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THEORY OF NEUTRON PRODUCTION IN QUASI-ELASTIC EXCHANGE COLLISIONS

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THEORY OF NEUTRON PRODUCTION IN QUASI-ELASTIC EXCHANGE COLLISIONS

Edward U. Vaughan

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ABSTRACT

Neutron production by protons of energy between 100 Mev and 300 Mev is investigated by means of the impulse approximation. It is found that a condition for the analysis to be possible without knowledge of the phases of the neutron-proton interaction is that this interaction be a function only of the momentum transfer in the scattering process, independent of the energy of the collision. Evidence is presented that the condition holds approximately for the features of neutron-proton scattering that have been investigated hitherto. The analysis of neutron production suggests that the impulse approximation is of doubtful validity at 100 Mev for nuclei having $A \geqslant 7$, but improves with increasing energy up to 300 Mev. Some of the main features of the process appear to be reproduced by a theory which takes very little account of the structure of the target nucleus, beyond the momentum distribution of its neutrons and the effects of the exclusion principle involving its protons. Effects of attenuation appear to be less important. In the course of the argument a method of representing effects of the exclusion principle is developed, in which excited states associated with a given ground state, and thus orthogonal to it, are approximated by a variational modification of plane waves which preserves their mutual orthogonality.

1. INTRODUCTION

The ease with which a neutron beam can be produced by exchange reactions in high-energy cyclotrons has led to extensive experimental work on the scattering of neutrons, especially by protons.[1-12] A necessary preliminary to this work has been the investigation of the energy spectrum of the neutron beam itself. As a result, there is now considerable information on the spectrum of neutrons produced in the forward direction by protons incident with high energy on light nuclei (see References 1, 2, 5, 6, and 10-15). On the other hand, no such motivation has promoted studies of the yield and angular distribution of these neutrons, and consequently these features have received relatively little attention.[16-21]

The forward spectrum changes only moderately with changes in proton energy or target element, exhibiting none of the highly specific features characteristic of low-energy nuclear reactions. This suggests an investigation of the possibility that well-known general features of nuclear theory, rather than imperfectly known details, may dominate the process, and may be adequate to account for the observations. Such a program receives some encouragement from a theoretical investigation of the process by Mandl and Skyrme,[22] which agrees in some respects with the experiments, and is so simple that its improvement may not lead to insuperable complexity.

The most promising method of treating scattering problems with complex nuclei as targets is the impulse approximation of Chew.[23] It appears that the necessity of knowing the phases of the elementary scattering matrix elements used in this approximation can frequently be avoided by some means; in such cases, sufficient information about the elementary scattering process is contained in its cross section. The scattering from the complex target is then determined, after the elimination of the incident proton and product neutron, by matrix elements of the target nucleus of simple structure independent of the nuclear interactions that give rise to the scattering. The problem that remains is that of representing the nuclear states in a manner simple enough to permit the necessary integrations, yet capable of yielding results of reasonable accuracy.

Applications of the impulse approximation have been restricted, for the most part, to the deuteron[24]--whose simplicity permits adequate representation of its states--or else to processes that may be treated by the use of sum rules.[26,28] As the problem treated here cannot be simplified by either of these methods, the nuclear model used will have to be oversimplified. A form of independent-particle model has accordingly been adopted; calculations become very difficult for anything more elaborate, and the success of the nuclear shell model gives empirical support to this choice.

The states occupied by these individual nucleons remain to be determined. A certain amount of information is available on the momentum distribution of nucleons in nuclear ground states,[25-30] but for excited states nothing better than plane waves seems to be known. These states have a serious deficiency in that their lack of orthogonality to the ground states creates difficulties in treating the exclusion principle. (The corresponding objection to the use of plane waves for the incident proton and product neutron, as required by the impulse approximation, is much less serious, because these nucleons have much greater momenta--and thus much shorter wave lengths--than the target nucleons). Now, the exclusion principle is a general quantum mechanical feature, which may be expected to act similarly in many nuclei, rather than one of the specific details whose neglect is suggested by the experiments. There is reason to believe that it is in fact quite important in inelastic scattering processes; for example, it is invoked to account for the fact that nuclei are more transparent than they would be if their constituent nucleons were as effective attenuators as free nucleons.[31,32] It would thus be advantageous to improve upon the plane-wave excited states at least enough to achieve consistency with the exclusion principle.

In view of the present fragmentary state of knowledge of nuclear forces and nuclear states, it appears that the above-mentioned ground-state momentum distributions constitute the only positive information available. Now, there exists in the theory of metals a method of constructing excited states in terms of lower-lying states.[33] This amounts to subtracting from plane waves their overlap onto the lower-

energy states. The only property of the excited states used in this procedure is their orthogonality to the lower states. Their mutual orthogonality within a single band is ensured by their having different propagation vectors. Higher bands can be obtained by repetition of the process, with the inclusion of the just-completed band among the states to which the states of the new band must be orthogonal.

The method cannot be applied in this simple form to the construction of nuclear states. There is no lattice structure, and consequent propagation vector, to ensure mutual orthogonality within bands, while the inductive method of successive construction of states is difficult to apply because the spectrum is continuous. The method has accordingly been generalized by replacing the use of plane waves by the variational condition for free-particle states--a condition that would be equivalent to using plane waves, in the absence of supplementary conditions. The reason for using this device is that it ensures the mutual orthogonality of the resulting states. The orthogonality to the ground states is readily achieved by imposing it as a supplementary condition, and this condition enforces a departure from plane-wave states.

Since the states thus constructed are related only indirectly to actual conditions in the nucleus, their use is legitimate only if the results do not depend critically upon detailed properties of the states. The purpose of the elaborate definition is merely to ensure consistency, and of course the consistency condition does not determine the states uniquely. In order to check the supposed insensitivity to the choice of states, a separate calculation has been performed by use of harmonic oscillator states instead of plane waves. These are easier to work with, as the required orthogonality is ensured if the ground state of the harmonic oscillator is given the same spread in momentum space as that observed in nuclear ground states. The harmonic oscillator states are less realistic than the modified plane waves, but may well be suitable for the proposed use in checking, because they are so different from plane waves. In particular, they have a discrete spectrum, while plane waves have a continuous spectrum.

Even such simple states as these become very complicated in practice if the target nucleus is larger than an α particle. The fact that experiments show no great changes in neutron spectrum associated with the transition from light to heavy nuclei encourages the attempt to seek further simplification. Accordingly, the calculations have been performed for a single α particle as target, in the expectation of predicting results for other light nuclei by adding (without interference) the results from their constituent α particles. It may be added that the α-particle model is supported by previous successes, such as explanations of the π^+/π^- ratio in meson production by nucleons,[34] and of the character of the stars arising from the capture of π^- mesons by nuclei.[35]

Since some important targets cannot be considered to be composed entirely of α particles, it is necessary to add contributions from odd nucleons. The importance of the odd neutron in Be^9 is very apparent from a comparison of the neutron spectrum and neutron yield from Be with those from C, as has been noted by previous investigators.[18, 21] Here again it is proposed to ignore interference between contributions from the odd nucleon and the core of α particles.

Thus, the odd particles are treated as independent particles, and each α particle is also treated independently--moreover, each α particle is itself treated by an independent-particle model. In the last step, the center-of-mass recoil of the α particle as a whole has been ignored.

In atomic physics it is customary to justify each approximation employed by applying certain validity criteria. Such criteria are known for some of the above-discussed approximations,[36] but cannot be applied rigorously because the nuclear forces are not sufficiently well known. The danger of relying on deductive discussion of validity of approximations is shown by the success of the nuclear shell model, despite grave a priori doubts as to the correctness of an independent-particle approach to nuclear problems. It seems that the best way to test approximations in nuclear theory is by comparing their consequences with experiment. Thus, in this work, the validity of the approximations is a hypothesis. This justifies the use of whatever consistent

simplifications may be necessary to make the calculations feasible. The resulting approximations are necessary to make the calculations possible, rather than demonstrably sufficient to give adequate accuracy.

2. NUCLEAR MATRIX ELEMENTS

The scattering process to be considered is an exchange collision, in which an incident proton loses its charge in the target, and is detected as a product neutron. Since, however, this nucleon will be considered to have retained its identity, while its i-spin state has changed, it will always be designated by the same subscript 0. As this use will be in connection with operators, there should be no confusion with the use of the same subscript with states to indicate initial states. This nucleon may be referred to as the fast particle.

As a consequence of the impulse approximation, one of the target neutrons differs from the others by being the one struck. Since this is the one that receives the charge lost by the fast particle, it will sometimes be referred to, when in its final state, as the residual proton.

The initial and final states of the target nucleus may be called ψ_0 and ψ_f respectively. In its final state the nucleus may be referred to as the residual nucleus; in this usage, the residual proton is considered as a part of the residual nucleus.

Following the above conventions, the spin and i-spin of the fast particle are called $\vec{\sigma}_0$ and $\vec{\tau}_0$ respectively. Since the impulse approximation implies the use of plane waves for the states of the fast nucleon, the initial and final momenta of this nucleon are c-numbers, which may be called $\vec{k}_0$ and $\vec{k}_f$ respectively. The momentum $\vec{k}_0 - \vec{k}_f$ transferred to the target is designated $\vec{\Delta k}$. The initial and final states of spin and i-spin will be denoted by χ_0 and χ_f respectively.

The operators for target nucleons are $\vec{k}_i$, $\vec{\sigma}_i$, and $\vec{\tau}_i$. The initial and final values of the momentum $\vec{k}_i$ are $\vec{k}_i'$ and $\vec{k}_i''$ respectively. As a consequence of the conservation of momentum, the difference $\vec{\Delta k}_i = \vec{k}_i'' - \vec{k}_i'$, while vanishing for most nucleons, must have the value $\vec{\Delta k}$ for the struck nucleon.

As a consequence of (Galilean) relativity and spatial homogeneity, the interaction between the fast nucleon and the struck nucleon requires for its specification not four independent momenta (initial and final

for each nucleon), but only two. These may be taken to be the momentum transfer $\overrightarrow{\Delta k}$ and the complementary momentum transfer $\overrightarrow{q_i}$ resulting from the interchange of the final states of the two particles designated 0 and i. The latter momentum is defined by the relations

$$\overrightarrow{q_i} = \overrightarrow{k_0} - \overrightarrow{k_i''} = \overrightarrow{k_f} - \overrightarrow{k_i'}.$$

That this operator is defined only for the struck nucleon is indicated clearly by the equality on the right, which represents the conservation of momentum for this nucleon.

The interaction between the fast nucleon and the target may be written, according to the impulse approximation, in the form

$$R = \sum_{i=1}^{A} \tau_0^- \tau_i^+ R(0,i) \; \delta(\overrightarrow{\Delta k_i} - \overrightarrow{\Delta k}).$$

The δ function expresses conservation of momentum, while the factor $\tau_0^- \tau_i^+$ changes the fast particle from proton to neutron, and the struck particle from neutron to proton, as required.

The cross section is controlled by a matrix element of the form

$$M_{f0} = \left(\psi_f \chi_f, \left[\tau_0^- \sum_{i=1}^{A} \tau_i^+ R(0,i) \; \delta(\overrightarrow{\Delta k_i} - \overrightarrow{\Delta k})\right] \chi_0 \psi_0\right).$$

The elementary scattering operator R(0, i) has been investigated generally by Wolfenstein and Ashkin,[37] for the case in which the struck nucleon is free, rather than bound into a nucleus. This case will be sufficient for the present purpose, since the idea of the impulse approximation includes ignoring the effects of binding on the scattering operator. The result obtained by Wolfenstein and Ashkin may be expressed in the form

$$R(0,i) = a(\Delta k^2, q_i^2) + b(\Delta k^2, q_i^2)\overrightarrow{n_i} \cdot (\overrightarrow{\sigma_i} + \overrightarrow{\sigma_0}) + \mathcal{M}_{\lambda\mu}(\overrightarrow{\Delta k}, \overrightarrow{q_i}) \; \sigma_i^{\lambda} \sigma_0^{\mu},$$

where

$$\overrightarrow{n_i} = \overrightarrow{\Delta k} \times \overrightarrow{q_i}.$$

The "normal vector" $\vec{n}_i$ used here differs from that used by Wolfenstein and Ashkin, being in fact just half of their $\vec{n}$. These authors included in their interaction a term in $\vec{n}_i \cdot (\vec{\sigma}_0 - \vec{\sigma}_i)$, but this has been suppressed as incompatible with the well-established charge symmetry of nuclear forces. They also give a more explicit form of the tensor $\mathcal{M}_{\lambda\mu}$, but it will be sufficient for the present purpose to note that there is no loss of generality in requiring it to be symmetric in λ and μ.

The following notation will be employed in the treatment of this interaction:

$$S_i = a_i + b_i \vec{n}_i \cdot \vec{\sigma}_i, \qquad S = \sum_{i=1}^{A} \tau_i^+ S_i \; \delta(\Delta\vec{k}_i - \Delta\vec{k}),$$

$$V_i^\lambda = b_i n_i^\lambda + \mathcal{M}_{\lambda\mu} \sigma_i^\mu, \qquad \vec{V} = \sum_{i=1}^{A} \tau_i^+ \vec{V}_i \; \delta(\Delta\vec{k}_i - \Delta\vec{k}).$$

Thus, the key equations take the forms

$$R(0, i) = S_i + \vec{V}_i \cdot \vec{\sigma}_0,$$

$$R = \tau_0^- [S + \vec{V} \cdot \vec{\sigma}_0],$$

$$M_{f0} = (\psi_f \chi_f, \tau_0^- [S + \vec{V} \cdot \vec{\sigma}_0] \chi_0 \psi_0)$$

It is relatively simple to get a form for the matrix element in which the fast-particle operators no longer appear. In the first place, the i-spin states of this particle are known, so that the states χ_0 and χ_f take the respective forms $p(0)\, \chi_0(\vec{\sigma}_0)$ and $n(0)\, \chi_f(\vec{\sigma}_0)$. Here $p(0)$ and $n(0)$ are the two possible i-spin states, proton and neutron, while $\chi_0(\sigma_0)$ and $\chi_f(\sigma_0)$ are spin (but not i-spin) states. Since $(n(0), \tau_0^- p(0)) = 1$, the matrix element becomes

$$M_{f0} = \left(\psi_f \chi_f(\vec{\sigma}_0), [S + \vec{V} \cdot \vec{\sigma}_0] \chi_0(\vec{\sigma}_0) \psi_0\right).$$

In contrast to the i-spin states, the spin states of the fast particle are entirely unrestricted. It is thus necessary to square the matrix element and sum over final states and average over initial states. As there are two initial spin states, the average requires a factor 1/2, and the fact that initial and final states form complete

sets results in the reduction of the sum to a trace:

$$\sum_{\sigma_0} |M_{f0}|^2 = 1/2 \operatorname*{Tr}_{(\sigma_0)} [(\psi_0, \{S^* + \vec{V}^* \cdot \vec{\sigma_0}\} \psi_f)(\psi_f, \{S + \vec{V} \cdot \vec{\sigma_0}\} \psi_0)].$$

The trace is evaluated immediately by means of the well-known relations

$$(\vec{\sigma} \cdot \vec{a})(\vec{\sigma} \cdot \vec{b}) = a \cdot b + i \vec{\sigma} \cdot \vec{a} \times \vec{b},$$

$$(1/2) \operatorname*{Tr}_{(\sigma)} [1] = 1, \qquad \operatorname*{Tr}_{(\sigma)} [\vec{\sigma}] = 0.$$

The result is

$$\sum_{\sigma_0} |M_{f0}|^2 = (\psi_0, S^* \psi_f)(\psi_f, S \psi_0) + (\psi_0, \vec{V}^* \psi_f) \cdot (\psi_f, \vec{V} \psi_0)$$

$$= |S_{f0}|^2 + |V_{f0}|^2.$$

The fast-particle operators have now been suppressed, and the matrix element has been expressed in terms of the purely nuclear matrix elements

$$S_{f0} = (\psi_f, S \psi_0) \quad \text{and} \quad \vec{V}_{f0} = (\psi_f, \vec{V} \psi_0).$$

3. EXPRESSION OF NUCLEAR MATRIX ELEMENTS IN TERMS OF FORM FACTORS

The three components of the vector nuclear matrix element $\vec{V}_{f0}$ have the same structure as the scalar matrix element S_{f0}; consequently, it will be sufficient to treat the latter. It may be written in the form

$$S_{f0} = \left(\psi_f, [\sum_{i=1}^{A} \tau_i^+ \; S_i \; \sigma(\vec{\Delta k}_i - \vec{\Delta k})] \, \psi_0\right) .$$

It is now necessary to consider the nature of the states ψ_0 and ψ_f. Despite the simplifications of the impulse approximation, it will not be possible to evaluate the matrix elements without extreme simplification of these states.

In the first place, the target nucleus will be considered to be an aggregate of α particles. Interference between the constituent α particles will be ignored. Odd-odd nuclei, and nuclei with excess protons, will not be treated; consequently residual nucleons too few in number to form an α particle will constitute either a single neutron or a triton. The targets Li^7, Be^9, and C^{12} may all be treated by this scheme, using for ψ_0 and ψ_f states of four particles or fewer. Moreover, ψ_0 will be the ground state of this set of nucleons, and consequently its spin and i-spin dependence can be included in a separate factor,

$$\psi_0(1 \ldots A) = \chi_0(\ldots \sigma_i, \tau_i \ldots) \, \phi_0(\ldots \vec{k}_i \ldots) .$$

In the second place, the new simple targets will be treated by means of the Hartree or independent-particle model. That is, the nucleons will have their individual states, and the state of the system will in general be a determinant or antisymmetrized product of these states. For ψ_0, of course, the factor χ_0 alone will be antisymmetrized, and the factor ϕ_0 will be a simple product whose symmetry is ensured by putting all nucleons in the same single-particle state.

This approximation does not treat correctly the center-of-mass motion of the target. In effect, instead of binding the nucleons together, it binds each of them to a fixed center of infinite mass. Since the particles treated are in fact bound to the rest of the actual target nucleus, the principal error involved is the neglect of the recoil of

this nucleus. This recoil may not be uncomfortably large, for targets of seven or more nucleons.

A more serious error is that effects of other correlations between the nucleonic motions are also lost. This, however, is the point of the approximation, because such correlations are unknown, and possibly very complicated.

The correct states will be linear combinations of the states here employed, and vice versa. Use of the incorrect states will yield tolerably good results provided the component correct states lie within a range of energy not larger than the resolution of the experiments, which is perhaps 5 Mev, and provided interference effects resulting from the superposition are random.

If ψ_0 is properly antisymmetrized, then the symmetry of the interaction ensures that only antisymmetric states ψ_f will contribute. It is thus unnecessary to write ψ_f as a determinant--a simple product will suffice. In order for this to be true, however, it is necessary that the individual particle states used be mutually orthogonal, to avoid the appearance of new normalizing factors containing the non-vanishing overlap integrals. If orthogonality holds, the spin and i-spin factors of the product state may be collected separately:

$$\psi_f = \chi_f(\ldots \sigma_i, \tau_i \ldots)\, \phi_f(\ldots \vec{k}_i'' \ldots) \quad .$$

The matrix element now takes the form

$$S_{f0} = \left(\chi_f\, \phi_f, \left[\sum_{i=1}^{A} \tau_i^{+}\, S_i\, \delta(\vec{\Delta k}_i - \vec{\Delta k})\right] \phi_0 \quad \chi_0\right).$$

Since the term containing S_i is the identity operator for all nucleons except that indicated by the index i, the state ϕ_f can differ from the state ϕ_0 only with respect to this nucleon. Thus, the state ϕ_f must have all particles except particle i in their ground states $\phi_0(j)$, and particle i in one of the excited single-particle states $\phi_{\vec{k}}(i)$--$\vec{k}$ being an index which specifies the particular state. The index f thus includes specification of the excited particle i, and the summation over i is suppressed. Then S_{f0} takes the form

$$S_{f0} = \left(\chi_f, \left[\tau_i^{+} \left(\phi_{\vec{k}}(i),\ S_i\ \delta(\vec{\Delta k}_i - \vec{\Delta k})\ \phi_0(i)\right)\right]\ \chi_0\right).$$

The fact that the operator S_i depends on the operator $\vec{q}_i = \vec{k}_0 - \vec{k}''_i = \vec{k}_f - \vec{k}'_i$, and thus on the momentum operator $\vec{k}_i$, makes the result of further reduction of this expression for S_{f0} too complicated to be useful. If, however, the state $\phi_0(i)$ has a sufficiently narrow spread in momentum space, it is permissible to replace $\vec{q}_i$ by $\vec{k}_f$. This approximation will also be satisfactory if S_i is sufficiently insensitive to $\vec{q}_i$.

Comparison of the results of neutron-proton scattering experiments near 180° at various energies shows that the cross sections for the same momentum transfer do not differ greatly. For example, the differential cross section at several energies is plotted against exchange momentum transfer in Fig. 1. The fact that the curves for different energies are not far apart when Δk is not too large shows that the cross section is not sensitive in this range to any variable except Δk. In particular, it is not sensitive to q. Accordingly, the use of the mean value $\vec{k}_f$ for $\vec{q}_i$ may not lead to unduly large errors. The argument is of course incomplete, since it applies to the entire cross section, and would fail to disclose large variations of the separate terms if they were in opposite directions.

It is true that evidence for $\vec{q}$-dependence of the neutron-proton interaction is provided by the experiments that have been performed[39] on polarization in neutron-proton scattering. In fact, the polarization is parallel to the normal vector $\vec{n} = \Delta\vec{k} \times \vec{q}$, and thus cannot fail to vary with $\vec{q}$. Examination of these experiments, nevertheless, is not entirely discouraging. It appears that the polarization vanishes near 90° in the center-of-mass system, as well as at 180°, and in the intervening range of angles has a maximum of the order of 20 %. This range of angles corresponds to the range of Δk within which q-independence of the cross section is demonstrated in Fig. 1. Such q-dependence of individual terms as may be associated with the possibility of polarization is seen to be moderately small in the interesting range, and may reasonably be treated less carefully than the main (unpolarized) contribution.

The result of replacing $\vec{q}_i$ by $\vec{k}_f$ in S_i will be denoted by $\bar{S}_i$. As $\bar{S}_i$ is independent of the momentum $\vec{k}_i$, the matrix element is

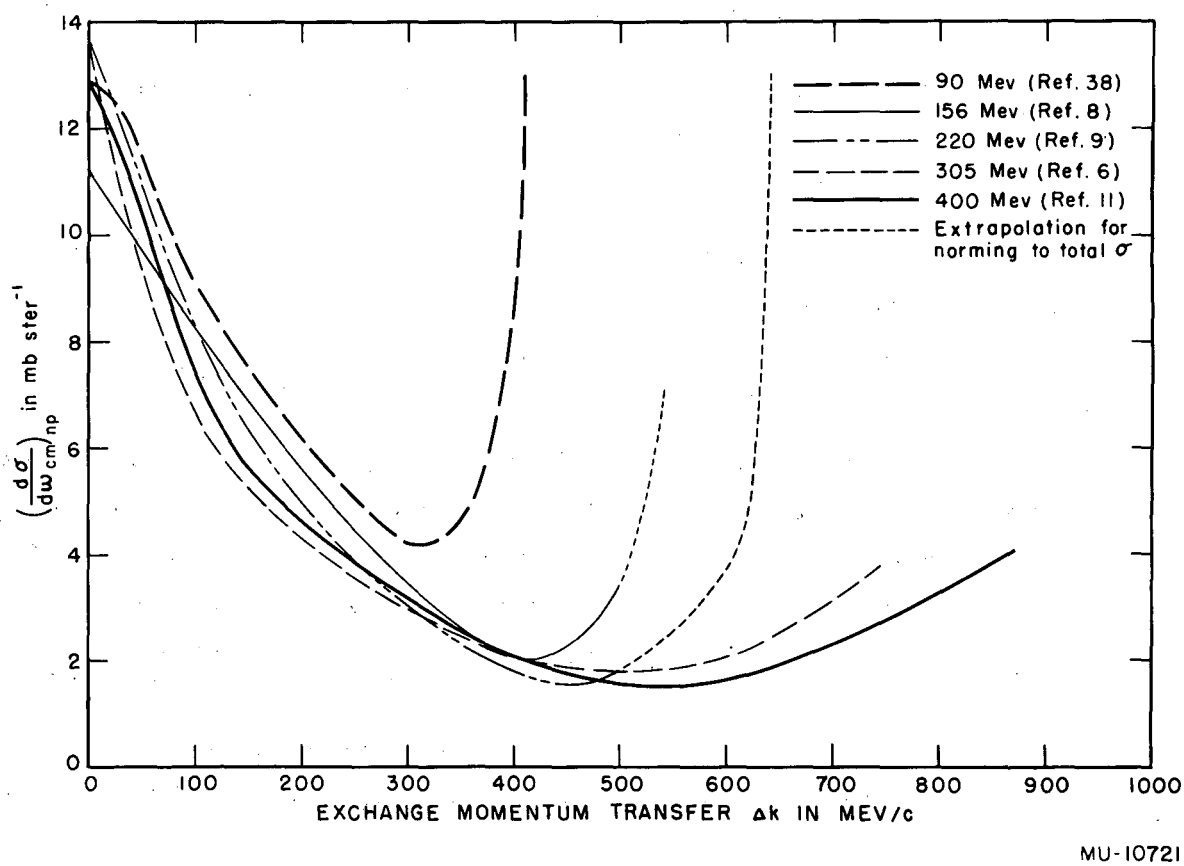

Fig. 1. Neutron-proton differential cross section as a function of momentum transfer.

considerably simplified:

$$S_{f0} = \left(\phi_{\vec{k}}(i),\ \delta(\vec{\Delta k}_i - \vec{\Delta k})\ \phi_0(i)\right)(\chi_f, [\tau_i^+\ \bar{S}_i]\ \chi_0).$$

The factor in the ϕ's is independent of the index i, and depends only on the momentum transfer $\vec{\Delta k}$, and on the quantum number $\vec{k}$ which specifies the final state. It is convenient to introduce the definition

$$F_{\vec{k}}(\vec{\Delta k}) = \left(\phi_{\vec{k}}(i),\ \delta(\vec{\Delta k}_i - \vec{\Delta k})\ \phi_0(i)\right)$$

$$= \left(\phi_{\vec{k}}(i),\ e^{i\vec{\Delta k}\cdot\vec{x}_i}\ \phi_0(i)\right),$$

in which the last step is an application of the representation of the momentum transfer operator $\delta(\vec{\Delta k}_i - \vec{\Delta k})$ in configuration space as $\delta(\vec{x}_i'' - \vec{x}_i')\ e^{i\vec{\Delta k}\cdot\vec{x}_i'}$. Substitution now yields

$$S_{f0} = F_{\vec{k}}(\vec{\Delta k})\,(\chi_f, \tau_i^+\ \bar{S}_i\ \chi_0).$$

It is now easy to sum the square of S_{f0} over the states χ_f, since these states constitute a complete set:

$$\sum_{\sigma,\tau} |S_{f0}|^2 = |F_{\vec{k}}(\vec{\Delta k})|^2\ (\chi_0, \tau_i^-\ \tau_i^+\ \bar{S}_i^*\ \bar{S}_i\ \chi_0).$$

The significance of the index i is that it specifies which of the target nucleons is excited in the final state. Which one it is makes no difference to the energy; consequently it is necessary to sum over all of them to get the entire contribution to the spectrum:

$$\sum_{i,\sigma,\tau} |S_{f0}|^2 = |F_{\vec{k}}(\vec{\Delta k})|^2 (\chi_0, [\sum_{i=1}^{A} \tau_i^-\ \tau_i^+\ \bar{S}_i^*\bar{S}_i]\ \chi_0).$$

The further reduction requires the explicit exhibition of $\bar{S}_i$, in order to exploit the form of its spin dependence. Consider first the definition

$S_i = a_i + b_i\ \vec{n}_i\cdot\vec{\sigma}_i$. Since a_i, b_i, and $\vec{n}_i$ depend on the nucleon i only through the operator $\vec{q}_i$ --which, however, is to be replaced by $\vec{k}_f$, and thus loses its dependence on the nucleon i --the index i can be suppressed in the notation for the mean values of these operators. There results

$$\bar{S}_i = \bar{a} + \bar{b}\,\vec{\bar{n}}\cdot\vec{\sigma}_i\,,$$

$$\bar{S}_i^{\,*}\,\bar{S}_i = |\bar{a}|^2 + |\bar{b}|^2\,\bar{n}^2 + (\bar{a}^*\bar{b} + \bar{b}^*\bar{a})\,\vec{\bar{n}}\cdot\vec{\sigma}_i\,.$$

If the state χ_0 is not a singlet in the spins, it will be necessary to employ the average of $\sum_{i,\sigma,\tau} |S_{f0}|^2$ over all orientations of the initial total spin. The result will then always be spherically symmetric in the spin space; thus, the term in $\vec{\bar{n}}\cdot\vec{\sigma}_i$ cannot contribute. Consequently, an expression independent of the spins is obtained:

$$\sum_{i,\sigma,\tau} |S_{f0}|^2 = |F_{\vec{k}}(\overrightarrow{\Delta k})|^2\,\{\,|\bar{a}|^2 + |\bar{b}|^2\,\bar{n}^2\}(\chi_0,[\sum_{i=1}^{A}\tau_i^-\,\tau_i^+]\,\chi_0).$$

The operator $\tau_i^-\,\tau_i^+$ annihilates a proton, but gives unity when applied to a neutron. It is thus the projection operator onto neutrons, and the sum $\sum_{i=1}^{A}\tau_i^-\,\tau_i^+$ is the number of neutrons N in the initial state of the target nucleus. It follows that

$$\sum_{i,\sigma,\tau} |S_{f0}|^2 = N\,\{\,|\bar{a}|^2 + |\bar{b}|^2\,\bar{n}^2\}\,|F_{\vec{k}}(\overrightarrow{\Delta k})|^2.$$

The quantity $\bar{n}^2$ may be evaluated as follows:

$$\bar{n}^2 = (\overrightarrow{\Delta k}\times\vec{k}_f)^2 = \overrightarrow{\Delta k}\cdot\vec{k}_f\times(\overrightarrow{\Delta k}\times\vec{k}_f) = \overrightarrow{\Delta k}\cdot[k_f^2\,\overrightarrow{\Delta k} - (\vec{k}_f\cdot\overrightarrow{\Delta k})\vec{k}_f]$$

$$= k_f^2\,\Delta k^2 - (\vec{k}_f\cdot\overrightarrow{\Delta k})^2\,.$$

For a free-nucleon target, $\vec{k}_f\cdot\overrightarrow{\Delta k} = 0$. In order to employ the results of free-nucleon scattering experiments, it is therefore necessary to ignore the term in $(\vec{k}_f\cdot\overrightarrow{\Delta k})^2$. This is somewhat unsatisfactory in the forward direction, where $\vec{k}_0$, $\vec{k}_f$, and $\overrightarrow{\Delta k}$ are all parallel, for then $\vec{k}_f\cdot\overrightarrow{\Delta k} = k_f\,\Delta k$ holds, rather than $\vec{k}_f\cdot\overrightarrow{\Delta k} << k_f\,\Delta k$. Fortunately, this difficulty affects only the term in $|\bar{b}|^2$. It is probable, from the previously mentioned experiments on polarization in neutron-proton exchange collisions,[39] that b does not exceed 20% of the other coefficients in the relevant range of Δk. Thus, $|\bar{b}|^2$ contributes less than 4% of the total cross section, and neglect of the term in $(\overrightarrow{\Delta k}\cdot\vec{k}_f)^2$ leads to a relatively small error. The sum now appears as

$$\sum_{i,\sigma,\tau} |S_{f0}|^2 = N\,\{\,|\bar{a}|^2 + |\bar{b}|^2\,\Delta k^2 k_f^2\,\}\,|F_{\vec{k}}(\overrightarrow{\Delta k})|^2\,.$$

The result of a similar evaluation of $|V_{f0}|^2$ must be

$$\sum_{i,\sigma,\tau} |V_{f0}|^2 = N \{ |\bar{b}|^2 \Delta k^2 k_f^2 + \bar{\mathcal{M}}_{\lambda\mu}^* \bar{\mathcal{M}}_{\lambda\mu} \} \; |F_{\vec{k}}(\vec{\Delta k})|^2 .$$

In this expression also, any dependence of $|\bar{\mathcal{M}}|^2$ on $\vec{\Delta k} \cdot \vec{k}_f$ must be ignored. Invariance under time reversal requires that such dependence be approximately <u>even</u> in any case, since $\vec{\Delta k} \cdot \vec{k}_f$ is an approximation to $\vec{\Delta k} \cdot \vec{q}$, which changes sign under time reversal.

The cross section depends on the sum of the scalar and vector contributions. In this sum, $N|F_{\vec{k}}(\vec{\Delta k})|^2$ is a common factor. It is thus convenient to introduce a new definition:

$$|\bar{R}|^2 = |\bar{a}|^2 + 2|\bar{b}|^2 k_f^2 \Delta k^2 + |\mathcal{M}|^2 .$$

The sum then takes the simple form

$$\sum_{i,\sigma,\tau} |M_{f0}|^2 = N|\bar{R}|^2 |F_{\vec{k}}(\vec{\Delta k})|^2 .$$

In the case of a free nucleon target, $|F_{\vec{k}}(\vec{\Delta k})|^2$ reduces to the δ-function that imposes conservation of momentum. Thus $|\bar{R}|^2$ is the matrix element for neutron-proton scattering, and accordingly is observable experimentally. In particular, thanks to the ignoring of $\vec{k}$-dependence, knowledge of the <u>phases</u> in the interaction is not needed.

The square of the matrix element has not yet been summed over all the states that contribute to the spectrum at a given energy. Some type of angular degeneracy exists for the states $\phi_{\vec{k}}(i)$, and this has not yet been summed over. In fact, $\vec{k}$ represents a set of three quantum numbers, of which the energy of the state constitutes only one. The summation over the remaining two may be denoted $\Sigma_{\hat{k}}$, where $\hat{k}$ denotes the unit vector $\vec{k}/k$. No cases in which the sum is an integral, $\Sigma_{\hat{k}}$ will be interpreted to signify $\int d\hat{k}$; a factor such as $k^2/(2\pi)^3$, which might have been expected, will be included later with the density of final states.

When $|F_{\vec{k}}(\vec{\Delta k})|^2$ is summed over $\hat{k}$, no vector remains other than $\vec{\Delta k}$. Consequently, the result cannot depend on the direction of this vector, but only on its magnitude. A suitable notation is then

$|F_k(\Delta k)|^2 = \Sigma_{\hat{k}} |F_{\vec{k}}(\vec{\Delta k})|^2$. The neutron intensity is thus proportional to the quantity

$$\sum_f |M_{f0}|^2 = N|\bar{R}|^2 |F_k(\Delta k)|^2 .$$

4. CONSTRUCTION OF MODIFIED PLANE-WAVE STATES

Although the use of an individual-particle model for the nucleus leads to certain general relations, specific predictions become possible only when the individual particle states are known or assumed. Perhaps the best-known assumption of this kind is the use of plane-wave states, which are often used for the sake of simplicity even when their validity is doubtful.

It is to be noted, however, that plane waves are unsuitable as bound states of the nucleons, and are available as approximations only for states in the continuum. Since the concept of a transition, which is fundamental in scattering problems, has no meaning except between orthogonal states, it follows that the states used in the continuum must be orthogonal to the bound states. The latter, however, are not orthogonal to plane waves; such waves must therefore be modified. Moreover, it is necessary that the transitions be mutually exclusive, which means that the continuum states must be mutually orthogonal.

It is possible to construct a set of states, orthogonal to given states and to each other, yet retaining some of the simplicity of plane waves. The procedure is to set up a variational condition and boundary conditions which together define plane waves, and then to impose the orthogonality to the given states as supplementary conditions. These conditions are introduced into the variation problem by Lagrangian multipliers, which are subsequently determined so that the conditions will be satisfied.

A certain awkardness arises from the fact that those states to which orthogonality is required are most conveniently represented in polar coordinates, whereas such coordinates are inconvenient for treating plane waves. It is necessary to expand the plane waves in spherical waves, and to express the various conditions in terms of these expansion functions.

The argument calls, at various points, for the use of certain properties of the spherical Bessel functions. The required properties, with notations and definitions, are collected at this point for reference.

Any spherical Bessel function $Z_\ell(kr)$, which belongs with angular momentum ℓ and has radial momentum k, satisfies the differential equation

$$\frac{d^2[\, r\, Z_\ell(kr)]}{dr^2} + [\, k^2 - \frac{\ell(\ell+1)}{r^2}]\;\; r\, Z_\ell(kr) = 0.$$

The proposed program will exploit the fact that this equation is equivalent to the variational condition

$$\delta(k^2) = 0,$$

where

$$k^2 = \frac{\int_0^\infty dr\; \{(\frac{d[r\, Z_\ell(kr)]}{dr})^2 + \ell(\ell+1)\, Z_\ell^{\,2}(kr)\}}{\int_0^\infty dr\, r^2\, Z_\ell^{\,2}(kr)}$$

The function regular at the origin will be denoted by $j_\ell(kr)$. The notation $h_\ell^{\pm}(kr)$ will be employed for the "spherical Hankel functions," which behave asymptotically like

$(kr)^{-1}\, e^{\pm i\,(kr - \frac{\pi}{2}\ell)}$ respectively. The other real function will be called $n_\ell(kr)$. The relations between these functions are:

$$h_\ell^{\pm}(kr) = n_\ell(kr) \pm i\, j_\ell(kr) \qquad n_\ell(kr) = (1/2)[h_\ell^{+}(kr) + h_\ell^{-}(kr)]$$

$$j_\ell(kr) = (1/2i)[h_\ell^{+}(kr) - h_\ell^{-}(kr)]$$

The definition of the spherical Neumann functions n_ℓ and of the spherical Hankel functions $h_\ell^{\pm}$ differ by factors -1 and $\pm i$ respectively from those given by Morse and Feshbach[40] and by Sommerfeld.[41]

Explicit representations for the case $\ell = 0$ are

$$h_0^{\pm}(kr) = \frac{e^{\pm i\, kr}}{kr}\;, \qquad j_0(kr) = \frac{\sin kr}{kr}\;, \qquad n_0(kr) = \frac{\cos kr}{kr}\;.$$

Finally, the relation between plane and spherical waves can be written in the equivalent forms

$$e^{i\vec{k}\cdot\vec{r}} = \sum_{\ell=0}^{\infty} (2\ell+1)\, i^{\ell}\, j_{\ell}(kr)\, P_{\ell}(\hat{k}\cdot\hat{r}),$$

$$4\pi j_{\ell}(kr) = \int d\hat{k}\, e^{i(\vec{k}\cdot\vec{r} - \frac{\pi}{2}\ell)}\, P_{\ell}(\hat{k}\cdot\hat{r}) \quad .$$

In these expressions, $\hat{k}$ and $\hat{r}$ are unit vectors in the directions $\vec{k}$ and $\vec{r}$ respectively. Their components are direction cosines, and they may as well be called <u>direction vectors</u>. The notation $\int d\hat{k}$ is used for integration with respect to direction over the entire solid angle.

By analogy with the expansion of plane waves in terms of spherical waves, the desired modified plane-wave state may be expanded in the form

$$\psi_{\vec{k}}(\vec{r}) = \sum_{\ell=0}^{\infty} (2\ell+1)\, i^{\ell}\, P_{\ell}(\hat{k}\cdot\hat{r})\, f_{\ell}(k,r) \quad .$$

In order to complete the specification, the boundary condition at inifinite distance is required. There is a certain amount of arbitrariness about this condition. A suitable choice is the one used for scattering problems, which is that the incident flux be that of a plane wave, so that the state differs from a plane wave only in the outward flux. This condition can be expressed in terms of the radial functions $f_{\ell}(k,r)$ in the form

$f_{\ell}(k,r) - j_{\ell}(k\, r) \sim A_{\ell}\, e^{ikr}/kr$. Here A_{ℓ} is the amplitude of the scattered wave with spherical harmonic of order ℓ, and is to be determined in the course of the argument.

The functions $f_{\ell}(k,r)$ are now required to satisfy the same variational condition as the functions $j_{\ell}(kr)$:

$\delta(k^2) = 0,$

$$k^2 = \frac{\int_0^{\infty} dr\, \left\{\left(\frac{d\,[r f_{\ell}(k,r)]}{dr}\right)^2 + \ell(\ell+1)\, f_{\ell}^{\,2}(k,r)\right\}}{\int_0^{\infty} dr\, r^2\, f_{\ell}^{\,2}(k,r)}$$

Finally, the functions $f_\ell(k, r)$ are to be distinguished from the function $j_\ell(kr)$ by requiring them to be orthogonal to certain functions. The states to which orthogonality is imposed may be represented in spherical coordinates, and have the form

$$Y_\ell^m(\hat{r})\, \phi_{\ell,n}(r) \quad .$$

Of course, the number of values of n for any value ℓ may in principle be any positive integer, but this integer will naturally be small in practical cases.

The orthogonality conditions can now be written readily:

$$\int_0^\infty r^2\, dr\ \phi_{\ell,n}(r)\, f_\ell(k, r) = 0.$$

Introducing a Lagrangian multiplier $\beta_{\ell,n}(k)$ for each of these conditions, and performing the standard variational manipulations, one finds the resulting Euler-Lagrange equation to be an inhomogeneous differential equation,

$$\frac{d^2[rf_\ell(k, r)]}{dr^2} + [k^2 - \frac{\ell(\ell+1)}{r^2}]\ rf_\ell(k, r) = r \sum_n \beta_{\ell,n}(k)\, \phi_{\ell,n}(r).$$

The solution of this equation can be obtained in terms of the Green's function defined by the boundary conditions and the corresponding homogeneous equation. This solution is

$$f_\ell(k, r) = j_\ell(k\ r) - \sum_n \alpha_{\ell,n}(k)\, v_{\ell,n}(k, r),$$

$$v_{\ell,n}(k, r) = \int_0^\infty \phi_{\ell,n}(r')\, G_\ell(kr, kr')\, r'^2\, dr' \ .$$

The coefficients $\alpha_{\ell,n}(k)$ are proportional to the multipliers $\beta_{\ell,n}(k)$. Since the β's are not of interest in themselves, there is no need to determine the factors of proportionality.

This solution may now be introduced into the orthogonality conditions, in order to determine the values of the α's. The following notation will be convenient:

$$a_{\ell,n}(k) = \int_0^\infty j_\ell(kr)\, \phi_{\ell,n}(r)\, r^2\, dr,$$

$$b_{\ell,nn'}(k) = \int_0^\infty v_{\ell,n'}(k,r)\, \phi_{\ell,n}(r)\, r^2\, dr.$$

The orthogonality conditions take the form

$$\sum_{n'} b_{\ell,nn'}(k)\, \alpha_{\ell,n'}(k) = a_{\ell,n}(k) .$$

The solution for the coefficients $\alpha_{\ell,n}(k)$ is now obtained in terms of the matrix reciprocal to the matrix of the integrals $b_{\ell,nn'}(k)$,

$$\alpha_{\ell,n}(k) = \sum_{n'} b^{-1}_{\ell,nn'}(k)\, a_{\ell,n'}(k) .$$

There remains the problem of determining the required Green's function $G_\ell(kr, kr')$. In the first place, it enters linearly into $v_{\ell,n}(k,r)$, and thus also into $b_{\ell,nn'}(k)$. It consequently enters inversely into $b^{-1}_{\ell,nn'}(k)$ and into $\alpha_{\ell,n}(k)$. Hence, the function $f_{\ell,n}(k,r)$ is homogeneous of order zero in the Green's function, and there is no need to normalize it. Since the homogeneous equation is that for the spherical Bessel functions, it is necessary only to find expressions in terms of such functions, which satisfy the boundary conditions of being regular at the origin and reducing to outgoing waves at infinity. These conditions determine the function at once:

$$G_\ell(kr, kr') = \begin{cases} j_\ell(kr)\, h^+_\ell(kr'), & r < r' \\ j_\ell(kr')\, h^+_\ell(kr), & r > r' \end{cases} .$$

This completes the determination of the modified plane-wave states in terms of the states to which they are required to be orthogonal. The results thus obtained have greater generality than will be required, however. The case that will be of interest later will be that in which there is only one state $\phi_{\ell,n}$, having $\ell = 0$, and representing the state of a nucleon in an "α particle." A simpler notation is possible in which the index n is suppressed; for example, the state of interest can be called $\phi_0(r)$. For this case, the results obtained above take the forms

$$\psi_{\vec{k}}(\vec{r}) = e^{i\vec{k}\cdot\vec{r}} - \cdot \frac{a_0(k)v_0(k,r)}{b_0(k)} ,$$

$$v_0(k,r) = \int_0^\infty \phi_0(r')\, G_0(kr, kr') r'^2\, dr' = \left| G_0(kr, kr' \mid \phi_0(r')\right),$$

$$a_0(k,r) = \int_0^\infty j_0(kr)\, \phi_0(kr)\, r^2 dr = \left(j_0(kr),\ \phi_0(r)\right),$$

$$b_0(k) = \int_0^\infty v_0(k,r)\, \phi_0(r)\, r^2\, dr$$

$$= \int_0^\infty r^2\, dr\, \phi_0(r) \int_0^\infty r'^2 dr'\, \phi_0(r')\, G_0(kr, kr')$$

$$= \left(\phi_0(r) \mid G_0(kr,\ kr') \mid \phi_0\ (r')\right),$$

$$G_0(kr, kr') = \begin{cases} j_0(kr)\ h_0^{+}(kr') = \dfrac{\sin kr \cdot e^{ikr'}}{k^2 rr'} & r < r' \\ j_0(kr')h_0^{+}(kr) = \dfrac{\sin kr' \cdot e^{ikr}}{k^2 rr'} & r > r' \end{cases} .$$

5. FORM FACTOR FOR MODIFIED PLANE-WAVE STATES

When modified plane waves are introduced into the expression for the nuclear form factor, the result is

$$F_{\vec{k}}(\vec{\Delta k}) = \left(e^{i\vec{k}\cdot\vec{r}} - \frac{a_0(k)v_0(k,r)}{b_0(k)}, e^{i\vec{\Delta k}\cdot\vec{r}}\phi_0(\vec{r})\right) .$$

If the spherically symmetric state $\phi_0(\vec{r})$ is written in the form $(4\pi)^{-1/2}\phi_0(r)$, then $\phi_0(r)$ will be a normalized function of the radius r. The integration over angles can be performed at once; it picks out the spherically symmetric parts of the exponentials, and introduces a factor 4π:

$$F_{\vec{k}}(\vec{\Delta k}) = (4\pi)^{1/2}\left[\left(j_0(|\vec{k} - \vec{\Delta k}|\, r), \phi_0(r)\right) - \frac{a_0(k)}{b_0(k)}\left(v_0(k,r), j_0(r\Delta k)\,\phi_0(r)\right)\right].$$

Since $a_0(k) = \left(j_0(kr), \phi_0(r)\right)$, it is proper to write

$$\left(j_0(|\vec{k} - \vec{\Delta k}|r),\ \phi_0(r)\right) = a_0(|\vec{k} - \vec{\Delta k}|).$$

The remaining integral in $F_{\vec{k}}(\vec{\Delta k})$ will be denoted by $C_0(k, \Delta k)$:

$$C_0(k, \Delta k) = \left(v_0(k,r),\ j_0(r\Delta k)\ \phi_0(r)\right)$$

$$= \left(\phi_0(r')\ |G_0(kr, kr')|\ j_0(r\Delta k)\ \phi_0(r)\right).$$

Then the relation

$$b_0(k) = \left(\phi_0(r')\,|G_0(kr, kr')\ |\phi_0(r)\right) = C_0(k, 0)$$

reduces the number of independent integrals. The form factor may now be written

$$F_{\vec{k}}(\vec{\Delta k}) = (4\pi)^{1/2}\, a_0(k)\left[\frac{a_0(|\vec{k} - \vec{\Delta k}|)}{a_0(k)} - \frac{C_0(k, \Delta k)}{C_0(k, 0)}\right] .$$

Since the direction $\hat{k}$ of the vector $\vec{k}$ is not observed, it is necessary to square the form factor and integrate over this direction. The result, which still depends on the magnitude of $\vec{k}$, will no longer contain any direction at all, and so is sufficiently represented by the

notation $|F_k(\Delta k)|^2$. Its evaluation begins as follows:

$$|F_k(\Delta k)|^2 = \int d\hat{k}\, |F_{\vec{k}}(\overrightarrow{\Delta k})|^2 =$$

$$= 4\pi\, |a_0(k)|^2 \int d\hat{k} \left| \frac{a_0(|\vec{k} - \overrightarrow{\Delta k}|)}{a_0(k)} - \frac{C_0(k, \Delta k)}{C_0(k, 0)} \right|^2$$

$$= |4\pi\, a_0(k)|^2 \cdot (1/2) \int_{-1}^{1} d\mu \left| \frac{a_0(\sqrt{k^2 + (\Delta k)^2 - 2\mu k \Delta k})}{a_0(k)} - \frac{C_0(k, \Delta k)}{C_0(k, 0)} \right|^2.$$

At this point it becomes necessary to introduce an explicit form for the ground-state radial factor $\phi_0(r)$. Since it is simple, the Gaussian form will be used, and, as will be seen in Section 8, this choice is not contradicted by experiment. With proper choice of units, this state is then

$$\phi_0(r) = 2\pi^{-1/4}\, e^{-(1/2)r^2}.$$

Then $a_0(k)$ has the value

$$a_0(k) = \int_0^\infty \frac{\sin kr}{kr}\, \phi_0(r)\, r^2\, dr = (4\pi)^{1/4}\, e^{-(1/2)k^2},$$

and the form factor becomes

$$|F_k(\Delta k)|^2 = (4\pi)^{5/2}\, e^{-k^2} \cdot (1/2) \int_{-1}^{1} d\mu \left\{ e^{2\mu k \Delta k - (\Delta k)^2} + \left| \frac{C_0(k, \Delta k)}{C_0(k, 0)} \right|^2 \right.$$

$$\left. - 2R \left(\frac{C_0(k, \Delta k)}{C_0(k, 0)} \right) e^{\mu k \Delta k - (1/2)(\Delta k)^2} \right\}$$

$$= (4\pi)^{5/2}\, e^{-k^2} \left\{ \left(\frac{\sinh 2k\Delta k}{2k\Delta k}\, e^{-(\Delta k)^2} + \left| \frac{C_0(k, \Delta k)}{C_0(k, 0)} \right|^2 \right. \right.$$

$$\left. - 2R \left(\frac{C_0(k, \Delta k)}{C_0(k, 0)} \right) \frac{\sinh k\Delta k}{k\Delta k}\, e^{-(1/2)(\Delta k)^2} \right\}.$$

The double integral $C_0(k, \Delta k)$ remains to be evaluated. Writing out the defining equation at length, and introducing the explicit form of the Green's function, one obtains

$$C_0(k, \Delta k) = \int_0^\infty r^2 dr\, \phi_0(r) \int_0^\infty r'^2 dr' \phi_0(r') G_0(kr, kr') j_0(r'\Delta k)$$

$$= \int_0^\infty r^2 dr \phi_0(r)\, h_0^+(kr) \int_0^r r'^2 dr' \phi_0(r')\, j_0(kr')\, j_0(r'\Delta k)$$

$$+ \int_0^\infty r'^2 dr'\, \phi_0(r')\, j_0(r'\Delta k)\, h_0^+(kr') \int_0^{r'} r^2 dr\, \phi_0(r)\, j_0(kr).$$

In each of the two terms, the first integral (the one with variable upper limit) has an <u>even</u> integrand. The ranges of these integrals can therefore be extended symmetrically to negative values of the respective variables of integration, as indicated in Fig. 2, provided factors of 1/2 are introduced to restore the original values. But when this has been done, <u>odd</u> integrands cannot contribute at all. Since the irregular spherical Bessel function $n_0(kr)$ is odd, and in view of the relation $h_0^+ = n_0 + i\, j_0$, the replacement of j_0 by $-i\, h_0^+$ will not affect the values of these integrals. The integral takes the form

$$C_0(k, \Delta k) = \frac{-i}{2} \int_0^\infty r^2 dr\, \phi_0(r)\, h_0^+(kr) \int_{-r}^{r} r'^2 dr'\, \phi_0(r')\, h_0^+(kr')\, j_0(\Delta k r')$$

$$\frac{-i}{2} \int_0^\infty r'^2 dr'\, \phi_0(r')\, j_0(\Delta k r')\, h_0^+(kr') \int_{-r'}^{r'} r^2 dr\, \phi_0(r)\, h_0^+(kr).$$

As a result of these manipulations, the two terms now have identical integrands, and are extended over mutually exclusive domains which together constitute the upper right half plane. Moreover, since the two terms arose from a single integral, there is no inconsistency in the sign of the element of area to prevent them from being recombined:

$$C_0(k, \Delta k) = \frac{-i}{2} \int_{r+r'>0} r^2 dr \int r'^2 dr'\, h_0^+(kr)\, h_0^+(kr')\, \phi_0(r)\, \phi_0(r')\, j_0(r'\Delta k).$$

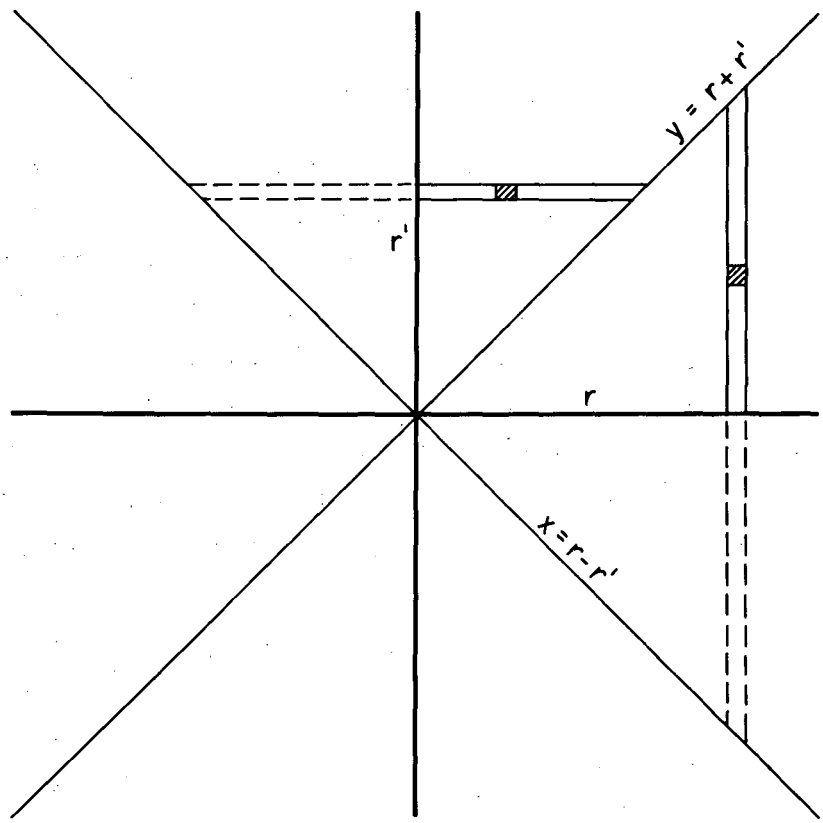

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Fig. 2. Domains and coordinate systems for the integral $C_0(k, \Delta k)$.

Since only the ratio $\frac{C_0(k, \Delta k)}{C_0(k, 0)}$ is to be used, it is unnecessary to substitute normalized functions for $\phi_0(r)$ and $\phi_0(r')$. Substitution leads to

$$C_0(k, \Delta k) = \frac{-i}{2k^2 \Delta k} \int_{r + r' > 0} dr \int dr' \, (r \sin r'\Delta k) \, e^{\frac{-r^2 + r'^2}{2} + ik(r + r')}$$

$$= -\frac{1}{2k^2 \Delta k} \Big[\frac{d}{dk} \int_{r + r' > 0} dr \int dr' \sin r'\Delta k \cdot e^{-(1/2)(r^2 + r'^2) + ik(r + r')}$$

$$+ i \frac{d}{d\Delta k} \int_{r + r' > 0} dr \int dr' \cos r'\Delta k \cdot e^{-(1/2)(r^2 + r'^2) + ik(r + r')} \Big].$$

The limits can now be simplified, as suggested in Fig. 2, by the transformation

$$\left\{ \begin{array}{l} x = r - r' \\ y = r + r' \end{array} \right\} \quad \text{or} \quad \left\{ \begin{array}{l} r = 1/2(y + x) \\ r' = 1/2(y - x) \end{array} \right\},$$

whose Jacobian is 1/2. The result is

$$C_0(k, \Delta k) = -\frac{1}{4k^2 \Delta k} \Big[\frac{d}{dk} \int_{-\infty}^{\infty} dx \int_0^{\infty} dy \; e^{-(1/4)(x^2 + y^2) + iky}$$

$$\left(\cos \frac{x\Delta k}{2} \cdot \sin \frac{y\Delta k}{2} - \sin \frac{x\Delta k}{2} \cdot \cos \frac{y\Delta k}{2}\right)$$

$$+ i \frac{d}{d\Delta k} \int_{-\infty}^{\infty} dx \int_0^{\infty} dy \; e^{-(1/4)(x^2 + y^2) + iky}$$

$$x \left(\cos \frac{x\Delta k}{2} \cdot \cos \frac{y\Delta k}{2} + \sin \frac{x\Delta k}{2} \cdot \sin \frac{y\Delta k}{2}\right) \Big].$$

To this expression the terms in the integrand which are <u>odd in x</u> cannot contribute. Moreover, those <u>even in x</u> can be evaluated by completing the square in the exponent, or by reference to Peirce, Formula 508. It is found that

$$\int_{-\infty}^{\infty} dx \; e^{-(1/4)x^2} \cos \frac{x\Delta k}{2} = 2\sqrt{\pi} \, e^{-(\Delta k/2)^2} .$$

Thus, the double integral is reduced to a single integral,

$$C_0(k,\Delta k) = \frac{-\sqrt{\pi}}{2k^2\Delta k}\left[\frac{d}{dk}\left\{e^{-(\Delta k/2)^2}\int_0^\infty dy\, e^{-(1/4)y^2+iky}\sin\frac{y\Delta k}{2}\right\}\right.$$

$$\left. + i\,\frac{d}{d\Delta k}\left\{e^{-(\Delta k/2)^2}\int_0^\infty dy\, e^{-(1/4)y^2+iky}\cos\frac{y\Delta k}{2}\right\}\right].$$

Unlike the x integration, the y integration cannot be performed in terms of elementary functions. The required new transcendental funtion will be taken in the standard form

$$\Phi(u) = (1/2)\int_0^\infty dt\, e^{-(1/4)t^2}\sin(ut).$$

Then

$$\int_0^\infty e^{-(1/4)y^2 + y(k\pm\Delta k/2)} = \sqrt{\pi}\; e^{-(k\pm\Delta k/2)^2} + 2i\Phi(k\pm\Delta k/2).$$

Although the y integral can now be expressed in terms of $\Phi(k\pm\Delta k/2)$, the expression for $C_0(k,\Delta k)$ is not very useful unless the differentiations with respect to k and Δk can be performed. This requires a knowledge of the properties of the function Φ, which will now be investigated:

$$\Phi(u) = \frac{1}{4i}\left[\int_0^\infty dt\; e^{-(1/4)t^2 + i\,tu} - \int_0^\infty dt\, e^{-(1/4)t^2 - i\,tu}\right]$$

$$= (1/2)e^{-u^2}\left[\int_{-u}^{i\infty} dq\, e^{q^2} - \int_u^{i\infty} dq\, e^{q^2}\right] = (1/2)e^{-u^2}\int_{-u}^{u} dq\, e^{q^2}$$

$$= e^{-u^2}\int_0^u dq\, e^{q^2}.$$

On p. 32 of Jahnke-Emde[42] appears a table of $\int_0^u e^{q^2}\,dq$ for the range $u < 2$. A simple method of numerical evaluation for larger arguments is to integrate the following differential equation on a differential analyzer. The equation follows by direct differentiation of the last result,

$$\frac{d\Phi}{du} = -2u\Phi + 1.$$

This relation permits evaluation of the derivatives in $C_0(k, \Delta k)$. Its solution as a differential equation requires an additional condition. This is provided by the obvious relation

$$\Phi(0) = 0.$$

It now follows that $\Phi'(0) = 1$. Higher derivatives can be evaluated, and a power series set up, but these will not be needed. Further qualitative information on the behavior of the function is contained in the asymptotic expression resulting from integration by parts of the original integral with infinite upper limit:

$$\Phi(u) = (1/2)\left[-e^{-(1/4)t^2} \frac{\cos(ut)}{u} \Big|_0^\infty - \frac{1}{2u} \int_0^\infty dt\, e^{-(1/4)t^2} \cdot t \cos(ut)\right]$$

$$\rightarrow \frac{1}{2u} \qquad \text{as} \qquad u \rightarrow \infty .$$

This asymptotic approximation, and the obvious approximate value $\Phi(u) \approx u$ for small values of u, are shown together with $\Phi(u)$ itself in Fig. 3.

The evaluation of $C_0(k, \Delta k)$ is now straightforward:

$$C_0(k, \Delta k) = \frac{-\sqrt{\pi}}{4k^2 \Delta k} \Big[2 \left(\frac{d}{dk} - \frac{d}{d\Delta k} \right) \{ e^{-(\Delta k/2)^2} \Phi(k + \Delta k/2) \}$$

$$-2 \left(\frac{d}{dk} + \frac{d}{d\Delta k} \right) \{ e^{-(\Delta k/2)^2} \Phi(k - \Delta k/2) \}$$

$$+ i\sqrt{\pi} \left(-\frac{d}{dk} + \frac{d}{d\Delta k} \right) e^{-(\Delta k/2)^2 - (\Delta k/2 + k)^2}$$

$$+ i\sqrt{\pi} \left(\frac{d}{dk} + \frac{d}{d\Delta k} \right) e^{-(\Delta k/2)^2 - \left(\frac{\Delta k}{2} - k\right)^2} \Big]$$

$$= \frac{\sqrt{\pi}}{2k\Delta k} e^{-(\Delta k/2)^2} \Big[\{ \Phi(k + \Delta k/2) - \Phi(k - \Delta k/2) \}$$

$$+ i\sqrt{\pi} \sinh k\Delta k \cdot e^{-k^2 - (1/4)(\Delta k)^2} .$$

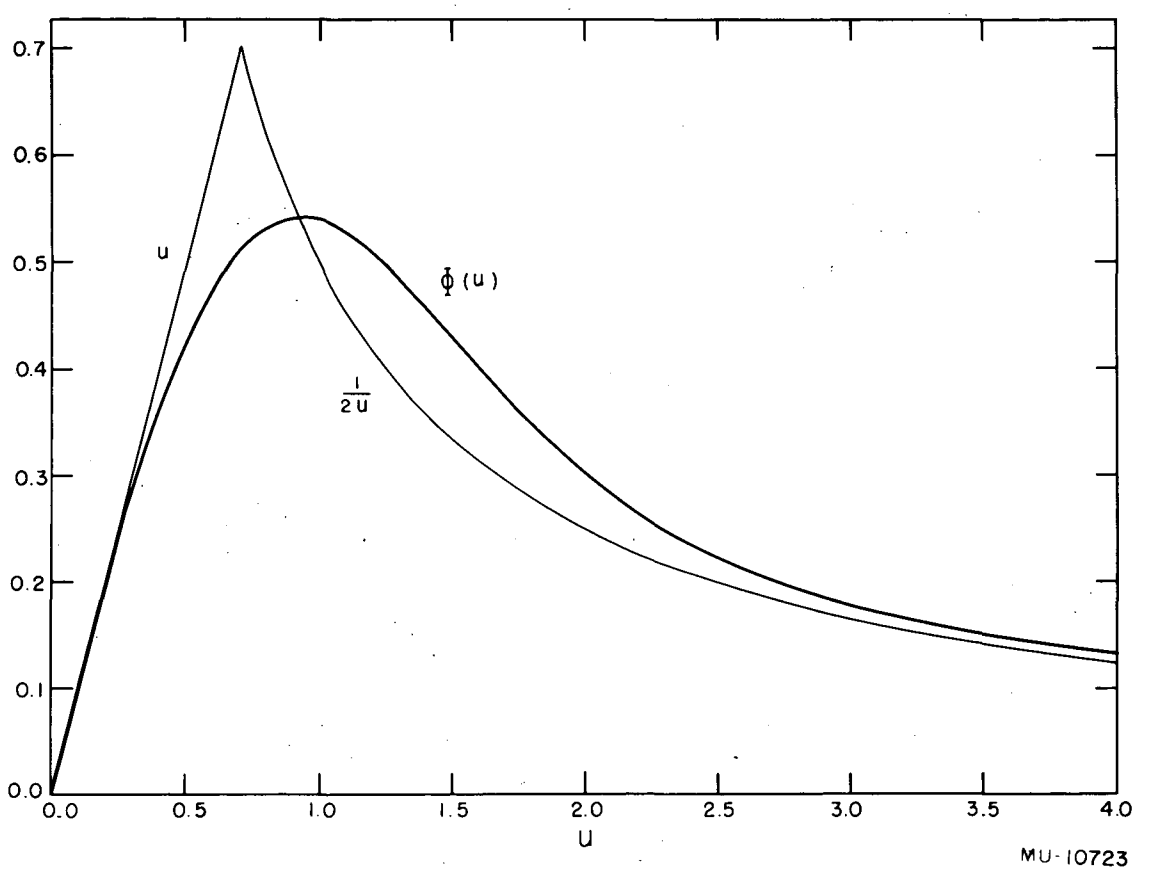

Fig. 3. The transcendental function $\Phi(u) = e^{-u^2} \int_0^u e^{q^2}\, dq$.

In the limit $\Delta k \to 0$, the function becomes

$$C_0(k, 0) = \frac{\sqrt{\pi}}{2k} \left[\Phi'(k) + i \sqrt{\pi}\, k\, e^{-k^2} \right]$$

$$= \frac{\sqrt{\pi}}{2k} \left[1 - 2k\, \Phi(k) + i \sqrt{\pi}\, k\, e^{-k^2} \right].$$

The evaluation of the form factor can now be completed without any further difficulty. In the first place

$$\frac{C_0(k, \Delta k)}{C_0(k, 0)} =$$

$$\frac{e^{-(\Delta k/2)^2}}{\Delta k} \; \frac{\{\Phi(k + \Delta k/2) - \Phi(k - \Delta k/2)\} + i\sqrt{\pi} \sinh k\Delta k \cdot e^{-k^2 - (1/4)(\Delta k)^2}}{\{1 - 2k\, \Phi(k)\} + i\sqrt{\pi}\, k\, e^{-k^2}}.$$

From this there follow

$$\left| \frac{C_0(k, \Delta k)}{C_0(k, 0)} \right|^2 =$$

$$\frac{e^{-(1/2)(\Delta k)^2}}{(\Delta k)^2} \; \frac{\{\Phi(k + \Delta k/2) - \Phi(k - \Delta k/2)\}^2 + \pi \sinh^2 k\Delta k \cdot e^{-2k^2 - (1/2)(\Delta k)^2}}{\{1 - 2k\, \Phi(k)\}^2 + \pi k^2 e^{-2k^2}},$$

$$R\left(\frac{C_0(k, \Delta k)}{C_0(k, 0)} \right) =$$

$$\frac{e^{-(\Delta k/2)^2} \{1 - 2k\Phi(k)\}\{\Phi(k + \Delta k/2) - \Phi(k - \Delta k/2)\}}{\Delta k[\{1 - 2k\, \Phi(k)\}^2 + \pi k^2 e^{-2k^2}]}$$

$$+ \frac{\pi k \sinh k\Delta k \cdot e^{-2k^2 - (\Delta k)^2/2}}{\Delta k[\{1 - 2k\, \Phi(k)\}^2 + \pi k^2 e^{-2k^2}]}.$$

The form factor can now be written as

$$|F_k(\Delta k)|^2 = (4\pi)^{5/2}\, e^{-k^2-(1/2)(\Delta k)^2}\Big\{e^{-(1/2)(\Delta k)^2}\,\frac{\sinh 2k\Delta k}{2k\Delta k}$$

$$+\ \frac{\{\Phi(k+\Delta k/2)-\Phi(k-\Delta k/2)\}^2+\pi\sinh^2 k\Delta k\cdot e^{-2k^2-(1/2)(\Delta k)^2}}{(\Delta k)^2[\{1-2k\,\Phi(k)\}^2+\pi k^2 e^{-2k^2}]}$$

$$-\ \frac{2\sinh k\Delta k}{k(\Delta k)^2}\ .$$

$$\frac{e^{-(\Delta k/2)^2}\{1-2k\,\Phi(k)\}\{\Phi(k+\Delta k/2)-\Phi(k-\Delta k/2)\}+\pi k\sinh k\Delta k\cdot e^{-2k^2-(1/2)(\Delta k)^2}}{\{1-2k\,\Phi(k)\}^2+\pi k^2 e^{-2k^2}}\ .$$

It is convenient to introduce the notations

$$\alpha(k,\Delta k) = \Phi(k+\Delta k/2) - \Phi(k-\Delta k/2),$$

$$\beta(k) = \Phi'(k) = 1 - 2k\,\Phi(k).$$

The second and third terms in the expression for $|F_k(\Delta k)|^2$ have a common denominator. The numerators contain terms in α^2 and $\alpha\beta$ respectively, which suggests attempting to complete the square. The result is a simplification of the expression for the form factor:

$$|F_k(\Delta k)|^2 = (4\pi)^{5/2}\, e^{-k^2-(\Delta k)^2}\Big\{\frac{\sinh 2k\Delta k}{2k\Delta k} - \frac{\sinh^2 k\Delta k}{k^2(\Delta k)^2}$$

$$+\ \frac{\left(\alpha e^{(\Delta k/2)^2} - \beta\,\frac{\sinh k\Delta k}{k}\right)^2}{(\Delta k)^2[\beta^2+\pi k^2 e^{-2k^2}]}\ \Big\}$$

$$= (4\pi)^{5/2}\, e^{-k^2-(\Delta k)^2}\ \Big\{\frac{\sinh 2k\Delta k}{2k\Delta k}\left(1-\frac{\tanh k\Delta k}{k\Delta k}\right)$$

$$+\ \frac{\left(\frac{\alpha}{\Delta k}\, e^{(\Delta k/2)^2} - \beta\,\frac{\sinh k\,\Delta k}{k\Delta k}\right)^2}{\beta^2+\pi k^2 e^{-2k^2}}\ \Big\}\ .$$

For small values of Δk, the numerator of the last term cannot be evaluated accurately without a very accurate table of the function Φ. It can be handled by an expansion in powers of Δk, however. The condition for this is that not only Δk, but also $k\Delta k$, must be small. The expansion gives

$$\Phi(k \pm \Delta k/2) = \Phi(k) \pm \Phi'(k)(1/2\Delta k) + (1/2)\Phi''(k)\left((1/2)\Delta k\right)^2 \pm \frac{1}{3!}\Phi'''(k)(\Delta k/2)^3 + \ldots,$$

from which follows

$$\frac{a(k, \Delta k)}{\Delta k} = \beta(k) + \frac{1}{3!}\beta''(k)(\Delta k/2)^2 + \frac{1}{5!}\beta^{iv}(k)(\Delta k/2)^4 + \ldots.$$

Successive applications of the differential equation for Φ lead to the result

$$\beta''(k) = 2 + [(2k)^2 - 3!]\,\beta(k).$$

Thus,

$$\frac{a}{\Delta k}\, e^{(\Delta k/2)^2} = \beta + \frac{1}{3}(\Delta k/2)^2\,[1 + 2k^2\beta] + \ldots .$$

Now

$$\beta\,\frac{\sinh k\Delta k}{k\Delta k} = \beta\cdot\left[1 + \frac{1}{3!}(k\Delta k)^2 + \frac{1}{5!}(k\Delta k)^5 + \ldots\right].$$

This leads to the required expansion

$$\frac{a(k, \Delta k)}{\Delta k}\, e^{(\Delta k/2)^2} - \beta\,\frac{\sinh k\Delta k}{k\Delta k} = \frac{1}{3}(\Delta k/2)^2 + \ldots .$$

The higher terms in the series will not be needed.

6. FORM FACTOR FOR HARMONIC OSCILLATOR STATES

In order to determine whether the results are sensitive to the nature of the final states used for the proton, the calculation has been repeated using for these states the excited states of the harmonic oscillator whose ground state is $\phi_0(\vec{r})$. This ensures the desired orthogonality between the various states used.

Instead of an index $\vec{k}$, the final state is specified by the values of a set of three integers. The notation adopted is

$$\phi_f(\vec{r}) \rightarrow \phi_{\ell mn}(\vec{r}) = u_\ell(x)\, u_m(y)\, u_n(z),$$

$$\phi_0(\vec{r}) \rightarrow \phi_{000}(\vec{r}) .$$

The energy of these states is

$$E_{\ell mn} = E_\ell + E_m + E_n = (\ell + m + n + \tfrac{3}{2})\,\omega$$
$$= (N + \tfrac{3}{2})\,\omega,$$

where

$$N = \ell + m + n.$$

When these expressions are introduced, the form factor becomes

$$F_{\ell mn}(\overrightarrow{\Delta k}) = \left(\phi_{\ell mn}(\vec{r}),\ e^{i\overrightarrow{\Delta k}\cdot\vec{r}}\ \phi_{000}(\vec{r})\right) .$$

Then

$$\left| F_N\ (\Delta k)\right|^2 = \sum_{\ell+m+n=N} \left| F_{\ell mn}(\overrightarrow{\Delta k})\right|^2$$ will replace the

previously used quantity

$$\left| F_k\ (\Delta k)\right|^2 = \int \left| F_{\vec{k}}(\overrightarrow{\Delta k})\right|^2 \hat{dk}$$ in determining transition rates.

It is apparent that

$$F_{\ell mn}(\overrightarrow{\Delta k}) = M_\ell(\Delta k_x)\cdot M_m(\Delta ky)\cdot M_n(\Delta kz) \quad ,$$

in which, for example

$$M_\ell(q) = \int_{-\infty}^{\infty} u_\ell(x)\, e^{i\,qx}\, u_0(x)\, dx.$$

The summed form factor thus takes the form

$$\left| F_N(\Delta k) \right|^2 = \sum_{\ell+M+N=N} \left| M_\ell(\Delta k_x) \cdot M_m(\Delta ky) \cdot M_n(\Delta k_z) \right|^2 .$$

It is now necessary to introduce explicit expressions for the harmonic oscillator eigenstates $u_n(x)$:

$$u_n(x) = N_n H_n(x)\, e^{-(1/2)x^2} ,$$

where

$$H_n(x) = (-1)^n \frac{\partial}{\partial x^n} e^{-x^2} ,$$

and

$$N_n = \frac{1}{\sqrt{2^n\, n!\, \sqrt{\pi}}} .$$

The evaluation of $M_\ell(q)$ proceeds as follows:

$$M_\ell(q) = \frac{(-1)^\ell}{\sqrt{\pi \cdot 2^\ell \cdot \ell!}} \int_{-\infty}^{\infty} dx\, e^{iqx} \frac{d^\ell}{dx^\ell} e^{-x^2} =$$

$$\frac{1}{\sqrt{\pi\, 2^\ell \cdot \ell!}} \int_{-\infty}^{\infty} dx\, e^{-x^2} \frac{d^\ell}{dx^\ell} e^{iqx} =$$

$$\frac{(iq)^\ell}{\sqrt{\pi \cdot 2^\ell \cdot \ell!}} \int_{-\infty}^{\infty} dx\, e^{-x^2 + iqx} = \frac{(iq)^\ell}{\sqrt{2^\ell\, \ell!}} e^{-(1/4)q^2} .$$

Substitution in the expression for $| F_N(\Delta k) |^2$ yields the result

$$\left| F_N(\Delta k) \right|^2 = e^{-(1/2)(\Delta k)^2} \sum_{\ell+m+n=N} \frac{(\Delta k_x)^{2\ell}}{2^\ell\, \ell!} \cdot \frac{(\Delta k_y)^{2m}}{2^m m!} \cdot \frac{(\Delta k_z)^{2n}}{2^n\, n!}$$

$$= \frac{1}{N!} \frac{(\Delta k)^{2N}}{2^N} e^{-(1/2)(\Delta k)^2} .$$

If N = 0, the right side of this equation reduces to $e^{-(1/2)(\Delta k)^2}$. The fact that this approaches unity as Δk approaches zero checks the normalization for the case N = 0.

The case N = 0 is not of any special importance, however; in fact, it cannot occur at all for a saturated target. Moreover, the condition $\Delta k = 0$ is of less interest than the condition that the neutron production be in the forward direction. A convenient approximation to the latter condition can be found as follows:

$$2M\Delta E = k_0^2 - k_f^2 = (k_0 + k_f)(k_0 - k_f) \approx 2k_0 \Delta k \ .$$

The relation $\Delta E = N\omega$ leads to an expression for $(\Delta k)^2$ in terms of N:

$$(\Delta k)^2 \approx \frac{M^2(\Delta E)^2}{k_0^2} = \frac{M\omega^2}{2\,E_0} \quad N^2 \ .$$

The form factor can now be written

$$\left| F_N(\Delta k) \right|^2 \approx \frac{1}{N!} \left(\frac{\omega}{E_0} \frac{N^2}{4} \right)^N e^{-\frac{\omega}{E_0} \frac{N^2}{4}} \ .$$

Not only is this expression confined to the forward direction, but also the approximations fail unless $N\,\omega/E_0 << 1$.

7. CROSS SECTION FORMULAE

It is still necessary to express the cross section in terms of the scattering matrices that have been found. The application of such expressions to the case of neutron-proton scattering will provide the means for evaluating the function $|R(\Delta k, k_f)|^2$ which represents the elementary scattering. The basis of the discussion will be the usual quantum-mechanical formula for transition rates. If the rate is designated as w, and units are used in which $\hbar = c = 1$, this formula is

$$w = 2\pi \, |M_{f0}|^2 \, \rho(E_f).$$

Here $\rho(E_f)$ is the density of final states. It takes, for the present purpose, two different forms, corresponding respectively to excitation of discrete and continuous states of the residual nucleus.

Discrete Level

In this case the density of final states is

$$\rho(E_f) = \frac{k_f^2 dk_f \, d\omega_f}{8\pi^3 dE_f} = \frac{k_f^2 \, d\omega_f}{8\pi^3 v_f} .$$

In this expression, k_f, v_f, and ω_f are momentum, velocity, and solid angle, respectively, of the fast particle in its final state. If v_0 is used for the initial velocity of this particle, the cross section can be expressed as

$$\frac{d\sigma}{d\omega_f} = \frac{w}{v_0 \, d\omega_f} = \frac{k_f^2}{(2\pi)^2 v_0 v_f} \sum_f |M_{f0}|^2 .$$

With the value of $\sum_f |M_{f_0}|^2$ given at the end of Section 3, there follows

$$\frac{d\sigma}{d\omega_f} = \frac{v_f}{v_0} \left(\frac{k_f}{2\pi v_f}\right)^2 . \; N \, |R|^2 \, |F_k(\Delta k)|^2$$

$$= N \frac{v_f}{v_0} \left(\frac{M}{2\pi}\right)^2 |R(\Delta k, k_f)|^2 \, |F_k(\Delta k)|^2 .$$

The last step depends upon the nonrelativistic relation $k_f = Mv_f$, in which M is the nucleon mass.

This formula is applicable to neutron-proton scattering, since the recoiling proton, with respect to its internal degrees of freedom, is always in a "discrete level." In this case, $|F_k(\Delta k)|^2 = 1$ and $N = 1$ hold. It will be most convenient to handle this case in the center-of-mass system, in which $v_f = v_0$, and $k_f = \mu v_f$ (μ is the reduced mass). The result is

$$\left(\frac{d\sigma}{d\omega_{cm}}\right)_{np} = \left(\frac{\mu}{2\pi}\right)^2 |R(\Delta k, k_f)|^2 .$$

Note that the use of the argument k_f in R is not now an approximation.

Since $\mu = M/2$, this equation permits the evaluation of the factor $|R|^2$ in the cross-section formula in terms of

$$\left(\frac{d\sigma}{d\omega_{cm}}\right)_{np} .$$

It is most convenient, in referring to the experimental data, to have this cross section in terms of the energy of the incident fast nucleon in the laboratory system. For this purpose, the energy relation for neutron-proton scattering is employed in the form

$$k_0^2 = k_f^2 + (\Delta k)^2 .$$

The equivalent nucleon-scattering energy may therefore be defined by

$$E_{np} = [(\Delta k)^2 + k_f^2]/2M.$$

Then

$$\frac{d\sigma}{d\omega_f} = 4\frac{v_f}{v_o} N \left(\frac{d\sigma(\Delta k, E_{np})}{d\omega_{cm}}\right)_{np} |F_k(\Delta k)|^2 .$$

This expression is valid in the nonrelativistic approximation. It is understood that the variables employed are those of the actual scattering, and that these are used to evaluate E_{np}. The angle of neutron-proton scattering for which

$$\left(\frac{d\sigma}{d\omega}\right)_{np}$$

is evaluated is just

$$2 \sin^{-1} \left(\frac{\Delta k}{\sqrt{2 M E_{np}}} \right) .$$

Continuum Level

The expression to be derived will not apply to the most general case, but rather to the case found earlier to be of interest, in which a single nucleon--called the residual proton--is excited into a continuum level, while the rest of the nucleons are left in their ground states. Variables without subscripts will be used for the residual proton. The states of this particle will be specified by means of the asymptotic momentum k. The density of final states then takes the form

$$\rho(E_f) = \frac{k_f^2 dk_f \, d\omega_f k^2 \, dk \, d\omega}{(2\pi)^6 \, dE} .$$

The cross section, which is now also a spectral distribution function, has the form

$$\frac{d\sigma}{d\omega_f \, dE_f} = \int \left(\frac{w}{V_0 \, d\omega_f \, dE_f \, d\omega} \right) d\omega = \frac{k_f^2 \, k^2}{(2\pi)^5 v_f v_0 v} \sum_f |M_{f0}|^2$$

$$= \frac{k_f^2 \, k^2}{(2\pi)^5 v_f v_0 v} N |R|^2 |F_k(\Delta k)|^2$$

$$= N \frac{Mk}{(2\pi)^3} \cdot 4 \frac{v_f}{v_0} \left(\frac{d\sigma(\Delta k, E_{np})}{d\omega_{cm}} \right)_{np} \cdot |F_k(\Delta k)|^2 .$$

Remarks

It is instructive to write the preceding results in a somewhat different form.

Discrete level: $$\frac{\left(\frac{d\sigma}{d\omega_f} \right)}{N \left(\frac{d\sigma}{d\omega_{cm}} \right)_{np}} \frac{v_0}{4 v_f} = |F_k(\Delta k)|^2$$

Continuum level: $$\frac{\left(\frac{d\sigma}{d\omega_f dE_f}\right)}{N\left(\frac{d\sigma}{d\omega_{cm}}\right)_{np}} \frac{v_0}{4v_f} = \frac{kM}{(2\pi)^3} \mid F_k(\Delta k) \mid^2 .$$

The feature to be noticed is that the left sides are to be evaluated experimentally as functions of three arguments: the energy E_0 of the incident protons, the energy E_f of the product neutrons, and the momentum transfer Δk. The right sides, on the other hand, must be investigated theoretically by means of nuclear models, and are functions of only two arguments, the momentum transfer Δk and the energy transfer $\Delta E = E_0 - E_f$, but not of E_0 or E_f separately. Thus, independently of any model, the experimental quantities on the left must also be functions only of these two variables.

The experimental verification of this relation would constitute a check on the hypotheses required for its deduction. These are the validity of the impulse approximation, and the neglect of the q-dependence of the elementary scattering matrix. That departures from the energy shell produce insignificant changes in this scattering matrix may be considered as part of the impulse approximation. The actual deduction has been based, in addition, on an independent-particle model, but it is probable that this may be replaced by a weaker condition, and thus not be concerned in the proposed test.

The data available at present are somewhat meager for the proposed test. There is extensive information at 100 Mev from Harvard,[21] but spectra with known absolute intensities are available at other energies in the forward direction only. The Harvard results for carbon show a spectrum that is rather insensitive to ΔE and Δk, except for a monotonic decrease to zero in the neighborhood of the threshold locus. At higher energies, such as are available at Harwell, Rochester, and elsewhere, the spectrum shows a strong maximum not far from threshold. This difference, between a flat spectrum and a strongly peaked one, represents a failure of the test for independence of E_0, and thus a failure of the hypotheses. It is to be hoped that this failure applies to the experiments performed at the Harvard energy of about 100 Mev, rather than generally. In fact, as seen in Fig. 1, this

energy is too low to escape the q-dependence of the neutron-proton scattering cross section. It is known, also, that the index of refraction and the opacity of nuclear matter both decrease substantially with increasing energy in the range from 100 Mev to 150 Mev, thus producing more favorable conditions for the impulse approximation at the higher energies.

If the Harvard work is omitted from consideration, the remaining data on spectra provide no variation of Δk independently of ΔE and E_0. Consequently, the proposed verification cannot be made directly. It will be seen later that some available work on angular distributions--which represent integrals of the spectra, rather than the spectra themselves--provides the means to test a similar relation.

8. THE GROUND STATE

The ground state of the nucleons has been assigned the Gaussian form $e^{-(1/2)r^2}$, but it is still necessary, in order to complete the specification of this state, to determine the unit of length in which r is to be measured. This unit determines the size of the nucleus, and also, by a Fourier transformation, the momentum distributions of the nucleons. Accordingly, experiments on nuclear sizes and momentum distributions must be called upon in the evaluation of this parameter.

It is, however, somewhat difficult to interpret experiments on nuclear sizes in terms of a Gaussian distribution. Moreover, the experiments that yield momentum distributions are high-energy scattering experiments of the type presently under consideration. Consequently, only experiments on momentum distribution have been drawn upon.

Henley[29] has used a Gaussian momentum distribution in discussing the production, at 90° to the beam, of positive π mesons by 345-Mev protons (nominal energy) impinging on carbon. He chose the parameter of the Gaussian distribution so as to fit the experiments by Hadley and York on deuteron formation by pickup,[25] using the general theory of Chew and Goldberger.[26] This Gaussian gave a rough, but not unsatisfactory, fit to the meson production experiments also, showing itself superior for this purpose to the Fermi-gas momentum distribution, and to the distributions used by Chew and Goldberger themselves in treating pickup. The Gaussian used is the one that results in a mean kinetic energy of 19.3 Mev.

Wolff[28] has interpreted experiments by Cladis[27] on the inelastic scattering of protons by means of a Gaussian momentum distribution. Instead of using the mean kinetic energy, he reports the value of the energy at which the probability density is reduced by a factor e^{-1} from its maximum value. A suitable designation of this parameter is ϵ; its value is found by Wolff to be 16 Mev. In order to compare this with Henley's result, use is made of the fact that this value would be the mean value of the energy if the momentum space had two dimensions instead of three. Consequently, the two energies

are in the ratio of two to three, and Henley's result takes the form of a value $\epsilon = \frac{2}{3} \times 19.3$ Mev = 12.9 Mev.

Very direct information on the momentum distribution has been obtained by coincidence experiments on quasi-elastic scattering. The nature of the process has been elucidated through the angular correlation experiments by Chamberlain and Segre.[43] The possibility of such a process of quasi-elastic scattering is itself encouraging with regard to the applicability of the impulse approximation. Actual study of momentum distributions by this method has been carried out by Wilcox.[30] He finds the momentum distribution of the protons in Be to be close to Gaussian, with ϵ = 20 Mev.

It is apparent that the three values of ϵ are not concordant. They are, however, of somewhat low precision. The experiments of Cladis allow to ϵ any value between 14 Mev and 19 Mev, while those of Wilcox allow any value between 16 Mev and 25 Mev. The experiments on pickup and meson production seem to be no better in this respect. Thus, the failure of the numbers to agree is not a serious inconsistency, but a manifestation of the fact that the experiments determine the value of the parameter only within rather wide limits.

As units have been adopted in which $\hbar = c = 1$, the choice of the unit for some one quantity will fix the units for all quantities. The unit of energy ϵ fixes a unit of momentum $1/\rho = \sqrt{2M\epsilon}$, and a unit of length ρ. The Henley, Wolff, and Wilcox values of ϵ fix values of $1/\rho$ which are 155 Mev, 173 Mev, and 194 Mev respectively.

From the considerable range of possibilities, the value $1/\rho$ = 170 Mev has been chosen. Then $\rho = 1/170$ Mev $= 1.15 \times 10^{-13}$ cm, and ϵ = 15.4 Mev. This choice is not a critical one, as the main features of the work are fairly insensitive to it. It is apparent that the state $\psi = e^{-(1/2)r^2}$ has the property $|\psi|^2 = e^{-1}$ for r = 1; consequently, ρ is the required unit of length.

9. KINEMATICAL CONSIDERATIONS

The use of the impulse approximation depends upon a kinematical similarity between scattering from a complex nucleus and scattering from a single nucleon. The similarity is not an actual equivalence, however; consequently some care is called for in setting up the relation between the two processes. The principal difficulty has been dealt with by introducing the energy E_{np}. Certain difficulties remain, however, connected with the influence of the nuclear Coulomb field and of the binding energy of a nuclear nucleon; neither of these effects is present in the case of nucleon-nucleon scattering.

It is necessary to consider the following variables:

M = nucleon mass of about 939 Mev,

D = Coulomb barrier height,

$\vec{k}_0$ = momentum of incident proton,

E_0 = energy of incident proton,

$\vec{k}_f$ = momentum of product neutron,

E_f = energy of product neutron,

θ = angle between directions of $\vec{k}_0$ and $\vec{k}_f$,

$\Delta\vec{k} = \vec{k}_0 - \vec{k}_f$ = momentum transfer,

$\Delta E = E_0 - E_f$ = energy transfer,

$E_{np} = [k_f^2 + (\Delta k)^2]/2M = E_f + \Delta k^2/2M$.

The complementary momentum transfer $\vec{q}$ cannot be considered a kinematical variable, since it involves the momentum of a target nucleon, and such momenta are distributed in a manner controlled by the state of the target nucleus.

The variables in this list are not all independent. The principal relations between them are:

$$E_0 = k_0^2/2M, \quad E_f = k_f^2/2M,$$

$$(\Delta k)^2 = k_0^2 + k_f^2 - 2\,\vec{k}_0 \cdot \vec{k}_f = k_0^2 + k_f^2 - 2\,k_0\,k_f \cos\theta.$$

It is desirable to precede discussion of these relations with consideration of the manner in which the Coulomb barrier is to be handled. The height D of this barrier is of course small compared to the initial and final energies, E_0 and E_f, of the fast nucleon. However,

it is not necessarily small compared to their difference, the energy transfer ΔE, and must accordingly be given consideration in the definitions of E_0 and E_f. Now, in view of the high energies involved, the Coulomb barrier will usually be surmounted, and events that involve penetration of the barrier will be unimportant. The energy E_0 should therefore be considered as less than the energy of the proton beam by the amount D lost in climbing the barrier. The neutron, on the other hand, has no barrier to cross in escaping from the nucleus; thus E_f is the experimental neutron energy. It follows that the energy transfer ΔE is smaller by the amount D than the experimentally measured difference between proton and neutron energies.

It is now necessary to distinguish between the two models--the harmonic oscillator model and the modified plane-wave model. In the former model, the spectrum of levels available to the residual nucleus is discrete, and the levels are uniformly spaced with an interval ω. This unit is closely related to the unit ϵ introduced in discussing the nuclear ground state,

$$\omega = 2\epsilon = 30.8 \text{ Mev.}$$

It will be convenient to use this unit of energy even in treating the modified plane-wave model, in order to facilitate comparisons between the two models.

The modified plane-wave model has a continuous rather than a discrete energy spectrum. The characteristic energy that requires determination in this case is the depth B of the initial state below the lowest level of the continuum of final states. This energy is the threshold for the neutron production process. The energy B having been absorbed in freeing the neutron, the remainder of the energy transfer, $E^* = \Delta E - B$, is available as excitation of the residual nucleus. If $E^* = 0$, then $\Delta E = B$ holds. Thus, B is a possible energy transfer, and like ΔE itself must be reduced from the conventional value by the amount D. Aside from this correction, it is just the difference between the binding energies of the target and residual nuclei.

This definition may be illustrated by numerical evaluation of B in the case of carbon, which will be required later. Hollander et al., in their isotope table,[44] give the maximum energies of the positron

from $N^{12}(\beta^+)C^{12}$ as 16.6 Mev, while that of the positron from $N^{13}(\beta^+)\,C^{13}$ is 1.2 Mev. Now, the pair N^{13}-C^{13} is a mirror pair, whose energy difference is all Coulombic except for the neutron-proton mass difference. As the latter difference is the same for the other pair N^{12}-C^{12}, and the Coulomb energy also must be very nearly equal in the two cases, the difference between the maximum energies of the two positrons is just the desired quantity B. Thus, B = 16.6 Mev - 1.2 Mev = 15.4 Mev.

A similar calculation can be made based on the maximum energy, 13.4 Mev, of the electron from $B^{12}(\beta^-)C^{12}$. This time the estimate of the Coulomb energy is obtained from the maximum energy, 1.0 Mev, of the positron from $C^{11}(\beta^+)\,B^{11}$. Because β^- and β^+ emissions are compared this time, there is a correction 2 x 0.5 Mev = 1.0 Mev for the mass of two electrons. Thus, the result is B = 13.4 Mev + 1.0 Mev + 1.0 Mev = 15.4 Mev.

The agreement of the two estimates to one-tenth Mev is in part fortuitous, as the estimate of the barrier height D is shown by studies of isotopic multiplets to be reliable only to perhaps 0.2 Mev. Also, the fact that $2B = \omega$(or $B = \epsilon$) is a coincidence, since it would not hold for some other nucleus such as He^4 or O^{16}.

The residual proton energy $E^* = \Delta E - B$ is to be understood as being measured at the top of the Coulomb barrier, since the correction -D applied to ΔE is equivalent to taking the zero for the energy of the last proton at the top of the barrier. This proton will gain the further energy D in leaving the nucleus and escaping to infinite distance. The energy at infinite distance is of no interest, however, since this proton is not detected in the experiments. On the other hand, the momentum of this proton just outside the nucleus is of interest, as it is used to specify the plane wave whose "modification" produces the state of the proton. This momentum is

$$k = \sqrt{2ME^*} \quad .$$

There may appear to be a question as to whether k ought to be measured inside or outside of the nuclear potential well. The model used has, however, the characteristic that it never introduces nuclear forces at all; consequently there is no well, and such a question cannot

arise. Or it may be said that the forces have been schematically replaced by the condition of orthogonality to the ground state, and hence cannot be introduced again in the form of a well-depth correction.

With these definitions settled, it is possible to carry out the necessary kinematical calculations. As an example, consider the transformation from the experimentally determined parameters, E_0, E_f, and θ, to the parameters used in the cross-section formulae, Δk, ΔE, and E_{np}:

$$\Delta E = E_0 - E_f,$$

$$(\Delta k)^2 = k_0^2 + k_f^2 - 2\, k_0\, k_f \cos\theta = 2\, ME_0 + 2\, ME_f - 4M\sqrt{E_0 E_f}\cos\theta,$$

$$E_{np} = \frac{k_f^2 + (\Delta k)^2}{2M} = E_0 + 2E_f - 2\sqrt{E_0 E_f}\cos\theta.$$

The inverse of this is

$$E_f = E_{np} - \frac{(\Delta k)^2}{2M},$$

$$E_0 = \Delta E + E_f = E_{np} + \Delta E - (\Delta k)^2/2M,$$

$$\cos\theta = \frac{E_0 + E_f - (\Delta k)^2/2M}{2\sqrt{E_0 E_f}} =$$

$$\frac{2\, E_{np} + \Delta E + 3(\Delta k)^2/2M}{2\sqrt{[E_{np} + \Delta E - (\Delta k)^2/2M][E_{np} - (\Delta k)^2/2M]}}.$$

The form factor for modified plane-wave states has been evaluated as a function of k and Δk. It is therefore desirable to consider these as independent variables. Since E_0 is (ideally) constant in any one experiment, it is a convenient third independent variable. Since experiments involve definite angles, it will be necessary to consider loci of constant θ (and E_0) in the k - Δk plane. The most important of these loci is that for $\theta = 0$. Its equation is

$$\Delta k = k_0 - k_f = k_0 - \sqrt{2ME_0 - 2MB - k^2},$$

or equivalently

$$(\Delta k - k_0)^2 + k^2 = 2M(E_0 - B) = k_0^2 - 2MB.$$

This equation represents a circle in the $k - \Delta k$ plane, having its center at $\Delta k = k_0$, $k = 0$, and its radius of length $\sqrt{2M(E_0 - B)} = \sqrt{k_0^2 - 2MB}$. Since the radius is less than k_0, the circle does not intersect the axis $\Delta k = 0$. Because of the simplicity and importance of these loci, they are presented, for the energies E_0 of several cyclotrons, in Fig. 4.

The loci of constant θ are determined, in general, by the parametric equations

$$(\Delta k)^2 = 2ME_0 + 2ME_f - 4M\sqrt{E_0E_f}\cos\theta,$$

$$E_f = E_0 - B - k^2/2M.$$

Elimination of E_f leads to the single equation

$$[2E_0 - B - k^2/2M - (\Delta k)^2/2M]^2 = 4E_0\cos^2\theta(E_0 - B - k^2/2M).$$

Since $\cos\theta > 1$ is impossible, the circle $\theta = 0$ represents a cutoff--all loci of constant θ (for given E_0) must lie inside this circle. There is also a cutoff at the threshold $k = 0$; thus, only the semicircle having $k \geqslant 0$ is required. This is the reason for the importance of the case $\theta = 0$.

The nature of the curves for general θ can be determined by considering their derivatives

$$[2E_0\sin^2\theta - B - k^2/2M - (\Delta k)^2/2M]\,k\,dk + [2E_0 - B - k^2/2M - (\Delta k)^2/2M]\,\Delta k\cdot d\Delta k = 0.$$

The condition for a vertical tangent is found to be either $\Delta k = 0$ or

$$2E_0 = B + k^2/2M + (\Delta k)^2/2M.$$

The first condition is never applicable, since the curves are all inside the circle $\theta = 0$, while the axis $\Delta k = 0$ is entirely outside this circle. The second condition, however, together with the equation of the curve, determines a point where the tangent is vertical,

$$\Delta k = k_0, \qquad k = \sqrt{k_0^2 - 2MB}.$$

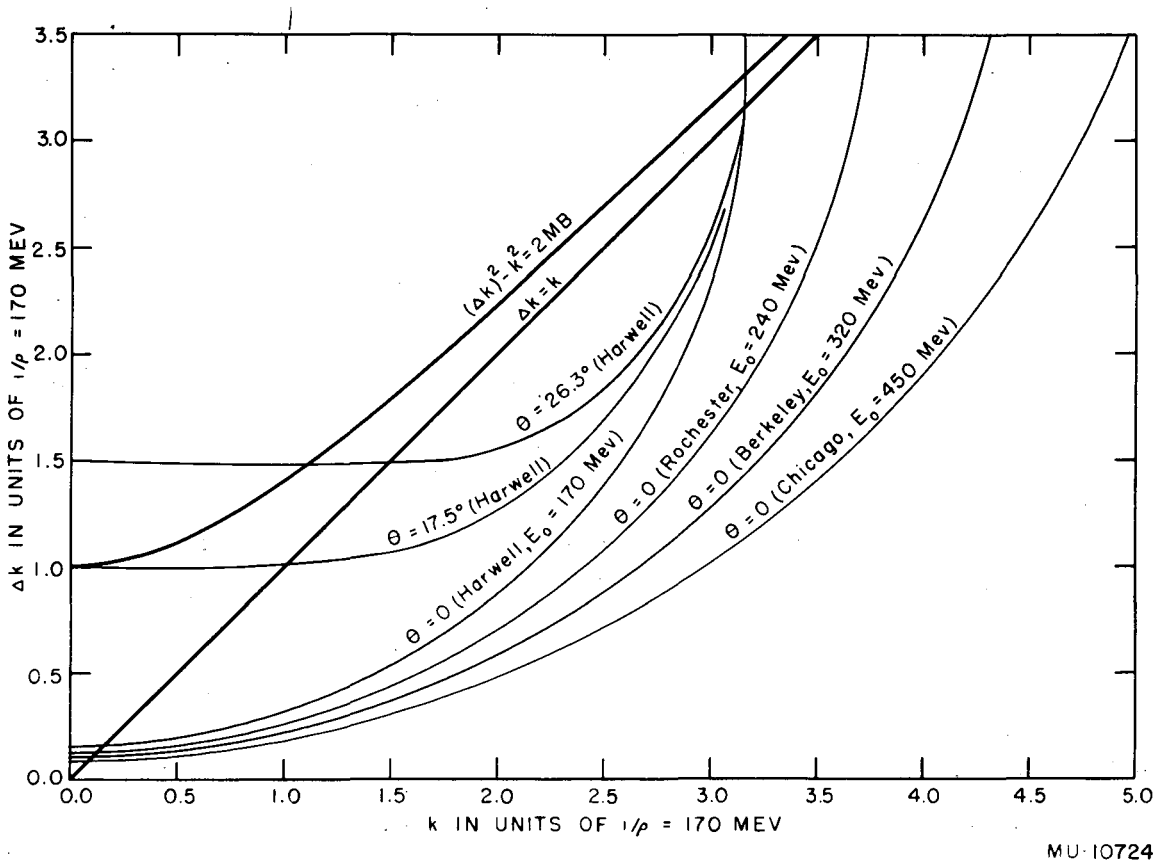

Fig. 4. Loci in the k-Δk plane.

This point is independent of θ. Thus, all loci of constant θ are mutually tangent at this point. It is apparent that the point is indeed the point where a circle of radius

$$\sqrt{k_0^{\,2} - 2MB},$$

with its center at $\Delta k = k_0$, $k = 0$, has a vertical tangent; thus, the situation is transparent for the case $\theta = 0$, at least.

Taking up next the case of a horizontal tangent leads to the condition

$$\text{either} \quad k = 0$$

$$\text{or } 2E_0 \sin^2\theta = B + k^2/2M + (\Delta k)^2/2M.$$

The first condition shows that the curves of constant θ intersect the axis $k = 0$ with horizontal tangents. The second condition, together with the equation of the curve, determines a second point where the curve is horizontal,

$$\Delta k = k_0 \sin\theta,$$

$$k = \sqrt{k_0^{\,2} \sin^2\theta - 2MB} \quad .$$

This point exists if and only if k is real, which requires

$$\sin\theta \geqslant \frac{\sqrt{2MB}}{k_0} = \sqrt{\frac{B}{E_0}} \quad .$$

It is possible to eliminate θ from the expressions for the point, thus obtaining

$$(\Delta k)^2 - k^2 = 2MB.$$

This is a condition satisfied by all points where the second condition for a horizontal tangent holds. It is thus the equation of a locus that is intersected with horizontal tangent by the curves of constant θ. It represents a hyperbola asymptotic to the lines $\Delta k = \pm k$, which intersects the axis $k = 0$ at $\Delta k = \sqrt{2MB}$. It is when θ is so small that the locus of constant θ intersects the axis $k = 0$ at a smaller value of Δk that the second horizontal tangent fails to exist. It may be remarked that the point where the tangent is vertical also lies on the hyperbola.

Since the kinematical relations for a free-nucleon target include

$$(\Delta k)^2 = k^2 = 2M\Delta E,$$

the condition for quasi-elastic scattering may reasonably be taken as either

$$(\Delta k)^2 = k^2$$

or

$$(\Delta k)^2 = 2M\Delta E = k^2 + 2MB.$$

Thus, the region between the hyperbola and its asymptote will be considered to be the quasi-elastic region. It is apparent, however, that the concept of quasi-elastic scattering is a natural one only where the hyperbola approaches the asymptote.

A locus of constant θ which intersects the hyperbola (with zero slope) where it is near the asymptote clearly intersects the asymptote with a small slope. The fact that a locus of constant θ which has only one horizontal tangent is confined between those which have two and the circle $\theta = 0$ prevents its slope also from having a large value at its intersection with the asymptote. Consequently, the loci of constant θ all have small slopes throughout the quasi-elastic region.

It is expected -- and will later be found to be predicted by the present theory -- that scattering in the quasi-elastic region is more aboundant than scattering elsewhere. It follows that the most important part of a locus of constant θ is the part traversing this region, where its slope has just been shown to be small. In this important region, therefore, it is a fair approximation to substitute a locus of constant Δk for a locus of constant θ. This substitution will be exploited to such an extent that the loci of constant θ themselves will not be needed. Two of them are shown in Fig. 4, however, to illustrate the above remarks; the hyperbola and its asymptote are also included.

10. QUANTITATIVE RESULTS OF FORM-FACTOR CALCULATIONS

Since the nuclear form factor, in the case of modified plane-wave states, is a function only of the two variables Δk and k, it may be thought of as the altitude of a surface above the plane of these two variables. This plane is the one depicted in Fig. 4. The sections of such a surface, taken on loci of constant θ, are closely related to the energy spectra of the product neutrons. The relation can be made closer by the use of $(1/2)k^2$ as the independent variable along such a section, in view of the relation

$$E_f = E_0 - B - k^2/2M$$

which holds between this variable and the neutron energy E_f. Moreover, since the principal quantum number of a harmonic-oscillator state is

$$N = B + k^2/2M,$$

the use of $(1/2)k^2$ as independent variable facilitates the simultaneous presentation of modified-plane-wave and harmonic-oscillator results.

Such simultaneous presentation is further simplified by consideration of $(2\pi)^{-3}k \cdot |F_k(\Delta k)|^2$, instead of $|F_k(\Delta k)|^2$ itself. The new factor $(2\pi)^{-3}k$ is the density-of-states factor for the residual proton, which has been (in effect) included in $|F_N(\Delta k)|^2$ in performing the sum over ℓ, m, and n subject to $\ell + m + n = N$. When $|F_N(\Delta k)|^2$ is plotted as a histogram, the areas under the histogram and under the curve of $(2\pi)^{-3}k \cdot |F_k(\Delta k)|^2$ have the same singnificance-- as shown in Section 7, the significance is that multiplication by $k_f(d\sigma/d\omega_{cm})_{np}$ gives (aside from a constant factor, the same in both cases) the number of neutrons produced in the corresponding range of E_f.

It is also convenient to consider a third function. This is the form factor for unmodified plane waves, obtained from that for modified plane waves by suppressing the correction terms arising from the modification. This function is called $N_0(k, \Delta k)$, and the function explicitly considered is $(2\pi)^{-3}k \cdot N_0(k, \Delta k)$. The comparison of modified and unmodified plane-wave form factors gives desirable insight into the effect of the modification, and thus of the exclusion principle.

In Figs. 5 through 8 appear graphs of the form factors, constructed in the manner detailed above. It will be noted, however, that the sections are, for the most part, on loci of constant Δk, rather than of constant θ. This is simpler, gives a better idea of the behavior of the functions, and is nearly as closely related to experiment. Moreover, loci of constant θ change as the proton beam energy is changed; thus it has been necessary to choose explicitly the energy of the Harwell cyclotron (170 Mev) in order to fix the locus of $\theta = 0$ on which a section is taken. In examining these curves, it is convenient to keep in mind that the independent variable, $(1/2)k^2$, is a measure of the product neutron energy (with reversed sign and shifted origin) in units of 30.8 Mev.

These curves show that the surface for $(2\pi)^{-3} k \cdot N_0(k, \Delta k)$ is a ridge near the locus $k = \Delta k$, high and sharp for small k(and Δk), and getting lower and broader with increasing k. The surfaces for $(2\pi)^{-3} k \cdot |F_k(\Delta k)|^2$ and $|F_N(\Delta k)|^2$, however, behave in this manner only for large k; for small k the ridges retain much of their breadth, and decrease in height instead of increasing. This difference is to be ascribed to the operation of the exclusion principle.

The form factors (other than that for unmodified plane waves) may be seen in Fig. 5 to be small on the forward locus for the Harwell proton beam. From Fig. 4 it appears that the forward loci for proton beams of higher energy lie farther from the quasi-elastic region $k \approx \Delta k$, and thus in regions where the form factors are even smaller than in Fig. 5. In considering a forward locus it should be noted that the theory cannot be applied throughout its length; when Δk becomes large, the q-dependence of the neutron-proton interaction (as seen in Fig. 1) becomes significant, rendering invalid the concept of a form factor. The line at $(1/2)k^2 = 2.20$, in Fig. 5, represents the approximate upper limit of $(1/2)k^2$ for which the curves have any significance.

These results may profitably be compared with those of Wolff,[28] which have been drawn upon in connection with the ground-state momentum distribution. Wolff used an approximation which, as he showed, gives correctly the lowest three moments (namely, those of order 0, 1, and 2) of the spectrum of inelastically scattered protons

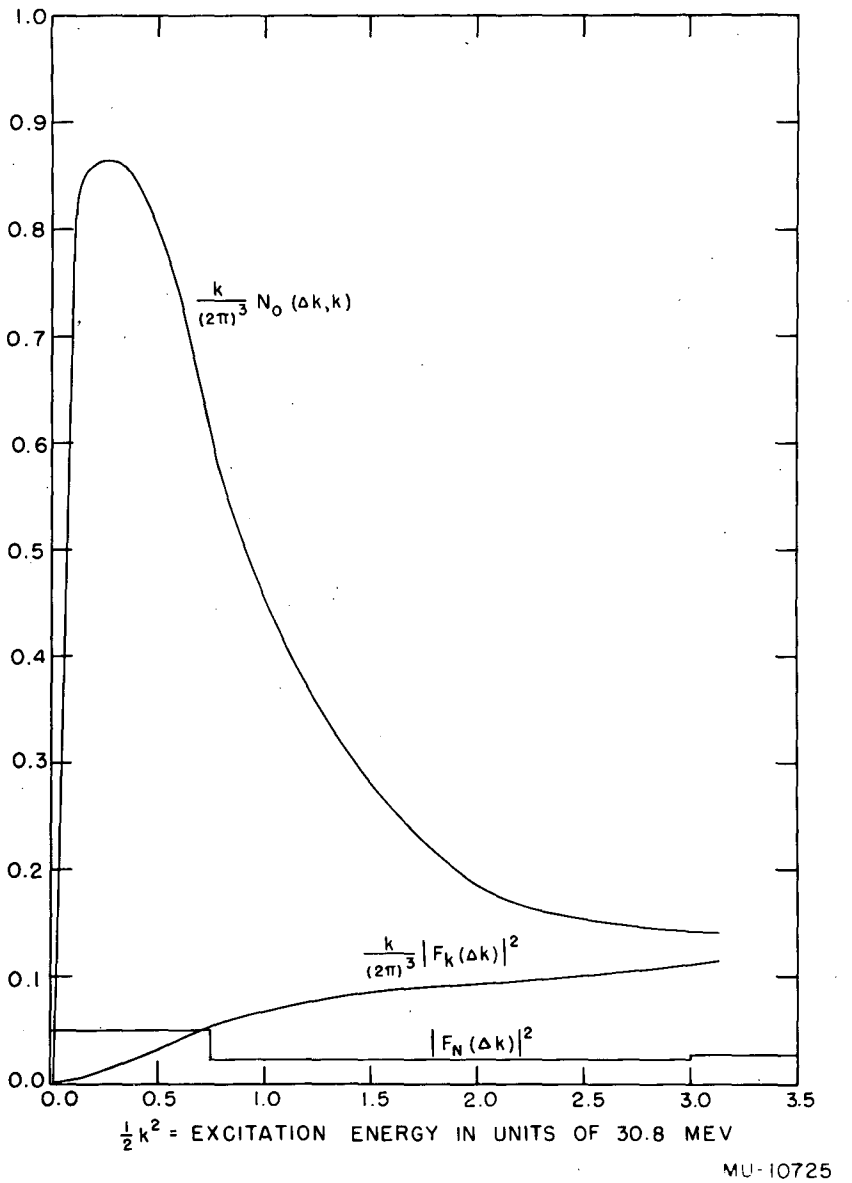

Fig. 5. Form factors on the locus $\theta = 0$ for 170-Mev protons.

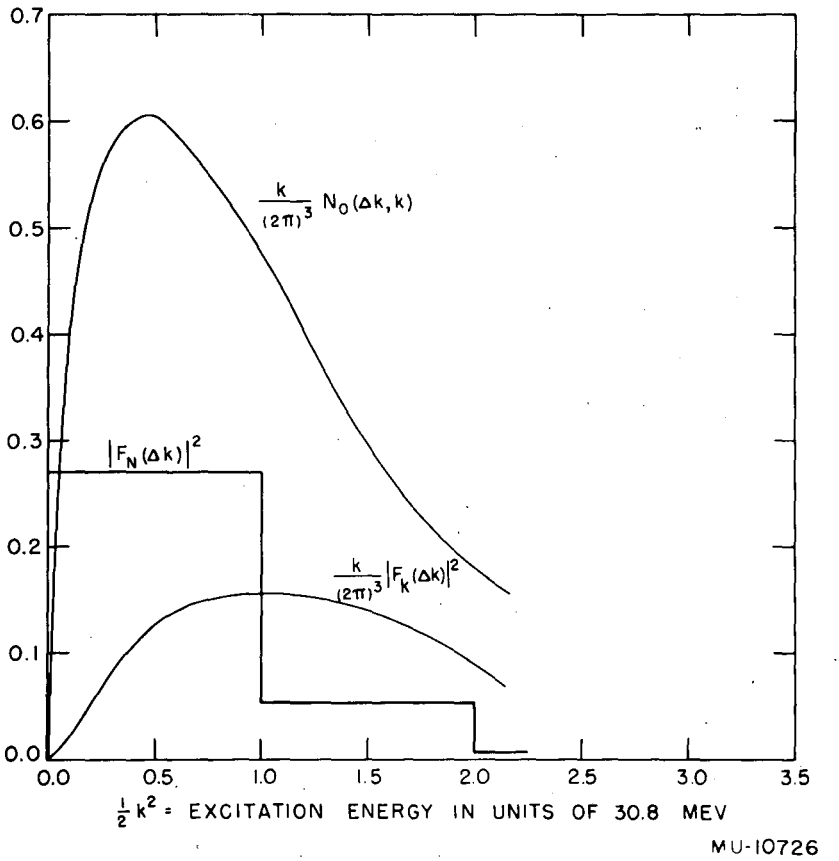

Fig. 6. Form factors on the locus $\Delta k = 0.896$ for 170-Mev protons.

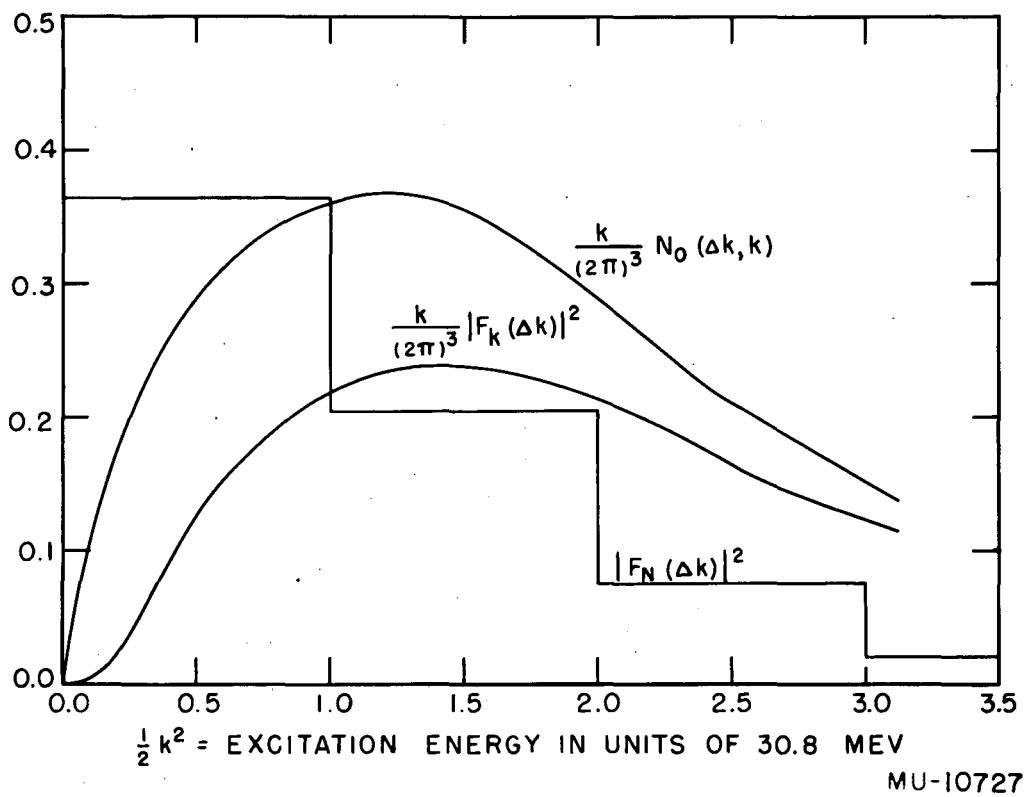

Fig. 7. Form factors on the locus $\Delta k = 1.50$ for 170-Mev protons.

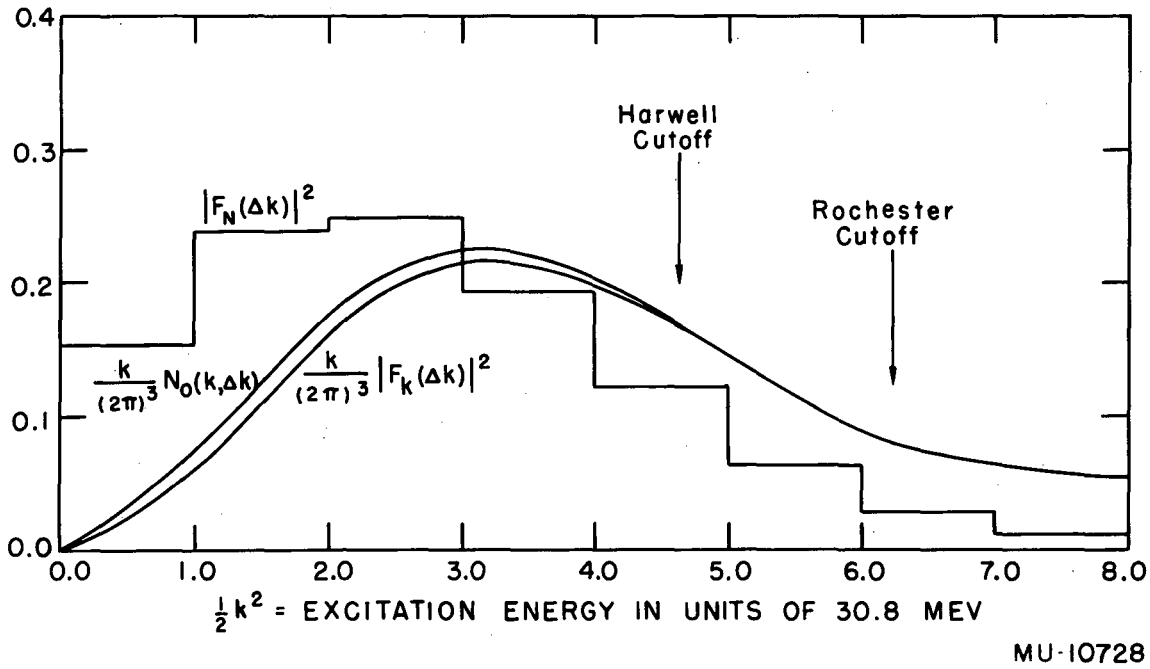

Fig. 8. Form factors on the locus $\Delta k = 2.50$ for 170-Mev protons.

at a given angle θ. The results are independent of the final states. He also showed that the approximation is a good one in the case of the harmonic oscillator. In this work, he ignored certain interference terms, which represent the effect of the exclusion principle. The work reported herein is not quite directly comparable, since it deals with the form factors instead of the spectra and with loci of constant Δk instead of constant θ; use of neutrons instead of protons makes less difference. Nevertheless, it can be seen that the general similarity of harmonic-oscillator and plane-wave results for large Δk shows that his approximation is tolerably good for plane waves, while the differences between harmonic-oscillator and plane-wave results indicate the type of variations that may arise as effects of the inaccurately treated higher moments. Cladis et al[27] found that Wolff's location of the peak of the proton spectrum differed from experiment by 12 Mev and 27 Mev at 30° and 40° respectively. At these angles, the momentum transfers at the peaks of the spectra (in the present units) were 2.26 and 2.82. The harmonic-oscillator and plane-wave form factors are found from Fig. 8 to have peaks at energies differing by about 27 Mev, which is indeed of the same order of magnitude as the difference between theory and experiment.

It may be noted that the general agreement of harmonic-oscillator and modified plane-wave behavior, especially as contrasted with the unmodified plane waves, tends to justify extension of Wolff's assumption of the relative unimportance of higher moments to conditions where the exclusion principle is effective.

11. SUM RULE FOR ANGULAR DISTRIBUTION

The completeness property of the states of the residual proton can be used to evaluate the integral of the form factor over all energies at a fixed value of the momentum transfer. The integral can be written

$$\int d(k^2/2)\ k/(2\pi)^3\ |F_k(\Delta k)|^2 = \int \frac{k^2 dk\, d\omega}{(2\pi)^3}\ |F_{\vec{k}}(\overrightarrow{\Delta k})|^2$$

$(\Delta k = \text{const.})$ $\qquad$ $(\Delta k = \text{const.})$

$$= \int \frac{k^2\, dk\, d\omega}{(2\pi)^3}\ (\phi_{\vec{k}}, e^{i\overrightarrow{\Delta k}\cdot\vec{r}}\phi_0)|^2 = \sum_f{}' \ |(\phi_f, e^{i\overrightarrow{\Delta k}\cdot\vec{r}}\phi_0)|^2 .$$

$(\Delta k = \text{const.})$

The notation $\sum_f{}'$ indicates that the sum is to be extended over all states ϕ_f available to the residual proton. Although the discussion has been made as if the states ϕ_f were in the continuum, the result is clearly valid for discrete states also.

The completeness theorem cannot yet be applied because, since the energy of the state ϕ_f restricts the direction of $\overrightarrow{\Delta k}$ through the relation

$$2M\Delta E = k_0^2 - k_f^2 = k_0^2 - (\vec{k}_0 - \overrightarrow{\Delta k})^2 = 2\vec{k}_0 \cdot \overrightarrow{\Delta k} - (\Delta k)^2,$$

the operator

$$e^{i\overrightarrow{\Delta k}\cdot\vec{r}}$$

is not the same for all states ϕ_f. This difficulty may be overcome by noting that the factor $|F_k(\Delta k)|^2$ of the original integral is actually independent of the direction of $\overrightarrow{\Delta k}$. (This is not inconsistent, because the restriction of the direction of $\overrightarrow{\Delta k}$ is actually a restriction only on $\vec{k}_0 \cdot \overrightarrow{\Delta k}$, and $\vec{k}_0$ is not otherwise involved in the integral.) The value of the integral is thus unchanged if $\overrightarrow{\Delta k}$ is replaced by a fixed vector $\overrightarrow{\Delta k'}$ of the same magnitude. The result is

$$\int d(k^2/2)\cdot k/(2\pi)^3 \cdot |F_k(\Delta k)|^2 = \sum_f{}' \left| (\phi_f, e^{i\overrightarrow{\Delta k'}\cdot\vec{r}}\phi_0) \right|^2 ,$$

$(\Delta k = \text{const.})$

with

$$(\Delta k')^2 = (\Delta k)^2.$$

It is apparent that the variation of the operator with the states ϕ_f can be avoided in this manner only in the case of integration at constant Δk.

If there are no protons in the target, the available states ϕ_f constitute a complete set. Consequently, in this case,

$$\int \frac{k^2\,dk}{(2\pi)^3} \mid F_k(\Delta k) \mid^2 = (\phi_0,\ e^{-i\overrightarrow{\Delta k'}\cdot\vec{r}}\, e^{i\overrightarrow{\Delta k'}\cdot\vec{r}}\, \phi_0) = 1 \ .$$

(Δk=const.)

If the target is saturated, however, the state ϕ_0 is not included among the possible final states ϕ_f, because it already has its complement of protons. In this case the argument takes the form

$$\int \frac{k^2\,dk}{(2\pi)^3} \mid F_k(\Delta k)\mid^2 = \sum_f \mid(\phi_f,\ e^{i\overrightarrow{\Delta k'}\cdot\vec{r}}\,\phi_0)\mid^2 - \mid(\phi_0,\ e^{i\overrightarrow{\Delta k'}\cdot\vec{r}}\,\phi_0)\mid^2$$

(Δk=const.)

$$= 1 - e^{-(1/2)(\Delta k')^2} = 1 - e^{-(1/2)(\Delta k)^2} \ ,$$

where the Gaussian initial state, $\phi_0 = e^{-(1/2)r^2}$, has been used for the evaluation of

$$(\phi_0,\ e^{i\overrightarrow{\Delta k'}\cdot\vec{r}}\ \phi_0).$$

The above results apply to a free neutron, or to a neutron in an α particle. They can be extended to many other nuclei by using the α-particle nuclear model, and neglecting interference between constituent α particles or between α particles and odd neutrons. The cases of a constituent deuteron or triton are not covered, however. A different approach would be required for the deuteron, but the triton may be treated by a simple extension of the above discussion. In this case the transition to the ground state of He^3 may be considered from the standpoint of hole theory; a proton hole in the state ϕ_0 is transformed into a neutron hole in the same state, and the matrix element for this must be

$$(\phi_0,\ e^{i\overrightarrow{\Delta k}\cdot\vec{r}}\ \phi_0) = e^{-(1/4)(\Delta k)^2} \ .$$

The square of this quantity constitutes the entire contribution of the ground-state transition to the integral of $N|F_k(\Delta k)|^2$. Since there are two neutrons in the triton, the corresponding contribution of the excited-state transitions is

$$2[1 - e^{-(1/2)(\Delta k)^2}].$$

The result is, finally,

$$\int \frac{k^2 dk}{(2\pi)^3} \cdot N|F_k(\Delta k)|^2 = 2 - e^{-(1/2)(\Delta k)^2}.$$

(Δk=const.)

The form of this result reflects the presence in the triton of two neutrons, and of a proton that excludes the residual proton from one of two otherwise possible spin states in the case of a ground-state transition.

It is convenient to introduce the notation

$$\overline{|F(\Delta k)|^2} = \int_{(\Delta k=const.)} \frac{k^2 dk}{(2\pi)^3} \cdot N|F_k(\Delta k)|^2.$$

A rule for the value of this quantity, which applies to all cases except odd-odd targets and targets with negative neutron excess, is: every neutron contributes unity, while every proton contributes $-e^{-(1/2)(\Delta k)^2}$. The rule is represented by the equation

$$\overline{|F(\Delta k)|^2} = N - Z\, e^{-(1/2)(\Delta k)^2}.$$

It is not apparent that this result has any direct experimental significance. It will now be shown that it has, however, in that it provides a fairly good estimate of the angular distribution of the product neutrons.

The correct angular distribution is obtained by integrating the neutron yield at constant θ, rather than the form factor at constant Δk:

$$\frac{d\sigma}{d\omega_f} = N \int_{(\theta=const.)} 4 \frac{k_f}{k_0} \left(\frac{d\sigma}{d\omega_{cm}}\right)_{np} |F_k(\Delta k)|^2 \frac{k^2\, dk}{(2\pi)^3}.$$

It has already been noted that loci of constant θ and of constant Δk are nearly coincident in the quasi-elastic region, and that it is just in this region that large values of the form factor are possible. Thus the main contribution to the integral comes from a region in which integration at constant θ differs but little from integration at constant Δk.

As further support for this point, Table I has been prepared. This table presents, for both harmonic-oscillator and modified plane-wave form factors, the fraction of the sum-rule limit obtained by integrating only up to the forward locus (θ=0) for 170-Mev incident protons.

Table I

Comparison of sum-rule limit with integrated form factors at 170 Mev.

		Harmonic Oscillator		Modified Plane Wave	
Δk	Sum-Rule Limit	Integral	Fraction of Sum-Rule Limit	Integral	Fraction of Sum-Rule Limit
0.896	0.330	0.318	96.4%	0.232	70.4%
1.500	0.675	0.665	98.4%	0.547	81.1%
2.500	0.956	0.884	92.5%	0.626	65.5%

It is true that the loci of constant θ cross the quasi-elastic region again, this time with nearly vertical tangents quite different from loci of constant Δk. This happens, however, where k is near its maximum, and thus corresponds to production of low-energy neutrons. The experiments, however, do not detect low-energy neutrons; moreover, the q-dependence of the neutron-proton cross section excludes them from consideration in the theory presented here. The integral for the angular distribution must accordingly be terminated before the locus of constant θ re-enters the quasi-elastic region. It should be remarked that if θ is too large there are no high-energy neutrons, and the theory is not applicable at all.

The case of small θ also constitutes an exception. As see in Fig. 5, in this case there is no strong peak in the spectrum to justify neglect of the difference between the paths of integration. Since, however, the yield for small θ is small, the use of the sum rule will replace one small value by another--namely, by $1 - e^{(1/2)(\Delta k)^2}$, with small Δk. This degree of agreement will be sufficient for many purposes.

It is therefore justifiable, to a considerable extent, to write

$$\frac{d\sigma}{d\omega_f} \approx N \int_{(\Delta k = \text{const.})} 4 \frac{k_f}{k_0} \left(\frac{d\sigma}{d\omega_{cm}}\right)_{np} |F_k(\Delta k)|^2 \frac{k^2 dk}{(2\pi)^3} .$$

On loci of constant Δk, with Δk not so large that the q-dependence of the neutron-proton cross section invalidates the theory, the factor

$$\left(\frac{d\sigma}{d\omega_{cm}}\right)_{np}$$

is a function only of Δk, and thus can be removed from the integration. The factor k_f, which is proportional to the square root of the energy E_f of the product neutrons, varies over a limited range because of the exclusion of low-energy neutrons from consideration. For example, in the Harwell experiments, the ratio of the smallest and largest values of k_f is about 0.7. When, therefore, it is taken outside the integral by an application of the mean-value theorem, its mean value $\bar{k}_f$ is fixed within quite narrow limits. The desired result now follows:

$$\frac{d\sigma}{d\omega_f} \approx 4 \frac{\bar{k}_f}{k_0} \left(\frac{d\sigma}{d\omega_{cm}}\right)_{np} \cdot N \int_{(\Delta k = \text{const.})} \frac{k^2 dk}{(2\pi)^3} |F_k(\Delta k)|^2 =$$

$$= 4 \frac{\bar{k}_f}{k_0} \left(\frac{d\sigma}{d\omega_{cm}}\right)_{np} \overline{|F(\Delta k)|^2}$$

$$= 4 \frac{\bar{k}_f}{k_0} \left(\frac{d\sigma}{d\omega_{cm}}\right)_{np} [N - Z e^{-(1/2)(\Delta k)^2}] .$$

It is still necessary to choose the value of Δk at which to integrate in estimating the neutron production at a given angle θ, and also to fix the best value of $\bar{k}_f$. To the extent that these choices are arbitrary, they are not critical; if they were, the approximations could not have been justified. A simple choice is to evaluate all parameters at the intersection of the locus of constant θ with the quasi-elastic locus

$\Delta k = k$; at this point the form factor is near its maximum, and the loci of constant θ and constant Δk are nearly tangent.

In general,

$$(\Delta k)^2 = 2M(E_0 + E_f) - 4M\sqrt{E_0 E_f} \cos\theta,$$

and

$$E_f = E_0 - B - k^2/2M.$$

But now, in addition,

$$\overline{\Delta k} = \overline{k}.$$

Thus

$$\overline{\Delta k}^2 = 2M(2E_0 - B - \overline{\Delta k}^2/2M) - 4M\cos\theta\sqrt{E_0(E_0 - B - \overline{\Delta k}^2/2M)}.$$

$$\approx (4M E_0 - 2MB - \overline{\Delta k}^2)(1 - \cos\theta).$$

The coefficient of $\overline{\Delta k}^2$ is $1 + (1 - \cos\theta)$. Its reciprocal is $1 - (1-\cos\theta) + (1 - \cos\theta)^2 - \ldots$, or nearly $\cos\theta$ if θ is not too large. Thus

$$\overline{\Delta k}^2 \approx 2(4ME_0 - 2MB)\cos\theta \sin^2\frac{\theta}{2},$$

and

$$\overline{\Delta k} \approx 2k_0 \sin\frac{\theta}{2}\sqrt{\left(1 - \frac{B}{2E_0}\right)\cos\theta}$$

$$\approx 2k_0 \sin\frac{\theta}{2}\sqrt{\cos\theta}\left(1 - \frac{B}{4E_0}\right).$$

Then

$$\overline{k_f}^2 = 2ME_f = 2M(E_0 - B) - \overline{\Delta k}^2$$

$$\approx 2M(E_0 - B) - (4ME_0 - 2MB)\cdot 2\sin^2\frac{\theta}{2}\cos\theta$$

$$= 2M(E_0 - B)\cos^2\theta + 2ME_0(\sin^2\theta - 4\sin^2\frac{\theta}{2}\cos\theta)$$

$$- 2MB(\sin^2\theta - 2\sin^2\frac{\theta}{2}\cos\theta)$$

$$= 2M(E_0 - B)\cos^2\theta + \mathcal{O}(\theta^4) + \mathcal{O}(B\theta^2),$$

and

$$\overline{k_f} \approx k_0\cos\theta\left(1 - \frac{B}{2E_0}\right).$$

It will be recalled that the neutron-proton cross section is to be evaluated at the proton energy $E_{np} = [k_f^2 + (\Delta k)^2]/2M$. Thus there follows

$$\overline{E}_{np} = (\overline{k_f}^2 + \overline{\Delta k}^2)/2M = E_0 - B.$$

For the larger angles, the simple approximate formulae are inadequate, and the exact expressions must be employed. In any case, the final result takes the form

$$\frac{k_0}{4\overline{k_f}} \frac{\left(\frac{d\sigma(E_0,\theta)}{d\omega_f}\right)}{\left(\frac{d\sigma(\overline{E}_{np}, \overline{\Delta k})}{d\omega_{cm}}\right)_{np}} = \overline{|F(\overline{\Delta k})|^2} = N - Z\, e^{-(1/2)\overline{\Delta k}^2}.$$

This relation is subjected in Section 14 to an experimental check similar to the one applied to a related equation in Section 7.

12. INTERPRETATION OF EXPERIMENTS. I: EFFECTS OF ATTENUATION

The theory has now been carried to a point where comparison with experiment is possible. Such a comparison will be more illuminating, however, if previous theoretical considerations of the neutron-production process are kept in mind.

Mandl and Skyrme[22] have proposed a theory of this process, which has had some success in accounting for experimental results. Their theory, like that developed herein, may be thought of as an improvement of a simpler theory in which all influence of the structure of the target nucleus is ignored except that of the momentum distribution of the struck nucleon. However, instead of attempting to evaluate the effect of the exclusion principle while ignoring possible corrections to the impulse approximation, these authors proceed in a complementary manner by ignoring the exclusion principle while attempting to correct the approximation for the effects of the attenuation of the fast nucleon beam in its traversal of the nucleus.

The result of their calculation is an energy spectrum in the forward direction of the form

$$\frac{d\sigma(0^\circ)}{d\omega_f\, dE_f} = f(KR)\ \frac{d\sigma_1}{dE_f} + \frac{d\sigma_2}{dE_f} .$$

The second term is an additional correction for double scattering, and contributes less than 20% of the neutron yield. The dominant first term contains a factor f (KR) which corrects for attenuation, and which is a function only of the product of the attenuation constant K and the nuclear radius R; the other factor represents the result obtained with neglect of attenuation, and is similar to the result of an "uncorrected plane wave" calculation of the type made in the preceding sections.

The argument leading to a sum rule for the total neutron yield applies to this theory, despite the fact that $\theta = 0$, because with uncorrected plane waves the quasi-elastic peak in the form factor exists for small angles: witness the plot of $N_0(k, \Delta k)$ in Fig. 5. This sum rule fixes the integral of $d\sigma_1/dE_f$, nearly independently of the

assumed momentum distribution of the target nucleons. Since the second term is small, the attenuation factor in the first term is left in control of the neutron yield. The main features of the theory can then be inferred by representing this yield in the approximate form

$$\frac{d\sigma(0^o)}{d\omega_f} \approx f(KR)\cdot N \left(\frac{d\sigma(0^o)}{d\omega_{lab}}\right)_{np} = f(KR)\cdot 4N \left(\frac{d\sigma(0^o)}{d\omega_{cm}}\right)_{np} .$$

The original calculations were made with the value $K = 3.6 \times 10^{12}\ cm^{-1}$. This value results from work on neutron total cross sections by Taylor et al.,[7] but has been corrected by these authors (see their Erratum) to $K = 2.4 \times 10^{12} cm^{-1}$. With this correction, K is quite insensitive to energy in the range from 150 Mev to 340 Mev; thus, the factor f(KR) is the same for all these energies. (The insensitivity to energy also obviates the need to distinguish between the values of K for the incident proton and product neutron, at least provided the target has a sufficiently small neutron excess.) This simplifies the application of the theory to the neutron-yield experiments of Knox,[18] performed at 340 Mev.

In order that the double-scattering term may be corrected as well as possible without the extensive calculations required for its accurate evaluation, the corrections to the Mandl and Skyrme results have been computed from the formula

$$\frac{d\sigma(0^o)}{d\omega_f} \approx \frac{d\sigma'(0^o)}{d\omega_f}\,\frac{f(KR)}{f(K'R)} \left(\frac{d\sigma}{d\omega_{cm}}\right)_{np} \Big/ \left(\frac{d\sigma'}{d\omega_{cm}}\right)_{np} .$$

In this formula, the primed values are those used by Mandl and Skyrme, and the unprimed ones are the correct ones for the case considered. The ratio of the neutron-proton cross sections differs from unity only for the comparison with the experiments at 340 Mev.

The original and recalculated theoretical results are presented in Table II, together with the experimental data. The ratios of the yields to the yield from C have been included explicitly, because such ratios can be determined experimentally more reliably than absolute yields. (Multiple traversals of the target by the beam make serious difficulties in measuring the effective beam intensity.) While the theory is perhaps reasonably close to the experiments at 170 Mev,

Table II

Theory of Mandl and Skyrme: neutron yields

Target	170-Mev Protons Theory (Original[a] corrected)		170-Mev Protons Experiment[b]	340-Mev Protons Theory (Corrected)	340-Mev Protons Experiment[c]
C(mb ster^{-1})	68	113	(81 ± 15)	135	(55 ± 17)
Be(mb ster^{-1})	66	103	81 ± 15	123	82 ± 26
Be/C		0.93	1.01±0.06	0.93	1.5
Al(mb ster^{-1})	105	195	192 ± 35	235	115 ± 36
Al/C		1.70	2.39± 0.12	1.70	2.1

[a] Ref. 22

[b] Ref. 13

[c] Ref. 18

it is apparent that the predicted yields are far too high at 340 Mev, and also that the predicted Be/C ratio is much too low. The former discrepancy is weakened by the existence of evidence, reported by Schecter et al.,[19] for yields at 340 Mev higher than those of Knox; in a later section, however, reasons are presented for preferring the lower yields.

It appears, therefore, that a theory including attenuation while neglecting the exclusion principle is not impressively successful in interpreting experimental neutron yields. There is some indication of the nature of the corrections required. The difference in structure between Be and C is not sufficiently represented in the model used, and some additional means of reducing the forward yields is required. Both these objects are, in fact, achieved by including the effects of the exclusion principle.

On the other hand, complete neglect of attenuation is also difficult to justify. The product 2KR has a value of about 1.3 for Be, indicating substantial attenuation even in this light nucleus. For heavy nuclei, Hofmann and Strauch (at about 100 Mev) and Knox (at about 340 Mev) have demonstrated that the forward neutron yield varies nearly as $A^{2/3}$, thus showing that attenuation is significant for the specific process being considered. It is true that Cladis et al.[27] have reported a ratio 4/3 between the yields of inelastic protons from O^{16} and C^{12}, but the statistical uncertainties in the work seem to make insignificant the difference between $4/3 = 1.33$ and $(4/3)^{2/3} = 1.21$. Only the ratio of about 19 between neutron yields from Pb and C at 170 Mev,[13] which nearly agrees with the ratio of the neutron numbers (21 for Pb^{208}, for example), suggests small attenuation; but this is only one fact, and is difficult to reconcile with results at 100 Mev and 340 Mev quoted above, which include data on Pb.

Nevertheless, attenuation receives very little consideration in subsequent discussions. The main reason for this neglect is convenience, but it can be rationalized to some extent. The concept of an attenuated beam is hardly applicable in a light nucleus, where the nucleons interacting with the attenuated beam on the far side of the nucleus are the same ones whose interaction produced the attenuation

on the near side. Moreover, the loss arising from attenuation must, except in the forward direction, be largely restored by multiple collisions, the argument being the well-known one which concludes that attenuation experiments in poor geometry do not detect scattering. This argument applies in the case discussed here because the neutron yield does not vary rapidly with scattering angle.[16, 17, 20]

13. INTERPRETATION OF EXPERIMENTS II: ODD NEUTRON AND SATURATED CORE

In reporting the fact that the neutron yield from Be is greater than that from C at 340 Mev, Knox[18] takes notice of a suggestion by Chew that the cause is the presence of an odd neutron in Be^9, contrasted to the saturated character of C^{12}. According to a model employed by Mullin and Guth[45] in treating the photodisintegration of Be^9, the loosely bound last neutron may be considered to move in the field of the Be^8 core; in the application to high-energy processes, it must then behave nearly like a free neutron. In discussing their experiments on neutron production at 95 Mev, Hofmann and Strauch[21] adopt this model of Be^9. In accordance with the $A^{2/3}$ dependence of neutron yields which they found in heavy elements, these authors estimate the contribution of the Be^8 core to the neutron yield as $(2/3)^{2/3}$ that of C^{12}; they assign to the odd neutron the difference between the observed yield from Be and this estimated yield from the core, and proceed to show that the spectrum and angular distribution of this difference are indeed very like those expected from a free neutron target.

Application of this method to other data is limited to cases where information on the relative yields from Be and C is available. Such cases are found in the experiments of Randle et al.[13] at 170 Mev, and of Knox at 340 Mev. By use of the data of Table II, the forward neutron yields from the odd neutron of Be have been calculated by this method, with and without attenuation, and the results (corrected to the center-of-mass system by division by 4) compared in Table III with the neutron-proton forward (exchange) cross section. It appears that the agreement is satisfactory only at the higher energy, and that allowance for attenuation makes little difference. The success at 340 Mev is a point favoring the Knox yield measurement over that of Schecter et al.

Table III

Proton energy	Odd-neutron yields $(1/4[Be - (\frac{2}{3})^{2/3}C]$	$(1/4[Be - \frac{2}{3}C]$	$\left(\frac{d\sigma(0^o)}{d\omega_{cm}}\right)_{np}$
Mev	mb sterad^{-1}	mb sterad^{-1}	mb sterad^{-1}
170	4.8 ± 1.0	6.8 ± 1.2	11.2 ± 0.7
340	10 ± 3.2	11.4 ± 3.6	13.7 ± 2.1

Since the neutron spectrum from C has not been studied at 340 Mev, it is only at 170 Mev that this subtraction method can be used to estimate the spectrum of the odd neutron yield in the forward direction. In Fig. 9 are presented the Be and C spectra and estimated incident proton spectrum, as reported by Randle et al. The difference Be - $\frac{2}{3}$ C is also shown, and is seen to be very much like the proton spectrum, as would be expected if the neutron were free.

The poor agreement between odd-neutron and free-neutron yields at 170 Mev is puzzling, in view of the similarity of the spectral distributions at that energy, and in consideration of the good agreement between the yields both at 95 Mev and at 340 Mev. The evidence appears, on the whole, favorable to the extended applicability of the idea that the odd neutron in Be^9 behaves as if free.

The idea of the similarity of the contribution from the Be^8 core to that from C^{12} is supported, to some extent, by the experiments of Cladis et al[27] on inelastic proton scattering. The energy distributions of protons scattered from C^{12} and O^{16} are found by these investigators to be the same, the yields being in the ratio three to four.

Hofmann and Strauch have also investigated Li, and find that the neutron spectrum is similar to that of Be, but the yield is greater. They suggest that this results from the last <u>two</u> neutrons in Li^7, acting as if only partially free. The partial character of the freedom, associated with binding energies of 6 or 7 Mev instead of less than 2 Mev as in Be, prevents fruitful application of their subtraction method. Other evidence relating to Li is provided by work of Goodell et al.,[15] who find the neutron spectrum from Li at the Columbia

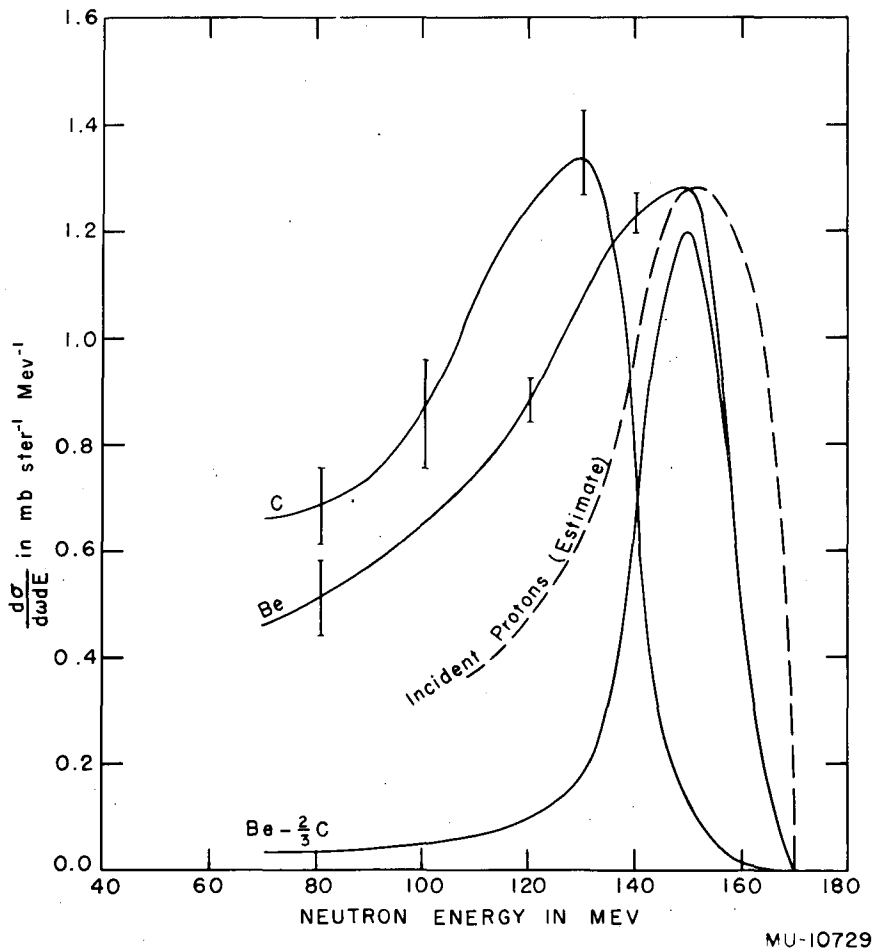

Fig. 9. Odd-neutron spectrum at 170 Mev. Vertical lines on the curves for C and Be indicate roughly the accuracy of the measurements.

cyclotron to be quite similar to that from Be. Also, Ball[5] and DePangher[6] have found the spectrum from LiD at Berkeley to be distinctly narrower than that from Be; it appears unlikely that this effect can be ascribed entirely to the deuteron, since any possible difference in the momentum distributions of the loose neutrons in D and Be must be too small to account for the observed difference in width.

The treatment of the triton, in the derivation of its sum rule in an earlier section, suggests that Li^7 may best be handled by separating the (essentially elastic) transition to the ground state of Be^7--$Li^7(p,n)Be^7$--from transitions to excited states. The same may be done for Be^9, for comparison; it is then apparent that the ground-state transitions must be nearly the same for these two targets, and that differences arise from transitions to excited states. The ground-state transition produces neutrons in the shadow-diffraction pattern (Babinet's principle), and thus is the source of that part of the strong peak found in Be which lies close to the forward direction. This accounts for much of the forward neutron production from Li, but leaves unexplained how the four neutrons in Li yield, by transitions to excited states, more product neutrons in the forward direction than the four neutrons of the Be^8 core.

The fact that two of the neutrons in Li^7 are bound with about half the energy of the other two, or of the four in the Be^8 core, is suggestive. In the spirit of the impulse approximation, it is even more suggestive to find, in the work of Wilcox[30] on quasi-elastic scattering, that one of the protons in Li^7 has a much narrower momentum distribution than the other two. It may be supposed that the two less tightly bound neutrons also have the narrow momentum distribution. Half the yield from these two, together with the contribution from the ground-state transition, may be expected to be similar to the yield from a free neutron or the odd neutron in Be; the remaining two neutrons ought to behave like those of O^{16}, C^{12}, or the core of Be; the peculiarities of Li are then to be ascribed to the other half of the excited-state contribution from the loose neutrons. The high yield found by Hofmann and Strauch is explained by saying this last

contribution is larger than that of a core neutron, presumably because of the narrower momentum distribution, but not so large as that from a free neutron because of the effect of the exclusion principle.

The success of these discussions, even when not quantitative, indicates clearly the value of considering the specific properties of individual nuclei, rather than representing all nuclei as larger or smaller samples of the same nuclear matter after the manner of Mandl and Skyrme.

14. INTERPRETATION OF EXPERIMENTS III: FORM-FACTOR ANALYSIS

The special features of neutron production from unsaturated targets having been accounted for to some extent, the case of a saturated target remains to be discussed. It is convenient to consider the angular distribution first, because this does not require knowledge of the incident proton spectrum; moreover, there is little information on spectra except in the forward direction, where the analysis of the spectrum will prove to be unnecessary because the theory of the angular distribution is already inadequate. Unfortunately, the data on the angular distribution from C are limited to a statement, in the paper by Schecter et al.[19] that at 340 Mev it is similar to those found for Be, Al, Cu, and U at this energy by Miller, Sewell, and Wright.[16]

Nevertheless, it is possible to test the theory of the angular distribution from a saturated target by considering the Be^8 core of Be^9. This is less satisfactory than C, because the extra neutron obscures matters, for example by contributing to statistical errors. However, there are not enough data to make an investigation feasible except in the case of Be. In addition to the work of Miller et al., referred to above, there is an investigation by DeJuren[17] over a wider range of angles; it is worth noting that the former authors detected neutrons by the $C^{12}(n, 2n)C^{11}$ reaction, having a 20-Mev threshold, while the latter author employed Bi fission, with a 50-Mev threshold. Finally, there is the work of Snowden[20] at 170 Mev, in which several neutron energy thresholds were, in effect, employed.

These data may be compared among themselves, and also with the result obtained theoretically by the use of the completeness theorem, by first writing the latter result (of Section 11) explicitly for the case of Be^9 in the form

$$\frac{k_0}{4\bar{k}_f} \frac{\left(\frac{d\sigma}{d\omega_f}\right)}{\left(\frac{d\sigma}{d\omega_{cm}}\right)_{np}} = \overline{|F(\overline{\Delta k})|^2} = 5 - 4\, e^{-(1/2)\overline{\Delta k}^2} .$$

The general form of this relation is tested by the agreement between measurements of the initial expression as a function of $\overline{\Delta k}$ at 170 Mev and at 340 Mev; the specific form of $\overline{|F(\Delta k)|^2}$ is tested by its agreement with the final expression. The results are presented in Fig. 10, which includes not only the above-mentioned data on the angular distributions, but also the normalization of these data to the forward yields measured by Randle et al.[13] at 170 Mev, and by Knox[18] at 340 Mev. With these yields, the various curves in Fig. 10 agree as to order of magnitude, at least.

This agreement would be lost if the neutron yield reported by Schecter et al.[19] had been used instead of that of Knox. It is convenient to collect at this point the evidence that the Knox result is to be preferred, beginning with a more complete account of the data themselves.

The contributions from Knox are the value 1.5 for the ratio of the neutron yields from Be and C as measured by Bi fission detection, together with an expression, found by $C^{12}(n, 2n)C^{11}$ detection, for the yield from Be in terms of the excitation function of his neutron detector. The latter result can be written

$$\frac{d\sigma(0^\circ)}{d\omega_f} = (1 \pm 0.30) \times 42 \text{ mb sterad}^{-1} \times \frac{\text{yield of } C^{12}(n, 2n)C^{11} \text{ at 270 Mev}}{\text{yield of } C^{12}(p, pn)C^{11} \text{ at 340 Mev}} .$$

The first contribution of Schecter et al. is a value of the required yield ratio in the C detector, based on data that had appeared in the literature subsequent to the work of Knox. The value is

$$\frac{\text{yield of } C^{12}(n, 2n)C^{11} \text{ at 270 Mev}}{\text{yield of } C^{12}(p, pn)C^{11} \text{ at 340 Mev}} = (1 \pm 0.10) \times 1.95.$$

They also measured the absolute yield, this time for C, finding

$$\frac{d\sigma(0^\circ)}{d\omega_f} = (220 \pm 50) \text{ mb sterad}^{-1}.$$

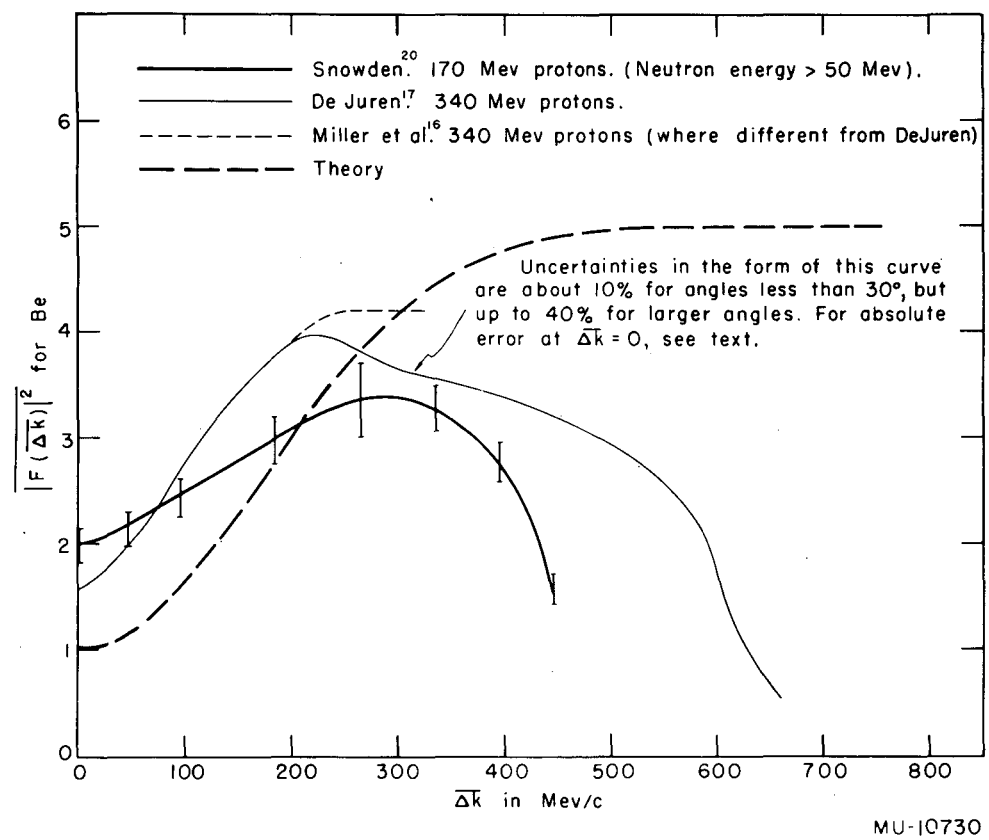

Fig. 10. Analysis of neutron angular distributions.

Substituting the quoted yield ratio in the result from Knox gives the value, for Be,

$$\frac{d\sigma(0^{\circ})}{d\omega_f} = (82 \pm 26) \text{ mb sterad}^{-1} .$$

This is the basis for the values that have been quoted herein as due to Knox.

Any quantity that has been reported on the basis of the Knox value can be changed to what it would be on the basis of the Schecter et al. value by multiplying by the ratio

$$\frac{\left.\frac{d\sigma(0^{\circ})}{d\omega_f}\right|_S}{\left.\frac{d\sigma(0^{\circ})}{d\omega_f}\right|_K} = 1.5 \times \frac{220 \pm 50}{82 \pm 26} = 4.0 \pm 1.5 .$$

This ratio ought to be unity, but differs from unity by more than the probable error. Unless the probable errors have been underestimated, there is a systematic error in the yield determinations. The most apparent source of error is that the Be-to-C ratio of 1.5 has been measured for neutrons of energy above 50 Mev, while the absolute yields being compared have been measured for neutrons whose energy may extend as low as 20 Mev.

The curves of Fig. 10 show that increasing the data obtained at 340 Mev by a factor 4.0 would not merely spoil the rough agreement, but would indicate yields more copious than from free neutrons (given by the high-energy theoretical value)--several times as copious, at some angles. Even at the "low" threshold of 20 Mev, it seems difficult to account for so many neutrons. This difficulty is to be constrasted with the fair degree of agreement visible in the curves as they stand, and also with the approximate agreement of the odd-neutron yield with that from a free neutron, as found in the preceding section.

That the value found by Schecter et al. is too large is also suggested by the total inelastic cross section for neutrons on C. This value has been determined by Ball[5] by two different methods. He

obtains (203 ± 33) mb directly by attenuation in poor geometry, and 209 mb as the difference between the total (elastic plus inelastic) cross section of 288 mb found by DeJuren and Moyer[4] by attenuation in good geometry, and the elastic cross section of 79 mb resulting from the integration over all (significant) angles of his own measured differential elastic cross section. In the absence of an estimate of the error of the second method, it may be considered as a check on the first; the discussion will accordingly be based on the value

$$\sigma_{in} = (203 \pm 33) \text{ mb sterad}^{-1}.$$

The neutron yield leads, on the other hand, to an estimate of a part of this inelastic cross section. In the first place, DeJuren[17] reports that his angular distribution of neutrons from Be implies (through integration over the angles) that the total neutron production cross section is 1.05 times the forward differential neutron production cross section. Since the angular distribution from C is unlikely to be narrower than that from Be, the corresponding factor for C is probably not less than 1.05. Thus, a lower bound on the total neutron-production cross section in C implied by the yield of Schecter et al. is

$$\sigma_{ex} = (220 \pm 50) \times 1.05 \text{ mb sterad}^{-1} = (230 \pm 52) \text{ mb sterad}^{-1}.$$

The charge symmetry of nuclear interactions leads to the conclusion that the last number is also a lower bound on the proton-production cross section of incident neutrons, which is only a part of the neutron inelastic cross section. In fact, the cross section for inelastic scattering of protons ought to be of comparable magnitude. The difference of the quoted values, however, is only

$$(-27 \pm 62) \text{ mb sterad}^{-1}.$$

Thus, 35 mb sterad^{-1}, at most, is available for other inelastic processes. This is to be compared with the result from the Knox yield, which reduces σ_{ex} by a factor 4.0 to 58 ± 13 mb sterad^{-1}, leaving 144 ± 40 mb sterad^{-1} to accommodate the remaining inelastic processes.

The large neutron production found by Schecter et al. could conceivably be accounted for if the incident proton were able to produce

several neutrons having energies above 20 Mev. The data for lower-energy protons, however, do not suggest such multiple production; and the fact that the attenuation constant in nuclear matter does not increase with increasing energy makes unlikely any considerable increase, with increasing energy, of multiplicity from light nuclei.

The significance of the curves of Fig. 10 may now be considered in more detail. Aside from their mutual agreement as to order of magnitude, their most notable common feature is the small values at small $\overline{\Delta k}$. This feature of the experimental curves is evidence for the operation of the exclusion principle, which is known to be the cause of a similar feature of the theoretical curve. However, the experimental values at small $\overline{\Delta k}$ are not so small as the theory requires. Moreover, the experiments fall below the theory at large values of $\overline{\Delta k}$; the experiments even disagree with each other, in that this decrease occurs at smaller values of $\overline{\Delta k}$ at 170 Mev than at 340 Mev.

The break in the high-energy curve at $\overline{\Delta k}$ = 575 Mev/c is probably related to the minimum in the neutron-proton cross section (see Fig. 1) at this value of the momentum transfer, as the assumed independence of q fails at and above this minimum. The corresponding point on the low-energy curve is at $\overline{\Delta k}$ = 400 Mev/c, where no break occurs; nevertheless the greatest decrease occurs for $\overline{\Delta k} > 400$ Mev/c. Thus, much of the difference between the two experimental curves can probably be explained by the difference in $\overline{\Delta k}$ at the minima of the corresponding neutron-proton cross sections.

On the other hand, the reason the theoretical prediction is too high for a range of lower values of $\overline{\Delta k}$ (e.g. 300 Mev/c to 550 Mev/c for the 340-Mev curve) is presumably to be sought among the deficiencies of the impulse approximation, such as its neglect of attenuation. Similarly, the failure of the low-energy curve to rise so high as the high-energy one may be ascribed to a greater influence of such deficiencies at the lower energy; this influence appears to make the approximation largely useless at still lower energies.

The discussion of the extra yield at small $\overline{\Delta k}$ may be assisted by expressing in terms of the variables now being used the discussion of yields from core and odd neutron given previously. For example, since C and Be yields at $\theta = 0$ are equal at 170 Mev, the value 1.97 of the 170-Mev curve at $\overline{\Delta k} = 0$ applies to C as well, and 2/3 of this, or 1.31, to the Be^8 core of Be^9, leaving 0.66 for the odd neutron. At 340 Mev, the fact that the C yield is 2/3 that of Be leads to the partition of the value 1.55 at $\overline{\Delta k} = 0$ into contributions of 0.69 and 0.86 from the Be^8 core and the odd neutron, respectively. It is now evident that, while the odd-neutron yields are of the right order of magnitude, the core yields are much too large.

It is true that the theory does not, strictly, predict zero yield from the core at $\theta = 0$; this result is a consequence of further approximations made in order to get a simple form for the angular distribution. The yield at $\theta = 0$ is small, however, and decreases rapidly with increasing energy because the forward locus (Fig. 4) moves into regions in which the form factor is small. This source of neutrons probably can at most explain the difference between the core yields at the lower and higher energies.

Another possibility is that the neutrons in the forward direction arise from multiple scattering. The work of Goldberger[31] on multiple scattering, and the work of Mandl and Skyrme[22] on double scattering, however, agree in indicating that this process produces relatively few neutrons in the forward direction.

The character of the difficulty in the forward direction is to be appreciated by realizing that the small yield predicted at $\theta = 0$ arises from an interference effect. The free-particle scattering is corrected by a term, arising through the exclusion principle, from the protons in the state from which the neutron is ejected. In the 340-Mev data, the interference is not quite perfectly destructive at $\theta = 0$, but reduces the contribution from 4.00 to 0.69. The effect of the exclusion principle is estimated incorrectly by 0.69/4.00, or about 17%. An error of this size may well be expected to arise from the q-dependence of the neutron-proton interaction, or from corrections to the impulse approximation. For example, polarizations of this order of magnitude are found in neutron-proton scattering.

An interesting related feature is that the curve for 340 Mev is parallel to the theoretical curve in the range $0 < \overline{\Delta k} < 200$ Mev/c, but displaced to the left by 70 to 100 Mev/c. If this type of behavior is general, it may help to explain the behavior of Li^7 discussed in the preceding section. It was pointed out that two of the neutrons in Li^7 have a markedly narrow momentum distribution; these neutrons presumably give an angular distribution theory like the one in Fig. 10, but correspondingly narrower. With such a curve, the strong suppression of neutrons at small Δk would be largely washed out by a shift to the left as large as that noted at the beginning of the paragraph. This would account for the relatively large neutron yield from Li.

This narrowing of the neutron-deficient region at small angles has the appearance of a diffraction effect; the exclusion principle produces a kind of "shadow," into which some diffraction may occur. The primary diffraction effect is already included in the form of the effect of the momentum distribution in defining the theoretical curve itself. There is a secondary effect, however, which would appear--for example--in a second-order perturbation theory of the process; this is the elastic scattering of the incident-proton and product-neutron waves arising from the attenuation of these waves, or what is usually called "shadow scattering." It is difficult to estimate whether this effect would be large enough to explain the data.

15. SUMMARY

The theoretical treatment of high-energy processes in complex targets has been faced for some time with the difficulty, emphasized by the work of Goldberger,[31] that the exclusion principle should tend to suppress inelastic events in the forward direction, but that experiments have failed to demonstrate this effect. It now appears probable that this is in part related to the failure of the impulse approximation at 100 Mev, where many of the experiments have been made, and Goldberger's explicit calculations apply. The effect is actually demonstrable at higher energy, in the form of an angular distribution of neutrons, produced by quasi-elastic exchange collisions in Be, which is nearly flat at small angles, rather than a steep function of the angle as is the neutron-proton cross section. It seems likely that suitable experiments would demonstrate an actual neutron deficiency in the forward direction. The experiments should be performed on C at a sufficiently high energy, and might consist of determinations of neutron spectra at several angles. Such an experiment has actually been performed with the Rochester cyclotron by Nelson, Guernsey, and Mott,[14] but unfortunately the beam monitoring was not consistent at the two angles investigated, and consequently the relative yields at these angles were left undetermined.

The few neutrons that occur "illegitimately" in the forward direction are not more numerous than imperfections of the theory might lead one to expect. They may arise in part from elastic scattering of the incident proton before, or of the product neutron after the exchange collision; in part from an uncancelled residue in the interference between bare-neutron scattering and effects of neighboring protons. The spectrum in the former case would presumably be that corresponding to the angle of inelastic scattering, as given by the theory presented herein; in the latter case, the spectrum would perhaps be like that arising from uncorrected plane waves, determined by the form factor $N_0(k, \Delta k)$, but possibly modified if the perfection of the interference varies considerably over the spectrum. It is thus not possible to predict the neutron spectrum in the forward direction with any certainty. The form

factors $N_0(k, \Delta k)$ in the forward direction, and $|F_k(\Delta k)|^2$ for $\Delta k = 0.896$, represent possibilities; indeed, they seem to be of suitable width to account roughly for the C spectra reported from Harwell and Rochester,[13,14] when folded into the incident-proton spectra. The broad neutron spectra observed at Berkeley[1,2,12] and Columbia,[15] which contrast not only with those mentioned previously but also with those found at still higher energies,[10,11] are probably to be ascribed to correspondingly broad proton spectra in the internal beams of the cyclotrons at these two laboratories.

The analysis leading to these conclusions has been based on the impulse approximation, which appears to be fruitful--though perhaps not perfect--if the energy is sufficiently high. To escape the limitations resulting from lack of knowledge of the phases of the neutron-proton scattering matrix, it has been found necessary to subject all arguments to the condition that this matrix be a function only of the momentum transfer in the scattering process, but not of the energy. Fortunately, such evidence as is available suggests that this condition is approximately satisfied within a substantial range of momentum transfers; however, small departures within the range, and large ones outside it, may be responsible for some of the divergence between the theory and the experiments.

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APPENDIX: COMPLETENESS THEOREM FOR MODIFIED PLANE WAVES

The first step in establishing the desired completeness relation is to reduce the problem from three dimensions to one dimension by expressing the properties of the modified plane waves in terms of corresponding properties of modified spherical waves. The hypotheses are

$$e^{i\vec{k}\cdot\vec{x}} = \sum_{\ell=0}^{\infty} (2\ell+1)\, P_\ell(\hat{k}\cdot\hat{x})\, j_\ell(kr) \quad ,$$

$$\psi_{\vec{k}}(\vec{x}) = \sum_{\ell=0}^{\infty} (2\ell+1)\, P_\ell(\hat{k}\cdot\hat{x})\, f_\ell(k,r) \quad ,$$

$$\int_0^\infty r^2\, dr\, f_\ell(k,r)\, f_\ell(k',r) = \int_0^\infty r^2\, dr\, j_\ell(kr)\, j_\ell(k'r),$$

$$\int_0^\infty k^2\, dk\, f_\ell(k,r)\, f_\ell(k,r') = \int_0^\infty k^2\, dk\, j_\ell(kr)\, j_\ell(kr') \quad .$$

The first two equations relate plane waves and modified plane waves to spherical waves and modified spherical waves, while the last two assert that the modified spherical waves have the same orthogonality and completeness properties, and the same normalization, as do the unmodified spherical waves. The notations $\hat{k}$ and $\hat{x}$ represent unit vectors; thus $\hat{k}\cdot\hat{x}$ is the cosine of the angle between $\vec{k}$ and $\vec{x}$.

By the addition theorem for spherical harmonics,

$$P_\ell(\vec{k}'\cdot\vec{x}) = P_\ell(\hat{k}'\cdot\hat{k})\, P_\ell(\hat{k}\cdot\hat{x}) + \text{terms containing azimuth angles.}$$

Moreover, the azimuth-dependent terms are such that their integrals over all azimuths vanish. There follows from this

$$(\psi_{\vec{k}}(\vec{x}),\ \psi_{\vec{k}'}(\vec{x})) = 4\pi \sum_{\ell=0}^{\infty} (2\ell+1) P_\ell(\hat{k}\cdot\hat{k}') \int_0^\infty r^2\, dr\, f_\ell(k,r)\, f_\ell(k',r)$$

$$= 4\pi \sum_{\ell=0}^{\infty} (2\ell+1) P_\ell(\hat{k}\cdot\hat{k}') \int_0^\infty r^2\, dr\, j_\ell(kr) j_\ell(k'r)$$

$$= (e^{i\vec{k}\cdot\vec{x}},\ e^{i\vec{k}'\cdot\vec{x}}) .$$

In a similar way, the addition theorem, in the form $P_\ell(\hat{k}\cdot\hat{x}') = P_\ell(\hat{x}\cdot\hat{x}')\, P_\ell(\hat{k}\cdot\hat{x})$ + azimuth-dependent terms, can be used to show that

$$(\psi_{\vec{k}}(\vec{x}),\ \psi_{\vec{k}}(\vec{x}')) = (e^{i\vec{k}\cdot\vec{x}},\ e^{i\vec{k}\cdot\vec{x}'}) .$$

Since the plane waves $e^{i\vec{k}\cdot\vec{x}}$ constitute a complete orthonormal set, it follows that the modified plane waves $\psi_{\vec{k}}(\vec{x})$ also constitute such a set, provided the functions $f_\ell(k,x)$ satisfy the conditions of the hypothesis.

The one-dimensional problem has a continuous eigenvalue spectrum. This creates difficulties which can be avoided by imposing an upper limit on the variable r. The problem then has a discrete eigenvalue spectrum, yet passes into the original problem when the bound is increased indefinitely. The usual custom will be followed of supposing that completeness of the eigenfunctions defined by the problem with bound implies completeness of the eigenfunctions defined by the problem without bound, without attempting a rigorous treatment of the passage from one problem to the other.

The comparison functions $j_\ell(kr)$ are defined by the variational condition

$\delta\ \{(j_\ell(kr), \Omega_\ell j_\ell(kr))/(j_\ell(kr), j_\ell(kr))\} = 0$, in which the operators Ω_ℓ have the form $-d^2/dr^2 + \ell(\ell+1)/r^2$. Instead of this explicit formulation, it will be more convenient to introduce the eigenfunctions ψ_k and eigenvalues k^2 of an operator Ω, using the conditions

$$\delta \left\{ \frac{(\psi_k, \Omega \psi_k)}{(\psi_k, \psi_k)} \right\} = 0 \quad , \qquad \frac{(\psi_k, \Omega \psi_k)}{(\psi_k, \psi_k)} = k^2 .$$

These conditions will be said to constitute the "free problem." The following properties, common to the operators Ω_ℓ, will be used:

(a) the operator Ω is Hermitian; thus, its eigenfunctions constitute a complete set;

(b) its eigenvalue spectrum is discrete;

(c) its eigenvalues are not degenerate;

(d) its eigenvalue spectrum is bounded below by zero, but is unbounded above.

The functions $f_\ell(k, r)$ will correspondingly be replaced by the eigenfunctions ϕ_q of a "constrained problem" defined by the same variational condition

$$\delta \left\{ \frac{(\phi_q, \Omega \phi_q)}{(\phi_q, \phi_q)} \right\} = 0,$$

supplemented by the orthogonality condition

$$(f, \phi_q) = 0 .$$

The function f will be subjected to the following conditions:

$$(f, \psi_k) \neq 0, \quad \text{all } \psi_k ;$$

$$(f, f) = 1.$$

The first condition could perhaps be relaxed, but simplifies the argument without affecting the applications that are contemplated. The second condition entails no loss of generality.

The usual variational argument leads to the equations

$$\Omega \psi_k = k^2 \psi_k \quad ,$$

$$\Omega \phi_q = q^2 \phi_q + \beta_q f,$$

in which k^2, q^2, and β_q are Lagrangian multipliers. From these equations follows not only the usual expression for the eigenvalue of the free problem,

$$k^2 = \frac{(\psi_k, \Omega \psi_k)}{(\psi_k, \psi_k)} \quad ,$$

but also, in view of the orthogonality of ϕ_q and f, the same expression for the eigenvalue of the constrained problem,

$$q^2 = \frac{(\phi_q, \Omega\, \phi_q)}{(\phi_q, \phi_q)} .$$

It is known that the expression $(g, \Omega g)/(g, g)$ is minimized by the choice for g of the eigenfunction of the free problem corresponding to the lowest eigenvalue. Consequently, the lowest eigenvalue of the free problem is a lower bound for the eigenvalues q^2 of the constrained problem; and the known lower bound, zero, of the eigenvalues of the free problem is also a lower bound for those of the constrained problem.

The orthogonality of f and ϕ_q also permits application, to the functions ϕ_q, of the argument by which the orthogonality of the functions ψ_k is usually inferred from the Hermiticity of Ω. The finite range of r permits normalization of these functions. Thus,

$$(\psi_k, \psi_{k'}) = \delta_{kk'} ,$$

$$(\phi_q, \phi_{q'}) = \delta_{qq'} .$$

If now h is a function orthogonal to f, and also to all functions ϕ_q belonging to eigenvalues $q^2 < Q^2$, then the ratio $(h, \Omega h)/h, h)$ is minimized by the choice $h = \phi_Q$, and its minimum value is thus Q^2. Any such h whatever, therefore, has the property

$$(h, h) \leq (h, \Omega h)/Q^2 .$$

Since the completeness property is associated with expansions, it is necessary to introduce an arbitrary function g, together with its partial expansion

$$\sum_Q = \sum_{q^2 < Q^2} \phi_q (\phi_q, g) .$$

Since the functions ϕ_q are all orthogonal to f, it is essential to add the function f to the set ϕ_q. The remainder ρ_Q in the expansion of g is thus to be defined by the equation

$$g = f(f, g) + \sum_Q + \rho_Q \quad .$$

It now follows readily from the orthonormal property of the functions f and ϕ_q that ρ_Q is orthogonal to f and to all the functions ϕ_q included in $\sum_Q$. It is thus of the type of the function h in the preceding result, and consequently has the property

$$(\rho_Q, \rho_Q) \leqslant (\rho_Q, \Omega \rho_Q)/Q^2.$$

This inequality relates the norm of ρ_Q to the quantity $(\rho_Q, \Omega \rho_Q)$. The latter quantity may be investigated by considering the expression

$$\big([\, g-f(f, g)], \Omega [\, g-f(f, g)]\big) = \big([\, \Sigma_Q + \rho_Q], \Omega [\, \Sigma_Q + \rho_Q]\big)$$

$$= (\Sigma_Q, \Omega \Sigma_Q) + (\Sigma_Q, \Omega \rho_Q) + (\rho_Q, \Omega \Sigma_Q) + (\rho_Q, \Omega \rho_Q) \; .$$

Several of the terms on the right can be evaluated. Thus

$$(\rho_Q, \Omega \Sigma_Q) = \sum_{q^2<Q^2} (\rho_Q, \Omega \phi_q)(\phi_q, g)$$

$$= \sum_{q^2<Q^2} (\rho_Q, [\, q^2 \phi_q + \beta_q f])(\phi_q, g) = 0,$$

and similarly

$$(\Sigma_Q, \Omega \rho_Q) = 0.$$

Moreover, since

$$(\phi_q, \Omega \phi_{q'}) = (\phi_q, [\, q'^2 \phi_{q'} + \beta_{q'} f\,]) = q'^2 \delta_{qq'} \quad ,$$

there follows

$$(\Sigma_Q, \Omega \Sigma_Q) = \sum_{q^2<Q^2} \sum_{q'^2<Q^2} (\phi_q, \Omega \phi_{q'})(g, \phi_q)(\phi_{q'}, g)$$

$$= \sum_{q^2<Q^2} q^2 (g, \phi_q)(\phi_q, g) \geqslant 0 \quad ;$$

in which the last step depends on the fact that zero is a lower bound for q^2.

As a consequence of these relations, ρ_Q has the property

$$(\rho_Q, \Omega \rho_Q) \leq ([g - g(f,g)], \Omega [g - f(f,g)]) .$$

Combining this with the inequality for the norm of ρ_Q, it is found that

$$(\rho_Q, \rho_Q) \leq ([g - f(f,g)], \Omega [g - f(f,g)])/Q^2.$$

Since the numerator on the right is a constant, and since the completeness property is just that the norm of the remainder ρ_Q can be made as nearly zero as desired by including more terms in the expansion, it is apparent that completeness will follow if the eigenvalues q^2 (of which Q^2 is one) are unbounded above.

To show that the eigenvalues have this property, it is necessary to return to the relations $\Omega \psi_k = k^2 \psi_k$ and $\Omega \phi_q = q^2 \phi_q + \beta_q f$, which have been obtained previously. From them follow the equations

$$(\Omega \psi_k, \phi_q) - k^2(\psi_k, \phi_q) = 0,$$

$$(\psi_k, \Omega \phi_q) - q^2 (\psi_k, \phi_q) = \beta_q (\psi_k, f).$$

Because of the Hermiticity of Ω, these equations lead to an equation for (ψ_k, ϕ_q) of the form

$$(k^2 - q^2) (\psi_k, \phi_q) = \beta_q (\psi_k, f) .$$

Since by hypothesis $(\psi_k, f) \neq 0$, the vanishing of $k^2 - q^2$ implies that of β_q. Thus, $\Omega \phi_q = k^2 \phi_q$, and since the eigenvalues of the free problem are not degenerate, ϕ_q can only be ψ_k itself, which contradicts the orthogonality condition $(\phi_q, f) = 0$. The contradiction can be avoided only by forbidding $k^2 = q^2$. It is then permissible to divide by $k^2 - q^2$ to obtain an expression for (ψ_k, ϕ_q). There then follows from the completeness of the set of functions ψ_k the expression

$$\phi_q = \sum_k \psi_k (\psi_k, \phi_q) = \beta_q \sum_k \frac{\psi_k(\psi_k, f)}{k^2 - q^2} .$$

The supplementary condition now leads to an equation for the eigenvalues q^2:

$$0 = (f, \phi_q) = \beta_q \sum_k \frac{(f, \psi_k)(\psi_k, f)}{(k^2 - q^2)} .$$

When this equation is differentiated with respect to q^2, the denominators take the positive definite form $(k^2 - q^2)^2$. Thus, the slope of the right side of the equation is negative everywhere. The expression has simple poles at the points $q^2 = k^2$; clearly it descends from $+\infty$ at each pole to $-\infty$ at the next, passing through zero once and only once on the way. That is, one and only one eigenvalue of the constrained problem occurs between each pair of successive eigenvalues of the free problem.

Since the eigenvalues of the free problem are unbounded above, the eigenvalues of the constrained problem also have this property. This completes the proof of the completeness of the set constituted by the function f and the eigenfunctions ϕ_q of the constrained problem associated with f. It may be noted that the eigenvalue spectrum of the constrained problem is discrete.

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