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DESTRUCTION OF STORED CYCLOTRON BEAM BY COULOMB SCATTERING

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Frank S. Crawford, Jr.

July 8, 1956

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ABSTRACT

We consider the time dependence of the probability distribution function $G(A, t)$ for the amplitude A of the vertical oscillation of a charged particle circulating at a radius R in a cyclotron, under the influence of multiple Coulomb scattering by residual gas atoms. If the amplitude A becomes equal to the dee half aperture d the particle strikes the dee and is lost from the circulating beam. It is found that G obeys the diffusion-type differential equation

$$\frac{\partial G}{\partial t} = D_2 \frac{1}{A} \frac{\partial G}{\partial A} + D_1 \frac{\partial^2 G}{\partial A^2} .$$

The diffusion constants D are calculated and it is found that we have

$$D_1 = D_2 = \frac{R^2 \left(\langle \Theta^2(t) \rangle / t\right)}{4n} .$$

where $n = -(R/B)\partial B/\partial R$ is the usual magnetic field index and $\langle \Theta^2(t) \rangle$ is the mean-square multiple-scattering angle, projected on the vertical plane, accumulated during the time t. It is found that an equilibrium is established, after which $G(A, t)dA = \text{const} \times J_0 \left[2.40\, A/(d - \Delta d)\right] \exp(-\lambda t)$, where J_0 is the zeroth-order Bessel function, and $\Delta d \sim 0.05d$ is a small correction to the effective dee aperture due to the divergence of the single Coulomb scattering cross section at zero scattering angle. The decay constant λ is given by

$$\lambda = 1.15\, (2.40)^2\, D_1/d^2,$$

where the factor 1.15 is a correction factor arising from single Coulomb scattering. Good agreement is found between the theoretical prediction for λ and the experimental values found during beam storage experiments by Crawford and Stubbins.

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I. INTRODUCTION

The storage of large amounts of circulating beam is of current interest in the design of some new types of accelerators.[1] With a different motivation, beam storage experiments were carried out in 1954 by the author and Warren F. Stubbins.[2] During those experiments, the rate of destruction of stored beam by gas scattering was measured for various storage radii. It was found that gas scattering caused the amount of stored circulating beam to decay exponentially as a function of time, with a mean life of a few seconds. No deviation from a simple exponential decay law was ever observed, although the decay of stored beam was often followed for over two decades. This fact at first suggested that single catastrophic collisions were responsible, since it is well known that exponential decay laws usually represent "one-shot" processes. Consideration of possible single collision processes (nuclear scattering, Rutherford scattering, and electron pickup) showed that they were all too improbable to account for the observed mean lives. The calculation below shows that a combination of small-angle multiple and single Coulomb collisions accounts quantitatively for the observed storage-lifetime data. The results of the calculation are contained in Eq. (62), which expresses the mean life of the stored beam as function of the relevant parameters.

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[1] D. W. Kerst et al., Phys. Rev. 102, 590 (1956); C. K. O'Neil, Phys. Rev. 102, 1418 (1956).

[2] F. S. Crawford, Jr., and W. F. Stubbins, Beam Storage in the 184-Inch Cyclotron, UCRL-3463, July 1956.

II. EQUILIBRIUM DISTRIBUTION OF VERTICAL OSCILLATION AMPLITUDES

We consider a group of particles that has been accelerated to a radius R and left circulating there by turning off the rf accelerating voltage. The rf voltage remains turned off in what follows. The particles start out at time $t = 0$, rf turnoff time, distributed in vertical oscillation amplitude A in some manner determined by their past history. Each particle retains its initial oscillation state until it makes a collision with a gas molecule. In general, a collision introduces a sudden change in both the phase constant δ and the amplitude A that characterizes the oscillation. In any one collision, the amplitude A can either increase or decrease, so that, for small-angle scatters where A is not changed appreciably by one collision, we expect each particle to execute a random walk in amplitude A. Whenever A becomes as large as the half aperture d of the dee, the particle is lost, so that we might describe the process as a "random walk near a cliff."

After a certain amount of time, an equilibrium distribution of vertical oscillation amplitudes is established, the shape of which remains constant. Particles slide down the slope of the distribution and "over the cliff" (hit the dee) at a constant rate, so that the whole distribution decays without changing its shape, and therefore decays exponentially in time.

The results of this section are embodied in Eqs. (14) and (15).

Let $G(A,t)dA$ be the number of beam particles having vertical oscillation amplitude A, in the increment dA, at the time t. Then by Taylor's expansion and a suitable averaging process we obtain[3]

$$t\partial G(A,0)/\partial t = \left\langle A(t) - A(0)\right\rangle \partial G(A,0)/\partial A + \left\langle \left(\frac{A(t) - A(0)}{2}\right)^2 \right\rangle \partial^2 G(A,0)/\partial A^2, \quad (1)$$

where the angular parenthesis represents an average over the small-angle scatters which occur during the time t.

For our particular problem, we show in the next section that

$$D_1 \equiv \left\langle (A(t) - A(0))^2 \right\rangle / 2t \quad (2)$$

and

$$D_2 \equiv A(0) \left\langle A(t) - A(0) \right\rangle / t \quad (3)$$

are independent of the time t, and of the initial amplitude $A(0)$, for small t and arbitrary $A(0)$, so that they may be introduced as constants in Eq. (1). After introducing these constants (whose values we will later calculate explicitly), we obtain

$$\partial G(A,0)/\partial t = (D_2/A(0))\, \partial G(A,0)/\partial A + D_1\, \partial^2 G(A,0)/\partial A^2.$$

---

[3] Joos, "Theoretical Physics" (Hafner Publishing Co., first edition), p. 563.

Since there is now nothing special about the time t = 0, we may rewrite this as

$$\partial G(A,t)/\partial t = D_2(1/A)\partial G(A,t)/\partial A + D_1 \partial^2 G(A,t)/\partial A^2. \tag{4}$$

Equation (4) is a form of the usual one-dimensional diffusion or heat-flow equation. D_1 is the usual diffusion constant. Since $D_2 \neq 0$, the "flow" is nonisotropic.[3] Aside from this fact, our problem corresponds closely to that of a conducting bar whose two ends, located at $A = \pm d$, are maintained at a temperature $G = 0$. (There are no particles with vertical oscillation amplitude A greater than the dee half aperture d.)

Strictly speaking, we expect D_1 and D_2 to be, not constant, but slowly varying in time, since the stored beam gradually loses energy by ionizing collisions, and D_1 and D_2 will turn out to be energy-dependent. Practically speaking, however, D_1 and D_2 may be regarded as constants.

Equation (4) is, as usual, solved by separation of variables. We let

$$G(A,t) = y(A)T(t) \tag{5}$$

and find a solution

$$T(t) = \exp(-\lambda t). \tag{6}$$

Then y(A) satisfies

$$Ay''(A) + (D_2/D_1)y'(A) + (\lambda/D_1)Ay(A) = 0, \tag{7}$$

where a prime represents differentiation with respect to A.

It will be seen in the next section that in our problem,

$$D_1 = D_2 \equiv D. \tag{8}$$

If we substitute $(\lambda/D)A^2 \equiv x^2$ and Eq. (8) into Eq. (7), we obtain

$$xy''(x) + y'(x) + xy(x) = 0, \tag{9}$$

where the prime now represents differentiation with respect to x. This is just Bessel's equation of zeroth order. Therefore we have the solution

$$y(A) = C\,J_0(x) = C\,J_0\left[\left(\frac{\lambda}{D}\right)^{1/2} A\right], \tag{10}$$

where C is a constant, and J_0 is a Bessel function of the first kind, of order zero. (Since y(A) is finite at A = 0, the second solution of Bessel's equation is ruled out.)

The boundary conditions on the solution are such that there are no particles with amplitude greater than A = d. That is,

$$y(d) = CJ_0 \left[(\lambda/D)^{1/2} d\right] = 0. \tag{11}$$

Equation (11) is satisfied by $\lambda = \lambda_n$, where λ_n is given by

$$(\lambda_n/D)^{1/2} d = z_n, \quad n = 1, 2, 3, \ldots, \tag{12}$$

where $z_1 = 2.40$, $z_2 = 5.52$, $z_3 = 8.65$, . . . , and $J_0(z_n) = 0$.

The solution also depends on the initial distribution G(A, 0). If we take an initial square distribution G(A, 0) = 1, we find[4]

$$G(A,t) = 1.60\, J_0(z_1 A/d)\exp(-\lambda_1 t) - 1.06\, J_0(z_2 A/d)\exp(-(z_2^2/z_1^2)\lambda_1 t)$$
$$+ 0.86\, J_0(z_3 A/d)\exp(-(z_3^2/z_1^2)\lambda_1 t) - \ldots \tag{13}$$

Since $z_2^2/z_1^2 = 5.3$, and $z_3^2/z_1^2 = 12.9$, we see that all terms except the first one in Solution (13) soon become negligible. Since G = 0 at A = d after equilibrium has been established, the effect of the transition between the square wave G(A, 0) = 1 and the equilibrium shape

$$G(A,t) = 1.60\, J_0(2.40A/d)\exp(-\lambda_1 t)$$

is most noticeable for large A. If we examine the rate of approach to equilibrium of G(A, t) at the point A = 0.8d, we find that G(A, t), as given by Eq. (13), is within 25% of the first (equilibrium) term after $\lambda_1 t = 0.3$, within 10% at $\lambda_1 t = 0.5$, and within 1% after $\lambda_1 t = 1.0$.

In summary, we find that, independent of the initial shape, when "equilibrium" has been achieved the stored beam has a distribution of vertical oscillation amplitudes A given by

$$y(A)dA = \text{const} \times J_0(2.40A/d)dA, \tag{14}$$

and the total number of particles decays exponentially with a mean life τ given by

$$\tau^{-1} = \lambda_1 = (2.40)^2 D/d^2, \tag{15}$$

where D is given by Eqs. (2), (3), and (8). We will later find that single Coulomb scattering, which does not contribute directly to the above "diffusion" process, increases the decay rate by approximately 15% over that given by Expression (15).

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[4] Churchill, Fourier Series and Boundary Value Problems (McGraw-Hill, 1941) 1st edition, page 167, problem 1.

III. CALCULATION OF DIFFUSION CONSTANTS

In order to find numerical answers for the decay rate of the stored beam given by Expression (15), we must evaluate the diffusion constants D_1 and D_2, defined by Eqs. (2) and (3), respectively, and must verify Eq. (8). The results of this section are contained in Eq. (35).

By squaring out the term in parenthesis in Eq. (2), we find

$$D_1 = (\langle A^2(t)\rangle - A^2(0))/2t - D_2. \tag{16}$$

This is a more convenient expression for D_1, to which we return in Eq. (27).

The vertical oscillation of one particle (between gas collisions) is given as a function of time by

$$Z = A \sin (n^{1/2}\omega_0 t + \delta), \tag{17}$$

where $\omega_0 t$ is the azimuth of the particle, $n = -\frac{R}{B}\frac{\partial B}{\partial r}$ is the magnetic-field index, and A and δ are the amplitude and phase constant of the oscillation. Let θ_z be the angle between the particle orbit and its projection on the median plane, at any instant. Since $\theta_z << 1$, we can write

$$\theta_z = \frac{dz}{ds} = (n^{1/2} A/R) \cos (n^{1/2} \omega_0 t + \delta), \tag{18}$$

where $s = R\,\omega_0\, t$.

Let
$$n^{1/2}\,\omega_0\, t \equiv \phi \tag{19}$$

and
$$n^{1/2}/R \equiv 1/a. \tag{20}$$

Then we can rewrite (17) and (18) as

$$Z = A \sin (\phi + \delta) \tag{21}$$

and
$$a\,\theta_z = A \cos (\phi + \delta)\ . \tag{22}$$

The treatment of scattering is complicated by the fact that the new amplitude A and phase constant δ generated by a given scatter depend on the phase $\phi + \delta$ at which the scatter occurs, as well as on the scattering angle. For instance, an "upwards" scatter through the angle A/a generates a new oscillation having $A' = 0$ provided the particle was passing "downwards" through $Z = 0$ at the time of the scatter, but instead generates an oscillation with $A' = 2A$ provided the particle was passing "upwards" through $Z = 0$ when scattered.

We now consider in detail the changes in amplitude A and phase constant δ resulting from an arbitrary sequence of successive small-angle collisions, m in number. After obtaining a general expression for A(m), we average over suitable probability distributions for the scattering angles and for the phases $\phi + \delta$ at which the collisions occur, and finally obtain expressions for D_1 and D_2.

Let us specify Eqs. (21) and (22) more completely by writing A(m) and $\delta(m)$, for the amplitude and phase, respectively, after m collisions. Let Z_m and ϕ_m be the values of displacement and of $n^{1/2}\omega_0 t$ respectively at the instant of the mth collision, and let θ_m be the scattering angle (projected on the vertical plane) of the mth collision. For brevity, we have dropped the subscript z from θ_z. During the (m + 1) st collision, Z remains fixed, and the angle is changed by θ_{m+1}. That is, from Eqs. (21), and (24), respectively, we have

$$A(m) \sin(\phi_{m+1} + \delta(m)) = Z_{m+1} = A(m+1) \sin(\phi_{m+1} + \delta(m+1))$$

and

$$A(m) \cos(\phi_{m+1} + \delta(m)) + a\,\theta_{m+1} = A(m+1) \cos(\phi_{m+1} + \delta(m+1)).$$

Squaring and adding, we obtain

$$A^2(m+1) = A^2(m) + a^2\theta_{m+1}^2 + 2a\theta_{m+1} A(m) \cos(\phi_{m+1} + \delta(m)). \tag{23}$$

We now use Eq. (23) as a recursion formula to obtain, by repeated application,

$$\begin{aligned} A^2(m) &= A^2(0) + a^2(\theta_1^2 + \theta_2^2 + \ldots + \theta_m^2) + \\ &+ 2a\Big\{\theta_1 A(0) \cos\left[\phi_1 + \delta(0)\right] + \theta_2 A(1) \cos\left[\phi_2 + \delta(1)\right] + \ldots \\ &\quad + \theta_m A(m-1) \cos\left[\theta_m + \delta(m-1)\right]\Big\} \qquad (24) \\ &\equiv A^2(0) + x^2, \end{aligned}$$

where x^2 is defined by Eq. (24). Equation (24) specifies in complete detail the development of A(m) during the m collisions. So far, there has been no averaging.

We now average both sides of Eq. (24) over a probability distribution for collisions such that positive and negative scatters are equally likely, and such that all scattering events are independent of one another. We also neglect the effect of energy loss in changing the scattering cross section. Then we have

$$\langle\theta_k\rangle = 0, \text{ and } \langle\theta_1^2\rangle = \langle\theta_2^2\rangle = \ldots = \langle\theta_m^2\rangle, \text{ so that}$$

$$\langle A^2(m)\rangle = A^2(0) + ma^2 \langle\theta_1^2\rangle. \tag{25}$$

All the terms in Eq. (24) enclosed in the braces average to zero. For example,

$$\langle\theta_2 A(1) \cos\left[\phi_2 + \delta(1)\right]\rangle = \langle\theta_2\rangle\langle A(1) \cos\left[\phi_2 + \delta(1)\right]\rangle = 0, \text{ since } \langle\theta_2\rangle = 0.$$

That is, A(1) does not depend on θ_2, although it does depend on θ_1. This is obvious from the physical meaning: the amplitude after one collision does not depend on the second collision, which has not yet occurred.

We now recognize that $m \left\langle \theta_1^2 \right\rangle = \left\langle \Theta^2(m) \right\rangle$, (26)

where $\left\langle \Theta^2(m) \right\rangle$ is the mean-square multiple-scattering angle after m scatters. We can now replace m by the time t during which the m scatters occur. Equation (25) becomes

$$\left\langle A^2(t) \right\rangle - A^2(0) = a^2 \left\langle \Theta^2(t) \right\rangle . \tag{27}$$

Substituting (27) into Eq. (16), we obtain

$$D_1 = a^2 \left\langle \Theta^2(t) \right\rangle / 2t - D_2 . \tag{28}$$

Since $\left\langle \Theta^2(t) \right\rangle$ is proportional to the time t, we see that the first term in Eq. (28) is a constant.

To obtain D_2, defined in Eq. (3) we must find $\left\langle A(m) \right\rangle$. To do this, we first take the square root of both sides of Eq. (24), and then expand in a power series, to obtain

$$\begin{aligned} A(m) &= \left[A^2(0) + x^2 \right]^{1/2} \\ &= A(0) \left[1 + \frac{x^2}{2A^2(0)} - \frac{x^4}{8A^4(0)} \right. \\ &\quad \left. + \frac{x^6}{16A^6(0)} - \frac{5\,x^8}{128A^8(0)} + \ldots \right] . \end{aligned} \tag{29}$$

Here we assume that we have $A(0) \neq 0$, and that m is small enough so that x^2 is small compared to $A^2(0)$. The expansion (29) carries us through terms of order θ_1^4, which is found to be sufficient for our purpose.

We first find $\left\langle A(m) \right\rangle$ only to order $\left\langle \theta_1^2 \right\rangle$. We later show that terms of order $\left\langle \theta_1^4 \right\rangle$ are negligible. To order $\left\langle \theta_1^2 \right\rangle$ Eq. (25) becomes

$$\left\langle x^2 \right\rangle = m\,a^2 \left\langle \theta_1^2 \right\rangle = a^2 \left\langle \Theta^2 \right\rangle . \tag{30}$$

To order $\left\langle \theta_1^2 \right\rangle$,

$$\left\langle x^4 \right\rangle = (2a)^2 A^2(0) \left\langle \theta_1^2 \right\rangle \left\{ \cos^2 \left[\theta_1 + \delta(0) \right] + \ldots + \cos^2 \left[\phi_m + \delta(m-1) \right] \right\} .$$

Here we have used the facts that $\left\langle (\theta_k A(k-1))^2 \right\rangle = \left\langle \theta_k^2 \right\rangle A^2(0)$, to order $\left\langle \theta_k^2 \right\rangle$, and that $\left\langle \theta_k^2 \right\rangle = \left\langle \theta_1^2 \right\rangle$ for all k. Now average over the phases $\phi_k + \delta(k-1)$ of the collisions, assuming randomness, so that $\cos^2 \left[\phi_k + \delta(k-1) \right]$ is replaced by 1/2. Then, since the braces in $\left\langle x^4 \right\rangle$ contain

m terms, we obtain

$$\langle x^4 \rangle = 2a^2 A^2(0) m \langle \theta_1^2 \rangle = 2a^2 A^2(0) \langle \Theta^2 \rangle . \tag{31}$$

The remaining contributions to the expansion (29), $\langle x^6 \rangle$, $\langle x^8 \rangle$, etc. do not contain any terms of order $\langle \theta_1^2 \rangle$.

Inserting Eqs. (30) and (31) into (29), we obtain, to order $\langle \theta_1^2 \rangle$,

$$A(0) \langle A(m) - A(0) \rangle = a^2 \langle \Theta^2 \rangle /2 - 2a^2 \langle \Theta^2 \rangle /8 = a^2 \langle \Theta^2(m) \rangle /4. \tag{32}$$

Or, replacing the collision number m with the corresponding time t, we have

$$A(0) \langle A(t) - A(0) \rangle = a^2 \langle \Theta^2(t) \rangle /4. \tag{33}$$

Inserting (33) into Eq. (3), we then obtain

$$D_2 = a^2 \langle \Theta^2(t) \rangle /4t. \tag{34}$$

Comparing Eqs. (34) and (28), we find that, to order $\langle \theta_1^2 \rangle$,

$$D_1 = D_2 = a^2 \langle \Theta^2(t) \rangle /4t. \tag{35}$$

We now show that higher terms through $\langle \theta_1^5 \rangle$ make a negligible contribution to $\langle A(m) \rangle$, and therefore to D_1 and D_2. Because positive and negative scatters are equally likely, $\langle \theta_1^3 \rangle$ and $\langle \theta_1^5 \rangle$ are zero, and we need only consider terms of order $\langle \theta_1^4 \rangle$.

We find, for the terms of $\langle A(m) \rangle$ of order $\langle \theta_1^4 \rangle$,

$$\langle x^4 \rangle = a^4 \left[m \langle \theta_1^4 \rangle + 2m (m - 1) \langle \theta_1^2 \rangle^2 \right], \tag{36}$$

$$\langle x^6 \rangle = 6a^4 A^2(0) \left[m \langle \theta_1^4 \rangle + m (m - 1) \langle \theta_1^2 \rangle^2 \right], \tag{37}$$

$$\langle x^8 \rangle = 6a^4 A^4(0) \left[m \langle \theta_1^4 \rangle + 2m (m - 1) \langle \theta_1^2 \rangle^2 \right]. \tag{38}$$

Inserting Eqs. (36), (37), and (38) into (29), and collecting terms, we find

$$\langle A(m) - A(0) \rangle (\text{order } \theta_1^4) = (a^4/64A^3(0)) \left[m \langle \theta_1^4 \rangle + 3m(m-1) \langle \theta_1^2 \rangle^2 - 25m(m-1) \langle \theta_1^2 \rangle^2 \right]. \tag{39}$$

We can show that

$$\langle \Theta^4 (m) \rangle = m \langle \theta_1^4 \rangle + 3m (m - 1) \langle \theta_1^2 \rangle^2, \tag{40}$$

so that, letting $m - 1 \approx m$ in the last term of Eq. (39), we obtain

$$\langle A(m) - A(0) \rangle (\text{order } \Theta^4) = (a^4/64A^3(0)) (\langle \Theta^4(m) \rangle - 25 \langle \Theta^2(m) \rangle^2). \tag{41}$$

We now assume that the distribution law for multiple scattering is given by a Gaussian curve. Then one can show that

$$\left\langle \Theta^4 \right\rangle = 3 \left\langle \Theta^2 \right\rangle^2 . \tag{42}$$

We now replace m by t, in Eq. (41), insert (42) into (41), multiply the fourth-order expression (41) by A(0), and add it to the second-order expression (33), divide by the time t, and obtain an expression for D_2 through the fourth order, given by

$$D_2 = \frac{a^2 \left\langle \Theta^2(t) \right\rangle}{4t} \left(1 - \frac{11}{8} \frac{a^2 \left\langle \Theta^2 \right\rangle}{A^2(0)}\right) \tag{43a}$$

$$= \frac{a^2 \left\langle \Theta^2(t) \right\rangle}{4t} \left(1 - \frac{11}{8} \frac{\left\langle x^2 \right\rangle}{A^2(0)}\right) . \tag{43b}$$

But expansion (29) depends for its validity on the condition that x^2 be small compared with $A^2(0)$, so that the correction term $(11/8) \left\langle x^2 \right\rangle / A^2(0)$ in (43b) can be consistently neglected. Its neglect corresponds to the neglect of $(t^2/2)\ \partial^2 G(A, t)/\partial t^2$ compared with $t \partial G(A, t)/\partial t$ in the derivation[3] of the differential equation (4).

At the boundary point A(0) = 0, expansion (29) does not hold. This is also a singular point for the differential equation (4). From physical considerations, we do not expect this to influence the result (35).

We may summarize and combine the results of the last two sections by inserting Eqs. (35) and (20) into (15) to obtain the partial decay constant λ(m. s.) due to multiple scattering (m. s.) alone,

$$\lambda(\text{m. s.}) = \lambda_1 = (2.40)^2 R^2 \left\langle \Theta^2(t) \right\rangle / 4d^2 nt, \tag{44}$$

where 2.40 is the first root of the zeroth-order Bessel function, R is the storage radius, d is the dee half aperture, n is the magnetic-field index, t is an arbitrary time interval, and $\left\langle \Theta^2(t) \right\rangle$ is the mean-square multiple-scattering angle accumulated during the time t; $\left\langle \Theta^2(t) \right\rangle / t$ is independent of t.

IV. CONTRIBUTION OF SINGLE COULOMB SCATTERING

A single scatter of sufficient magnitude can cause a particle to strike the dee. We find that single Coulomb (Rutherford) scattering of particles having small vertical oscillation amplitudes (compared with the dee half aperture) contributes an additional 5% to the rate of loss of stored beam, compared with the loss rate due to multiple scattering. Single scattering of particles having vertical oscillation amplitudes almost equal to the dee half aperture contributes another 10%.

If we square both sides of Eqs. (21) and (22), and then add the results, we find, after solving for θ_z,

$$\theta_z (A, Z) = \pm (1/a) (A^2 - Z^2)^{1/2}. \tag{45}$$

For a given Z, the largest value of θ_z consistent with missing the dee is obtained by setting A = d, to obtain

$$\theta_z(d, Z) = \pm (1/a) (d^2 - Z^2)^{1/2}. \tag{46}$$

The plus and minus signs correspond to "upwards" and "downwards" motion, at a given Z.

The largest single scatter (projected on the vertical plane) consistent with missing the dee, for a given initial amplitude A and displacement Z, is obtained by subtracting Eq. (45) from (46). We omit nothing essential if we consider the particle to have initially $Z > 0$, and $\theta_z > 0$. Then there are just two maximum scatters $\Delta\theta_z(\max)$ possible, corresponding to scattering up, and scattering down, to hit the dee. The absolute magnitudes of these limits are given by $\Delta\theta_z(\max) = \theta_z(\text{up})$, and $\Delta\theta_z(\max) = \theta_z(\text{down})$, where $\theta_z(\text{up})$ and $\theta_z(\text{down})$ are given by

$$\theta_z (\text{up}) = \theta_z(d, Z) - \theta_z(A, Z) = a^{-1} \left[(d^2 - Z^2)^{1/2} - (A^2 - Z^2)^{1/2}\right], \tag{47a}$$

and

$$\theta_z(\text{down}) = \theta_z(d, Z) + \theta_z(A, Z) = a^{-1} \left[(d^2 - Z^2)^{1/2} + (A^2 - Z^2)^{1/2}\right], \tag{47b}$$

respectively.

We now consider the scattering cross section for particles to hit the dee and be lost. We consider Z and A fixed, in the following. The single Coulomb (Rutherford) cross section may be written, for small-angle scatters,

$$d\sigma(\theta, \phi) = B^2 \frac{d\phi d\theta}{\theta^3} = B^2 d\phi d \left(\frac{-1}{2} \theta^{-2}\right), \tag{48a}$$

where

$$B = 2Ze^2/pv, \tag{48b}$$

and where θ is the polar scattering angle, ϕ is the azimuthal scattering angle, Ze is the electric charge of the target nucleus, and p and v are the proton momentum and velocity. We call $\phi = 0$ for a scatter upwards with no radial component to the scatter.

Consider a fixed value of ϕ. Scatters through a polar angle θ hit the dee for θ greater than

$$\theta(\max) = \Delta\theta_z(\max)/\cos\phi.$$

Holding ϕ fixed, we integrate Eq. (48a) over all polar angles greater than $\theta(\max)$, and obtain

$$d\sigma\,(\theta > \theta(\max),\ \phi) = \frac{B^2 d\phi}{2\,\theta(\max)^2} = \frac{B^2 d\phi \cos^2\phi}{2\left[\Delta\theta_z(\max)\right]^2}, \tag{49}$$

where $\Delta\theta_z(\max)$ is equal to $\theta_z(\text{up})$ over half of the azimuth ϕ, and to $\theta_z(\text{down})$ over the other half. We now integrate over ϕ, noticing that we can merely replace $\cos^2\phi$ by its average value of 1/2, to obtain the cross section for hitting the dee, for a given initial Z and A,

$$\sigma\,(Z, A) = \frac{\pi}{4}\ B^2\,(\theta_z(\text{up})^{-2} + \theta_z(\text{down})^{-2}), \tag{50}$$

where $\theta_z(\text{up})$ and $\theta_z(\text{down})$ are given by Eq. (47). Since, for a given amplitude A, the collision can occur at any displacement Z during the oscillation, we must average the expression (50) over $0 < Z < A$. Since the collisions occur randomly in time, the proper weighting factor is proportional to dt; and since we have considered all Z to be >0, and all $\theta_z > 0$, we should average over one quarter cycle. The probability of a collision at Z is then given by

$$dP = dt/4T = (2/\pi)\,dZ\,(A^2 - Z^2)^{-1/2}. \tag{51}$$

where T is the period of vertical oscillation. We thus find the time-averaged cross section for single Coulomb scattering loss, for a given vertical oscillation amplitude A, by averaging Eq. (50) over the distribution (51), to obtain

$$\sigma\,(A) = \int_{z=0}^{A} dP\sigma\,(Z, A) = \left(\frac{\pi}{2}\frac{B^2 a^2}{d^2}\right)(1 - A^2/d^2)^{-2} \tag{52}$$

$$= \sigma(0)\,(1 - A^2/d^2)^{-2},$$

where

$$\sigma(0) = \frac{\pi}{2}\,\frac{B^2 a^2}{d^2} \tag{53}$$

is the cross section for striking the dee when $A = 0$.

To obtain the total contribution of the single scattering, we must average the cross section (52) over the equilibrium distribution of vertical oscillations given by Eq. (14). We first observe that if the single scattering is anywhere large enough to attenuate the beam appreciably in a time that is short compared with the "equilibrium time" $T_e \approx 0.5\ \lambda^{-1}$ (m. s.), then Eq. (14) does not hold, and the equilibrium shape readjusts to something different. This is in fact what happens, although we shall see that the consequences are not very important for the total attenuation rate. We first notice that the single Coulomb cross section $\sigma(A)$ given by Eq. (52) becomes infinite for $A = d$. Even when we multiply by the equilibrium distribution function (14) and integrate over A, the weighted cross section still diverges, despite the fact that G goes to zero at $A = d$. Of course this does not mean that the decay time should become very short, because only a small part of the beam sees this "infinite" cross section. What happens is that if we start out with a distribution given by Eq. (14) and then suddenly "turn on" the single scattering, the portion of the distribution very near to $A = d$ is wiped out almost immediately, and a new equilibrium distribution is reached, in a time of the order of T_e. Since the single-scattering cross section quickly becomes reasonably small, away from zero scattering angles, the new distribution should look very similar to the old one, except that in Eq. (14), we should replace the gap half aperture d by an "effective gap aperture" $d' = d - \Delta d$. In the region between d' and d, there will be a "transition fillet." That is, the new distribution does not actually go all the way to zero at $A = d'$, since the cross section is not yet actually infinite. In terms of the heat-flow analogy, the effect of the divergence of the single scattering near zero scattering angle is to change the boundary condition from one that merely holds the ends of the bar at zero temperature, to one in which, in addition, the region <u>near</u> the ends is maintained at a temperature close to zero.

In order to see how the decay time is affected we must estimate d', which enters the multiple-scattering decay constant through Eq. (43). After this, we have an additional contribution to the decay obtained by averaging the single-scattering cross section over the new equilibrium distribution, inside $A = d'$.

The boundary d' is located roughly at the point where the scattering out due to the single scattering can just keep up with the "flow rate" from the diffusion of particles. Put differently, the single-scattering cross section must be able to dispose of any particle that finds itself inside Δd, in a time of the order of the equilibrium time T_e, since otherwise the region would start filling up. This condition may be written

$$N_v \ \sigma(d')\ T_e\ v = \epsilon, \tag{54}$$

where ϵ gives the probability of removing a particle during T_e, N_v is the number of gas atoms per unit volume, $\sigma(d')$ is given by Eq. (52) with $A = d'$, and v is the proton velocity.

The equilibrium time T_e is given by

$$T_e = f/\lambda \text{ (m. s.)}, \tag{55}$$

where $f \approx 0.5$, according to the discussion following Eq. (13). The boundary d' is obtained by setting $\epsilon \approx 1$.

Setting $A = d' = d - \Delta d$, in (52), and assuming $\Delta d << d$, we can write

$$\sigma(d') = \sigma(0)\, d^2/4\,(\Delta d)^2. \tag{56}$$

We also write, for the projected mean square scattering angle,[5]

$$\left\langle \Theta^2(x) \right\rangle = \left[(15)^2/(pv)^2\right] (\ell/X_0), \tag{57}$$

where pv is in Mev, ℓ is the amount of gas traversed in g cm^{-2}, and X_0 is the radiation length in g cm^{-2}.[6] Then we can combine Eqs. (54), (55), (56), (57), (48'), (43), and (20) to obtain

$$\Delta d/d = 7.6 \times 10^{-3}\, (fX_0 Z^2/\epsilon W)^{1/2}, \tag{58}$$

where W is the atomic weight. We see that the effective fractional narrowing of the gap is independent of the gap, of the gas pressure, and the energy of the protons, and depends only weakly on the atomic number of the gas. Choosing $f = 1/2$, and $\epsilon = 1$, and using $X_0 = 58$ g cm^{-2} for hydrogen and 38 g cm^{-2} and nitrogen, we find $\Delta d/d = 0.041$ and 0.062 for hydrogen and nitrogen, respectively. Since the decay constant, according to Eq. (43), goes as d^{-2}, the decay-rate corrections are roughly 8% and 12% for hydrogen and nitrogen, respectively. Since this calculation does not provide the exact values to choose for f and ϵ, we neglect the small dependence on the gas and say that the correction due to single scattering of particles having large vertical oscillation amplitudes is about 10%.

Finally, if we average the single-scattering cross section σ (A) over the modified vertical oscillation amplitude distribution $J_0(2.40A/d')dA$, we obtain

$$\left\langle \sigma(A) \right\rangle \approx 2.1\, \sigma(0), \tag{59}$$

where $\sigma(0)$ is given by (53).

Let

$$\lambda(ss) = N_v \left\langle \sigma(A) \right\rangle v \tag{60}$$

be the corresponding single scattering (ss) partial decay constant. Then, using almost the same procedure as was used to obtain (58), we find

$$\lambda(ss)/\lambda(ms) = 4.7 \times 10^{-4}\, (X_0 Z^2/W). \tag{61}$$

Result (61) is independent of the gas pressure, the dee gap, and the proton energy, and is almost independent of the gas, for any gas except hydrogen. The corresponding correction to λ(ms) amounts to just 2.7% for hydrogen, and 6.3% for nitrogen.

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[5] E. Segrè, Experimental Nuclear Physics, Vol I (John Wiley and Son, Inc., New York, 1953), p. 285.

[6] ibid., Table 8, p. 266.

Because of the approximate nature of the correction, we can neglect the small difference between hydrogen and other gases, and combine the corrections due to (58) and (61) to obtain an over-all correction of about 15% to $\lambda(ms)$, due to single Coulomb scattering.

We summarize all of the results of this report by writing the corrected decay constant for Coulomb scattering of stored beam,

$$\tau^{-1} = \lambda = (1.15)\,(2.40)^2\,R^2\,\left\langle \Theta^2(t)\right\rangle / 4d^2 nt \qquad (62)$$

where the symbols are as defined after Eq. (44).

V. COMPARISON WITH EXPERIMENT

In one measurement by Crawford and Stubbins[2], the quantity ℓ in Eq. (57) was measured indirectly by essentially measuring dE/dx at the particle orbit. This was done by measuring the time for protons stored intially at a known radius to lose sufficient energy by ionization collisions to spiral inward to a slightly smaller known radius. Measurement of dE/dx yields the electron density N_e, independent of the type of gas, except for a small correction which we call C, and which is due to the Z dependence of the ionization potential. The above measurement yielded

$$N_e = C \times (1.3 \pm 0.2)\ 10^{13} \text{ electrons cm}^{-3}, \qquad (63)$$

where C = 1.0 and 0.84 for air (or oil vapor) and hydrogen, respectively.

The conditions corresponding to (63) were given by p = 690 Mev/c, $\beta = 0.59$. Inserting these numbers and (63) into (57) yields

$$\left\langle \Theta^2(t)\right\rangle / t = 5.25 \times 10^{-4}\ CW/ZX_0 \text{ radian}^2 \text{ sec}^{-1}. \qquad (64)$$

Inserting the corresponding storage radius R = 63.5 in., field index n = 0.029, dee half aperture d = 2.5 in., and (64) into Eq. (62), we obtain the mean life

$$\tau = 0.052\ \ ZX_0/CW. \qquad (65)$$

Assuming that the main residual gas was air (~nitrogen), we insert Z = 7, $X_0 = 38$ g cm^{-2}, C = 1.0, and W = 14 into (20) to obtain

$$\tau(N_2) = (1.0 \pm 0.2) \text{ sec.}$$

If, instead, the gas was pure hydrogen, we have Z = 1, $X_0 = 58$ g cm^{-2}, C = 0.84, W = 1, and therefore

$$\tau(H_2) = (3.6 \pm 0.7) \text{ sec.}$$

Similarly, if the gas was pure oil vapor ($\sim CH_2$), we have

$$\lambda\,(CH_2) = (6/8)\,\lambda\,(C) + (2/8)\,\lambda(H_2), \text{ which yields}$$

$$\tau(\text{oil}) = (1.3 \pm 0.3)\text{ sec.}$$

The experimental mean life obtained under these conditions was

$$\tau(\text{exp.}) = (2.5 \pm 0.5)\text{ sec.},$$

where the error represents lack of reproducibility in measurements taken on different days, rather than errors in individual measurements.

We see that the experimental mean life lies between the calculated extremes, so that we can say that the experiment agrees with the theory. The experimental and theoretical values coincide if we assume that 85% of the gas molecules at the proton orbit consisted of hydrogen molecules, and 15% were air molecules. Alternatively, we may assume 75% hydrogen molecules and 25% oil vapor "CH_2 units."

The relative amounts of hydrogen, air, and oil vapor were not known; however, the reading on an ion gauge located at one side of the vacuum tank ordinarily is lowered about 20% when the hydrogen supply to the ion source is shut off.[7] The gauge is half as sensitive to hydrogen as to air, so that the above gauge observation represents a 40% effect. Since the proton orbits are closer to the hydrogen source than is the gauge, while the gauge is correspondingly closer to the air leaks, it is not unreasonable to assume 80% hydrogen at the orbit.

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[7] (J. Vale, private communication).