar and similarily for the primed quantities. The condition that the Dirac particle must be definitely in a positive energy state for definitely in a negative energy state in the asymptotic region may be expressed by the equations

$$\rho(\underline{f}) = \gamma(\underline{f})\,\rho(\underline{f})\,\gamma(\underline{f})$$
$$\rho'(\underline{f}') = \gamma(\underline{f}')\,\rho'(\underline{f}')\,\gamma(\underline{f}') . \tag{29}$$

By a treatment very similar to the reduction of the form of S_q in section one the density matrices may now be reduced to the forms*

$$\rho(f) = (\tfrac{1}{2}\,\mathrm{Tr}\,\rho(f))\left\{\sum_\pm \Lambda^\pm(\underline{f})\,\lambda^\pm(1 + i\gamma_5\,\gamma\cdot\underline{p}^\pm)\right\}$$
$$\rho'(f') = (\tfrac{1}{2}\,\mathrm{Tr}\,\rho'(f'))\left\{\sum_\pm \Lambda^\pm(\underline{f}')\,\lambda^\pm(1 + i\gamma_5\,\gamma\cdot\underline{p}'^\pm)\right\} \tag{30}$$

where

$$\underline{p}\cdot\underline{f} = \underline{p}'\cdot\underline{f}' = 0 \tag{31}$$

and where

$$\lambda^\pm = \mathrm{Tr}(\rho(\underline{f})\,\Lambda^\pm(\underline{f}))/\mathrm{Tr}\,\rho(\underline{f}) = \langle\Lambda^\pm(\underline{f})\rangle$$

$$\lambda^\pm_p = \frac{\mathrm{Tr}(\rho(\underline{f})\,\Lambda^\pm(\underline{t})\,i\gamma_5\gamma)}{\mathrm{Tr}(\rho(\underline{f}))} = \langle\Lambda^\pm\, i\gamma_5\gamma\rangle$$

and similarly for the primed variables.

The value of $\lambda^\pm$ specifies the energy state. For a positive energy particle $\lambda^+ = 1$ and $\lambda^- = 0$ whereas for the negative energy particle $\lambda^+ = 0$ and $\lambda^- = 1$. The pseudovectors $\underline{p}$ are the

---

* This form has been used by Michel and Wightman.

relativistic generalizations of the polarization vectors of the non-relativistic treatment and describe the spin of the particle and anti-particle.

This form of the density matrix, used in conjunction with the form of the S-matrix developed in section one, will give a convariant description of polarization phenomena. In the following section this covariant treatment is applied to double and triple scattering experiments and relativistic corrections are obtained.

Section 3. Covariant Polarization Formalism.

To find the state of polarization of a nucleon after a single scattering, one may put the expressions for $S(\underline{f}', \underline{t}, \underline{f})$, $\rho(\underline{f})$ and $\rho'(\underline{f}')$ given in Eqs. (6'), (23) and (30) into Eq. (27), which relates $\rho(\underline{f})$ and $\rho'(\underline{f}')$. With the help of the relations $\gamma(\underline{u}) = \bar{\gamma}(\underline{u})$ and $\gamma(\underline{u}, \underline{w}) = \bar{\gamma}(\underline{u}, \underline{w})$ for time-like $\underline{u}$ and $\underline{w}$, one then obtains

$$\frac{\mathrm{Tr}\ \rho'(\underline{f}')}{\mathrm{Tr}\ \rho(\underline{f})} \left\{ \sum_{\pm} \Lambda^{\pm}(\underline{f}') \quad \lambda'^{\pm}(1 + i\,\gamma_5\,\underline{\gamma}\cdot\underline{p}'^{\pm}) \right.$$

$$= (\lambda(\underline{f}', \underline{t})\,\gamma(\underline{t})) \left\{ \sum_{\pm} \Lambda^{\pm}(\underline{t})(f^{\pm} + i\,g^{\pm}\gamma_5\,\underline{\gamma}\cdot\underline{n}) \right\}$$

$$\times\ (\gamma(\underline{t})\,\gamma(\underline{t}, \underline{f})) \left\{ \sum_{\pm} \Lambda^{\pm}(\underline{f})\ \lambda^{\pm}\ (1 + i\,\gamma_5\,\underline{\gamma}\cdot\underline{p}^{\pm}) \right.$$

$$\times\ (\gamma(\underline{f}, \underline{t})\,\gamma(\underline{t})) \left\{ \sum_{\pm} \Lambda^{\pm}(\underline{t})(\bar{f}^{\pm} + i\,\bar{g}^{\pm}\gamma_5\,\underline{\gamma}\cdot\underline{n}) \right.$$

$$\times\ (\gamma(\underline{t})\,\gamma(\underline{t}, \underline{f}')) \quad . \tag{40}$$

By reducing the right-hand side of this equation to the form appearing on the left, one may obtain the polarization $\underline{p}'$ of the final beam in terms of $\underline{p}$, the initial polarization, and $\underline{f}^{\pm}$ and $\underline{g}^{\pm}$, the scattering parameters. At the same time the differential cross section

$$I = \mathrm{Tr}\ \rho(\underline{f}')/\mathrm{Tr}\ \rho(\underline{f})$$

will be obtained. Before performing this reduction, however, it is convenient to transform the equation into a simpler form. In particular the equation may be separated into two equations, each of which involves only two by two matrices and refers to a single type of particle. This not only simplifies computations but allows a more direct comparison to the nonrelativistic formulation.

To obtain this simplification the relations

$$\gamma(\underline{u})\,\gamma(\underline{u}) = \gamma(\underline{u},\underline{w})\,\gamma(\underline{u},\underline{w}) = 1$$

may be used to first transform Eq. (40) into

$$I(\gamma(\underline{t})\,\gamma(\underline{f}',\underline{t})\left\{\sum \Lambda(\underline{f}')\,\lambda'(1+i\,\gamma_5\,\underline{\gamma}\cdot\underline{p}')\right\}(\gamma(\underline{f}',\underline{t})\,\gamma(\underline{t}))$$

$$= \left\{\sum \Lambda(\underline{t})(f + i\,g\,\gamma_5\,\underline{\gamma}\cdot\underline{n})\right.$$

$$x(\gamma(\underline{t})\,\gamma(\underline{f},\underline{t}))\left\{\sum \Lambda(\underline{f})\,\lambda(1+i\,\gamma_5\,\underline{\gamma}\cdot\underline{p})\right\}(\gamma(\underline{f},\underline{t})\gamma(\underline{f}))$$

$$x\left\{\sum \Lambda(\underline{t})(\bar{f} + i\,\bar{g}\,\gamma_5\,\underline{\gamma}\cdot\underline{n})\right\} \tag{41}$$

where the $\pm$ are now to be understood. Using the Lorentz transformations $L(t)$ and $\bar{L}(t) = L^{-1}(t)$, this may be written

$$I\,L(\underline{t})(\gamma(\underline{t})\,\gamma(\underline{f}',\underline{t}))\left\{\sum \Lambda(\underline{f}')\,\lambda'(1+i\,\gamma_5\,\underline{\gamma}\cdot\underline{p}')\right\}(\gamma(\underline{f}',\underline{t})\gamma(\underline{t}))\bar{L}(\underline{t})$$

$$= L(\underline{t})\left\{\sum \Lambda(\underline{t})(f + i\,g\,\gamma_5\,\underline{\gamma}\cdot\underline{n})\right\}\bar{L}(\underline{t})$$

$$x\,L(\underline{t})(\gamma(\underline{t})\,\gamma(\underline{f},\underline{t}))\left\{\sum \Lambda(\underline{f})\lambda(1+i\,\gamma_5\,\underline{\gamma}\cdot\underline{p})\right\}$$

$$x\,(\gamma(\underline{f},\underline{t})\,\gamma(\underline{t}))\,\bar{L}(\underline{t})$$

$$x\,L(\underline{t})\left\{\sum \Lambda(\underline{t})(\bar{f} + i\,\bar{g}\,\gamma_5\,\underline{\gamma}\cdot\underline{n})\right\}\bar{L}(\underline{t}). \tag{42}$$

The $L(\underline{t})$ has the property that

$$L(\underline{t})\,\gamma_\mu\,\bar{L}(\underline{t}) = a_{\mu\nu}(\underline{t})\,\gamma_\nu \tag{43}$$

where $a_{\mu\nu}(\underline{t})$ satisfies

$$x_\mu\,a_{\mu\nu}(\underline{t}) = (x_1)_\nu \tag{44}$$

$$x_\mu = a_{\mu\nu}(\underline{t})\,(x_1)_\nu\ ,$$

$(x_1)_\mu$ being the components of any aribtrary vector $\underline{x}$ in the center of mass frame. Using Eqs. (43), (44) and (12) one finds

$$L(\underline{t})(\gamma(\underline{t})\,\gamma(\underline{x},\,\underline{t}))\,\bar{L}(\underline{t}) = \gamma(\underline{t}_1)\,\gamma(\underline{x}_1,\,\underline{t}_1)$$

$$= L(\underline{x}_1) \tag{45}$$

where $\underline{x}$ may be $\underline{f}$ or $\underline{f}'$. Eq. (42) may then be written

$$\begin{aligned} I(L(f_1')L(\underline{t}))\left\{\sum \Lambda(\underline{f}')\,\lambda\,(1 + i\,\gamma_5\,\underline{\gamma}\cdot\underline{p}')\right\}(\bar{L}(\underline{t})\,\bar{L}(\underline{f}_1')) \\ = L(\underline{t})\left\{\sum \Lambda(\underline{t})(f + i\,g\,\gamma_5\,\underline{\gamma}\cdot\underline{n})\right\}\bar{L}(\underline{t}) \\ \times\,(L(\underline{f}_1)\,L(\underline{t}))\left\{\sum \Lambda(\underline{f})\,\lambda(1 + i\,\gamma_5\,\underline{\gamma}\cdot\underline{p})\right\}(\bar{L}(\underline{t})\,\bar{L}(\underline{f}_1)) \\ \times\,L(\underline{t})\left\{\sum \Lambda(\underline{t})(\bar{f} + i\,\bar{g}\,\gamma_5\,\underline{\gamma}\cdot\underline{n})\right\}\bar{L}(\underline{t})\ . \end{aligned} \tag{46}$$

With the introduction of the pure space rotation transformation

$$R(\underline{x}_1) \equiv L(\underline{x}_1)\,L(\underline{t})\,\bar{L}(\underline{x})\ , \tag{47}$$

one obtains

$$\begin{aligned} I\,R(\underline{f}_1')\,L(\underline{f}')\left\{\sum \Lambda(\underline{f}')\,\lambda'(1 + i\,\gamma_5\,\underline{\gamma}\cdot\underline{p}')\right\}\bar{L}(\underline{f}')\,\bar{R}(\underline{f}_1') \\ = L(\underline{t})\left\{\sum \Lambda(\underline{t})(f + i\,g\,\gamma_5\,\underline{\gamma}\cdot\underline{n})\right.\ \bar{L}(\underline{t}) \\ \times\,R(\underline{f}_1)\,L(\underline{f})\left\{\sum \Lambda(\underline{f})\ (1 + i\,\gamma_5\,\underline{\gamma}\cdot\underline{p})\right\}\bar{L}(\underline{f})\,\bar{R}(\underline{f}_1) \\ \times\,L(\underline{t})\left\{\sum \Lambda(\underline{t})(\bar{f} + i\,\bar{g}\,\gamma_5\,\underline{\gamma}\cdot\underline{n})\right\}\bar{L}(\underline{t})\ . \end{aligned} \tag{48}$$

Defining

$$\begin{aligned} P_\mu &\equiv p_\nu\,a_{\nu\mu}(\underline{f}) \\ P'_\mu &\equiv p'_\nu\,a_{\nu\mu}(\underline{f}') \\ N_\mu &\equiv n_\nu\,a_{\nu\mu}(\underline{t}) \\ \Lambda^{\pm}_{(0)} &\equiv \tfrac{1}{2}(1 \pm \beta) \end{aligned}$$

and using equations similar to Eqs. (43) and (44), one obtains

$$
\begin{aligned}
I\ R(\underline{f}_1')\left\{\sum \Lambda(0)\ \lambda'\ (1 + i\,\gamma_5\,\underline{\gamma}\cdot\underline{P}')\right\}\bar{R}(\underline{f}_1') \\
= \left\{\sum \Lambda(0)(f + i\,g\,\gamma_5\,\underline{\gamma}\cdot\underline{N})\right\} \\
\times R(\underline{f}_1)\left\{\sum \Lambda(0)\ \lambda\ (1 + i\,\gamma_5\,\underline{\gamma}\cdot\underline{P})R(\underline{f}_1)\right. \\
\times \left\{\sum \Lambda(0)(\bar{f} + i\,\bar{g}\,\gamma_5\,\underline{\gamma}\cdot\underline{N})\right\} .
\end{aligned}
\tag{49}
$$

According to their definitions the $\underline{P}$ $\underline{P}'$ and $\underline{N}$ are the values of $\underline{p}$ $\underline{p}'$ and $\underline{n}$ in the Lorentz frame where $\underline{f}$, $\underline{f}'$, and $\underline{t}$, respectively, are pure time-like. Thus from the conditions

$$\underline{p}\cdot\underline{f} = p'\,f' = \underline{n}\cdot\underline{t} = 0$$

the four-vectors $\underline{P}$, $\underline{P}'$ and $\underline{N}$ must have vanishing fourth components. Considered as three-vectors the vectors $\underline{P}$ and $\underline{P}'$ are, in fact, just the proper polarizations of the incident and final beams, and N is the normal to the scattering plane as measured in the center of mass frame. With the definition

$$R(\underline{x}_1)\,\gamma_i\,\bar{R}(\underline{x}_1) \equiv r_{ij}(\underline{x}_1)\,\gamma_j\ , \tag{50}$$

Eq. (49) reduces to

$$
\begin{aligned}
I\ \textstyle\sum_\pm \Lambda^\pm(0)\ \lambda^{\pm\prime}(1 + i\,\gamma_5\,P_i^{\pm\prime}\,r_{ij}(\underline{f}_1')\,\gamma_j) \\
= \left\{\textstyle\sum_\pm \Lambda^\pm(0)(f^\pm + i\,g^\pm\gamma_5\,\gamma_i\,N_i)\right\} \\
\times \left\{\textstyle\sum_\pm \Lambda^\pm(0)\,\lambda^\pm(1 + i\,\gamma_5\,P_i^\pm\ r_{ij}(\underline{f}_1)\,\gamma_j)\right\} \\
\times \left\{\textstyle\sum_\pm \Lambda^\pm(0)(\bar{f}^\pm + i\,\bar{g}^\pm\,\gamma_5\,\gamma_i\,N_i)\right\}
\end{aligned}
\tag{51}
$$

where i and j need be summed only from 1 to 3. Since $i\gamma_5\gamma_i = \beta\sigma_i$ (i = 1, 2, 3) this equation splits into two parts, each of which is an equation in two by two matrices which refers to a single type of particle.

For the cases $\lambda^+ = 1$ or $\lambda^- = 1$, the equations may be written

$$I^{\pm}(1 + \tilde{P}'_i\ \sigma_i) =$$

$$(f^{\pm} \pm g^{\pm} N_i\ \sigma_i)(1 \pm \tilde{P}_i^{\pm}\ \sigma_i)(\bar{f}^{\pm} \pm \bar{g}^{\pm} N_i \cdot \sigma_i)$$

thereby defining $I^{\pm}$. The σ_i are now the two by two Pauli matrices and the vectors $\tilde{P}$ and $\tilde{P}'$ are defined by

$$\tilde{P}_i = P_j\ r_{ji}(\underline{f}_1) \qquad\qquad \tilde{P}'_i = P'_j\ r_{ji}(\underline{f}'_1) \quad . \tag{52}$$

These equations are, except for a sign change in σ_i for the negative energy states, identical with the equations obtained from the nonrelativistic treatment, except that the vectors $\tilde{P}$ and $\tilde{P}'$ replace the polarization vectors of the nonrelativistic treatment. In the analysis of double and triple scattering experiments one may proceed much as in the nonrelativistic case, remembering, however, that it is the proper polarization vector P, rather than P, which is the same in the outgoing beam of one scattering as in the incoming beam for the next. The connection between the $\tilde{P}$ of one scattering and the $\tilde{P}'$ of the preceding scattering is

$$\tilde{P}_i^{(n)} = \tilde{P}_j'^{(n-1)}\ r_{jk}^{-1}(\underline{f}'_{n-1})\ r_{ki}(\underline{f}_n) \tag{53}$$

where Eq. (52) has been used in conjunction with the identity $P_i^{(n)} \equiv P_i'^{(n-1)}$. The superscript (n) will denote the quantities referring to the <u>n</u>th scattering and the subscript n on the four-momenta denotes their center of mass values. The rotations appearing on the left

of Eq. (53) will introduce certain differences between the relativistic and nonrelativistic treatments. These will be called the rotational corrections.

A second type of correction comes from the use of the relativistic transformation of momenta between the successive frames. Thus the relation between the incoming momentum for the nth scattering and the outgoing momentum for the preceding scattering as measured in their respective center of mass frames is

$$(\underline{f}_n)_\mu = (\underline{f}'_{n-1})_\lambda \; a^{-1}_{\lambda\mu} \; (\underline{t}^{(n-1)}) \; a_{\nu\mu} (\underline{t}^{(n)}) \quad . \tag{54}$$

The major portion of the transformation appearing here will, except for extreme relativistic cases, be given by the nonrelativistic Galilean transformation. The remainder will be called the kinematical corrections.

To analyze double and triple scattering experiments it appears most convenient to choose the laboratory as the basic reference frame. Assuming the target particles to be at rest in the laboratory one notices that

$$P^{(n)} = \tilde{P}^{(n)} \quad ;$$

since the three Lorentz transformations which give

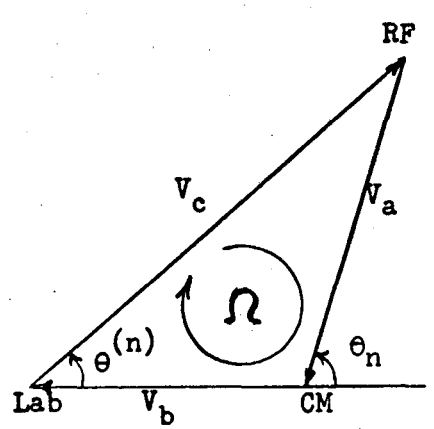
$$r_{\mu\nu} (\underline{f}_n) = a^{-1}_{\mu\eta} \; (\underline{f}^{(n)}) \; a_{\eta\lambda} \; (\underline{t}^{(n)}) \; a_{\lambda\nu} (\underline{f}_n) \; , \tag{55}$$

will be colinear and their product will be unity. For the scattered beam, however, the P' and $\tilde{P}'$ will differ. The formal manipulations in the relativistic treatment will, therefore, be identical with those of the nonrelativistic treatment except for the following two modifications: first, the connection between the momenta in the successive center of mass frames is given by Eq. (54); and second, an extra rotation

$r^{-1}_{\mu\nu}(\underline{f}_1')$ is applied to the polarization vector in the outgoing beam before it is interpreted as the incident polarization of the next scattering, or as the proper polarization. The rotation $r^{-1}_{\mu\nu}(\underline{f}_1')$ is the effect of the three successive Lorentz transformations which take a vector from its value in a rest frame of the scattered particle to the center of mass frame; then from center of mass to laboratory; and finally from laboratory back to a (new) rest frame of the scattered particle. This rotation may be specified by an axial vector $\vec{\Omega}$ which is given by the equation

$$\vec{\Omega} = (\vec{V}_a \times \vec{V}_b) \frac{1 + (\gamma)^{(a)} (\gamma)^{(b)} (\gamma)^{(c)}}{(1 + (\gamma)^{(a)})(1 + (\gamma)^{(b)})(1 + (\gamma)^{(c)})} \tag{56}$$

where $(\gamma)^{(a)}$, $(\gamma)^{(b)}$ and $(\gamma)^{(c)}$ are the Lorentz contraction factors associated with the three transformations listed above and $\vec{V}_a$, $\vec{V}_b$ and $\vec{V}_c$ are the space parts of the three relative velocities, respectively. The transformations and the corresponding rotation are schematically represented in the accompanying diagram, where $\theta^{(n)}$ and θ_n are the laboratory and center of mass scattering angles respectively. Since the rotation is about an axis perpendicular to the plane of scattering it may be neglected in the simple double scattering experiments and in the depolarization experiments: in these experiments the polarization vector is always perpendicular to the scattering plane and the rotation will not affect it.



In triple scattering experiments of the rotation category the polarization vector will have components in the plane of the second scattering. The asymmetry in the differential cross section after the third scattering will measure the component of proper polarization which is in the plane of the second scattering and which is perpendicular to the laboratory direction of the scattered beam. Both the kinematical and rotational effects will play a role. As an example, the important case in which the masses of the Dirac particle and the second target particle are equal will be treated. The considerations of the next section show that the results obtained here will be applicable to the case in which the second target is a Dirac particle.

Because of the kinematical corrections the second laboratory scattering angle $\theta^{(2)}$ is not $\theta_2/2$. The difference may be defined as

$$\alpha = \tfrac{1}{2}\theta_2 - \theta^{(2)} = \tfrac{1}{2}\theta_{CM} - \theta_{Lab} .$$

Since it is the component of polarization perpendicular to the laboratory direction of the scattered beam which is measured, there will, for a fixed θ_2, be a kinematical correction of the direction which specifies the component of polarization which is measured by the angle α. There will also be a rotational correction which changes the direction of the polarization vector by the angle $\delta = |\vec{\Omega}|$. The effect of this second correction may be accounted for by letting the polarization vector remain fixed but rotating the direction of the component which is in effect measured, by the angle $-\delta$. Taking the various senses into account the net effect of the two corrections is to rotate the direction of the effective component by $(\delta - \alpha)$ about the normal vector N. A calculation shows that $\delta = 2\alpha$, and the rotational effect just

reverses the kinematical correction. This has the simple physical consequence that the direction of the effective component makes an angle $\theta^{(2)}$ with the normal to the center of mass velocity. The relativistic expression for the rotation parameter* R in the P - P system, therefore, takes the relatively simple form

$$R = (|a|^2 - |m|^2)\cos(\theta_{CM} - \theta_{Lab}) - 4\,\mathrm{Re}\; g\, h^* \cos(\theta_{Lab})$$

$$+ 2\,\mathrm{Re}\; i\, c(a^* - m^*)\sin(\theta_{CM} - \theta_{Lab}) \tag{57}$$

where θ_{CM} and θ_{Lab} are the center-of-mass and laboratory angles at the second scattering. To obtain this last equation it was assumed that the prescription for extending the nonrelativistic formulas into the relativistic domain will continue to be valid when the target particle has internal coordinates. In the next section the case in which the target is another Dirac particle is considered and this assumption is validated.

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* This is the R parameter which will be measured in experiments in which magnetic fields are not used to rotate the directions of polarization vectors. See Reference 10.

Section 4. Polarization Formalism for Two Dirac Particles.

In the developments in the preceding sections it was assumed that the target particle had no internal coordinates. The form of the results suggests that the relativistic corrections involving the spin state of the first particle would not be changed if the second particle were to possess internal coordinates. Indeed, one finds that the manipulations involving the first particle spin state may be carried out almost unchanged if the second particle possesses spin. In this section the important case in which the second particle is also a Dirac particle is considered and the expected generalization is obtained. In this treatment it will be assumed that the two particles are distinguishable. Indistinguishable particles may then be treated by an appropriate anti-symmetrization of the results.

The S-matrix for the system of two Dirac particles may be expressed as a sum of terms, each of which is a product of an operator in the first spin space times an operator in the second spin space. Thus one may take all possible bilinear combinations of the matrices

$$(I^{(1)}, \gamma_\mu^{(1)}, \tfrac{1}{2}\sigma_{\mu\nu}^{(1)}, i\gamma_5^{(1)}\gamma^{(1)}, \gamma_5^{(1)}; I^{(2)}, \gamma_\mu^{(2)}, \tfrac{1}{2}\sigma_{\mu\nu}^{(2)}, i\gamma_5^{(2)}\gamma^{(2)}, \gamma_5^{(2)})$$

which are linear in the first and in the second subsets.

In exact analogy to the case treated above, the matrix $S_q(\underline{k}', \underline{t}, \underline{k})$ may be defined by

$$S(\underline{f}', \underline{h}', \underline{t}, \underline{f}, \underline{h}) = (\gamma^{(1)}(\underline{f}', \underline{t})\, \gamma^{(1)}(\underline{t}))(\gamma^{(2)}(\underline{h}', \underline{t})\, \gamma^{(2)}(\underline{t}))$$

$$\times\ S_q(\underline{k}', \underline{t}, \underline{k})(\gamma^{(2)}(\underline{t})\, \gamma^{(2)}(\underline{h}, \underline{t}))(\gamma^{(1)}(\underline{t})\, \gamma^{(1)}(\underline{f}, \underline{t})) \qquad (58)$$

where $\underline{h}$ and $\underline{h}'$ are the initial and final momenta of the second Dirac particle. The hole theory condition may be introduced and used in a manner analogous to the reduction to Eq. (7), with the results that:

$$\gamma^{(1)}(\underline{t})\, S_q(\underline{k}', \underline{t}, \underline{k})\, \gamma^{(1)}(\underline{t}) = S_q(\underline{k}', \underline{t}, \underline{k})$$

$$\gamma^{(2)}(\underline{t})\, S_q(\underline{k}', \underline{t}, \underline{k})\, \gamma^{(2)}(\underline{t}) = S_q(\underline{k}', \underline{t}, \underline{k}) \quad .$$

Consider now the term in $S_q(\underline{k}', \underline{t}, \underline{k})$ of the form

$$C_{\mu\nu\sigma\rho}\,(\tfrac{1}{2}\,\sigma^{(1)}_{\mu\nu})(\tfrac{1}{2}\,\sigma^{(2)}_{\sigma\rho}) \quad .$$

The condition that this term commutes with $\gamma^{(1)}(\underline{t})$ requires, in analogy to Eq. (17), that

$$t_\mu\, C_{\mu\nu\sigma\rho} = -\, t_\mu\, C_{\nu\mu\sigma\rho} = 0 \; .$$

Now applying the arguments which led to Eq. (18) one obtains

$$C_{\mu\nu\sigma\rho}\,(\tfrac{1}{2}\,\sigma^{(1)}_{\mu\nu}) = \gamma^{(1)}(\underline{t})(i\,\gamma^{(1)}_5\,\gamma^{(1)}_\lambda)\, C_{\lambda;\sigma\rho}$$

where $t_\lambda\, C_{\lambda;\sigma\rho} = 0$. The dependence on $\sigma^{(2)}_{\sigma\rho}$ may be similarily transformed to give

$$C_{\mu\nu\sigma\rho}\,(\tfrac{1}{2}\,\sigma^{(1)}_{\mu\nu})(\tfrac{1}{2}\,\sigma^{(2)}_{\sigma\rho}) = C_{\lambda\eta}\,\gamma^{(1)}(\underline{t})(i\,\gamma^{(1)}_5\,\gamma^{(1)}_\lambda)\,\gamma^{(2)}(\underline{t})(i\,\gamma^{(2)}_5\,\gamma^{(2)}_\eta)$$

where $C_{\lambda\eta}\, t_\eta = t_\lambda\, C_{\lambda\eta} = 0$. Eliminating all terms containing $\sigma_{\mu\nu}$'s in a similar manner one obtains

$$
\begin{aligned}
S_q(\underline{k}', \underline{t}, \underline{k}) = {} & f + f^{(1)}\gamma^{(1)}(\underline{t}) + f^{(2)}\gamma^{(2)}(\underline{t}) \\
& + g_\lambda^{(1)}(i\gamma_5^{(1)}\gamma_\lambda^{(1)}) + g_\lambda^{(2)}(i\gamma_5^{(2)}\gamma_\lambda^{(2)}) \\
& + c_\lambda^{(1)}\gamma^{(1)}(\underline{t})(i\gamma_5^{(1)}\gamma_\lambda^{(1)}) + c_\lambda^{(2)}\gamma^{(2)}(\underline{t})(i\gamma_5^{(2)}\gamma_\lambda^{(2)}) \\
& + g_{\lambda\rho}(i\gamma_5^{(1)}\gamma_\lambda^{(1)})(i\gamma_5^{(2)}\gamma_\rho^{(2)}) \\
& + c_{\lambda\rho}\gamma^{(1)}(\underline{t})(i\gamma_5^{(1)}\gamma_\lambda^{(1)})\gamma^{(2)}(\underline{t})(i\gamma_5^{(2)}\gamma_\rho^{(2)}) \\
& + f^{(3)}\gamma^{(1)}(\underline{t})\gamma^{(2)}(\underline{t}) \\
& + e_\lambda^{(1)}\gamma^{(2)}(\underline{t})(i\gamma_5^{(1)}\gamma_\lambda^{(1)}) + e_\lambda^{(2)}\gamma^{(1)}(\underline{t})(i\gamma_5^{(2)}\gamma_\lambda^{(2)}) \\
& + d_\lambda^{(1)}\gamma^{(2)}(\underline{t})\gamma^{(1)}(\underline{t})(i\gamma_5^{(1)}\gamma_\lambda^{(1)}) \\
& + d_\lambda^{(2)}\gamma^{(1)}(\underline{t})\gamma^{(2)}(\underline{t})(i\gamma_5^{(2)}\gamma_\lambda^{(2)}) \\
& + h_{\lambda\rho}^{(1)}(i\gamma_5^{(1)}\gamma_\lambda^{(1)})\gamma^{(2)}(\underline{t})(i\gamma_5^{(2)}\gamma_\rho^{(2)}) \\
& + h_{\lambda\rho}^{(2)}(i\gamma_5^{(2)}\gamma_\lambda^{(2)})\gamma^{(1)}(\underline{t})(i\gamma_5^{(1)}\gamma_\rho^{(1)}) \quad .
\end{aligned}
$$

The coefficients appearing here are functions of $\underline{k}'$, $\underline{t}$, $\underline{k}$ and are pseudovectors and tensors which are orthogonal to $\underline{t}$ on all indices. Thus, for example, $t_\lambda h'_{\lambda\rho} = t_\rho h'_{\lambda\rho} = 0$. Now the first two terms may be transformed into a more suitable form:

$$f + f'\gamma^{(1)}(t) = \sum_\pm \tfrac{1}{2}(1 \pm \gamma^{(1)}(\underline{t}))f^\pm$$

where

$$\tfrac{1}{2}(f^+ + f^-) = f \qquad\qquad \tfrac{1}{2}(f^+ - f^-) = f' \quad .$$

In the same way the rest of the terms may be grouped in pairs to give

$$S_q(\underline{k}', \underline{t}, \underline{k}) = \sum_{\pm} \tfrac{1}{2}(1 \pm \gamma^{(1)}(\underline{t}))\Big\{ f^{\pm} + g_\lambda^{(1)} \; i\,\gamma_5^{(1)}\gamma_\lambda^{(1)}$$

$$+ f^{(2)\pm}\gamma^{(2)}(\underline{t}) + g_\lambda^{(2)\pm}(i\,\gamma_5^{(2)}\gamma_\lambda^{(2)})$$

$$+ s_\lambda^{(2)\pm}\gamma^{(2)}(\underline{t})(i\,\gamma_5^{(2)}\gamma_\lambda^{(2)}) + g_{\lambda\rho}^{\pm}(i\,\gamma_5^{(1)}\gamma_\lambda^{(1)})(i\,\gamma_5^{(2)}\gamma_\lambda^{(2)})$$

$$+ h_{\lambda\rho}^{(1)\pm}(i\,\gamma_5^{(1)}\gamma_\lambda^{(1)})\,\gamma^{(2)}(\underline{t})(i\,\gamma_5^{(2)}\gamma_\rho^{(2)})$$

$$+ e_\lambda^{(1)\pm}\gamma^{(2)}(\underline{t})(i\,\gamma_5^{(1)}\gamma_\lambda^{(1)})\Big\} .$$

Performing the analogous grouping relative to $\gamma^{(2)}(\underline{t})$ one obtains

$$S_q(\underline{k}', \underline{t}, \underline{k}) = \sum_{\pm\pm}\Big\{\tfrac{1}{2}(1 \pm \gamma^{(1)}(\underline{t}))\tfrac{1}{2}(1 \pm \gamma^{(2)}(\underline{t}))\Big\}$$

$$\times\Big\{ f^{\pm\pm} + g_\lambda^{(1)\pm\pm}(i\,\gamma_5^{(1)}\gamma_\lambda^{(1)}) + g_\lambda^{(2)\pm\pm}(i\,\gamma_5^{(2)}\gamma_\lambda^{(2)})$$

$$g_{\lambda\rho}^{\pm\pm}(i\,\gamma_5^{(1)}\gamma_\lambda^{(1)})(i\,\gamma_5^{(2)}\gamma_\rho^{(2)})\Big\} ,$$

where

$$t_\lambda g_\lambda^{(1)\pm\pm} = t_\lambda g_\lambda^{(2)\pm\pm} = t_\lambda g_{\lambda\rho}^{\pm\pm} = g_{\lambda\rho}^{\pm\pm} t_\rho = 0 . \tag{59}$$

The $g_\lambda^{(1)\pm\pm}$ and $g_\lambda^{(2)\pm\pm}$ must be pseudovectors and may therefore be written

$$g_\lambda^{(1)\pm\pm} = g^{(1)\pm\pm} n_\lambda \qquad\qquad g_\lambda^{(2)\pm\pm} = g^{(2)\pm\pm} n_\lambda$$

where n_λ are the components of the only available unit pseudovector, that is,

$$n_\lambda \propto k'_\rho \; k_\sigma \; t_\mu \; \varepsilon_{\rho\sigma\mu\lambda}$$

The tensors $g_{\lambda\rho}^{\pm\pm}$ are, on the other hand, not restricted to a single type of term. The classification of possible tensor terms is facilitated by introducing the normalized vectors

$$s_\lambda = N_s\left[k_\lambda + k'_\lambda - t_\lambda\left\{t_\rho\,(k_\rho + k'_\rho)\right\}(t\cdot t)^{-1}\right]$$

$$d_\lambda = N_d\left[k_\lambda - k'_\lambda\right] \quad .$$

The vectors $\underline{t}$, $\underline{n}$, $\underline{s}$, $\underline{d}$ form an orthogonal set. The condition in Eq. (59) limits the possible terms in the $g_{\lambda\rho}^{\pm\pm}$ to those bilinear in the components of $\underline{n}$, $\underline{s}$ and $\underline{d}$. Invoking the requirement of invariance under spacial reflections the $g_{\lambda\rho}^{\pm\pm}$ reduce to the form

$$\begin{aligned} g_{\lambda\rho}^{\pm\pm} = {} & c^{\pm\pm} n_\lambda n_\rho + d^{\pm\pm} s_\lambda s_\rho + e^{\pm\pm} d_\lambda d_\rho \\ & + g'^{\pm\pm}(s_\lambda d_\rho + d_\lambda s_\rho) \\ & + g^{\pm\pm}(s_\lambda d_\rho - d_\lambda s_\rho) \ . \end{aligned}$$

Just as in the nonrelativistic case the required of invariance under time inversion[10] removes the last two terms since d_λ retains its sign under time inversion whereas s_λ changes sign. Thus the $S_q(\underline{k}', \underline{t}, \underline{k})$ finally takes the form

$$\begin{aligned} S_q(\underline{k}', \underline{t}, \underline{k}) = {} & \sum_{\pm\pm} \Lambda^{(1)\pm}(\underline{t})\, \Lambda^{(2)\pm}(\underline{t}) \\ & \times \Big[f^{\pm\pm} + g^{(1)\pm\pm} i\, \gamma_5^{(1)} \underline{\gamma}^{(1)}\cdot\underline{n} + g^{(2)\pm\pm} i\, \gamma_5^{(2)} \underline{\gamma}^{(2)}\cdot\underline{n} \\ & + (c^{\pm\pm} n_\lambda n_\rho + d^{\pm\pm} s_\lambda s_\rho + e^{\pm\pm} d_\lambda d_\rho) \\ & \times (i\,\gamma_5^{(1)} \gamma_\lambda^{(1)})(i\,\gamma_5^{(2)} \gamma_\rho^{(2)}) \Big] \ . \end{aligned} \tag{60}$$

In a very similar way the density matrix is reduced to the form

$$\rho(\underline{f}, \underline{h}) = \tfrac{1}{4} \operatorname{Tr} \rho(\underline{f}, \underline{h}) \Big[\sum_{\pm\pm} (\Lambda^{(1)\pm}(\underline{f}) \Lambda^{(2)\pm}(\underline{h})) \lambda^{\pm\pm} \times$$

$$\Big[1 + i \gamma_5^{(1)} \gamma_\mu^{(1)} p_\mu^{(1)\pm\pm} + i \gamma_5^{(2)} \gamma_\mu^{(2)} p_\mu^{(2)\pm\pm}$$

$$+ (i \gamma_5^{(1)} \gamma_\lambda^{(1)})(i \gamma_5^{(2)} \gamma_\eta^{(2)}) c_{\lambda\eta}^{\pm\pm} \Big] \tag{61}$$

where $p_\mu^{(1)\pm\pm}$, $p_\mu^{(2)\pm\pm}$ and $c_{\lambda\eta}^{\pm\pm}$ are the polarization and correlation parameters for the four types of systems, and satisfy

$$p_\mu^{(1)\pm\pm} f_\mu = p_\mu^{(2)\pm\pm} h_\mu = c_{\lambda\rho}^{\pm\pm} h_\rho = f_\lambda c_{\lambda\rho}^{\pm\pm} = 0 \quad .$$

These forms for ρ and S_q may now be substituted into

$$\rho'(\underline{f}', \underline{h}') = S(\underline{f}', \underline{h}', \underline{t}, \underline{f}, \underline{h},) \rho(\underline{f}, \underline{h}) \bar{S}(\underline{f}', \underline{h}', \underline{t}, \underline{f}, \underline{h}).$$

The transformations carried out in section three may then be performed upon the matrices in the two spin spaces independently and the equation will split into four equations in the two by two matrices each of which is identical in form to the nonrelativistic equations. The quantities appearing in the places of the nonrelativistic polarization and correlation components will be

$$\tilde{P}_i^{(1)} = P_j^{(1)} r_{ji}(\underline{f}_1)$$

$$\tilde{P}_i^{(2)} = P_j^{(2)} r_{ji}(\underline{h}_1)$$

$$\tilde{P}_i^{\prime(1)} = P_j^{\prime(1)} r_{ji}(\underline{f}_1')$$

$$\tilde{P}_i^{\prime(2)} = P_j^{\prime(2)} r_{ji}(\underline{h}_1')$$

$$\tilde{C}_{ij} = C_{km}\, r_{ki}(\underline{f}_1)\, r_{mj}(\underline{h}_1)$$

$$\tilde{C}'_{ij} = C'_{km}\, r_{ki}(\underline{f}'_1)\, r_{mj}(\underline{h}'_1) \tag{63}$$

where now the superscripts refer to the first or second particle and the $r_{\mu\nu}(\underline{x}_1)$ is defined as

$$r_{\mu\nu}(\underline{x}_1) = a^{-1}_{\mu\eta}(\underline{x})\, a_{\eta\lambda}(\underline{t})\, a_{\lambda\nu}(\underline{x}_1) \quad . \tag{64}$$

The modifications of the nonrelativistic formulas which the relativistic effects introduce are seen, now, to be completely parallel to those obtained when the target had no spin, and the assumption used at the end of section three is valid.

In the treatment of the correlation experiments the relativistic effects on both particles must be considered. In the C_{nn} type of correlation experiment, where the components of polarization perpendicular to the scattering plane are measured the rotations will again play no role. In the C_{KP} experiment the relativistic corrections will not vanish. The application of Eqs. (63) and (64) shows that the expression for the quantity measured in these experiments is in the relativistic region

$$C_{KP} = 4 \operatorname{Re} ich^* - 2 \operatorname{Re} g(a^* - m^*) \sin(\theta_{CM} - 2\,\theta_{Lab}) \quad . \tag{65}$$

PART III

Section 1. Polarization Formalism.

In this section the general formalism for the description of the nonrelativistic scattering of spin one particles by spin zero targets is developed. The treatment is along the same general lines as that used in the treatment of spin one-half particles in the earlier chapters, and is again based upon the use of the density matrix and the M matrix.

The M matrix which describes the scattering of a spin one particle by a target of zero spin will be three by three, and may be written in the following form:

$$M(\theta\phi) = A(\theta\phi) + B_i(\theta\phi)S_i + C_{ij}(\theta\phi)S_{ij} \ . \tag{1}$$

A summation convention is to be understood and i and j run over x, y and z. The S_i are the usual matrices

$$S_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad S_y = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$S_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} ,$$

while

$$S_{ij} \equiv \tfrac{1}{2}(S_i S_j + S_j S_i) - \frac{2}{3} I \delta_{ij} \ .$$

These matrices, together with the unit matrix, form a complete set in the space of three by three matrices. The $C_{ij}(\theta\phi)$ are made unique by imposing the condition that the matrix $C(\theta\phi)$ with elements $C_{ij}(\theta\phi)$ be symmetric and traceless.

The spin state of the scattered beam may be described by the density matrix $\rho(\theta\phi)$ which is defined in terms of ρ_{inc}, the density matrix before the scattering, by

$$\rho(\theta\phi) = M(\theta\phi)\,\rho_{inc}\,\overline{M}(\theta\phi)/\mathrm{Tr}\,\rho_{inc} \quad . \tag{2}$$

$\overline{M}$ is the hermitian conjugate of M. With this definition the differential cross section may be written[11]

$$I(\theta\phi) = \mathrm{Tr}\,\rho(\theta\phi) \tag{3}$$

and the average value of an operator A in the beam scattered by $\theta\phi$ is

$$\langle A\rangle_{\theta\phi} = \mathrm{Tr}\,\rho(\theta\phi)\,A/\mathrm{Tr}\,\rho(\theta\phi) \quad . \tag{4}$$

Using Eq. (3), the expansion of $\rho(\theta\phi)$ in terms of the S_i and S_{ij} may be written

$$\rho(\theta\phi) = I(\theta\phi)(\tfrac{1}{3} + \tfrac{1}{2}\,P_i(\theta\phi)\,S_i + T_{ij}(\theta\phi)\,S_{ij}) \tag{5}$$

where $T_{ij}(\theta\phi)$ will be taken to be traceless. From Eq. (4) one then finds that

$$\langle S_i\rangle_{\theta\phi} = P_i(\theta\phi) \tag{6}$$

$$\langle S_{ij}\rangle_{\theta\phi} = T_{ij}(\theta\phi) \tag{7}$$

where the elementary properties of the S_i and S_{ij} summarized in Table F have been used.

The P_i is therefore a measure of the spin angular momentum in the scattered beam, and will be called the vector polarization. This vector polarization, together with T_{ij}, which will be called the tensor polarization specifies the state of polarization of the particles

in the beam. In an unpolarized beam both P_i and T_{ij} are zero and the density matrix is a multiple of the unit matrix. If the incident beam is unpolarized ρ_{inc} may thus be taken as the unit matrix and

$$\rho(\theta\phi) = \frac{1}{3} M(\theta\phi)\, \bar{M}(\theta\phi) \quad .$$

Eqs. (4), (6) and (7) then give

$$P_i(\theta\phi) = \mathrm{Tr}\, \frac{1}{3} M(\theta\phi)\, \bar{M}(\theta\phi)\, S_i / \mathrm{Tr}\, \frac{1}{3} M(\theta\phi)\, \bar{M}(\theta\phi) \tag{8}$$

$$T_{ij}(\theta\phi) = \mathrm{Tr}\, \frac{1}{3} M(\theta\phi)\, \bar{M}(\theta\phi)\, S_{ij} / \mathrm{Tr}\, \frac{1}{3} M(\theta\phi)\, \bar{M}(\theta\phi) \quad . \tag{9}$$

These equations will be used below.

In a double scattering experiment, the beam which is first scattered thru (θ, ϕ) is then scattered thru a second angle which will be called $\theta'\phi'$. The M matrix corresponding to the second scattering is $M'(\theta'\phi')$ and the density matrix after the second scattering is accordingly

$$\begin{aligned} \rho'(\theta'\phi') &= M'(\theta'\phi')\, \rho(\theta\phi)\, \bar{M}'(\theta'\phi') / \mathrm{Tr}\, \rho(\theta\phi) \\ &= I'(\theta'\phi') \left(\frac{1}{3} + \frac{1}{2} P_i'(\theta'\phi') S_i + T_{ij}'(\theta'\phi')\, S_{ij}\right) \quad . \end{aligned} \tag{10}$$

The differential cross section after the second scattering is then

$$\begin{aligned} I'(\theta'\phi') &= \mathrm{Tr}\, \rho'(\theta'\phi') \\ &= \mathrm{Tr}\, \bar{M}'(\theta'\phi')\, M'(\theta'\phi') \left(\frac{1}{3} + \frac{1}{2} P_i(\theta\phi)\, S_i + T_{ij}(\theta\phi)\, S_{ij}\right) \\ &= \mathrm{Tr}(\bar{M}'(\theta'\phi')\, M'(\theta'\phi')) \\ &\quad \times \left\{ \frac{1}{3} + \frac{1}{2} P_i(\theta\phi) \frac{\mathrm{Tr}\, \frac{1}{3} M'(\theta'\phi')\, M'(\theta'\phi')\, S_i}{\mathrm{Tr}\, \frac{1}{3} M'(\theta'\phi')\, M'(\theta'\phi')} \right. \\ &\quad \left. + T_{ij}(\theta\phi) \frac{\mathrm{Tr}\, \frac{1}{3} M'(\theta'\phi')\, M'(\theta'\phi')\, S_{ij}}{\mathrm{Tr}\, \frac{1}{3} M'(\theta'\phi)\, M'(\theta'\phi)} \right\} \end{aligned} \tag{11}$$

If in analogy to Eqs. (8) and (9) one defines*

$$\tilde{P}'_i(\theta'\phi') \equiv \mathrm{Tr}\,\frac{1}{3}\,\bar{M}'(\theta'\phi')\,M'(\theta'\phi')\,S_i/\mathrm{Tr}\,\frac{1}{3}\,\bar{M}'(\theta'\phi')\,M'(\theta'\phi') \tag{12}$$

$$\tilde{T}'_{ij}(\theta'\phi') \equiv \mathrm{Tr}\,\frac{1}{3}\,\bar{M}'(\theta'\phi')\,M'(\theta'\phi')\,S_{ij}/\mathrm{Tr}\,\frac{1}{3}\,\bar{M}'(\theta'\phi')\,M'(\theta'\phi'); \tag{13}$$

the the differential cross section given in Eq. (11) may be written

$$I'(\theta'\phi') = 3\,I'_0(\theta'\phi')\left[\frac{1}{3} + \frac{1}{2}\,P_i(\theta\phi)\,\tilde{P}'_i(\theta'\phi') + T_{ij}(\theta\phi)\,\tilde{T}'_{ij}(\theta'\phi')\right] \tag{14}$$

where

$$I'_0(\theta'\phi') = \frac{1}{3}\,\mathrm{Tr}\,\bar{M}'(\theta'\phi')\,M'(\theta'\phi'). \tag{15}$$

The $P_i(\theta, \phi)$ and $T_{ij}(\theta, \phi)$ as given by Eqs. (8) and (9) are called the vector and tensor polarizabilities. The $\tilde{P}(\theta\phi)$ and $\tilde{T}(\theta\phi)$ given in Eqs. (12) and (13) may be called the vector and tensor analyzabilities, since they give the degree to which the vector and tensor polarizations of the incident beam affect the differential cross section after the scattering.

$I'_0(\theta'\phi')$ is the differential cross section if P_i and T_{ij}, the vector and tensor polarizations before the scattering, are zero. It will be called the unpolarized differential cross section and is independent of the azimuthal angle. This unpolarized differential cross section, the polarizabilities, and the analyzabilities can be expressed in terms of A, B_i and C_{ij}, the coefficients of the M matrix. Substituting Eq. (1) into Eqs. (15), (8), (9), (12) and (13), respectively, and making use of Table H, one obtains

---

* Notice the new ordering of $\bar{M}\,M$, however.

$$I_0 = \frac{1}{3}(3\,AA^* + 2\,B_i\,B_i^* + C_{ij}\,C_{ji}^*) \tag{16}$$

$$I_0\,P_n = \frac{1}{3}\left[4\,\mathrm{Re}(A\,B_n^*) + 2\,\mathrm{Re}\,C_{ni}\,B_i^* - \mathrm{Im}\,B_i\,B_j^*\,\varepsilon_{ijn} - \mathrm{Im}\,C_{ik}\,C_{kj}^*\,\varepsilon_{ijn}\right] \tag{17}$$

$$I_0\,\tilde{P}_n = \frac{1}{3}\left[4\,\mathrm{Re}(A\,B_n^*) + 2\,\mathrm{Re}\,C_{ni}\,B_i^* + \mathrm{Im}\,B_i\,B_j^*\,\varepsilon_{ijn} + \mathrm{Im}\,C_{ik}\,C_{kj}^*\,\varepsilon_{ijn}\right] \tag{18}$$

$$\begin{aligned} I_0\,T_{mn} = \frac{1}{3}\Big[& 2\,\mathrm{Re}\,AC_{mn}^* + \mathrm{Re}\,B_m\,B_n^* - \frac{1}{3}(B_iB_i^*)\,\delta_{mn} \\ & - \mathrm{Re}\,C_{mi}\,C_{in}^* + \frac{1}{3}(C_{ij}\,C_{ji}^*)\,\delta_{mn} \\ & - \mathrm{Im}(C_{mi}\,B_j^*\,\varepsilon_{ijn}) - \mathrm{Im}(\varepsilon_{mij}\,B_i\,C_{jn}^*)\Big] \end{aligned} \tag{19}$$

$$\begin{aligned} I_0\,\tilde{T}_{mn} = \frac{1}{3}\Big[& 2\,\mathrm{Re}\,AC_{mn}^* + \mathrm{Re}\,B_m\,B_n^* - \frac{1}{3}(B_i\,B_j^*)\,\delta_{mn} \\ & - \mathrm{Re}\,(C_{mi}\,C_{in}^*) + \frac{1}{3}(C_{ij}\,C_{ji}^*)\,\delta_{mn} \\ & + \mathrm{Im}\,(C_{mi}\,B_j^*\,\varepsilon_{ijn}) + \mathrm{Im}\,\varepsilon_{mij}\,B_i\,C_{jn}^*\Big] \end{aligned} \tag{20}$$

where ε_{ijk} is the usual vector product symbol and where primes and angles have been suppressed. One may easily verify that T_{ij} and $\tilde{T}_{ij}$ are indeed symmetric and traceless.

It will be noticed that the polarizabilities and analyzabilities are not identical. The terms which are different in the vector polarizability and analyzability will vanish, however, when the restrictions on the form of the M matrix which are implied by spacial symmetry and invariance under time reversal are imposed. On the other hand, the

tensor polarizability and analyzability will not become identical. To obtain these results the invariance arguments of Wolfenstein and Ashkin may be used to show that

$$A(\theta\phi) = a(\theta)$$
$$B_i(\theta\phi) = b(\theta)\, N_i \,, \tag{21}$$

while $C_{ij}(\theta\phi)$ must be a linear combination of the terms

$$C_N(\theta) \quad (N_i\, N_j - \tfrac{1}{3}\, \delta_{ij})$$
$$C_P(\theta) \quad (P_i\, P_j - \tfrac{1}{3}\, \delta_{ij})$$
$$C_K(\theta) \quad (K_i\, K_j - \tfrac{1}{3}\, \delta_{ij}) \,.$$

Here

$$\vec{N} = \vec{k}_{in} \times \vec{k}_{out} / |\vec{k}_{in} \times \vec{k}_{out}|$$
$$\vec{P} = \vec{k}_{out} + \vec{k}_{in} / |\vec{k}_{out} + \vec{k}_{in}|$$
$$\vec{K} = \vec{k}_{out} - \vec{k}_{in} / |\vec{k}_{out} - \vec{k}_{in}|$$

where the vectors $\hbar\, \vec{k}_{in}$ and $\hbar\, \vec{k}_{out}$ are the incident and final momenta. Since $(N_i\, N_j + P_i\, P_j + K_i\, K_j) = \delta_{ij}$, the matrix C_{ij} can be written as

$$C_{ij} = c(\theta)\, (N_i\, N_j - \tfrac{1}{3}\, \delta_{ij}) + d(\theta)\, (P_i\, P_j - K_i\, K_j)$$

and the M matrix may be expressed by

$$M(\theta\phi) = a(\theta) + b(\theta)\, N_i\, S_i + \Big\{ c(\theta)(N_i\, N_j - \tfrac{1}{3}\, \delta_{ij}) + d(\theta)\, (P_i\, P_j - K_i\, K_j) \Big\}\, S_{ij} \,. \tag{22}$$

The scalar coefficients $a(\theta)$, $b(\theta)$, $c(\theta)$, and $d(\theta)$ give a complete description of the scattering. The polarizability, analyzability and the differential cross sections may be expressed in terms of them. Carrying out the matrix multiplications in Eqs. (16) - (20) one obtains

$$I_0 = a\,a^* + \frac{2}{3}\,b\,b^* + \frac{2}{9}\,c\,c^* + \frac{2}{3}\,d\,d^* \tag{23}$$

$$I_0\,P_i = I_0\,\tilde{P}_i = \frac{2}{3}\left[2\,\mathrm{Re}\,b(a + \tfrac{1}{3}\,c)\right] N_i \tag{24}$$

$$\begin{aligned} I_0\,T_{ij} = \frac{1}{3}\Big[& \Big\{(a + \tfrac{1}{3}\,c)\,c^* + (a + \tfrac{1}{3})^*\,c - cc^* + dd^* + bb^*\Big\}(N_i N_j - \delta_{ij}) \\ & + \Big\{(a + \tfrac{1}{3}\,c)d^* + (a + \tfrac{1}{3}\,c)^*\,d\Big\}\,(P_i\,P_j - K_i\,K_j) \\ & + 2\,\mathrm{Im}\,db^*\,(P_i\,K_j + K_i\,P_j)\Big] \end{aligned} \tag{25}$$

$$I_0\,\tilde{T}_{ij} = \text{same except for sign of last term.} \tag{26}$$

These equations, when substituted into Eq. (14) will give the differential cross section after the second scattering, which is the quantity measured in the polarization experiments. Abbreviating Eqs. (24), (25) and (26) by

$$P_i = \beta\,N_i = \tilde{P}_i \tag{27}$$

$$T_{ij} = \eta(N_i\,N_j - \tfrac{1}{3}\,\delta_{ij}) + \kappa\,(P_i\,P_j - K_i\,K_j) + \tau\,(P_i\,K_j + K_i\,P_j) \tag{28}$$

$$\tilde{T}_{ij} = \eta\,(N_i\,N_j - \tfrac{1}{3}\quad{}_{ij}) + \kappa\,(P_i\,P_j - K_i\,K_j) - \tau\,(P_i\,K_j + K_i\,P_j)\ , \tag{29}$$

the expression in Eq. (14) for the differential cross section after the

second scattering becomes*

$$I' = I_0' \Big[1 + \frac{3}{2} \beta \beta' N_i N_i' \; 3 \Big\{ \eta (N_i N_j - \frac{1}{3} \delta_{ij}) + \kappa (P_i P_j - K_i K_j) + \tau (P_i K_j + K_i P_j) \Big\} \times \Big\{ \eta' (N_j' N_i' - \frac{1}{3} \delta_{ji}) + \kappa' (P_j' P_i' - K_j' K_i') - \tau' (P_j' K_j' + K_j' P_i') \Big\} \Big] \quad . \tag{30}$$

Upon performing the matrix multiplication this reduces to

$$I' = I_0' (1 + \tfrac{1}{2} tt' + \tfrac{2}{3} (u u' - v v') \cos \phi' + \tfrac{1}{6} w w' \cos 2 \phi') \tag{31}$$

where ϕ' is the azimuthal angle for the second scattering in the coordinate system in which the intermediate beam moves in the z direction and the normal for the first scattering is along the y axis. The t, u, v, w and I_0 are functions of the type of target, the energy and the scattering angle θ and the primes denote the second scattering. The coefficients in the equation are

$$t(\theta) = 3 \kappa \cos \theta + 3 \tau \sin \theta - \eta$$

$$u(\theta) = \frac{3}{2} \beta = \Big\{ 2 \operatorname{Re} b (a + \frac{1}{3} c)^* \Big\} I_0^{-1}$$

$$v(\theta) = 3 \kappa \sin \theta - 3 \tau \cos \theta$$

$$w(\theta) = - (3 \kappa \cos \theta + 3 \tau \sin \theta + 3 \eta) \tag{32}$$

---

* It will be noticed that since $\vec{k}_{out} = \vec{k}_{in}'$, the first $\{ \quad \}$ expression in Eq. (30) has the same functional dependence upon $\vec{k}_{in}$ that the second $\{ \quad \}$ expression has upon $\vec{k}_{out}'$.

where

$$3\,\kappa\, I_0 = 2\,\mathrm{Re}\, d(a + \tfrac{1}{3}c)^*$$

$$3\,\tau\, I_0 = 2\,\mathrm{Im}\, d\, b^*$$

$$3\,\eta\, I_0 = 2\,\mathrm{Re}\, c(a + \tfrac{1}{3}c)^* + dd^* + bb^* - cc^*$$

$$I_0 = aa^* + \tfrac{2}{3}bb^* + \tfrac{2}{9}cc^* + \tfrac{2}{3}dd^* . \qquad (33)$$

Eqs. (31), (32) and (33) give the explicit expression of the differential cross section after the second scattering in terms of the fundamental coefficients $a(\theta)$, $b(\theta)$, $c(\theta)$ and $d(\theta)$. The general form of the differential cross section after the second scattering given in Eq. (31) has also been derived by Lakin.[14]* In his method the explicit form of the M matrix is not used. He applies the invariance arguments directly to the quantity $M\bar{M}$ and obtains the form given in Eq. (31) where, however, the t, u, v, and w are given as the expectation values of certain operators after the scattering of an initially unpolarized beam. In particular he finds

$$t = 3\langle S_{zz}\rangle_{\theta\phi} = 3\,T_{zz}$$

$$u = \tfrac{3}{2}\langle S_y\rangle_{\theta\phi} = \tfrac{3}{2}P_y$$

$$v = -3\langle S_{xz}\rangle_{\theta\phi} = -3\,T_{xz}$$

$$w = 3\langle S_{xx} - S_{yy}\rangle_{\theta\phi} = 3\,T_{xx} - 3\,T_{yy} \qquad (34)$$

where z and y are the directions of the outgoing beam and the normal to the scattering plane respectively. One may easily verify these equations, here, by using Eqs. (27) and (28). Thus Eq. (27) says

* However the sign of one term in Lakin's formula is in error.

$P_y = \vec{P}\cdot\vec{N} = \beta$, and the u equations of (32) and (34) are equivalent. The T of Eq. (28) can be expressed as

$$T = \eta(E_{yy} - \tfrac{1}{3}) + (-(\sin\theta)\kappa + (\cos\theta)\tau)(E_{xz} + E_{zx})$$
$$((\cos\theta)\kappa + (\sin\theta)\tau)(E_{zz} - E_{xx}) \tag{35}$$

where E_{ij} are the unit tensors (i.e., $T \equiv \sum T_{ij} E_{ij}$). Using this expression one easily obtains the agreement between Eqs. (32) and (34).

Section 2. Analysis of the Experimental Data in Terms of the First Born Approximation.

The developments in section one apply to the most general type of interaction. It is of interest to see the form which the M-matrix will take if certain assumptions regarding the interaction are introduced. A common assumption is that the interaction between the deuteron and the spin zero target is the sum of the interactions of the individual nucleons which comprise the deuteron with the target. If these latter interactions are each the sum of a spin independent interaction and a spin-orbit interaction then the hamiltonian for the interaction of the deuteron with the target will, in lowest order, also be a sum of only a central force and a spin-orbit interaction.* Here the D-state contribution to the deuteron wave function is considered as a higher order effect. This and other higher order effects will be discussed in section four. Various authors[16,17, 18] have suggested that the radial dependence of the spin-orbit force be taken as the derivative of the spin independent potential. The interaction hamiltonian then takes the form

$$H = \left[U e^{i\delta} + \frac{V}{2} \left(\frac{\hbar c}{Mc^2}\right)^2 \vec{S}\cdot\vec{k} \times \nabla_f \right] f(r) \tag{36}$$

where M is the mass of the deuteron; $f(r)$ is the radial function normalized to $f(0) = -1$; $Ue^{i\delta}$ is the well depth of the central potential, where U is positive and real; V is the spin orbit well depth and may be written $g\,u\cos\delta$, where $g = 1$ corresponds to the pure Thomas term.[15] The ∇_f operates only on $f(r)$ whereas $\hbar k$ is the usual momentum operator. It is assumed that the phase δ which specifies the imaginary part of the potential is independent of position.

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* See Section IV below.

The matrix element of H between momentum states is

$$\langle H \rangle = \left[U e^{i\delta} + i \frac{V}{2} \vec{S}\cdot\vec{N} \sin\theta \left(\frac{pc}{Mc^2}\right)^2 \right] f(K) \tag{37}$$

where

$$\vec{N} = \vec{k}_{in} \times \vec{k}_{out} / \left|\vec{k}_{in} \times \vec{k}_{out}\right| , \tag{38}$$

$$K = \left|\vec{K}\right| = \left|\vec{k}_{in} - \vec{k}_{out}\right| , \tag{39}$$

$$P = \hbar \left|\vec{k}_{in}\right| = \hbar \left|\vec{k}_{out}\right|$$

and

$$f(K) = \int d\vec{r}\, f(r)\, e^{i\vec{K}\cdot\vec{r}} . \tag{40}$$

In the first Born approximation the M matrix is just a multiple of $\langle H \rangle$;

$$M(\theta\phi) = \frac{-2M}{4\pi\hbar^2} \langle H \rangle .$$

In this approximation the M-matrix given by Eq. (22) contains, therefore, only the spin independent and the vector type term; the $c(\theta)$ and $d(\theta)$ are zero. With the help of Eqs. (32) and (33) the vv' contribution to the differential cross section is found to vanish and the tt' and ww' contributions are of second order in both $\sin\theta$ and $\sin\theta'$ and will be expected to be small when either scattering angle is small. The experiments[13] have been unable to detect any contributions of these two types in the differential cross section.

If these terms are neglected the asymmetry defined as

$$e = (I(\phi=0) - I(\phi=\pi))/(I(\phi=0) + I(\phi=\pi))$$

may be expressed as $e = P_e P_e'$ where

$$P_e = \sqrt{\frac{2}{3}}\left(2 \text{ Re } ab^* / (aa^* + \tfrac{2}{3} bb^*)\right)$$

$$= \frac{\frac{2}{3} \; g \cos\delta \; \sin\delta \left(\frac{pc}{Mc^2}\right)^2 \sin\theta}{1 + \frac{1}{6}\left(g^2 \cos^2\delta \left(\frac{pc}{Mc^2}\right)^4 \sin^2\theta\right)} \tag{41}$$

and P_e' is the same function of the primed variables. The maximum value of P_e is $\sin\delta$, and it is obtained when the denominator is two. In so far as this approximation is valid the factor g is fixed by the slope of P_e at zero degrees and by the value of $(P_e)_{max} = \sin\delta$. Eq. (41) may be solved to give

$$g = \frac{(\sqrt{3/2})}{\cos\delta \; \sin\delta} \left(\frac{Mc^2}{pc}\right)^2 \left(\frac{dP_e}{d\theta}\right)_{\theta=0} . \tag{42}$$

In the polarization of 165 Mev deuterons by aluminum[13] the experimental values of P_e at angles less than 20° are consistent with a straight line fit passing through the origin. Using the value $P_e(18^\circ) = 46\%$ and $P_e(\max) = 85\%$ the value of g given by Eq. (42) is 23.5. A similar analysis can be carried out for proton scattering and one obtains as the analog of Eq. (42)

$$g_p = \frac{2}{\cos\delta \; \sin\delta} \left(\frac{mc^2}{pc}\right)\left(\frac{d\,P_e}{d\theta}\right)_{\theta=0} \tag{43}$$

where the small m is the proton mass. The g_p value given by the 300 Mev proton polarization data [1] is 20. The g value would appear, then, to be roughly the same for these two cases in which the proton energies differ by a factor of four. The value $g \sim 20$ is also

consistent with the low energy limit associated with the spin orbit coupling in the shell model.[18]

Using the values of $\sin\delta$ and g obtained above one may obtain estimates for the coefficients t and w which determine the $\cos 0\phi$ and $\cos 2\phi$ terms in the differential cross section. From Eqs. (32), (33) and (37), one finds

$$w = -bb^* / aa^* + \frac{2}{3} bb^* = 3t$$

$$= \frac{-g^2 \cos^2\delta\,(pc/Mc^2)^4 \sin^2\theta}{4 + \frac{2}{3} g^2 \cos^2\delta\,(pc/Mc^2)^4 \sin^2\theta}$$

$$\cong -.96 \sin^2\theta/(1 + .64 \sin^2\theta) \quad .$$

As a representative example the first and second scattering angles may be taken as 10°. One then finds

$$\frac{1}{6} ww' \cos 2\phi \cong \cos 2\phi/(6000)$$

$$\frac{1}{2} tt' \cong 1/(18000) \; .$$

The application of the theory developed in section one to the case of an aluminum target, as above, is not strictly permissible since the aluminum nucleus does not have spin zero. For carbon, however, to which the theory should apply, the experimental data does not agree at all well with the results of the Born approximation developed here. The polarization at angles less than 20° does not fit the predictions of the Born approximation, a sharp rise occurring between 20° and 24° in the experiments. The failure of the Born approximation to represent the experimental data may be due in part to an incorrect form for the

interaction. Even if this is correct, however, investigations[17] on the scattering of nucleons have shown that the Born approximation usually gives only the qualitative aspects of the results provided by more exact calculations. In the present case it is not clear that even the qualitative aspects will be provided by this approximation, for the approximation imposes upon the M-matrix a very special form which it is not in general required to have. In the treatment of polarization effects, particularly, the presence of the tensor terms in the M-matrix can be expected to have important consequences. These tensor terms will arise both from the higher order Born approximations based upon the interaction hamiltonian given in Eq. (36) and also from the inclusion of the D-state effects in the form of that hamiltonian. In order to obtain some basis for estimating these effects the contributions to the M-matrix from the second order Born approximation and from the inclusion of the deuteron D-state have been calculated and are discussed in the following two sections.

Section 3. The Second Born Approximation.

In the second Born approximation the M-matrix may be expressed as:

$$\mathcal{M} = \left(\frac{-2\,M}{4\pi\hbar^2} \right) \left[H_{fi} + H_{fm} \frac{1}{E - E_m + i\varepsilon} H_{mi} \right]$$

where the subscripts i, m, and f represent the initial, intermediate and final momentum states, and H_{ij} is the matrix element of the interaction hamiltonian given by Eq. (36). If the form factor f(r) is taken for simplicity to be $(-e^{-r^2/r_0^2})$, the various integrations may be performed and one obtains

$$\begin{aligned} \mathcal{M} = \left(\frac{-2\,M}{4\pi\hbar^2} \right) e^{-\alpha} \Bigg[& A + i\,B\,\vec{S}\cdot\vec{N}\,\sin\theta + A^2\tau \left(\frac{k^2\,T_1}{r_0^2\,a} \right) e^{\frac{1}{2}\alpha} \\ & + i\,B^2\,\tau\,\vec{S}\cdot\vec{N}\,(2\sin\theta/2) \left(\frac{r_0^2\,ak\,T_2 - k\,T_1}{(r_0^2\,a)^2} \right) e^{\frac{1}{2}\alpha} \\ & + B^2\tau \left(\frac{(2\Lambda_x + \Lambda_y + \Lambda_z)(r_0^2\,a\,T_2 - T_1)}{(r_0^2\,a)^3} - \frac{\Lambda_x\,T_3}{(r_0^2\,a)} \right) e^{\frac{1}{2}\alpha} \Bigg] \end{aligned} \tag{44}$$

where the following abbreviations have been used:

$$A \equiv -(r_0\sqrt{\pi})^3\,U\,e^{i\delta}$$

$$B \equiv -(r_0\sqrt{\pi})^3\,g\,U(\cos\delta)\,\tfrac{1}{2}(pc/Mc^2)^2$$

$$\alpha \equiv r_0^2\,k^2\sin^2\theta/2$$

$$\tau \equiv \frac{(Mc^2)}{2\pi^2 (pc)^2}$$

$$a \equiv k \cos \theta/2$$

$$\Lambda_x \equiv \sin^2 \theta/2 \quad S_N S_N$$

$$\Lambda_y \equiv \cos^2 \theta/2 \quad S_N S_N$$

$$\Lambda_z \equiv \cos^2 \theta/2 \quad S_K S_K - \sin^2 \theta/2 \quad S_P S_P - i \sin \theta/2 \cos \theta/2 \quad S_N$$

$$T_n \equiv \int_{-\infty}^{\infty} \frac{\rho^n \, d\rho \, e^{-\lambda^2(\rho - a)^2}}{k^2 - \rho^2 + i\varepsilon} . \tag{45}$$

The T_n may be expressed in terms of the tabulated[24] functions

$$F(x) = e^{-x^2} \left[\int_0^x e^{t^2} \, dt - \frac{i\sqrt{\pi}}{2} \right]$$

by

$$T_1 = \sqrt{\pi} \left\{ F(\lambda k - \lambda a) - F(\lambda k + \lambda a) \right\}$$

$$T_2 = \sqrt{\pi} \left\{ -\lambda^{-1} + k F(\lambda k - \lambda a) + k F(\lambda k + \lambda a) \right\}$$

$$T_3 = \sqrt{\pi} \left\{ -a\lambda^{-1} + k^2 F(\lambda k - \lambda a) - k^2 F(\lambda k + \lambda a) \right\}$$

where $\lambda = (r_0/\sqrt{2})$.

The real part of $F(x)$ has the asymptotic forms $F(x) \simeq x$ as $x \to 0$ and $F(x) \simeq \frac{1}{2} x^{-1}$ as $x \to \infty$, which are approximately correct for the regions $x \lesssim .3$ and $x \gtrsim 4$ respectively. In the intermediate region $F(x)$ rises to a maximum of about .542 near $x = .9$. For $\theta < 35^\circ$ the asymptotic forms may be inserted for

$F(\lambda k \pm \lambda a)$. If, in addition, certain small terms in $(1 - \cos\theta/2)$ and $(r_0 k)^{-2}$ are dropped, the M-matrix becomes

$$\cong \left(\frac{-2M}{4\pi\hbar^2}\right) e^{-\alpha}\Bigg[A + iB(\vec{S}\cdot\vec{N})\sin\theta$$

$$+ A\left(\frac{A\tau\sqrt{\pi}}{r_0^3}\right)\left(\frac{-1}{2\sqrt{2}} + \frac{r_0^2 k(k-a)}{\sqrt{2}} - \frac{(i\sqrt{\pi})\, r_0 k}{2}\right) e^{\frac{1}{2}\alpha}$$

$$+ iB\left(\frac{A\tau\sqrt{\pi}}{r_0^3}\right)(\vec{S}\cdot\vec{N})(2\sin\theta/2)\left(\frac{-3}{2\sqrt{2}} + \frac{r_0^2 k(k-a)}{\sqrt{2}} - \frac{(i\sqrt{\pi})r_0 k}{2}\right) e^{\frac{1}{2}\alpha}$$

$$+ B\left(\frac{B\tau\sqrt{\pi}}{r_0^3}\right)\Bigg\{\frac{(2\Lambda_x + \Lambda_y + \Lambda_z)}{(r_0 k)^2}\left(\frac{-3}{2\sqrt{2}} + \frac{r_0^2 k(k-a)}{\sqrt{2}} - \frac{(i\sqrt{\pi})r_0 k}{2}\right)$$

$$- \Lambda_x\left(\frac{-5}{2\sqrt{2}} + \frac{r_0^2 k(k-a)}{2} - \frac{(i\sqrt{\pi})r_0 k}{2}\right)\Bigg\} e^{\frac{1}{2}\alpha}\Bigg] \tag{46}$$

For 165 Mev deuterons on carbon the value of k is about four in units of 10^{13} cm^{-1}, while a reasonable choice for r_0 is[17] 1.9 in units of 10^{-13} cm^{-1}. Since then $(r_0 k) \simeq 7.6$, terms which were smaller by a factor $(r_0 k)^{-2}$ than others of the same form were neglected in Eq. (46). The tensor parts of the M-matrix are contained in the last term of Eq. (46). They are of a different character in the regions of large and small scattering angles, with the division coming at $\theta \sim 15^\circ$ (i.e., $r_0 k \sin\theta/2 = 1$). For small angles the dominant term comes from the $(\Lambda_y + \Lambda_z)$ contribution and is a multiple of $S_P S_P$. With this form of the tensor term the coefficients $c(\theta)$ and $d(\theta)$

are equal in magnitude but have opposite signs. Eqs. (32) and (33) then show that the w^2, and hence the $\cos 2\phi$ dependence, will, for small θ, not be affected in the order considered here. The main contribution to $d(\theta)$ will be negative imaginary and the tensor contributions to t will tend to cancel the vector contributions coming from the first Born approximation. These latter terms are small of order $\sin^2\theta$ whereas the tensor contributions are small of order $(U\tau/r_0 k)$. With $U = 85$ Mev* and $g = 20$ the tensor contributions to this term will be the dominant ones in the small angle region with the two terms canceling at $\sim 15^\circ$. In the region of large θ the dominant tensor term will come from the Λ_x contribution in Eq. (46). This is a term of the type $S_N S_N$ and $d(\theta) = 0$. The chief contribution to $c(\theta)$ will be negative imaginary and it will combine constructively in the expression $I_0 u = 2 \operatorname{Re} b(a + \frac{1}{3} c)^*$ with the imaginary part of a coming from the first Born approximation and the polarization will be enhanced. A much larger enhancement, however, will come from the second order contribution to b. The largest second order term in b will combine with the first order contribution in a to give maximum polarization independent of the phase angle δ. The interference between the first order term in b and the second order term in a may increase or decrease the polarization depending on the magnitude of δ. The second Born approximation indicates that the sudden increase in polarization around $\sim 12^\circ$ would more likely be due to these higher order effects in a and b rather than effects of the tensor coefficients c and d which

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* This value of U was used by Fernbach, Heckrotte and Lepore in their W.K.B. calculations of the proton-carbon polarization effects. Their cross section was too large by a factor of about two in the region between 11° and 19°. But the real part of the deuteron potential should be double the real part of the proton or neutron potential.

are quite small. The vanishing of $d(\theta)$ in the region $> 15^{\circ}$ means that there the t and w are both determined by the single parameter η and that, as in the first Born approximation, the relation $\frac{1}{2}\, tt' = \frac{1}{3}\left(\frac{1}{6}\, ww'\right)$ will still obtain. The sign of c is such that the η, and hence the $\cos 2\phi$ asymmetry, will be decreased in the region in which these tensor terms become important.

Section 4. Deuteron D-State Contributions.

In this section the hamiltonian describing the interaction of the deuteron with the spin zero target is calculated from the basic interactions between the two individual nucleons and the spin zero target which is taken to be a fixed scattering center. These will both be assumed to take the form

$$H'(r_i) = \frac{1}{2}\left[U + \frac{V}{2}\left(\frac{\hbar}{m\,c}\right)^2 \frac{1}{2}\,\vec{\sigma}^{(i)}\cdot\vec{k}^{(i)} \times \vec{\nabla}_f \right] f(r_i) \quad . \tag{47}$$

The indices $i = 1,2$ refer to the two nucleons, m is the mass of the nucleons and U, V and $f(r)$ are the same as in section two. With the introduction of the center-of-mass and relative coordinates

$$R = \tfrac{1}{2}(r_1 + r_2) \qquad\qquad r = (r_1 - r_2) \tag{48}$$

and the corresponding conjugate momenta

$$P = (p_1 + p_2) \qquad\qquad p = \tfrac{1}{2}(p_1 - p_2) \tag{49}$$

the total hamiltonian reduces to the form

$$H_{TOT} = H_0(R) + H_D(r) + H(R, r) \tag{50}$$

where $H_0(R) + H_D(r)$ is the hamiltonian for the free deuteron; $H_D(r)$ is the hamiltonian for the internal coordinates of the deuteron and

$$H(R, r) = \frac{1}{2}\,U(f(r_1) + f(r_2)) + \frac{V}{2}\left(\frac{\hbar}{Mc}\right)^2 \vec{\sigma}^{(1)}\cdot\vec{k}^{(1)} \times \vec{\nabla}(f(r_1))$$

$$+ \frac{V}{2}\left(\frac{\hbar}{Mc}\right)^2 \vec{\sigma}^{(2)}\cdot\vec{k}^{(2)} \times \vec{\nabla}(f(r_2)) \quad . \tag{51}$$

The expression appearing on the right-hand side of Eq. (51) is to be considered as a function of $\vec{R}$, $\vec{r}$ and the corresponding gradient operators. It is, of course, an operator in the spin space of the two nucleons.

The deuteron eigenfunctions are

$$\Psi^{M,K}(R, r) = e^{i\vec{K}\vec{R}}\psi^{M}(r) \tag{52}$$

where $\psi^{M}(r)$, the relative coordinate part of the deuteron wave function, satisfies

$$H_D(r)\psi^{M}(r) = E_D\psi^{M}(r) \quad ,$$

E_D being the deuteron energy. These $\psi^{M}(r)$ may be written

$$\psi^{M}(r) = \frac{1}{r\sqrt{4\pi}}\left(u(r) + w(r)\frac{S_{12}}{\sqrt{8}}\right)\chi^{M} \tag{53}$$

where the $\chi^{M} = Y^{M}_{101}(4\pi)^{\frac{1}{2}}$ are the usual triplet spin state vectors. The Y^{M}_{JLS} are the spin angle vectors defined as in Blatt and Weisskopf.[25] S_{12} is the tensor operator

$$S_{12} = 3(\vec{\sigma}^{(1)}\cdot\vec{r})(\vec{\sigma}^{(2)}\cdot\vec{r})(r)^{-2} - \vec{\sigma}^{(1)}\cdot\vec{\sigma}^{(2)}$$

and the spin angle function for the D-state may be written[26]

$$Y^{M}_{121} = \frac{S_{12}}{\sqrt{8}}\,Y^{M}_{101} \quad . \tag{54}$$

The Y^{M}_{JLS} also satisfy

$$\frac{S_{12}}{\sqrt{8}}\,Y^{M}_{121} = Y^{M}_{101} - \frac{1}{\sqrt{2}}\,Y^{M}_{121} \quad . \tag{55}$$

The $u(r)$ and $w(r)$ are the radial S and D wave functions and

$$\int_0^\infty dr\, u^2(r) = (\text{S - state probability}) \cong .96$$

$$\int_0^\infty dr\, w^2(r) = (\text{D - state probability}) \cong .04 .$$

In the first order Born approximation the M matrix for the scattering of the deuteron from the initial state Ψ^{MK} to the final state $\Psi^{M'K'}$ will be proportional to the matrix element

$$H^{M'K';MK}(R,r) \equiv \int d^3R\, d^3r\, \Psi^{*M'K'}(R,r)\, H(R, r)\, \Psi^{MK}(R\ r)$$

$$= \int d^3R\, e^{-iK'R}\, H^{M'M}(R)\, e^{iKR}$$

where

$$H^{M'M}(R) = \int d^3r\, \psi^{*M'}(r)\, H(R, r)\, \psi^{M}(r). \tag{56}$$

The matrix $H(R)$ with elements $H^{M'M}(R)$ is, therefore, the effective interaction hamiltonian for the collision for the deuteron with the spin zero target. More precisely, it is the effective interaction hamiltonian in the first Born approximation. In a higher order calculation it would be necessary, of course, to sum not only over all K, the total momenta of the deuteron in the intermediate states, but also over all of the unbound states in the internal variables. Were it not for this latter summation, the effective hamiltonian, $H^{M'M}(R)$, would be the exact hamiltonian for the interaction of the deuteron with the spin zero target.

As the first step in the calculation one may convert $H(R, r)$ into a function of the proper variables. This function is

$$H(R, r) = \tfrac{1}{2}(-U)\, e^{-R^2/r_0^2}\, e^{-r^2/4r_0^2}\, 2\cosh(R\, r\mu/r_0^2)$$

$$+ \frac{V}{2}\left(\frac{\hbar}{(Mc)^2}\right) e^{-R^2/r_0^2}\, e^{-r^2/4r_0^2}$$

$$\times \frac{1}{r_0^2}\Big\{ \tfrac{1}{2}(\vec{\sigma}_1 + \vec{\sigma}_2)\cdot \Big[(\vec{P} \times \vec{R} + \vec{p} \times \vec{r})(2\cosh Rr\mu/r_0^2$$

$$-(2\,\vec{p} \times \vec{R} + \tfrac{1}{2}\,\vec{P} \times \vec{r})(2\sinh Rr\mu/r_0^2)\Big]$$

$$+ \tfrac{1}{2}(\vec{\sigma}_1 - \vec{\sigma}_2)\cdot \Big[(2\,\vec{p} \times \vec{R} + \tfrac{1}{2}\,\vec{P} \times \vec{r})(2\cosh Rr\mu/r_0^2)$$

$$-(\vec{P} \times \vec{R} \quad \vec{p} \times \vec{r})(2\sinh Rr\mu/r_0^2)\Big]$$

In order to proceed with the calculation of $H(R)$ it is convenient to represent the functions $u(r)/r$ and $w(r)/r^3$ in the form

$$\frac{u(r)}{r} = A(r^2) = \int_0^\infty a(\xi)\, e^{-r^2\xi}\, d\xi$$

$$\frac{w(r)}{r^3} = B(r^2) = \int_0^\infty b(\xi)\, e^{-r^2\xi}\, d\xi$$

where $a(\xi)$ and $b(\xi)$ are the inverse Laplace transforms of A and B. A somewhat lengthy calculation then shows that $H(R)$ may be written

$$H(R) = -\int_0^\infty d\xi \int_0^\infty d\eta \Big[U\, A(R) + \frac{V}{4}\left(\frac{\hbar}{Mc}\right)^2 \Big\{ \vec{S} \times \vec{K},\ \vec{\nabla}\, A(R) \Big\}_+ \tag{57}$$

$$+ (\tfrac{3}{2}\, S_R\, S_R - 1)\ \frac{V}{2}\left(\frac{\hbar}{Mc}\right)^2 \left(\xi\, B(R) + \frac{C(R)}{(r_0^2 + D^2)} \right)$$

$$+ \frac{V}{4}\left(\frac{\hbar}{Mc}\right)^2 \Big\{ S \times K,\ \frac{\vec{R}\ E(R)}{(r_0^2 + D^2)} \Big\}_+ + \frac{V}{2}\left(\frac{\hbar}{Mc}\right)^2 \frac{F(R)}{(r_0^2 + D^2)} \Big]$$

where the following abbreviations are used:

$$A(R) = e^{-R^2/(r_0^2 + D^2)} \left(\frac{r_0^2}{r_0^2 + D^2} \right)^{3/2} \Bigg[a_1(\xi)\, a_1(\eta)$$

$$+ b_1(\xi)\, b_1(\eta) \left\{ \frac{r_0^4}{(r_0^2 + D^2)^2} + \frac{4}{3} \frac{R^2 r_0^2 D^2}{(r_0^2 + D^2)^3} + \frac{4}{15} \frac{R^4 D^4}{(r_0^2 + D^2)^4} \right\}$$

$$+ b_1(\xi)\, a_1(\eta)\, (\tfrac{3}{2} S_R S_R - 1) \left\{ \frac{8}{\sqrt{30}} \frac{R^2 D^2}{(r_0^2 + D^2)^2} \right\}$$

$$- b_1(\xi)\, b_1(\eta)\, (\tfrac{3}{2} S_R S_R - 1) \left\{ \frac{14}{15} \frac{R^2 r_0^2 D^2}{(r_0^2 + D^2)^3} + \frac{4}{15} \frac{R^4 D^4}{(r_0^2 + D^2)^4} \right\}$$

$$B(R) = e^{-R^2/(r_0^2 + D^2)} \left(\frac{r_0^2}{r_0^2 + D^2} \right)^{3/2} a_1(\xi)\, b_1(\eta)\, \frac{128}{\sqrt{30}} \frac{R^2 D^2}{(r_0^2 + D^2)^2}$$

$$C(R) = e^{-R^2/(r_0^2 + D^2)} \left(\frac{r_0^2}{r_0^2 + D^2} \right)^{3/2} b_1(\xi)\, b_1(\eta)$$

$$\times \left[-\frac{12}{5} \left(\frac{R^2}{r_0^2 + D^2} \right) - \frac{8}{5} \frac{R^4 D^2}{(r_0^2 + D^2)^3} + \frac{28}{5} \frac{R^2 D^2}{(r_0^2 + D^2)^2} \right]$$

$$E(R) = e^{-R^2/(r_0^2+D^2)} \left(\frac{r_0^2}{r_0^2+D^2}\right)^{3/2} b_1(\xi)\, b_1(\eta)$$

$$\times \; (-18/4) \left[\frac{-r_0^4}{(r_0^2+D^2)^2} + \frac{4}{3}\,\frac{r_0^2\, D^2}{(r_0^2+D^2)^2} - \frac{4\, R^2\, r_0^2\, D^2}{3\,(r_0^2+D^2)^3} \right.$$

$$+ \frac{8}{15}\,\frac{R^2\, D^4}{(r_0^2+D^2)^3} - \frac{4}{15}\,\frac{R^4\, D^4}{(r_0^2+D^2)^4} - \frac{1}{3}\,\frac{r_0^2}{(r_0^2+D^2)}$$

$$\left. - \frac{2}{15}\,\frac{R^2\, D^2}{(r_0^2+D^2)^2} \right]$$

$$F(R) = e^{-R^2/(r_0^2+D^2)} \left(\frac{r_0^2}{r_0^2+D^2}\right)^{3/2} b_1(\xi)\, b_1(\eta)$$

$$\times \left[\frac{4\, R^2}{(r_0^2+D^2)} - \frac{8\, R^2\, D^2}{(r_0^2+D^2)^2} - \frac{6\, r_0^2}{(r_0^2+D^2)} + \frac{8}{5}\,\frac{R^4\, D^2}{(r_0^2+D^2)^3} \right]$$

$$4\, D^2 = (\xi + \eta)^{-1}$$

$$\left\{ \vec{V}, \vec{W} \right\}_+ \equiv \vec{V}\cdot\vec{W} + \vec{W}\cdot\vec{V}$$

$$S_R = \vec{S}\cdot\hat{R} = \tfrac{1}{2}(\vec{\sigma}_1 + \vec{\sigma}_2)\cdot\hat{R}$$

$$a_1(\xi) = a(\xi)(2\, D^3 \sqrt{\pi})^{\frac{1}{2}}$$

$$b_1(\xi) = b(\xi)(2\, D^3 \sqrt{\pi})^{\frac{1}{2}}\, (D^2 \sqrt{60})$$

$$\hbar\vec{k} = \vec{P} = -i\,\hbar\,\vec{\nabla}_R \qquad (58)$$

The $a_1(\xi)$ and $b_1(\eta)$ are defined so that if $a(\xi)$ and $b(\xi)$ are multiples of the same delta function (so that the radial wave functions are Gaussian with the same range, multiplied by appropriate powers of r) then $a_1(\xi)$ and $b_1(\xi)$ are equal to this delta function multiplied by $(.96)^{\frac{1}{2}}$ and $(.04)^{\frac{1}{2}}$ respectively.

If one takes $b(\xi) = 0$ and adjusts $a(\xi)$ accordingly, then the limit in which there is only the S-state is obtained. If one also takes $D = 0$, which corresponds to taking a point deuteron, then the hamiltonian goes over to the one used in sections two and three. The effects of the finite size of the deuteron are contained in the dependence upon the parameter D which is effectively the size of the deuteron. One may note that if the deuteron size is about equal to r_0, the range of the force acting on the individual nucleons, then the range of the force acting on the deuteron itself is increased appreciably. This would tend to make the differential cross section for deuteron scattering squeeze into smaller angles, relative to the nucleon scattering at corresponding energies. Noticing that angles of the same momentum transfer should be compared, one finds this effect present in the experimental data.[27] If the D-state contributions are included but the deuteron size is taken small compared to r_0 then the hamiltonian again goes over to the one used in sections two and three except for certain terms which are quadratic in the D-state amplitude. Because of these latter terms the D-state contributions will play a role even for vanishingly small deuteron size.

In the actual physical case, in which the deuteron size is of the same order as r_0 there are a large number of terms which are essentially different from those appearing in sections two and three. Some of these terms are spin independent or are of the form of a spin orbit interaction.

Thus their effect is simply to alter somewhat the radial dependence of these terms as compared to those used in the earlier sections. In addition to these terms there are two new types of term which, independently of the particular form of the deuteron wave functions, have the forms

$$\left(\frac{3}{2} S_R S_R - 1 \right) f(R)$$

and

$$\left\{ \vec{S} \times \vec{K} , \vec{\nabla}\left(\left(\frac{3}{2} S_R S_R - 1 \right) g(R) \right) \right\}_{+} . \tag{59}$$

The contribution to the M-matrix from a term of the first type is $(-2M/4\pi\hbar^2)$ times

$$\int \left(\frac{3}{2} S_R S_R - 1 \right) f(R) \, e^{i\vec{K}\cdot\vec{R}} \, d^3R \tag{60}$$

where K is now the momentum transferred. Performing the angular integrations this expression becomes

$$\left(\frac{3}{2} S_K S_K - 1 \right) \int_0^\infty 4\pi R^2 \, dR \left(\frac{\sin KR}{KR} + \frac{3 \cos KR}{(KR)^2} - \frac{3 \sin KR}{(KR)^3} \right) f(R) . \tag{61}$$

The tensor contributions to the M-matrix from this term has therefore the same form, $\left(\frac{3}{2} S_K S_K - 1\right)$, for all values of the scattering angle. For small momentum transfer the term is of order θ^2 as may be seen with the help of Eq. (61). In contrast to the second Born approximation in which tensor contributions of the form $S_p S_p$ appeared, the coefficients $c(\theta)$ and $d(\theta)$ are now equal in both sign and magnitude, and the coefficient w in Eq. (32), and hence the $\cos 2\phi$ dependence of the differential cross section, will now be modified. The two

largest terms of the $(\frac{3}{2}\ S_R\ S_R - 1)\ f(R)$ type are the term multiplying $B(R)$ and the term in $UA(R)$ which contains the factor $8\cdot(30)^{-\frac{1}{2}}$. These terms have identical forms except that the factor U in the latter is replaced by $16\ V\ h^2\ \xi\ (2\ M^2\ c^2)^{-1}$ in the former. Letting $\xi = (1/32) \times 10^{26}\ cm^{-2}$, which corresponds to a deuteron radius of 4×10^{-13} cm, and taking $V = 20\ U$ one finds that the term not containing ξ is larger by an order of magnitude. The sign and phase of the $c(\theta)$ and $d(\theta)$ which arise out of this term will be the same as that of the lowest order contributions to $a(\theta)$ and Eqs. (31), (32) and (33) show, then, that the $\cos 2\phi$ asymmetry will be enhanced. To estimate the magnitude of this effect one may approximate $a(\xi)$ and $b(\eta)$ by multiples of a delta function. Then the integrations over ξ and η are trivial and the factor $a_1(\xi_0)\ b_1(\eta_0)$ becomes $(.96)^{\frac{1}{2}}(.04)^{\frac{1}{2}} \simeq 1/5$. Since

$$(\frac{3}{2}\ S_K\ S_K - 1) = -\frac{3}{4}\ (S_N\ S_N - 2/3) - \frac{3}{4}\ (S_P\ S_P - S_K\ S_K)$$

the contributions to $c(\theta)$ and $d(\theta)$ coming from this term are

$$c(\theta) = d(\theta) \cong \frac{1}{5}\ (-\frac{3}{4})\ (-2M/4\pi\hbar^2)\ U$$

$$\times\ (-1) \int_0^\infty 4\pi R^2\, dR\ \frac{R^2\ D^2}{(r_0^2 + D^2)^2}\ \frac{8}{\sqrt{30}} \left(\frac{r_0^2}{r_0^2 + D^2}\right)^{3/2} e^{-R^2/(r_0^2 + D^2)}$$

$$\times \left[\frac{\sin KR}{KR} + \frac{3\cos KR}{(KR)^2} - \frac{3\sin KR}{(KR)^3}\right]$$

$$= \frac{1}{5}\ (-\frac{3}{4})(\frac{-2\ M}{4\pi\hbar^2})\ 4\pi\left(\frac{-D^2\ U}{(r_0^2 + D^2)}\right)\frac{8}{\sqrt{30}}\left(\frac{r_0^2}{r_0^2 + D^2}\right)^{3/2} \frac{\sqrt{\pi}\,(r_0^2 + D^2)^{3/2}}{4}$$

$$e^{-K^2(r_0^2 + D^2)/4}\left[\frac{-K^2\ (r_0^2 + D^2)}{4}\right] .$$

The expression for $a(\theta)$, on the other hand, is given approximately as

$$a(\theta) = \left(\frac{-2M}{4\pi\hbar^2}\right) U(-1) \int d^3R \, \frac{r_0^2}{r_0^2 \quad D^2}^{3/2} \, e^{-R^2/r_0^2 + D^2} \, e^{iK\cdot R}$$

$$= \left(\frac{-2M}{4\pi\hbar^2}\right) (-U) \, (\pi)^{3/2} \, (r_0^2 + D^2)^{3/2} \left(\frac{r_0^2}{r_0^2 + D^2}\right)^{3/2} e^{-K^2(r_0^2 + D^2)/4} .$$

Thus

$$c(\theta) = d(\theta) = \frac{1}{5} \times \frac{3}{4} \, \frac{8}{\sqrt{30}} \, (k^2 D^2 \sin^2 \tfrac{1}{2}\theta) \, (a(\theta)) .$$

At the energy of the Berkeley cyclotron $k \sim 4 \times 10^{13}$ cm^{-1}. Taking $D = 1.4 \times 10^{-13}$ cm (see Catherine Way reference 32) one then obtains

$$c(\theta) = d(\theta) \cong \frac{32}{5} \, a(\theta) \sin^2 \tfrac{1}{2}\theta .$$

At $\theta \sim 15^\circ$, $c(\theta)$ and $d(\theta)$ are then ten percent of $a(\theta)$ and the value of $w(\theta)$ given in Eqs. (32) and (33) is about four-tenths. The $\cos 2\phi$ asymmetry is then, according to Eq. (31), about three percent. This is consistent with the experimental results which are that this asymmetry is less than four percent.

There is also the second term given in expression (59) to be considered. This has the form

$$\left\{ \vec{S} \times \vec{K}, \vec{\nabla}\left(\left(\tfrac{3}{2} \; S_R \; S_R \; - \; 1\right)(g(R))\right) \right\}_+ .$$

Its matrix element is proportional to

$$\int d^3R \; e^{-i\vec{k}_{out}\cdot\vec{R}} \left\{ \vec{S}, \vec{K} \times \vec{\nabla} \left(\left(\tfrac{3}{2} S_R \; S_R \; - \; 1\right) g(R)\right) \right\}_+ e^{ik_{in}\cdot R} .$$

An integration by parts on the operator $\vec{\nabla}$ allows this to be reduced to

$$\left\{ \vec{S}, \vec{N}\, i \sin\theta\, k^2 \int d^3\vec{R}\ \left(\frac{3}{2}\ S_R\, S_R\ -\ 1\right) g(R)\ e^{i\vec{K}\cdot\vec{R}} \right\}_{+}$$

which in turn reduces to

$$\left\{ S, N\ \left(\frac{3}{2}\ S_K\ S_K\ -\ 1\right) f(k, K) \right\}_{+} \quad .$$

But Table G shows that this is just a multiple of $S \cdot N$. Thus this second type of tensor term really reduces to a vector term and the only tensor term which arises is the one of the form $\left(\frac{3}{2}\ S_K\ S_K\ -\ 1\right)$ which was previously discussed.

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APPENDIX A

The Form and Definition of the S-Matrix

The S matrix used in the first section of part one was defined by

$$|f) = (S - 1)\,|f_0) \tag{A1}$$

where $(\theta\phi\,|\,f_0) \equiv f_0(\theta\phi)$ was the spin angle function describing the asymptotic outgoing part of the incident plane wave, and $(\theta\phi\,|\,f) \equiv f(\theta\phi)$ was the spin angle function describing the asymptotic scattered wave. That is,

$$\begin{aligned} \Psi^{(out)}_{inc}(r, \theta, \phi) &= \frac{\exp i\,Kr}{r}\, f_0(\theta\phi) \\ \Psi_{scat}(r, \theta, \phi) &= \frac{\exp i\,Kr}{r}\, f(\theta\phi)\,. \end{aligned} \tag{A2}$$

A more usual way of defining the S-matrix is to relate the incoming part of the incident plane wave to the scattered wave. Here some consideration must be given to phase factors and one may define S' by

$$|f') = (S' - 1)\,|f'_0) \tag{A3}$$

where $f'_0(\theta\phi)$ and $f'(\theta\phi)$ are defined by*

$$\begin{aligned} \left|\Psi^{(in)}_{inc}(r)\right) &= \left(\frac{1}{r}\right)\exp\ -i(Kr - \tfrac{1}{2}\pi L)\ \ |f') \\ \left|\Psi_{scat}(r)\right) &= -\left(\frac{1}{r}\right)\exp\ +i(Kr - \tfrac{1}{2}\pi L)\ \ |f'_0) \end{aligned} \tag{A4}$$

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* Here, the angular momentum, L, is considered as an operator; $L\,|\,L') = L'\,|\,L')$.

The S' defined in this second way is the quantity to which the time reversal arguments[12] may be directly applied and which obeys the reciprocity theorem $S'_{i,-j} = S'_{j,-i}$. Here the states referred to by $(-i)$ and $(-j)$ are the time inverses of the states (i) and (j), and they may be defined by

$$(L \;\; S \;\; J \;\; J_z \mid \phi'_{-j}) = (\phi'_j \mid L \;\; S \;\; J \;\; - \;\; J_z) \tag{A5}$$

where ϕ'_j are vectors in the spin angle space only. The reciprocity theorem, together with the spherical symmetry of the physical problem may be used to obtain the symmetry relation

$$S'_{ij*} = S'_{ji*} \tag{A6}$$

where the states ϕ_{i*} and ϕ_{j*} are defined by

$$(L \;\; S \;\; J \;\; J_z \mid \phi'_{j*}) = (\phi'_j \mid L \;\; S \;\; J \;\; J_z) \;\; . \tag{A7}$$

In a representation in which $\phi_j = \phi_{j*}$ one obtains the symmetry property

$$S'_{ij} = S'_{ji} \; . \tag{A8}$$

From this equation the symmetry of the S-matrix used in part one may be deduced.

The remainder of this appendix will be devoted to a discussion of the definition of the nuclear phase shifts in the presence of a coulomb field. This development will be prefaced by a few general remarks concerning the transformation properties of the symmetric S-matrix defined by Eqs. (A3) and (A6). This matrix will be considered as a matrix in a mixed representation, the basis vectors on the left and

right belonging to different sets. Now if the basis vectors $|\phi_i')$ are changed by phase factors to give $|\phi_i'') = e^{-i\alpha_i} |\phi_j')$ then $|\phi_{j*}')$ will be changed into

$$|\phi_{j*}'') = e^{i\alpha_j} |\phi_{j*}') .$$

The S' matrix element in the new representation becomes

$$S''_{ij*} = e^{i\alpha_i} S'_{ij*} e^{i\alpha_j} . \tag{A9}$$

The new matrix, considered as a matrix in the indices i and j, will continue to be symmetric; however, the unit matrix will not be transformed into itself. These properties are, of course, different from the transformation properties of a matrix in an ordinary (unmixed) representation. A corollary of these remarks is that the symmetry of the S' matrix will be retained when its definition is altered to

$$|f'') = (S'' - 1) |f_0'') \tag{A10}$$

where

$$\begin{aligned} |f'') &= e^{i\alpha} |f') \\ |f_0'') &= e^{-i\alpha} |f_0') . \end{aligned} \tag{A11}$$

However, the conditions under which $S' = 1$ will not be those for which $S'' = 1$. These remarks will give a background for the following discussion of the coulomb effects.

When there is a coulomb field present there is a progressive phase change at large r values and the definition of the S matrix must be modified. An appropriate generalization of Eqs. (A3) and (A4) is

$$|f^M) = (S^M - 1) |f_0^M) \equiv R^M |f_0^M) \tag{A12}$$

where

$$\left|\Psi^{(in)}_{inc}(r)\right) = +\left(\frac{1}{r}\right) \exp -i(Kr - \eta \ln 2Kr - \tfrac{1}{2}\pi L)\left|f_0^M\right) \tag{A13}$$

$$\left|\Psi_{scat}(r)\right) = -\left(\frac{1}{r}\right) \exp i(Kr - \eta \ln 2Kr - \tfrac{1}{2}\pi L)\left|f^M\right)$$

where $\eta = ZZ' e^2 \mu / h^2 K$ and μ is the reduced mass. In cases in which L is a constant of the motion the S^M matrix is diagonal in the $L\ S\ J\ J_z$ representation and the diagonal elements are expressed as $\exp 2i\delta_L$ (a possible J index is suppressed). Introducing the "nuclear" phase shifts

$$\delta_L^N = \delta_L - \sigma_L , \tag{A14}$$

where σ_L is the coulomb phase shift[29] $= \arg \Gamma(1 + L + i\eta)$, one may obtain

$$\begin{aligned}(S^M - 1) &= e^{2i\delta} - 1 \\ &= e^{2i\sigma}(e^{2i\delta^N} - 1) + (e^{2i\sigma} - 1)\end{aligned} \tag{A15}$$

where δ, σ and δ^N are considered as diagonal operators. With the definitions

$$R^N = S^N - 1 = (e^{2i\delta^N} - 1)$$

and

$$R^C = S^C - 1 = (e^{2i\sigma} - 1)$$

one may then write Eq. (A12) in the form

$$\left|f^M\right) = (e^{i\sigma} R^N e^{i\sigma} + R^C)\left|f_0^M\right) . \tag{A16}$$

An equivalent way of writing this equation is

$$|f^N) = R^N |f_0^N) + |f^C) \tag{A17}$$

where $|f^N)$ and $|f_0^N)$ are defined by

$$|\psi_{scat}(r)) = -\left(\frac{1}{r}\right) \exp i(Kr - \eta \ln 2Kr - \tfrac{1}{2}\pi L + i\sigma_L) \, |f^N)$$

$$|\psi^{(in)}_{inc}(r)) = +\left(\frac{1}{r}\right) \exp -i(Kr - \eta \ln 2Kr - \tfrac{1}{2}\pi L + i\sigma_L) \, |f_0^N) \tag{A18}$$

where $|f^C)$ is the value of $|f^N)$ when there is a pure coulomb interaction. The Eqs. (A17) and (A18) give a natural definition for the "nuclear" S matrix $S^N = R^N + 1$ when L is no longer a constant of the motion. It need hardly be mentioned that this "nuclear" S-matrix is not the same as the S-matrix which would describe the system if the coulomb interaction were simply removed, since it will be a function of e in general, but it does have part of the coulomb effect removed (i.e., the σ_L). Further, this matrix is symmetric and it becomes unity when the nuclear interaction vanishes. The S^M defined in Eq. (A12) is related to S^N according to

$$S^M = e^{i\sigma} S^N e^{i\sigma} . \tag{A19}$$

It is the S^M which was used in the phase shift analysis,* and as in part one, its triplet part corresponding to a given J value may be written[12]

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* More precisely $e^{-i\pi L/2} S^M e^{+i\pi L/2}$ may be identified with the S-matrix in part one.

$$S_J^M = \begin{pmatrix} \cos\varepsilon & -\sin\varepsilon \\ \sin\varepsilon & \cos\varepsilon \end{pmatrix} \begin{pmatrix} \exp 2i\,\delta_{J+1} & 0 \\ 0 & \exp 2i\,\delta_{J-1} \end{pmatrix} \begin{pmatrix} \cos\varepsilon & \sin\varepsilon \\ -\sin\varepsilon & \cos\varepsilon \end{pmatrix} \tag{A20}$$

where the J indices are suppressed on the right hand side. The corresponding $\delta^N_{J\pm 1}$ and ε^N for the "nuclear" S^N may be defined by the analogue of Eq. (A20), namely

$$S_J^N = \begin{pmatrix} \cos\varepsilon^N & -\sin\varepsilon^N \\ +\sin\varepsilon^N & \cos\varepsilon^N \end{pmatrix} \begin{pmatrix} \exp 2i\,\delta^N_{J+1} & 0 \\ 0 & \exp 2i\,\delta^N_{J-1} \end{pmatrix} \begin{pmatrix} \cos\varepsilon^N & +\sin\varepsilon^N \\ -\sin\varepsilon^N & \cos\varepsilon^N \end{pmatrix} \tag{A21}$$

Substituting Eqs. (A21) and (A20) into Eq. (A19) one may obtain the "nuclear" phase shifts $\delta^N_{J\pm 1}$ and admixture parameter ε^N in terms of the $\delta_{J\pm 1}$ and ε obtained in the phase shift analysis. The interesting feature is that the admixture parameter which measures the amount of mixing of the two angular momentum states is different when defined relative to S^M and S^N although the coulomb interaction causes no mixing.* The origin of this change is not in the physics, but rather in the way that the admixture parameter is defined. For if the physics were left unchanged but the phases of the basis vectors (ϕ_i) are changed then the admixture parameters ε' and ε'' defined as in Eq. (A20) for the symmetric matrices S'_{ij*} and S''_{ij*} would be different. The admixture parameter defined in the above way depends on the relative phases of the incident states as well as upon the amount of $J - 1$ wave which emerges when a $J + 1$ wave is incident and vice versa.

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* In this nonrelativistic treatment.

There is an alternative way of defining the admixture parameter which tells directly how an incident partial wave in one state divides into the two outgoing partial waves and which is independent of the phase factors contained in the basis vectors. One writes, instead of Eq. (A20), rather

$$S_J^M = \begin{pmatrix} e^{i\tilde{\delta}_{J+1}} & 0 \\ 0 & e^{i\tilde{\delta}_{J-1}} \end{pmatrix} \begin{pmatrix} \cos 2\bar{\varepsilon} & i \sin 2\bar{\varepsilon} \\ i \sin 2\bar{\varepsilon} & \cos 2\bar{\varepsilon} \end{pmatrix} \begin{pmatrix} e^{i\tilde{\delta}_{J+1}} & 0 \\ 0 & e^{i\tilde{\delta}_{J-1}} \end{pmatrix} \tag{A'20}$$

With this definition the relationship between the $\bar{\delta}_{J\pm1}$ and the $\tilde{\delta}^N_{J\pm1}$ defined by the analog Eq. (A21) is

$$\bar{\delta}_{J\pm1} = \tilde{\delta}^N_{J\pm1} + \sigma_{J\pm1} \tag{A22}$$

just as for the states for which L is conserved. The equations relating the two definitions of S_J^M are

$$\begin{aligned} \delta_{J+1} + \delta_{J-1} &= \bar{\delta}_{J+1} + \bar{\delta}_{J-1} \\ \tan 2\varepsilon &= \tan 2\bar{\varepsilon} / \sin(\bar{\delta}_{J+1} - \bar{\delta}_{J-1}) \\ \sin(\delta_{J+1} - \delta_{J-1}) &= \sin 2\bar{\varepsilon} / \sin 2\varepsilon \end{aligned} \tag{A23}$$

The phase shifts $\tilde{\delta}_L$ and the admixture parameter $\bar{\varepsilon}$ have a simple interpretation. The δ_L gives the shift in the phase relative to the unperturbed wave which the particle obtains in traveling the incoming leg of the scattering process. The scattering causes the incoming beam to divide between the two possible partial waves in a proportion fixed by $\bar{\varepsilon}$ and then the phase shift $\bar{\delta}_L$ or $\bar{\delta}_{L'}$ is added depending on whether the particle emerges in the original or in the other partial

wave. In so far as the entire coulomb effect can be considered as acting outside of the region around the origin in which the nuclear effects take place, the $\bar{\delta}^{N}_{LJ}$ and $\bar{\varepsilon}_{J}$ will be just the true nuclear phase shifts, those which obtain for the N - P scattering. This relationship is not true when the usual δ_{Lj} and ε_{J} are used because the shift in the phase induced by the coulomb effects becomes intimately incorporated into those phase shifts.

APPENDIX B

The Covariant Density Matrix.

In situations in which statistical mixtures of states are considered it is convenient to introduce the density matrix ρ which in an appropriate representation may be written[28]

$$\rho = |\psi_\alpha\rangle W_\alpha \langle\psi_\alpha|$$

where W_α is the probability that the system is in the state α, so that $\sum W_\alpha = 1$. The probability of finding the system in a region R may be written

$$w(R) = \mathrm{Sp}\,\rho\,\mathcal{P}$$

where Sp is the trace over both coordinate and spin variables, and $\mathcal{P}$ is the operator which projects onto the region R. If R is taken as the three dimensional momentum region $(dF) = df_1\, df_2\, df_3$ then

$$w(dF) = (dF)\,\mathrm{Tr}\,\rho_S(F)$$

where Tr is the trace in spin space and

$$\rho_S(F) = |a(F)|^2 \left\{ |u_\alpha(F))\, W_\alpha (u^*_\alpha(F)| \right\} .$$

The amplitude $a(F)$ is a function of the three momentum F defined in terms of $\psi_\alpha(F)$, the momentum space wave function, by

$$\psi_\alpha(F) = a_\alpha(F)\, |u_\alpha(F)) .$$

The $u_\alpha(F)$ are spinors which can be expressed as linear combinations of the $u_i(f)$ of section one, and like the $u_i(f)$ they may be defined in terms of their values in the frame in which $F = 0$ by the equation

$$u_\alpha(F) = L(\underline{f})\; u_\alpha(0) .$$

Then

$$w(dF) = (dF) \; |a(F)|^2 \; (\gamma)^F$$

where $(\gamma)^F$ is the Lorentz contraction factor. Since $(\gamma)^F(dF)$ is an invariant, the required invariance of $w(dF)$ requires that $|a(F)|^2$ is unchanged in a Lorentz transformation.

Notice that the density matrix and the volume element are not invariants separately. If, however, the particle is definitely in a positive energy state or definitely in a negative energy state one may write

$$\begin{aligned}
\rho_S(F) &= |a_\alpha(F)|^2 \; |u_\alpha(F))\, w_\alpha\, (u^*(F)| \\
&= |a_\alpha(F)|^2 \; |u_\alpha(F))\, w_\alpha\, (u^*_\alpha(F) \,|\, u_\alpha(F))(\pm)(\bar{u}_\alpha(F)| \\
&= (\gamma)^F \; |a_\alpha(F)|^2 \; |u_\alpha(F))(\pm w_\alpha)(\bar{u}_\alpha(F)| \\
&\equiv (\gamma)^F \; \rho(f) \; .
\end{aligned}$$

The $(\gamma)^F$ may now be put with the (dF) to form an invariant. The matrix $\rho\,(\underline{f})$, since its matrix elements

$$\rho_{ij}(\underline{f}) \equiv (\bar{u}_i(\underline{f}) \,|\, \rho\,(\underline{f}) \,|\, u_j(\underline{f}))$$

are invariants, must be of the form

$$\rho(f) = (\tfrac{1}{4}\,\mathrm{Tr}\,\rho\,(f)) \Big[1 + \lambda_\mu \gamma_\mu + \tfrac{1}{2} s_{\mu\nu}\, \sigma_{\mu\nu} + i\,\gamma_5\, \gamma_\mu\, p_\mu + q\,\gamma_5 \Big]$$

where the coefficients λ_μ, $s_{\mu\nu}$, p_μ and q transform in the evident manner.

The expectation value of the operator A over measurements in the region $R = (dF)$ is

$$\langle A \rangle (dF) = \mathrm{Sp}\, \rho\, A\, \rho \,/\, \mathrm{Sp}\, \rho\, \rho$$

$$= \frac{(\gamma)^F (dF)\ \mathrm{Tr}\ \rho\ (\underline{f})\ A}{(\gamma)^F (dF)\ \mathrm{Tr}\ \rho\ (\underline{f})} \quad .$$

If the region R restricts also the three momentum H of the second particle then the element (dH) will also appear in the invariant combination $(\gamma)^H (dH)$.

For the final state the matrix $\rho'(\underline{f}')$ is defined in the analogous way. It is related to $\rho(\underline{f})$ by the equation

$$\rho'(\underline{f}') = S(\underline{f}', \underline{t}, \underline{f})\, \rho(\underline{f})\, \bar{S}(\underline{f}', \underline{t}, \underline{f}).$$

Here the invariant elements $(\gamma)^K(dK)(\gamma)^T(dT)$ and $(\gamma)^T(dT)$ have been incorporated into the definitions of $\rho\ (\underline{f})$ and $\rho'(\underline{f}')$ respectively and the trivial integration over T and K performed, allowing these variable to be considered as fixed and discrete. The condition $\mathrm{Sp}\ \rho = 1$ becomes then $\mathrm{Tr}\ \rho\ (\underline{f}) = 1$, and the differential cross section is

$$I(\underline{f}') = |a(\underline{f}')|^2 = \mathrm{Tr}\ \rho'(\underline{f}') \quad .$$

TABLE A

$$M = a(\theta) + c(\theta)(\sigma_{1N} + \sigma_{2N}) + m(\theta)(\sigma_{1N}\sigma_{2N}) + g(\theta)(\sigma_{1P}\sigma_{2P} + \sigma_{1K}\sigma_{2K})$$
$$+ h(\theta)(\sigma_{1P}\sigma_{2P} - \sigma_{1K}\sigma_{2K})$$

$$I_0(\theta) = |a|^2 + |m|^2 + 2|c|^2 + 2|g|^2 + 2|h|^2$$

$$I_0 P(\theta) = 2 \operatorname{Re} c^*(a + m)$$

$$I_0(1 - D(\theta)) = 4|g|^2 + 4|h|^2$$

$$I_0 R_K(\theta) = \left[|a|^2 - |m|^2 - 4 \operatorname{Re} h g^*\right] \cos \theta/2 + 2 \operatorname{Re} i\, c(a-m)^* \sin \theta/2$$

$$I_0 C_{KP}(\theta) = 4 \operatorname{Re} i\, c\, h^*$$

$$I_0(1 - C_{NN}) = |a - m|^2 \quad 4|g|^2$$

. . . .

$$I_0 R_M(\theta) = (|a|^2 - |m|^2) \cos\left(\frac{\theta}{2} - \beta\right) - 4 \operatorname{Re} g h^* \cos\left(\frac{\theta}{2} + \beta\right)$$
$$+ 2 \operatorname{Re} i\, c(a - m)^* \sin\left(\frac{\theta}{2} - \beta\right)$$

$$I_0 C_{MM'}(\theta) = 2 \operatorname{Re} g(a - m)^* \cos(\beta - \beta') - 2 \operatorname{Re} h(a + m)^* \cos(\beta + \beta')$$
$$+ 4 \operatorname{Re} i\, c\, h^* \sin(\beta + \beta')$$

($\vec{M}$ is along $\vec{K}$ for $\beta = 0$, along $\vec{P}$ for $\beta = \pi/2$ and similarly for $\vec{M}$ and β'.)

TABLE B

$$a(\theta) = \tfrac{1}{4}(2\,M_{11} + M_{00} + M_{SS})$$

$$c(\theta) = \tfrac{1}{4}(i\sqrt{2})(M_{10} - M_{01})$$

$$m(\theta) = \tfrac{1}{4}(-\,2\,M_{1-1} + M_{00} - M_{SS})$$

$$g(\theta) = \tfrac{1}{4}(M_{11} + M_{1-1} - M_{SS})$$

$$h(\theta) = \tfrac{1}{4}(1/\cos\theta)(M_{11} - M_{1-1} - M_{00})$$

$$= \tfrac{1}{4}(\sqrt{2}/\sin\theta)(M_{10} + M_{01})$$

TABLE C

$$I_0 = \tfrac{1}{2}\,|M_{11}|^2 + \tfrac{1}{4}\,|M_{00}|^2 + \tfrac{1}{4}\,|M_{SS}|^2 + \tfrac{1}{2}\,|M_{10}|^2 + \tfrac{1}{2}\,|M_{01}|^2 + \tfrac{1}{2}\,|M_{1-1}|^2$$

$$I_0P = (\sqrt{2}/4)\ \mathrm{Re}\ i(M_{10} - M_{01})(M_{11} - M_{1-1} + M_{00})^*$$

$$I_0D = \tfrac{1}{2}\ \mathrm{Re}\left\{(M_{11} - M_{1-1})M_{00}^* + (M_{11} + M_{1-1})M_{SS}^* - 2\,M_{01}\,M_{10}^*\right\}$$

$$I_0R_K = \tfrac{1}{2}\left(\cos\frac{\theta}{2}\right)\mathrm{Re}\left\{\left(M_{00} + (\cos\theta - 1)\,\frac{\sqrt{2}\,M_{10}}{\sin\theta}\right)\left(M_{11} + M_{1-1} + M_{SS}\right)^* + \left(\frac{\sqrt{2}\,M_{10}}{\sin\theta} + \frac{\sqrt{2}\,M_{01}}{\sin\theta}\right)M_{SS}^*\right\}$$

$$I_0C_{KP} = (1/2\sin\theta)\left(|M_{01}|^2 - |M_{10}|^2\right)$$

$$I_0(1 - C_{NN}) = \tfrac{1}{2}\left(|M_{SS}|^2 + |M_{11} + M_{1-1}|^2\right)$$

TABLE D

$$M_{SS} = \left(\frac{-i\,2}{k}\right)\left\{\tfrac{1}{2} R^{0}_{00} + \frac{5}{4}(3\cos^2\theta - 1)\, R^{2}_{20}\right\}$$

$$M_{11} = \left(\frac{-i\,2}{k}\right)\left\{\cos\theta\left(\frac{3}{4} R^{1}_{11} + \frac{3}{4} R^{2}_{11} + \frac{\sqrt{6}}{4} R^{2}\right)\right.$$

$$\left. + (5\cos^3\theta - 3\cos\theta)\left(\tfrac{1}{4} R^{2}_{31} + \frac{7}{8} R^{3}_{31} + \frac{5}{8} R^{4}_{31} + \frac{\sqrt{6}}{8} R^{2}\right)\right.$$

$$M_{00} = \left(\frac{-i\,2}{k}\right)\left\{\cos\theta\left(\tfrac{1}{2} R^{0}_{11} + R^{2}_{11} - \frac{\sqrt{6}}{2} R^{2}\right)\right.$$

$$\left. + (5\cos^2\theta - 3\cos\theta)\left(\frac{3}{4} R^{2}_{31} + R^{4}_{31} - \frac{\sqrt{6}}{4} R^{2}\right)\right.$$

$$M_{01} = \left(\frac{-i\,2}{k}\right)\frac{\sin\theta}{\sqrt{2}}\left\{\left(\frac{3}{4} R^{1}_{11} - \frac{3}{4} R^{2}_{11} - \frac{\sqrt{6}}{4} R^{2}\right)\right.$$

$$\left. + (5\cos^2\theta - 1)\left(\tfrac{1}{2} R^{2}_{31} + \frac{7}{16} R^{3}_{31} - \frac{15}{16} R^{4}_{31} + \frac{\sqrt{6}}{4} R^{2}\right)\right\}$$

$$M_{10} = \left(\frac{-i\,2}{k}\right)\left(\frac{\sin\theta}{\sqrt{2}}\right)\left\{\left(-\tfrac{1}{2} R^{0}_{11} + \tfrac{1}{2} R^{2}_{11} - \frac{\sqrt{6}}{4} R^{2}\right)\right.$$

$$\left. + (5\cos^2\theta - 1)\left(-\frac{3}{4} R^{2}_{31} + \frac{3}{4} R^{4}_{31} + \frac{\sqrt{6}}{4} R^{2}\right)\right\}$$

$$M_{1-1} = \left(\frac{-i\,2}{k}\right)(5\cos^3\theta - 5\cos\theta)\left(-\tfrac{1}{4} R^{2}_{31} + \frac{7}{16} R^{3}_{31} - \frac{3}{16} R^{4}_{31} - \frac{\sqrt{6}}{8} R^{2}\right)$$

TABLE E

<u>N - P System</u>

$$I_0(\theta) = |a|^2 + |m|^2 + 2\,|c|^2 + 2\,|g|^2 + 2\,|h|^2 + 2\,|b|^2$$

$$I_0P(P, \theta) = 2\,\mathrm{Re}\; c(a + m)^* + 2\,\mathrm{Re}\; b(a - m)^*$$

$$I_0P(N, \theta') = 2\,\mathrm{Re}\; c(a + m)^* - 2\,\mathrm{Re}\; b(a - m)^*$$

$$I_0(1 - D(N, P, \theta)) = |a - m|^2 + 4\,|h|^2 + 4\,|b|^2$$

$$= I_0(1 - D(P, N, \theta'))$$

$$I_0(1 - D(P, P, \theta)) = 4\,|g|^2 + 4\,|h|^2$$

$$= I_0(1 - D(N, N, \theta'))$$

$$I_0R_K(P, P, \theta) = \cos\frac{\theta}{2}\left\{|a|^2 - |m|^2 - 4\,\mathrm{Re}\; g\,h^* - 4\,\mathrm{Re}\; c\,b^*\right\}$$

$$+ \sin\frac{\theta}{2}\left\{2\,\mathrm{Re}\; i\,c(a - m)^* + 2\,\mathrm{Re}\; i\,b(a + m)^*\right\}$$

$$I_0R_P(N, N, \theta') = \cos\frac{\theta}{2}\left\{2\,\mathrm{Re}\; i(a - m)c^* - 2\,\mathrm{Re}\; i(a + m)b^*\right\}$$

$$+ \sin\frac{\theta}{2}\left\{|a|^2 - |m|^2 + 4\,\mathrm{Re}\; g\,h^* + 4\,\mathrm{Re}\; c\,b^*\right\}$$

$$I_0R_K(N, P, \theta) = \cos\frac{\theta}{2}\left\{2\,\mathrm{Re}(a + m)g^* - 2\,\mathrm{Re}(a - m)h^*\right\}$$

$$+ \sin\frac{\theta}{2}\left\{4\,\mathrm{Re}\; i\,c\,g^* + 4\,\mathrm{Re}\; i\,b\,h^*\right\}$$

TABLE E (Cont.)

$$I_0 R_P(P, N, \theta') = \cos\frac{\theta}{2}\left\{ -4\,\mathrm{Re}\; i\, c\, g^* - 4\,\mathrm{Re}\; i\, b\, h^* \right\} + \sin\frac{\theta}{2}\left\{ 2\,\mathrm{Re}(a+m)g^* + 2\,\mathrm{Re}(a-m)h^* \right\}$$

$$I_0(1 - C_{NN}(\theta)) = |a-m|^2 + 4\,|g|^2 + 4\,|b|^2$$

$$I_0(C_{KP}(\theta)) = 4\,\mathrm{Re}\; i\, c\, h^* + 4\,\mathrm{Re}\; i\, b\, g^*$$

Note: The R experiments are those in which a neutron initially at rest in the laboratory frames scatters the proton through a center-of-mass angle of θ.

TABLE F

1. $\mathrm{Tr}\, S_i = \mathrm{Tr}\, S_{ij} = 0$

2. $\mathrm{Tr}\, S_i S_{jk} = 0$

3. $\mathrm{Tr}\, S_i S_j = 2\delta_{ij}$

4. $\mathrm{Tr}\, S_{ij} S_{k\ell} = \frac{1}{2}\left[\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}\right] - \frac{1}{3}\delta_{ij}\delta_{k\ell}$

5. $\mathrm{Tr}\, S_i S_j S_k = i\varepsilon_{ijk}$ (Prove using 6)

6. $S_j S_k = S_{jk} + \frac{i}{2} S_{j\times k} + \frac{2}{3}\delta_{jk}$

7. $$S_i S_j S_k = \frac{1}{2}\left[\delta_{ij} S_k + \delta_{jk} S_i\right] + \frac{i}{2}\left[S_{i\times j,k} + S_{j\times k,i} - S_{k\times i,j}\right] + \frac{i}{3}\varepsilon_{ijk} I$$

$S_{j\times k} \equiv \varepsilon_{ijk} S_i$ (not summed on i)

$S_{j\times k,\ell} \equiv \varepsilon_{ijk} S_{i,\ell}$ (not summed on i)

$$\varepsilon_{ijk} = \begin{cases} 1 & \text{if } i, j, k \text{ are cyclic} \\ -1 & \text{if } i, k, j \text{ are cyclic} \\ 0 & \text{if } i, j, k \text{ are not all different.} \end{cases}$$

TABLE G

$$S_i S_{jk} = \frac{i}{2}\left[S_{i\,x\,j,\,k} + S_{i\,x\,k,\,j}\right] + \frac{1}{4}\left[\delta_{ij} S_k + \delta_{ik} S_j - \frac{2}{3}\delta_{jk} S_i\right]$$

$$S_{ij} S_k = \frac{i}{2}\left[S_{j\,x\,k,\,i} + S_{i\,x\,k,\,j}\right] \frac{1}{4}\left[\delta_{jk} S_i + \delta_{ik} S_j - \frac{2}{3}\delta_{ij} S_k\right]$$

$$S_{ij} S_{k\ell} = \frac{1}{3}\left[\delta_{ij} S_{k\ell} + \delta_{k\ell} S_{ij} + \frac{1}{2}\delta_{ik}\delta_{j\ell} + \frac{1}{2}\delta_{i\ell}\delta_{jk} - \frac{1}{3}\delta_{ij}\delta_{k\ell}\right]$$

$$- \frac{1}{4}\left[\delta_{ik}(S_{j\ell} - \frac{i}{2} S_{j\,x\,\ell}) + \delta_{i\ell}(S_{jk} - \frac{i}{2} S_{j\,x\,k}) + \delta_{jk}(S_{i\ell} - \frac{i}{2} S_{i\,x\,\ell}) + \delta_{j\ell}(S_{ik} - \frac{i}{2} S_{i\,x\,k})\right]$$

TABLE H

$$\mathrm{Tr}\, S_{ij} S_k S_\ell = \mathrm{Tr}\, S_{ij} S_{k\ell} = \tfrac{1}{2}\delta_{ik}\delta_{j\ell} + \tfrac{1}{2}\delta_{i\ell}\delta_{jk} - \tfrac{1}{3}\delta_{ij}\delta_{k\ell}$$

$$\mathrm{Tr}\, S_{ij} S_{k\ell} S_m = \frac{i}{4}\left[\delta_{ik}\varepsilon_{j\ell m} + \delta_{i\ell}\varepsilon_{jkm} + \delta_{jk}\varepsilon_{i\ell m} + \delta_{j\ell}\varepsilon_{ikm}\right]$$

$$\mathrm{Tr}\, S_{ij} S_{k\ell} S_{mn} = -\frac{2}{9}\delta_{ij}\delta_{k\ell}\delta_{mn} + \left[\delta_{ij}\delta_{km}\delta_{\ell n}\right]_S - \left[\delta_{i\ell}\delta_{kn}\delta_{mj}\right]_S$$

$\mathrm{Tr}\, S_{ij} S_{k\ell} S_{mn}$ is independent of the order of the three factors. Each factor itself is unchanged by an interchange of the order of its two indices.

The $[\quad]_S$ means the contents of the bracket is to be symmetrized with respect to interchanges of the orders of each element of the pairs ij, $k\ell$ and mn and also with respect to interchanges of the pairs with each other. Thus

$$\left[\delta_{ij}\delta_{km}\delta_{\ell n}\right]_S = \frac{1}{6}\left[\delta_{ij}\delta_{km}\delta_{\ell n} + \delta_{ij}\delta_{kn}\delta_{\ell m}\right.$$

$$\left. + \delta_{\ell n}\delta_{ik}\delta_{jm} + \delta_{\ell n}\delta_{im}\delta_{jk} + \delta_{km}\delta_{i\ell}\delta_{jn} + \delta_{km}\delta_{in}\delta_{j\ell}\right]$$

$$\left[\delta_{i\ell}\delta_{kn}\delta_{mj}\right]_S =$$

$$\frac{1}{8}\left[\delta_{i\ell}\delta_{kn}\delta_{mj} + \delta_{j\ell}\delta_{kn}\delta_{mi} + \delta_{ik}\delta_{\ell n}\delta_{mj} + \delta_{jk}\delta_{\ell n}\delta_{mi}\right.$$

$$\left. + \delta_{i\ell}\delta_{km}\delta_{nj} + \delta_{j\ell}\delta_{km}\delta_{ni} + \delta_{ik}\delta_{\ell m}\delta_{nj} + \delta_{jk}\delta_{\ell m}\delta_{ni}\right].$$

TABLE I

Experimental and Theoretical Observables

					θ_j(Theo.)
θ_1	=	$\sigma_{Tot} \geqslant 20^\circ$	=	22.24 ± 0.70 mb	22.04 mb
θ_2	=	$I_o(90.0^\circ)$	=	3.72 ± 0.19 mb	3.71 mb
θ_3	=	$r(80.2^\circ)$	=	1.045 ± 0.039	0.997
θ_4	=	$r(71.4^\circ)$	=	0.971 ± 0.032	0.991
θ_5	=	$r(64.0^\circ)$	=	0.958 ± 0.032	0.987
θ_6	=	$r(60.8^\circ)$	=	1.013 ± 0.041	0.986
θ_7	=	$r(52.4^\circ)$	=	0.997 ± 0.035	0.989
θ_8	=	$r(44.8^\circ)$	=	1.008 ± 0.026	1.003
θ_9	=	$r(36.0^\circ)$	=	1.074 ± 0.040	1.034
θ_{10}	=	$r(31.9^\circ)$	=	1.031 ± 0.031	1.055
θ_{11}	=	$r(23.4^\circ)$	=	1.098 ± 0.033	1.098
θ_{12}	=	$s(76.2^\circ)$	=	0.613 ± 0.108	0.486
θ_{13}	=	$s(63.9^\circ)$	=	0.635 ± 0.068	0.559
θ_{14}	=	$s(53.4^\circ)$	=	0.633 ± 0.052	0.653
θ_{15}	=	$s(42.9^\circ)$	=	0.760 ± 0.040	0.761
θ_{16}	=	$s(32.3^\circ)$	=	0.837 ± 0.060	0.856
θ_{17}	=	$s(21.6^\circ)$	=	0.891 ± 0.067	0.924
θ_{18}	=	$t(23.0^\circ)$	=	0.245 ± 0.079	0.254
θ_{19}	=	$t(25.8^\circ)$	=	0.299 ± 0.055	0.315
θ_{20}	=	$t(36.5^\circ)$	=	0.456 ± 0.081	0.476
θ_{21}	=	$t(52.0^\circ)$	=	0.533 ± 0.060	0.490
θ_{22}	=	$t(65.2^\circ)$	=	0.503 ± 0.048	0.474
θ_{23}	=	$t(80.5^\circ)$	=	0.472 ± 0.063	0.516

TABLE I (Cont.)

					θ_j(Theo.)
θ_{24}	=	$u(22.3^\circ)$	=	-0.330 ± 0.142	-0.020
θ_{25}	=	$u(34.4^\circ)$	=	-0.175 ± 0.084	0.012
θ_{26}	=	$u(41.8^\circ)$	=	0.111 ± 0.076	0.076
θ_{27}	=	$u(54.1^\circ)$	=	0.322 ± 0.058	0.237
θ_{28}	=	$u(70.9^\circ)$	=	0.381 ± 0.088	0.443
θ_{29}	=	$u(80.1^\circ)$	=	0.752 ± 0.114	0.509

$$r(x) = \frac{I_0(x^\circ)}{I_0(90^\circ)}$$

$$s(x) = \frac{I_0\ P(x^\circ)}{I_0(x^\circ)\ \cos x\ \sin x}$$

$$t(x) = \frac{I_0\ D(x^\circ)}{I_0\ (x^\circ)}$$

$$u(x) = \frac{I_0\ R(x^\circ)}{I_0\ (x^\circ)\ \cos \frac{x}{2}}$$

TABLE J

Summary of Solutions from Second Run

$^1S_0$	$^1D_2$	$^3P_0$	$^3P_1$	$^3F_3$	$^3F_4$	Mixing between $^3P_2$ and $^3F_2$	$^3P_2$	$^3F_2$	Ref (10^0) $\times 10^{13}$ cm^{-1}	$\sum$(or M)
$^2\delta_0$	$^2\delta_2$	$^2\delta_{10}$	$^2\delta_{11}$	$^2\delta_{33}$	$^2\delta_{34}$	$-^2\varepsilon_M$	$^2\delta_{12}$	$^2\delta_{32}$		
-0.794	0.327	0.095	-0.805	-0.014	0.031	-0.272	0.734	-0.110	0.224	26.4
-1.224	0.064	-0.892	-0.310	0.032	0.125	0.516	0.806	-0.079	0.270	27.2
0.033	-0.295	0.544	1.209	-0.068	0.012	2.476	-0.386	0.210	0.113	29.6
1.519	-0.027	1.229	0.336	-0.018	0.024	2.250	-0.609	0.235	0.172	32.4
-1.487	0.147	-1.132	-0.248	0.104	0.096	0.460	0.659	-0.163	0.232	36.0
1.010	0.131	2.212	0.229	-0.318	0.055	-2.800	-0.372	0.057	0.001	36.0
-0.544	0.000	-2.333	-0.473	0.188	-0.022	-1.248	0.489	-0.185	0.017	40.6
0.530	-0.052	-0.148	-0.214	-0.024	0.590	0.761	-0.672	0.063	0.148	41.0
-1.435	-0.130	-1.704	-0.271	0.179	0.048	-0.818	0.478	-0.070	0.007	41.1

Table continued

TABLE J (Cont.)

$^1S_0$	$^1D_2$	$^3P_0$	$^3P_1$	$^3F_3$	$^3F_4$	Mixing between $^3P_2$ and $^3F_2$	$^3P_2$	$^3F_2$	Ref(10°) $\times 10^{13} cm^{-1}$	$\sum$(or M)
$^2\delta_0$	$^2\delta_2$	$^2\delta_{10}$	$^2\delta_{11}$	$^2\delta_{33}$	$^2\delta_{34}$	$-^2\varepsilon_M$	$^2\delta_{12}$	$^2\delta_{32}$		
-0.371	0.310	-0.550	-1.175	-0.138	0.051	0.209	0.419	-0.076	-0.123	20.6
1.497	-0.043	1.267	-0.016	-0.242	0.034	3.053	-0.622	0.139	-0.198	21.8
0.547	-0.294	0.025	0.670	0.136	0.014	2.933	-0.860	0.055	-0.157	22.4
-1.325	0.040	-1.378	-0.436	0.005	0.012	0.648	0.568	-0.252	-0.179	31.8
-0.088	-0.082	-1.202	-0.729	0.226	-0.027	-1.190	0.623	-0.337	-0.116	36.0
-0.175	0.307	-2.467	-0.668	-0.045	0.046	0.072	0.227	-0.077	-0.043	41.7
0.422	-0.045	-0.139	1.053	-0.234	0.043	-3.091	-0.637	0.053	-0.122	42.9
0.645	-0.008	2.647	0.383	-0.239	0.037	-2.448	-0.418	0.095	-0.059	43.7
1.333	-0.423	1.350	0.064	-0.105	-0.010	2.625	-0.443	0.364	-0.105	48.0

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