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ON THE COMMUTATION RELATIONS OF INTERACTING SPINOR FIELDS

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ABSTRACT

It is shown that the requirement that the Hamiltonian density commute with itself on a spacelike surface precludes the possibility that three or more different spinor fields, coupled to one another in Yukawa-type interactions, commute with each other. If the Hamiltonian contains only two such fields, however, they may be assumed to either commute or anticommute without violating this requirement.

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I. INTRODUCTION

The form of the commutation relations between field operators that represent physically different Fermi-Dirac particles has been recently investigated by Kinoshita.[1] He has shown that if the Lagrangian contains interaction terms that are bilinear in spinor fields, these fields must anticommute* in order that unique equations of motion be obtained from Schwinger's variational principle. However, if the equations of motion are obtained from the canonical commutation laws

$$-i\,\dot{\psi}^j = \left[H,\ \psi^j\right], \qquad -i\,\dot{\bar{\psi}}^j = \left[H,\ \bar{\psi}^j\right], \tag{1.1}$$

the results are unique regardless of whether the spinor fields commute or anticommute. Since self-consistent results are obtained from the canonical formalism, it is not clear whether the inconsistency obtained by Kinoshita reflects the impropriety of the commutation relations or the inapplicability of the variational principle in this case. It is of interest, therefore, to determine whether Kinoshita's conclusions can be obtained without recourse to the variation formalism.

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* As used in this paper, the expression "commuting spinor fields" will always refer to different spinor fields. For a single spinor field the usual anticommutation relations are assumed.

The question of whether different spinor fields commute or anticommute is of no practical importance when the Hamiltonian contains only two such fields, since the physical observables obtained using either choice of commutation relations are the same. On the other hand, the transition amplitude for a particular process involving three different spinor fields is calculated in Section 2 by the formal application of the Dyson expansion of the S matrix,[2] and the result is found to depend on the choice of commutation relations. However, it is shown in Section 4 that if three or more spinor fields interact with each other via Yukawa-type interactions,[‡] the assumption that they commute with one another is inconsistent with the requirement that the Hamiltonian density commute with itself at two points on a spacelike surface.[3] If the Hamiltonian contains only two different spinor fields, they may be assumed to either commute or anticommute without violating the above requirement, which will henceforth be referred to as Postulate (II).

The case of three or more interacting spinor fields is thus fundamentally different from that of only two such fields in that Postulate (II) places a restriction on the commutation relations in the former case but not in the latter. Section 5 contains some speculations concerning the apparent distinction between these two cases.

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‡ By "Yukawa-type interactions" we merely mean that an interaction term in the Hamiltonian contains the spinor fields bilinearly and the boson field linearly.

II. TRANSITION MATRIX ELEMENTS

In this section the transition matrix for a simple process is evaluated by the formal application of Dyson's S-matrix expansion. This example illustrates a difference between the cases in which the different spinor fields are assumed to commute or anticommute.

The Dyson expansion expansion of the S matrix[2] is given by

$$S = \sum_{n=0}^{\infty} (-i)^n \frac{1}{n!} \int_{-\infty}^{\infty} d_4x_1 \ldots \int_{-\infty}^{\infty} d_4x_n \, P \left\{ H_I(x_1), \ldots , H_I(x_n) \right\} , \tag{2.1}$$

where P is an operator which orders the factors chronologically so that time values decrease from left to right. The transition amplitude for the second-order process corresponding to the diagram in Fig. I will be calculated for the two cases,

Case (a) the commutators of different spinor fields vanish, and

Case (b) the anticommutators of different spinor fields vanish.

We shall see that in Case (a) the propagator for the virtual fermion of Type 2 is not the usual Feynman propagator.

The form of the interaction representation interaction Hamiltonian is chosen as

$$H_I(x) = g_1 \, \bar{\psi}^1(x) \, \psi^2(x) \, \phi^1(x) + g_2 \, \bar{\psi}^2(x) \, \psi^3(x) \, \phi^2(x) + \text{H.C.}, \tag{2.2}$$

where the ψ's are different spin-$\frac{1}{2}$ fields and the ϕ's are different real scalar fields. The term of the S matrix corresponding to Fig. I is

$$M^{(2)} = (-i)^2 \int d_4x_1 \, d_4x_2 \, P \left\{ H_I^{\,a}(x_2) \, , \; H_I^{\,b}(x_1) \right\} , \tag{2.3}$$

where

$$H_I^{\,a}(x) = g_1 \bar{\psi}^1(x)\, \psi^2(x)\, \phi^1(x) \quad , \tag{2.2a}$$

$$H_I^{\,b}(x) = g_2 \bar{\psi}^2(x)\, \psi^3(x)\, \phi^2(x) \quad . \tag{2.2b}$$

The expectation value of $M^{(2)}$ is taken between an initial state of the system containing fermion 3 and boson 2 and a final state containing fermion 1 and boson 1, all particles being in plane-wave states and the fermions being in definite spin states. Then we have

$$\begin{aligned} M_{FI}^{(2)} &= \langle \Psi_F \,|\, M^{(2)} \,|\, \Psi_I \rangle \quad , \\ &= \langle \Psi_F \,|\, M^{(2)} \,|\, \Psi_I \rangle \quad , \\ &= -g_1 g_2 \int_{-\infty}^{\infty} d_4x_1\, d_4x_2\, \langle \Psi_F \,|\, P\left\{\bar{\psi}^1(x_2)\psi^2(x_2)\phi^1(x_2), \bar{\psi}^2(x_1)\psi^3(x_1)\phi^2(x_1)\right\} |\, \Psi_I \rangle \\ &= -g_1 g_2 \int_{-\infty}^{\infty} d_4x_1\, d_4x_2\, \langle \Psi_F \,|\, P\ (x_1, x_2) \,|\, \Psi_I \rangle \ ; \end{aligned} \tag{2.4}$$

the last two lines are a definition of $P(x_1, x_2)$. In order to perform the time ordering we split the Feynman diagram of Fig. I into its two constituent parts corresponding to propagation by a particle and by an antiparticle, Fig. IIa and IIb, respectively. Then we have

$$\begin{aligned} P(x_1, x_2) = {} & \theta(x_2 - x_1)\bar{\psi}^1(x_2)\psi^2(x_2)\, \phi^1(x_2)\bar{\psi}^2(x_1)\psi^3(x_1)\, \phi^2(x_1) \\ & + \theta(x_1 - x_2)\bar{\psi}^2(x_1)\psi^3(x_1)\, \phi^2(x_1)\bar{\psi}^1(x_2)\psi^2(x_2)\, \phi^1(x_2) \quad , \end{aligned} \tag{2.5}$$

where

$$\theta(x) = \begin{cases} +1 & \text{for } x_0 > 0 \\ 0 & \text{for } x_0 < 0 \end{cases} , \tag{2.6}$$

$$\left.\begin{aligned} \theta(x) + \theta(-x) &= 1 \\ \theta(x) - \theta(-x) &= \epsilon(x) = \frac{x_0}{|x_0|} \end{aligned}\right\} . \tag{2.7}$$

Since ϕ^1 and ϕ^2 commute with each other (and, of course, with the fermion fields), Eq. (2.5) becomes

$$P(x_1, x_2) = \phi^1(x_2)\,\bar{\psi}_\alpha^{\,1}(x_2)\Big[\theta(x_2 - x_1)\,\psi_\alpha^{\,2}(x_2)\,\bar{\psi}_\beta^{\,2}(x_1) \mp \theta(x_1 - x_2)\,\bar{\psi}_\beta^{\,2}(x_1)\,\psi_\alpha^{\,2}(x_2)\Big]\,\psi_\beta^{\,3}(x_1)\,\phi^2(x_1) , \tag{2.8}$$

the upper sign applying if we have

$$\begin{aligned} \left[\psi_\beta^{\,3}(x_1),\ \bar{\psi}_\alpha^{\,1}(x_2)\right]_+ &= 0 , \\ \left[\bar{\psi}_\beta^{\,2}(x_1),\ \bar{\psi}_\alpha^{\,1}(x_2)\right]_+ &= 0 , \\ \left[\psi_\beta^{\,3}(x_1),\ \psi_\alpha^{\,2}(x_2)\right]_+ &= 0 , \end{aligned} \tag{2.9}$$

the lower sign applying if we have

$$\begin{aligned} \left[\psi_\beta^{\,3}(x_1),\ \bar{\psi}_\alpha^{\,1}(x_2)\right]_- &= 0 , \\ \left[\bar{\psi}_\beta^{\,2}(x_1),\ \bar{\psi}_\alpha^{\,1}(x_2)\right]_- &= 0 , \\ \left[\psi_\beta^{\,3}(x_1),\ \psi_\alpha^{\,2}(x_2)\right]_- &= 0 . \end{aligned} \tag{2.10}$$

It can be noted that the minus sign in front of the second term on the right side of Eq. (2.8) can also be obtained by requiring two of the commutators and one anticommutator to vanish. Making use of the usual Fourier decomposition of field operators, we obtain

$$M_{FI}^{(2)} = \int_{-\infty}^{\infty} d_4x_1\, d_4x_2\, N_{\alpha\beta}(x_1, x_2) \Big[\theta(x_2 - x_1) \langle \psi_\alpha^{\,2}(x_2)\, \overline{\psi}_\beta^{\,2}(x_1) \rangle_0$$

$$\mp\ \theta(x_1 - x_2) \langle \overline{\psi}_\beta^{\,2}(x_1)\, \psi_\alpha(x_2) \rangle_0 \Big]$$

$$= \int_{-\infty}^{\infty} d_4x_1\, d_4x_2\, N_{\alpha\beta}(x_1, x_2) \Big\{ \theta(x_2 - x_1) \Big[-iS_{\alpha\beta}^{\;+(m_2)}(x_2 - x_1) \Big]$$

$$\mp\ \theta(x_1 - x_2) \Big[-iS_{\alpha\beta}^{\;-(m_2)}(x_2 - x_1) \Big] \Big\} \quad , \tag{2.11}$$

where $N_{\alpha\beta}(x_1, x_2)$ is a c-number,

$$N_{\alpha\beta}(x_1, x_2) = -\frac{g_1 g_2}{2(2\pi)^6} \left(\frac{m_1 m_3}{\omega^{m_1}\, \omega^{m_3}\, \omega^{M_1}\, \omega^{M_2}} \right)^{\frac{1}{2}} u_\alpha^{\,s}(m_1, \vec{p}_1)\, u_\beta^{\,r}(m_3, \vec{p}_3)$$

$$\times\ e^{ip_3\cdot x_1}\, e^{-ip_1\cdot x_2}\, e^{iq_2\cdot x_1}\, e^{-iq_1\cdot x_2} \quad . \tag{2.12}$$

With the help of the relationships

$$S^{+}(x) = \tfrac{1}{2} \Big[S(x) - i\, S^{(1)}(x) \Big] \quad , \tag{2.13}$$

$$S^{-}(x) = \tfrac{1}{2} \Big[S(x) + i\, S^{(1)}(x) \Big] \quad , \tag{2.14}$$

$$S_F(x) = S^{(1)}(x) + i \epsilon(x) S(x) \quad , \tag{2.15}$$

$$S_T(x) = \epsilon(x) S_F(x) \quad , \tag{2.16}$$

where $S_F(x)$ is the Feynman propagator, Eq. (2.11) becomes, for Case (a),

$$M_{FI}^{(2)} = -\tfrac{1}{2} \int_{-\infty}^{\infty} d_4x_1 \, d_4x_2 \, N_{\alpha\beta}(x_1, x_2) S_{T\alpha\beta}^{(m_2)}(x_2 - x_1) \quad , \tag{2.17a}$$

and for Case (b),

$$M_{FI}^{(2)} = -\tfrac{1}{2} \int_{-\infty}^{\infty} d_4x_1 \, d_4x_2 \, N_{\alpha\beta}(x_1, x_2) S_{F\alpha\beta}^{(m_2)}(x_2 - x_1) \quad . \tag{2.17b}$$

For (b) we obtain, for the intermediate state, the Feynman propagator, which is a Green's function for the Dirac equation, i.e.,

$$(\gamma_\mu \partial_\mu + m) S_F(x) = 2 i \, \delta_4(x) \quad . \tag{2.18}$$

For (a) the propagator is the function S_T, which satisfies

$$(\gamma_\mu \partial_\mu + m) S_T(x) = -\frac{2}{\pi} \, \delta_3(\vec{x}) \, P \frac{1}{x_0} \quad , \tag{2.19}$$

in which P indicates that one must take the principal value when integrating over x_0.

From Eq. (2.11) we see that the two propagators differ only in the sign of the part corresponding to propagation by a negative-energy particle. This difference in sign is due to the odd number of transpositions of different spinor fields involved in going from Eq. (2.5) to (2.8). Thus,

the transition probability for the physical process that corresponds to Fig. 2.1 depends on the commutation relations of the different spinor fields. This dependence does not occur for all processes involving the Hamiltonian (2.2). An example of a transition probability the calculation of which involves an even number of transpositions of different spinor fields and which is therefore the same for Cases (a) and (b) is given in the next section.

III. VACUUM EXPECTATION VALVE

The probability that a vacuum state at $t = -\infty$ shall remain a vacuum for $t = \infty$ is

$$W_0 = \left| \langle S \rangle_0 \right|^2 = \left\langle P\left(e^{-i \int_{-\infty}^{\infty} H_I(x) d_4x}\right) \right\rangle_0 \left\langle P_-\left(e^{i \int_{-\infty}^{\infty} H_I(x) d_4x}\right) \right\rangle_0 , \tag{3.1}$$

where P_- is the operator that orders the factors in the opposite order of times to that of P.

To prove that the expansion in Eq. (2.1) yields the same result for W_0 whether the different spinor fields commute or anticommute, we merely show that the vacuum expectation value of each term in the expansion (2.1) of S and the corresponding expansion of $S^\dagger$ is the same for the two possibilities. The expression of interest is

$$\left\langle P(x_n) \right\rangle_0 = \left\langle P \left\{ H_I(x_1), \ldots, H_I(x_n) \right\} \right\rangle_0 .$$

After the time ordering is performed we have the product of n Hamiltonian densities. For convenience, the indices may be considered to be interchanged after the ordering is carried out so that $\left\langle P(x_n) \right\rangle_0$ becomes $\left\langle H_I(x_1) \ldots H_I(x_n) \right\rangle_0$, which is the sum of 4^n terms. However, only the terms which contain an even number of a given ϕ^i and which for every ψ^i have a corresponding $\bar{\psi}^i$ are nonzero. Thus $\left\langle P(x_n) \right\rangle_0$ is nonzero only if n is even. The order of the factors is now rearranged so that all the boson operators appear on the right. By splitting these up into positive- and negative-frequency parts and operating successively on the vacuum, we may replace them by c-numbers. Next, the following rearrangement is carried out. Call the operator on the extreme right ψ^{ia}. Pick

out a $\bar{\psi}^{ia}$ such that between it and ψ^{ia} there are equal numbers of $\bar{\psi}^{i}$ and ψ^{i}, and commute $\bar{\psi}^{ia}$ to the right until it is next to ψ^{ia}. Call $\bar{\psi}_{\alpha}^{ia}$ ψ_{β}^{ia} a factor pair. Repeat the procedure for the first ψ^{jb} to the left of the last factor pair formed until all the operators are in factor pairs, all pairs for a given i being grouped together. Now $\langle P(x_n) \rangle_0$ may be unambiguously replaced by c-numbers. Since in the formation of each factor pair and later in the regrouping of all pairs with a given i to stand together an even number of transpositions of different spinor fields is performed, the final result is the same for Case (a) as for (b). Similarly $\langle P_-(x_n) \rangle_0$ is the same for the two cases, and thus the result for the physical quantity W_0 is independent of whether the different spinor fields commute or anticommute.

IV. RESTRICTIONS IMPOSED BY COMMUTATIVITY CONDITION.

In this section it is shown that for Case (a), the Hamiltonian density does not commute with itself on a spacelike surface. We shall evaluate the commutator of the Hamiltonian densities for the interaction Hamiltonian (2.2), considering separately the two cases (a) and (b) discussed in Section 2.

Postulate (II) implies that

$$\langle \Psi' \mid \left[P(x', y') + Q(x', y') \right] \mid \Psi \rangle = 0 \quad , \tag{4.1}$$

where $|\Psi\rangle$ and $|\Psi'\rangle$ are any two state vectors (not necessarily physical ones), $x' = (\vec{x}, 0)$, $y' = (\vec{y}, 0)$, $(P + Q)$ is the commutator of the total Hamiltonian,

$$P(x, y) + Q(x, y) = \left[H(x), \ H(y) \right]_- \quad , \tag{4.2}$$

and

$$Q(x, y) = \left[H_I(x), \ H_I(y) \right]_- \quad ; \tag{4.3}$$

i.e., all terms in $P(x, y)$ involve the Hamiltonian of the free fields. For Case (a), with $g_1 = g_2 = 1$ for convenience, we have

$$\begin{aligned} Q(x, y) = {} & i\Delta^{(M_1)}(x - y)\left[\bar{\psi}^1(y)\psi^2(y) + \bar{\psi}^2(y)\psi^1(y) \right]\left[\bar{\psi}^1(x)\psi^2(x) + \bar{\psi}^2(x)\psi^1(x) \right] \\ & + i\Delta^{(M_2)}(x - y)\left[\bar{\psi}^2(y)\psi^3(y) + \bar{\psi}^3(y)\psi^2(y) \right]\left[\bar{\psi}^2(x)\psi^3(x) + \psi^3(x)\psi^2(x) \right] \\ & + i\,\phi^1(x)\phi^2(y)\left[\bar{\psi}^1(x)S^{(m_2)}(x - y)\psi^3(y) + \bar{\psi}^3(y)S^{(m_2)}(y - x)\psi^1(x) \right] \\ & + i\,\phi^1(y)\phi^2(x)\left[\bar{\psi}^3(x)S^{(m_2)}(x - y)\psi^1(y) + \bar{\psi}^1(y)S^{(m_2)}(y - x)\psi^3(x) \right] \end{aligned}$$

Cont.

$$-i\phi^1(x)\phi^1(y)\Big[\bar{\psi}^1(x)S^{(m_2)}(x-y)\psi^1(y) - \bar{\psi}^1(y)S^{(m_2)}(y-x)\psi^1(x)$$

$$+\bar{\psi}^2(x)S^{(m_1)}(x-y)\psi^2(y) - \bar{\psi}^2(y)S^{(m_1)}(y-x)\psi^2(x)\Big]$$

$$-i\phi^2(x)\phi^2(y)\Big[\bar{\psi}^2(x)S^{(m_3)}(x-y)\psi^2(y) - \bar{\psi}^2(y)S^{(m_3)}(y-x)\psi^2(x)$$

$$+\bar{\psi}^3(x)S^{(m_2)}(x-y)\psi^3(y) - \bar{\psi}^3(y)S^{(m_2)}(y-x)\psi^3(x)\Big]$$

$$+2\,\phi^1(x)\phi^2(y)\Big[\bar{\psi}^1(x)\psi^2(x)+\bar{\psi}^2(x)\psi^1(x)\Big]\Big[\bar{\psi}^2(y)\psi^3(y) + \psi^3(y)\psi^2(y)\Big]$$

$$+2\,\phi^1(y)\phi^2(x)\Big[\bar{\psi}^2(x)\psi^3(x) + \bar{\psi}^3(x)\psi^2(x)\Big]\Big[\bar{\psi}^2(y)\psi^1(y) + \bar{\psi}^1(y)\psi^2(y)\Big] \quad , \tag{4.4}$$

where

$$\Big[\phi^i(x),\ \phi^i(y)\Big]_- = i\Delta^{(M_i)}(x-y) \quad ,$$

$$\Big[\psi_\alpha{}^i(x),\ \bar{\psi}_\beta{}^i(y)\Big] = -i\,S_{\alpha\beta}{}^{(m_i)}(x-y) \ . \tag{4.5}$$

Since the Δ and S functions vanish for

$$(x-y)^2 = \Big[(\vec{x}-\vec{y})^2 - (x_0-y_0)^2\Big] > 0 \quad ,$$

only the terms which do not contain either the Δ or S function are nonzero in $Q(x', y')$.

It is convenient to choose

$$|\Psi\rangle = b_1^{+}(\vec{p}_1)\, b_2^{+}(\vec{p}_2)\, a_3^{+s}(\vec{q}_3)\, |\Psi_0\rangle ,$$
$$|\Psi'\rangle = a_1^{+r}(\vec{q}_1)\, |\Psi_0\rangle , \tag{4.6}$$

where $b_i^{+}(\vec{p}_i)$ creates a meson of type i with momentum $\vec{p}_i$, $a_i^{+s}(\vec{q}_i)$ creates a fermion of type i with momentum $\vec{q}_i$ and spin s, and $|\Psi_0\rangle$ is the vacuum state. For later convenience we set

$$\vec{p}_1 = \vec{q}_1 = \vec{p}_2 = -\vec{q}_3 = \vec{p} .$$

Evaluating Eq. (4.2), we obtain

$$\langle\Psi'|\, Q(x', y')\, |\Psi\rangle = -\frac{i}{(2\pi)^6}\left(\frac{m_1\, m_3}{\omega^{M_1}\, \omega^{M_2}\, \omega^{m_1}\, \omega^{m_3}}\right)^{\frac{1}{2}}$$

$$\times\Big[\bar{u}_\alpha^{\,r}(m_1, \vec{p})\, S_{\alpha\beta}^{+(m_2)}(x' - y')\, u_\beta^{\,s}(m_3, -\vec{p})$$
$$+ \bar{u}_\alpha^{\,r}(m_1, \vec{p})\, S_{\alpha\beta}^{-(m_2)}(y' - x')\, u_\beta^{\,s}(m_3, -\vec{p})\Big] , \tag{4.7}$$

where $\omega = +\sqrt{\vec{p}^{\,2} + m^2}$, u^j is a spinor, $\bar{u}^j$ its adjoint, and

$$S^{+}(\vec{x}, 0) = \tfrac{1}{2}\left[S(\vec{x}, 0) - i\, S^{(1)}(\vec{x}, 0)\right]$$
$$= -\tfrac{i}{2}\, S^{(1)}(\vec{x}, 0) \qquad \text{for } \vec{x} \neq 0 ,$$
$$S^{-}(\vec{x}, 0) = \tfrac{1}{2}\left[S(\vec{x}, 0) + i\, S^{(1)}(\vec{x}, 0)\right] \tag{4.8}$$
$$= \tfrac{i}{2}\, S^{(1)}(\vec{x}, 0) \qquad \text{for } \vec{x} \neq 0 ,$$

since

$$S(x) = \frac{i}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{d_3k}{\omega} e^{i\vec{k}\cdot\vec{x}} \left[\omega\gamma_4 \cos \omega x_0 - (\vec{k}\cdot\vec{\gamma} + i\, m)\sin \omega x_0\right] ,$$

$$S^{(1)}(x) = \frac{i}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{d_3k}{\omega} e^{i\vec{k}\cdot\vec{x}} \left[(\vec{k}\cdot\vec{\gamma} + i\, m)\cos \omega x_0 + \omega\gamma_4 \sin \omega x_0\right] . \tag{4.9}$$

Then we have

$$\langle\Psi'| Q(x', y') |\Psi\rangle = -\frac{1}{2(2\pi)^6}\left(\frac{m_1 m_3}{\omega^{M_1}\omega^{M_2}\omega^{m_1}\omega^{m_3}}\right)^{\frac{1}{2}}$$

$$\times\ \bar{u}_\alpha^{\,r}(m_1, \vec{p})\left[S_{\alpha\beta}^{(1)(m_2)}(\vec{x} - \vec{y}, 0) - S_{\alpha\beta}^{(1)(m_2)}(\vec{y} - \vec{x}, 0)\right] u_\beta^{\,s}(m_3, -\vec{p})$$

$$= -\frac{i}{(2\pi)^9}\left(\frac{m_1 m_3}{\omega^{M_1}\omega^{M_2}\omega^{m_1}\omega^{m_3}}\right)^{\frac{1}{2}} \bar{u}_\alpha^{\,r}(m_1, \vec{p})$$

$$\times\left[\int_{-\infty}^{\infty} \frac{d_3k}{\omega^{m_2}} e^{i\vec{k}\cdot(\vec{x} - \vec{y})}\, \vec{k}\cdot\vec{\gamma}_{\alpha\beta}\right] u_\beta^{\,s}(m_3, -\vec{p}) , \tag{4.10}$$

which is nonzero. Since the terms in $P(x, y)$ that involve both boson fields must contain one of them bilinearly, it is clear that we have $\langle\Psi'| P(x', y') |\Psi\rangle = 0$, and Eq. (4.2) is violated. For Case (b), on the other hand, all the terms in $Q(x, y)$ involve either the Δ or the S function and so we have $Q(x', y') = 0$. Also we have $P(x', y') = 0$, and thus the assumption that different spinor fields anticommute is the simplest one that satisfies Postulate (II).

It is an interesting fact that if the interaction involves only two different spinor fields that commute with each other Postulate (II) is not violated. This can be verified easily by direct calculation.

V. CONCLUDING REMARKS

It has been shown that the requirement that the Hamiltonian density commute with itself on a spacelike surface implies that spinor field operators representing different particles which interact with one another cannot be assumed to commute, but that this conclusion can be drawn only when there are three or more such fields. The distinction between the case of two fields and that of three fields is closely connected to a difference in the permutation properties of two and three or more elements. This suggests that for three or more fields the commutation relations may involve more than two field operators. The choice of the forms of these commutation relations can be determined by generalizing the consequences of the usual commutation relations for a single field. Since quantizing with commutators or anticommutators leads to ensembles of particles obeying Bose-Einstein or Fermi-Dirac statistics, and these are related respectively to the identical and the alternating representations of the symmetric group, the forms of the more complicated commutation relations should perhaps be similarly related to the higher-order irreducible representations of that group. In this connection we note that it is the distinctness of the two boson fields that destroys the symmetry of the Hamiltonian in the interchange of any two spinor fields, and permits nonzero transition amplitudes between initial and final states described by eigenfunctions belonging to different irreducible representations of the symmetric group. However, the requirement that the eigenfunctions of two physically realizable systems belong to definite representations of the symmetric group places

severe restrictions on the symmetry properties of the Hamiltonian with respect to interchanges involving the different spinor fields. This fact may perhaps serve as a guide in the further investigation of the interactions of several spinor fields.

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1. T. Kinoshita, Phys. Rev. 96, 199 (1954).

2. F. J. Dyson, Phys. Rev. 75, 486 (1949).

3. See, for example, W. Pauli, Prog. Theor. Phys. 5, 526 (1950). This is a special case of what Pauli refers to as Postulate (II): "Physical quantities (observables) commute with each other in two space-time points with a space-like distance." Strictly speaking, only the Hamiltonian density integrated over a finite volume is an observable. For this reason, in order to deal with physical quantities at two different points of spact-time, x_1 and x_2, one may integrate the densities over suitable regions of space R_1 and R_2, so that all points in (R_1, t_1) are spacelike with respect to all points in (R_2, t_2). The requirement that the Hamiltonian density commute with itself on a spacelike surface is also an integrability condition on the Tomonaga-Schwinger equation. In connection with this see K. Nishijima, Prog. Theor. Phys. 5, 187 (1950).

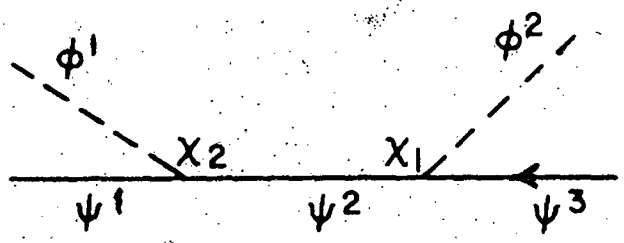

Fig. I

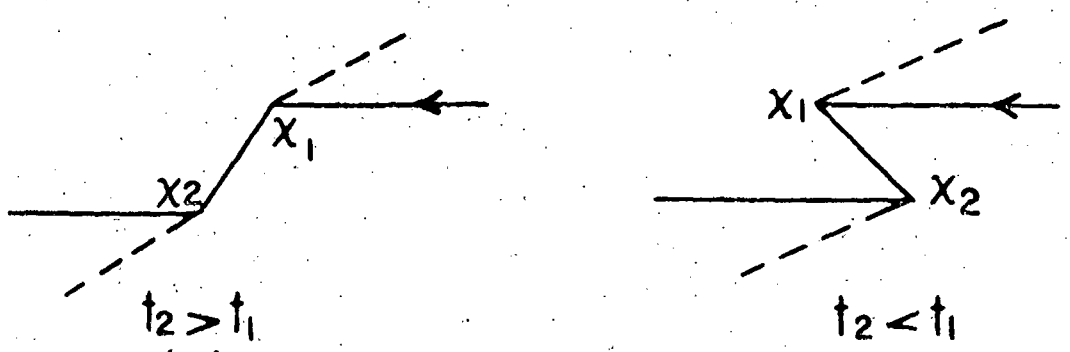

Fig. II

FIGURE CAPTIONS

Figure I: Feynman diagram for a second order process involving three different spinor fields.

Figure II: The Feynman diagram of Fig. I divided into its two constituent parts corresponding to propagation by a particle (a), and by an antiparticle (b).