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RELATIVISTIC ELECTRODYNAMICS OF ROTATING MEDIA

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John Ise, Jr.

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ABSTRACT

The electromagnetic field equations in rotating systems are derived from the generally covariant tensor formulation and used to calculate the emf in rotating material media. The electrostatic field outside a rotating bar magnet is shown to be zero, neglecting end effects. The emf in rotating magnetic insulators is also derived and compared to the value derived from the Minkowski equations for uniformly moving media. Some extra terms representing current and charge density appear in the ordinary vector formulation of the field equations in rotating coordinates, and the physical significance of these quantities is discussed.

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I. INTRODUCTION

The phenomena connected with rotating permanent magnets and rotating conductors in magnetic fields have fascinated physicists and electrical engineers since the time of Faraday, and many experiments, some quite ingenious, have been proposed and carried out to decide whether or not the "magnetic lines of force" rotate with the presumed sources of these lines. (For a list of these experiments, including a detailed analysis of several of the experimental results, both from the standpoint of the "fixed field" and "rotating field" theories, see E. G. Cullwick.[1]) Whether or not the magnetic lines rotate was shown to be meaningless in a comprehensive review of the electrodynamics of moving media by Swann, Tate, Bateman, and Kennard,[2] i.e., magnetic field "lines" have no physical reality independent of the electromagnetic field, or to quote Panofsky,[3] "Any observed phenomena which depend on a field description must be describable in terms of the behavior of the field quantities alone, independent of the nature of the mechanism which produces the field quantities."

The review article of Swann et al.[2] was apparently not given sufficient publicity, for experiments continued to be performed on rotating magnets, solenoids, and dielectrics, and some sharp interchanges were aired in the Journal of the Institute of Electrical Engineers (London) in the late 1930's (see Cullwick for references). The subject of "unipolar induction" was then more or less dropped until the study of moving plasmas in astrophysics, from the standpoint of acceleration of cosmic rays, revived it. Alfven[4] gives a concise review of the importance of the relativity of electric fields in cosmic physics and treats the problem of the rotating magnet from both the "fixed-field" and "moving-field" theories, showing that they lead to the same results for the voltage induced in a wire, one end of which is connected to the axis of rotation of the magnet, the other sliding on the outside surface.

All authors derive their results from the Lorentz transformation equations for electric field $\vec{E}$ and magnetic field $\vec{H}$ which hold only for uniform motion (see, for example, Alfven, p. 6; Panofsky, p. 149 and 343; Cullwick, p. 109; or Sommerfeld,[5] p. 360). Thus the radial electric field at a distance r from the axis of a rotating magnet is given by

$$\vec{E} = -(\vec{\omega} \times \vec{r}) \times \vec{B}. \qquad (1)$$

The various authors quoted above all draw attention to the fact that div $E = 2\vec{\omega}\cdot\vec{B} \neq 0$, and Backus[6] makes this the basis for his disagreement with Alfven's derivation of Eq. (1) above. At this point, most of the authors state that problems involving rotation cannot be discussed consistently without recourse to general relativity, and go no farther.

It is the purpose of this paper to set up the general relativistic electromagnetic field equations in rotating media and point out their application to some of the classical problems of spinning magnets and dielectric cylinders. It will be seen that some interesting differences arise between cylindrical and rectangular coordinates, even without reference to any rotation.

II. THE FIELD EQUATIONS IN ROTATING SYSTEMS

The Maxwell equations are written (in rationalized MKS units)

$$\begin{aligned} &\text{(a)}\quad \operatorname{div}\vec{B} = 0, \\ &\text{(b)}\quad \operatorname{curl}\vec{E} = -\frac{\partial\vec{B}}{\partial t}, \\ &\text{(c)}\quad \operatorname{curl}\vec{H} = \vec{j}_{true} + \frac{\partial\vec{D}}{\partial t}, \quad \text{and} \\ &\text{(d)}\quad \operatorname{div}\vec{D} = \rho_{true}, \end{aligned} \tag{2}$$

with the additional relations

$$\vec{D} = \epsilon_0\vec{E} + \vec{P}, \text{ and} \tag{3}$$

$$\vec{B} = \mu_0(\vec{H} + \vec{M}). \tag{4}$$

These can be concisely written in tensor notation

$$\frac{\partial F_{ij}}{\partial x^k} + \frac{\partial F_{jk}}{\partial x^i} + \frac{\partial F_{ki}}{\partial x^j} = 0, \text{ and} \tag{5}$$

$$\frac{\partial H^{ik}}{\partial x^i} = J^k, \tag{6}$$

where, in rectangular coordinates, the antisymmetric electromagnetic field tensors, F and H, are given by

$$F_{ij} = \begin{pmatrix} 0 & cB_z & -cB_y & E_x \\ & 0 & cB_x & E_y \\ & & 0 & E_z \\ & & & 0 \end{pmatrix}, \qquad F^{ij} = \begin{pmatrix} 0 & cB_z & -cB_y & -E_x \\ & 0 & cB_x & -E_y \\ & & 0 & -E_z \\ & & & 0 \end{pmatrix}, \tag{7}$$

$$H_{ij} = \begin{pmatrix} 0 & \frac{H_z}{c} & -\frac{H_y}{c} & D_x \\ & 0 & \frac{H_x}{c} & D_y \\ & & 0 & D_z \\ & & & 0 \end{pmatrix}, \quad H^{ij} = \begin{pmatrix} 0 & \frac{H_z}{c} & -\frac{H_y}{c} & -D_x \\ & 0 & +\frac{H_x}{c} & -D_y \\ & & 0 & -D_z \\ & & & 0 \end{pmatrix}, \tag{8}$$

and J^k is the current four-vector

$$J^k = (\vec{j}/c,\ \rho). \tag{9}$$

The relation between the contravariant components F^{ij} and the covariant components F_{ij} is

$$F_{ij} = g_{i\ell} g_{jk} F^{\ell k}, \tag{10}$$

where the g_{ij} are given by the line element

$$-ds^2 = dx^2 + dy^2 + dz^2 - c^2dt^2 = -g_{ik}dx^i dx^k, \tag{11}$$

so that

$$\begin{aligned} g_{ik} &= 0 \text{ for } i \neq k \\ &= -1 \text{ for } i = k = 1,\ 2,\ 3 \\ &= +1 \text{ for } i = k = 4\ . \end{aligned} \tag{12}$$

The relations (5) and (6), being tensor equations, are valid in all reference frames that preserve relations (11), which include all uniformly moving Lorentz frames, but they are not valid in stationary coordinate systems other than rectangular Cartesian.

For the purpose of treating problems in rotation, it is very convenient to work in cylindrical coordinates, therefore we shall now find the tensors equivalent to (7) and (8) and the relations analogous to (5) and (6). In polar coordinates r, θ, z, and ct, the line element is given by

$$-ds^2 = dr^2 + r^2 d\theta^2 + dy^2 - c^2dt^2, \tag{13}$$

so that

$$g_{ij} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -r^2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{14}$$

and, if we define g^{ij} as the conjugate minor of g_{ij} divided by the determinant g of g_{ij}, then

$$g^{ij} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1/r^2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix}. \tag{15}$$

Now we define

$$F_{ij} = \begin{pmatrix} 0 & crB_z & -cB_\theta & E_r \\ & 0 & crB_r & rE_\theta \\ & & 0 & E_z \\ & & & 0 \end{pmatrix}, \text{ and } F^{ij} = g^{il} g^{jm} F_{lm} = \begin{pmatrix} 0 & \frac{cB_z}{r} & -cB_\theta & -E_r \\ & 0 & \frac{cB_r}{r} & -\frac{E_\theta}{r} \\ & & 0 & -E_z \\ & & & 0 \end{pmatrix} \tag{16}$$

with corresponding relations for H_{ij} and H^{ij} in terms of $\vec{H}/c$ and $\vec{D}$ instead of $c\vec{B}$ and $\vec{E}$. Then Eq. (5) may be taken over without change, while (6) has to be written

$$\frac{\partial (rH^{ik})}{\partial x^i} = rJ^k. \tag{17}$$

The reason for this will appear shortly.

Now that we have expressions (15) for the field tensors in cylindrical coordinates, let us consider a transformation from an inertial frame R, Θ, Z, cT (unaccelerated with respect to the fixed stars) to another frame r, θ, z, ct rotating with uniform angular velocity ω, i.e.,

$$
\begin{aligned}
R &= r, \\
\Theta &= \theta + (\omega/c)ct, \\
Z &= z, \\
cT &= ct.
\end{aligned}
\tag{18}
$$

(For a discussion of the implications involved in setting T = t, see Møller,[7] p. 225.) In these new rotating coordinates, the line element is given by

$$
-ds^2 = -g_{ik}dx^i dx^k = dr^2 + d\theta^2 r^2 + dz^2 + 2\omega r^2 d\theta dt - (c^2 - r^2\omega^2)dt^2, \tag{19}
$$

where

$$
g_{ik} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -r^2 & 0 & -\frac{\omega r^2}{c} \\ 0 & 0 & -1 & 0 \\ 0 & -\frac{\omega r^2}{c} & 0 & 1 - \frac{r^2\omega^2}{c^2} \end{pmatrix} \tag{20}
$$

and

$$
g^{ik} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -\frac{\left(1 - \frac{r^2\omega^2}{c^2}\right)}{r^2} & 0 & -\frac{\omega}{c} \\ 0 & 0 & -1 & 0 \\ 0 & -\frac{\omega}{c} & 0 & +1 \end{pmatrix}. \tag{21}
$$

It can be shown in general relativity (Tolman,[8] p. 262; Møller,[7] p. 302) that the field equations can be expressed in a covariant form by means of Eqs. (5) and the extension of (6) to covariant differentiation, i.e.,

$$
\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^k} \left(\sqrt{-g}\, H^{ik}\right) = J^i. \tag{22}
$$

Because, from (14), we have $\sqrt{-g} = r$, we see that (22) gives Eq. (17). Now $\sqrt{-g}$ is equal to r even in the rotating coordinates, therefore our field equations become (5) and (17), where F_{ij} is given in (16), and H^{ik} is obtained from H_{ik} by means of the g^{ij} given in (15), hence

$$H^{ik} = g^{ij} g^{kl} H_{jl}. \tag{23}$$

Consequently we have

$$H_{jl} = \begin{pmatrix} 0 & \frac{rH_z}{c} & \frac{-H_\theta}{c} & D_r \\ & 0 & \frac{rH_r}{c} & rD_\theta \\ & & 0 & D_z \\ & & & 0 \end{pmatrix}, \tag{24}$$

and

$$H^{ij} = \begin{pmatrix} 0 & \frac{H_z}{c\gamma^2 r} + \frac{\omega}{c} D_r & \frac{-H_\theta}{c} & \frac{\beta}{c} H_z - D_r \\ & 0 & \frac{H_r}{c\gamma^2 r} - \frac{\omega D_z}{c} & \frac{-D_\theta}{r} \\ & & 0 & -\frac{\beta}{c} H_r - D_z \\ & & & 0 \end{pmatrix}, \tag{25}$$

in which we have used the abbreviations

$$\beta = \omega r/c \text{ and } 1/\gamma = \sqrt{1 - \beta^2}. \tag{26}$$

Equation (5) becomes

$$\text{(a)} \quad \operatorname{div} \vec{B} = 0$$

$$\text{(b)} \quad \operatorname{curl} \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \tag{27}$$

just as in stationary coordinates, while (22) becomes

$$\text{(a)}\quad \frac{1}{r}\frac{\partial H_z}{\partial \theta} - \frac{\partial H_\theta}{\partial z} = j^r + \frac{\partial D_r}{\partial t} - \omega\frac{\partial D_r}{\partial \theta} - \frac{\beta}{c}\frac{\partial H_z}{\partial t} + \frac{\beta^2}{r}\frac{dH_z}{\partial \theta},$$

$$\text{(b)}\quad \frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r} = rj^\theta + \frac{\partial D_\theta}{\partial t} - \frac{2\omega^2 rH_z}{c^2} + \omega r \operatorname{div}\vec{D} - \omega\frac{\partial D_\theta}{\partial \theta} + \beta^2\left(\frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r}\right),$$

$$\text{(c)}\quad \frac{1}{r}\left[\frac{\partial (rH_\theta)}{\partial r} - \frac{\partial H_r}{\partial \theta}\right] = j^z + \frac{\partial D_z}{\partial t} - \frac{\beta^2}{r}\frac{\partial H_r}{\partial \theta} + \frac{\beta}{c}\frac{\partial H_r}{\partial t} - \omega\frac{\partial D_z}{\partial \theta}, \qquad (28)$$

$$\text{(d)}\quad \operatorname{div}\vec{D} = \rho + \frac{2\omega H_z}{c^2} - \frac{\beta}{c}\left(\frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r}\right).$$

The first three of these can be written

$$\operatorname{curl}\vec{H} = \vec{j} + \frac{\partial \vec{D}}{\partial t} + \vec{a}_r\left(\frac{\beta^2}{r}\cdot\frac{\partial H_z}{\partial \theta} - \omega\frac{\partial D_r}{\partial \theta} - \frac{\beta}{c}\frac{\partial H_z}{\partial t}\right) + \omega r\rho\vec{a}_\theta$$

$$+ \vec{k}\left(\frac{\beta}{c}\frac{\partial H_r}{\partial t} - \omega\frac{\partial D_z}{\partial \theta} - \frac{\beta^2}{r}\frac{\partial H_r}{\partial \theta}\right), \qquad (29)$$

where $\vec{a}_r$, $\vec{a}_\theta$, and $\vec{k}$ are unit vectors, and where we have used (28) to simplify the coefficient of $\vec{a}_\theta$. In the particular case of steady rotation of axially symmetric systems, where all derivatives with respect to t and θ vanish, we have

$$\begin{aligned} &\operatorname{div}\vec{B} = 0,\\ &\operatorname{curl}\vec{E} = 0,\\ &\operatorname{div}\vec{D} = \rho + 2\omega H_z/c^2 - \beta/c(\operatorname{curl}\vec{H})_\theta,\\ &\operatorname{curl}\vec{H} = \vec{j} + \omega r\rho\vec{a}_\theta. \end{aligned} \qquad (30)$$

These equations are worth some scrutiny. The first two are unchanged from those in inertial systems, but the third and fourth involve respectively an "extra" charge density and current. The extra current, $\omega r\rho\vec{a}_\theta$, looks just like a circulating current due to motions of a volume charge density ρ at a velocity ωr. However, it must be remembered that ρ is the charge density measured in the rotating system, and at rest with respect to this system. If we assume that ρ and $\vec{j}$, the "true" charge density and current, both vanish, we can eliminate the "extra" current but not the extra charge density, and Eqs. (30) become

$$\begin{aligned} &\operatorname{div} \vec{B} = 0, \\ &\operatorname{curl} \vec{E} = 0, \\ &\operatorname{curl} \vec{H} = 0, \\ &\operatorname{div} \vec{D} = 2\omega H_z/c^2. \end{aligned} \tag{31}$$

III. MATERIAL MEDIA AND CONSTITUTIVE EQUATIONS

The equations relating $\vec{E}$, $\vec{B}$, $\vec{D}$, $\vec{H}$, and $\vec{j}$ are usually written in inertial systems

$$\vec{D} = K\epsilon_0\vec{E}, \qquad \vec{B} = \mu\vec{H}, \qquad \vec{j} = \sigma\vec{E}, \tag{32}$$

which may be expressed in covariant tensor formulation (Tolman,[8] p. 263)

$$H_{ij}dx^i/ds = K\epsilon_0 F_{ij}dx^i/ds, \tag{33}$$

$$(g_{ij}F_{kl} + g_{ik}F_{lj} + g_{il}F_{jk})dx^i/ds = \mu c^2(g_{ij}H_{kl} + g_{ik}H_{lj} + g_{il}H_{jk})dx^i/ds, \tag{34}$$

and

$$J^i + J_k dx^k/ds\, dx^i/ds = \sigma\, g_{kl}F^{li}\, dx^k/ds, \tag{35}$$

where again K, μ, and σ are, respectively, the dielectric constant, absolute magnetic permeability, and electrical conductivity of the material measured in the rotating system, and dx^i/ds is the macroscopic velocity of the medium at the point of interest. Note from (19) that for a point which is stationary in the rotating system, dx^i/ds is equal to γ when $i = 4$, and otherwise zero. We thus obtain from (33)

$$\vec{D} = K\epsilon_0\vec{E}, \tag{36}$$

which is the same as the first equation (32), while equation (34) becomes

$$\begin{aligned} &B_\theta = \mu H_\theta, \\ &B_z + \omega r E_r \gamma^2/c^2 = \mu(H_z + \gamma^2\omega r D_r), \\ \text{and} \quad &B_r - \gamma^2\omega r E_z/c^2 = \mu(H_r - \gamma^2\omega r D_z). \end{aligned} \tag{37}$$

These equations bear a striking resemblance to Minkowski's equation for uniformly moving media (cf Sommerfeld,[5] p. 281),

$$\vec{B} - \vec{v} \times \vec{E}/c^2 = \mu(\vec{H} - \vec{v} \times \vec{D}), \tag{38}$$

but there is a considerable difference in interpretation. The $\vec{v}$ in (38) is the velocity (uniform) of a material medium of permeability μ relative to the frame (usually the laboratory system) in which the field vectors are measured. In (37), on the other hand, ωr is the velocity of the material medium, relative to an inertial frame at rest with respect to the fixed stars, and the field vectors are measured in this moving system. In uniformly moving media, if we measure the field quantities in the moving medium, we always obtain exactly (32). We have already met a similar difference between uniformly moving media and accelerated systems in the extra-current term of (30). This is a consequence of the differences between relativity of uniform translational and uniform rotational motion.

The last constitutive Eq. (35) gives

$$j_r = \sigma E_r, \quad j_\theta = \sigma E_\theta / r\gamma^2, \quad j_z = \sigma E_z, \tag{39}$$

where the extra factor of $1/r$ in j_θ arises from the dimensional difference in (r, z) and θ. This is discussed more fully in a later section and involves the difference between covariant and contravariant vectors in all coordinate systems other than rectangular cartesian.

IV. PHYSICAL LAWS IN ROTATING SYSTEMS

It should be remembered, in regard to the physical significance of the quantities occurring in Eqs. (30) and (31), that so far their only meaning is that determined by their definition as the components of the tensors (16) and (25), which transform in a known way under the transformation of coordinates (18) (Appendix II). Thus it is not yet obvious that the quantity $\vec{E}$, for example, is the force acting on a stationary unit charge in the rotating system. To examine this question, we must consider the transformation laws for covariant and contravariant vectors. It is shown in Appendix I that in considering the work done in a displacement δx^i by a force, we must take the covariant components of F_i.

As shown in Møller,[7] p. 305, the electromagnetic four-force acting on a particle with charge e is

$$F_i = (e/c) F_{ik} U^k, \tag{40}$$

where U^k is the four-velocity

$$U^i \equiv \frac{dx^i}{d\tau} = \frac{c\,dx^i}{ds} = (\lceil u^a, c \rceil) \qquad (\tau = s/c), \tag{41}$$

$u^\alpha = dx^\alpha/dt$ being the contravariant components of spatial velocity, and

$$\frac{1}{\Gamma} = \left[\left\{\left(1 - \frac{r^2\omega^2}{c^2}\right)^{1/2} - \frac{\gamma_\alpha u^\alpha}{c}\right\}^2 - \frac{u^2}{c^2}\right]^{1/2} . \tag{42}$$

Here, we have

$$\gamma_\alpha = -g_{\alpha 4}/\sqrt{+g_{44}},$$

$$u^2 = \gamma_{\alpha\beta}(dx^\alpha/dt)(dx^\beta/dt), \text{ and} \tag{43}$$

$$-\gamma_{\alpha\beta} = g_{\alpha\beta} - \gamma_\alpha\gamma_\beta .$$

Thus for the rotating coordinate system in which

$$-ds^2 = dr^2 + r^2 d\theta^2 + dz^2 + \frac{2r\omega^2}{c} d\theta dt - c^2 dt^2/\gamma^2, \tag{19}$$

we have

$$\begin{aligned} \gamma_\alpha &= (0,\ +\beta\gamma r,\ 0), \\ \gamma_{11} &= +1, \\ \gamma_{22} &= +\gamma^2 r^2, \\ \gamma_{33} &= +1, \text{ and} \\ \gamma_{\alpha\beta} &= 0 \text{ for } \alpha \neq \beta, \end{aligned} \tag{44}$$

where β and γ are defined in (26).

We thus have, for the components of the four-force,

$$\begin{aligned} F_1 &= \Gamma e(rB_z\dot{\theta} - B_\theta\dot{z} + E_r), \\ F_2 &= r\,\Gamma e(-B_z\dot{r} + B_r\dot{z} + E_\theta), \\ F_3 &= \Gamma e(B_\theta\dot{r} - rB_r\dot{\theta} + E_z), \text{ and} \\ F_4 &= -\Gamma e(E_r\dot{r} + rE_\theta\dot{\theta} + E_z\dot{z}) \end{aligned} \tag{45}$$

where

$$\frac{1}{\Gamma} = \left[\left\{\sqrt{1 - \frac{r^2\omega^2}{c^2}} - \frac{\beta\gamma r\dot{\theta}}{c}\right\}^2 - \frac{u^2}{c^2}\right]^{1/2} \quad \text{and}$$

$$u^2 = \dot{r}^2 + r^2\dot{\theta}^2\gamma^2 + \dot{z}^2,$$

the dots referring to time-differentiation.

For a charge at rest in the rotating frame we have $1/\Gamma = 1/\gamma$, and

$$\left.\begin{aligned} F_1 &= \gamma e E_r \\ F_2 &= \gamma e r E_\theta \\ F_3 &= \gamma e E_z \\ F_4 &= 0 \end{aligned}\right\} = (\gamma F_\alpha, 0),$$

and we see that the quantities E_r, E_θ, and E_z are indeed the components of the electric field strength in the moving system, because the work done by the force $\vec{F}_\alpha$ (real three-dimensional) in a displacement δr, $\delta\theta$, δz is

$$W = F_\alpha \delta x^\alpha = e(E_r \delta r + r E_\theta \delta\theta + E_z \delta z).$$

Similarly $\vec{B}$ is the magnetic induction, and if we consider the forces on a hypothetical magnetic pole, $\vec{H}$ and $\vec{D}$ can be shown to represent the usual magnetic field intensity and electric displacement vectors. As a check, if we let $\dot{r} = \dot{z} = 0$ and $\dot{\theta} = -\omega$, which thus represents a particle at rest in the laboratory frame, we have, using (26),

$$1/\Gamma = (1/\gamma + \beta^2\gamma^2)^2 - \beta^2\gamma^2 = 1,$$

and the four-force becomes

$$F_1 = e(E_r - \omega r B_z),$$

$$F_2 = r e E_\theta,$$

$$F_3 = e(E_z + \omega r B_r).$$

Comparison with Eqs. (A22) in Appendix II shows that this is the right force law, in the laboratory system, if $\vec{E}$ and $\vec{B}$ are the usual electric and magnetic field strengths.

V. SOLUTION FOR ROTATING MAGNETIC INSULATOR

Now that we have identified the vectors $\vec{E}$, $\vec{B}$, $\vec{D}$, and $\vec{H}$ in (30) with the usual electric and magnetic field vectors, our problem reduces to the solution of these equations with the constitutive Eqs. (36), (37), and (39), and the transformation back to the laboratory system. Consider an infinitely long, cylindrical magnetic insulator with a uniform field $\vec{B}_z$ parallel to the axis of rotation. All derivatives with respect to z and θ thus vanish. We have, from (37),

$$B_r - \mu H_r = \gamma^2 \omega r E_z (K\epsilon_0\mu - 1/c^2)$$
$$B_z - \mu H_z = -\gamma^2 \omega r E_r (K\epsilon_0\mu - 1/c^2) , \tag{46}$$

and, from (31d),

$$\frac{1}{r} \frac{\partial}{\partial r} (rD_r) = \frac{2\omega H_z}{c^2}$$

or (47)

$$\frac{1}{r} \frac{\partial}{\partial r} (rE_r) = \frac{2\omega H_z}{Kc^2} ,$$

so that

$$H_z = \frac{\epsilon_0 K c^2 E_r}{\omega r} . \tag{48}$$

Substituting in the second Eq. (46) (reverting to our original use of primed coordinates to typify the rotating system of Appendix II),

$$E_r' = \frac{c\beta B_z'}{c^2 \left[\epsilon_0\mu K + \gamma^2\beta^2(\epsilon_0 K\mu - \frac{1}{c^2})\right]} . \tag{49}$$

Now, in the laboratory frame, we have for the tensor F_{ij}, from Appendix II

$$F_{ij} = \begin{pmatrix} 0 & crB_z' & -cB_\theta' & E_r' - \omega r B_z' \\ & 0 & crB_r' & rE_\theta' \\ & & 0 & E_z' + \omega r B_r' \\ & & & 0 \end{pmatrix} ,$$

so that

$$E_r^{(lab)} = E_r' - \omega r B_z$$

$$= -\omega r B_z \left[1 - \frac{1}{c^2 \left\{ \epsilon_0 \mu K + \gamma^2 \beta^2 \left(\epsilon_0 \mu K - \frac{1}{c^2} \right) \right\}} \right] . \qquad (50)$$

This is to be compared with the corresponding term derived from the Lorentz transformation of special relativity (Appendix III),

$$E_r = \omega r B_z (1 - 1/\epsilon_0 \mu K c^2). \qquad (51)$$

Note that, for free space, $\epsilon_0 K \mu \rightarrow 1/c^2$ in our units.

VI. FIELD OF A ROTATING MAGNET

Consider now a permanently magnetized spinning cylinder, of radius a, either conducting or insulating. We shall assume that the magnetization $\vec{M}$ is uniform and in the z direction. Then from the first Eq. (30), B is uniform and in the z direction, and thus $\vec{H}$ is likewise, because $\rho = 0$ throughout the body of the cylinder. (This says nothing about any possible polarization charge that may exist on the surface of the cylinder.) If this cylinder is uncharged, we may integrate the third Eq. (30) over the whole volume enclosed by a cylindrical surface infinitesimally larger than the rotating magnet and of unit height in the z direction; we then have

$$\int \text{div}\, D' d\tau' = \int D_r' \, dS' = \int \rho' \, d\tau' + 1/c^2 \int 2\omega H_z' \, d\tau',$$

$$D_r' = \omega a H_z'/c^2, \qquad (52)$$

just outside the spinning magnet. Because in the vacuum, $D' = \epsilon_0 E'$, and $B' = \mu_0 H'$, we have

$$E_r' = \omega a B_z',$$

and from (A22)

$$E_r = E_r' - \omega a B_z' = 0. \qquad (53)$$

This proves the more or less self-evident assertion that there is no electric field outside a spinning magnet, measured in the laboratory frame, and neglects any end effects of "dipole-type" polarization-charge distributions in magnets of finite size.

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APPENDIX I

Consider a general transformation of coordinates in space time,

$$x'^i = x'^i(x^k) \tag{A1}$$

or

$$\left.\begin{aligned} dx'^i &= a^i_k dx^k = dx^k\, \partial x'^i/\partial x^k \\ dx^i &= \breve{a}^i_k dx'^k = dx'^k\, \partial x^i/\partial x'^k \end{aligned}\right\} . \tag{A2}$$

A contravariant four-vector a^i is defined as a collection of four quantities that transform according to the equation

$$a'^i = a^i_k a^k, \tag{A3}$$

and a covariant vector a_i is a similar collection in which the transformation is according to the equation

$$a'_i = \breve{a}^k_i a_k. \tag{A4}$$

To give a specific example, let us consider the transformation from rectangular coordinates x, y, z, ct to polar coordinates r, θ, z, ct. We have (letting the primed system refer to the polar coordinates)

$$\begin{aligned} x^1 &= x = r\cos\theta = x'^1 \cos x'^2, \\ x^2 &= y = r\sin\theta = x'^1 \sin x'^2, \\ x^3 &= z = x'^3, \\ x^4 &= ct = x'^4. \end{aligned} \tag{A5}$$

Thus the transformation matrix a^i_k is given by

$$a^i_k = \begin{pmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta/r & \cos\theta/r & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{A6}$$

and

$$\breve{a}^{\text{row}}_{\text{col}} = \begin{pmatrix} \cos\theta & -r\sin\theta & 0 & 0 \\ \sin\theta & r\cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{A7}$$

Let us see how a four-vector in rectangular coordinates transforms into polar coordinates. Specifically, let us consider the four-velocity

$$U^i = dx^i/d\tau = (\gamma\vec{u},\ c\gamma), \tag{A8}$$

where $\vec{u}$ is the spatial velocity vector with components dx/dt, dy/dt, and dz/dt, and

$$\gamma = 1/\sqrt{1 - u^2/c^2}$$

is the Lorentz contraction factor. The covariant components of the vector are given by

$$U_i = g_{ij}U^j = (-\gamma\vec{u}, c\gamma). \tag{A8'}$$

In polar coordinates, the contravariant velocity components become

$$\begin{aligned} U'^1 &= U^r = \alpha_k^1\, U^k = \gamma(u_x \cos\theta + u_y \sin\theta) \text{ and} \\ U'^2 &= U^\theta = \alpha_k^2\, U^k = \gamma(-u_x \sin\theta/r + u_y \cos\theta/r), \end{aligned} \tag{A9}$$

while the covariant components of velocity are

$$\begin{aligned} U_r &= \gamma(u_x \cos\theta + u_y \sin\theta) \text{ and} \\ U_\theta &= \gamma(-r \sin\theta\, u_x + r \cos\theta\, u_y). \end{aligned} \tag{A10}$$

The polar components of velocity in elementary mechanics are

$$\begin{aligned} u_r &= u_x \cos\theta + u_y \sin\theta \quad \text{and} \\ u_\theta &= -u_x \sin\theta + u_y \cos\theta, \end{aligned} \tag{A11}$$

and these do not agree with either of the above, (A9 or A10) being a sort of geometrical mean of the two. However, if our four-vector had been a force, instead of a velocity, and we had considered the work done in an elementary displacement δs (the θ-component of which is $r\delta\theta$), it is seen that the covariant force components F_i, when multiplied by the corresponding displacement δx^i, give us the correct value for the work done. As pointed out by Eddington, the elementary definitions of physical quantities refer in general only to rectangular components, and must be supplemented before we can decide whether the physical vector is covariant or contravariant. Thus, if we define force as "the product of mass and acceleration," the force turns out to be contravariant, while if we define force by "work equals force multiplied by displacement," the force is covariant, as we have just seen. This is expressed mathematically by

$$F^i = c^2\, d/ds(m_0 dx^i/ds) \tag{A12}$$

and

$$\delta W = F_i\, \delta x^i = \text{invariant}. \tag{A13}$$

To complete our notation, we define a contravariant tensor F^{ij} as a collection of 16 quantities that transform (under the coordinate transformation (2)) according to the equation

$$F'^{ij} = \alpha^i_l \, \alpha^j_m F^{lm}, \tag{A14}$$

and similarly for a covariant tensor

$$F'_{ij} = \alpha^l_i \, \alpha^m_j F_{lm}. \tag{A15}$$

APPENDIX II

We shall now derive the transformation coefficients between two systems rotating with respect to each other with a uniform angular velocity ω. Let Σ be an inertial system (fixed with respect to the distant stars) R, Θ, Z, ct, and let Σ' be a system r, θ, z, ct rotating with respect to Σ.

Thus if $x^i = R, \Theta, Z, ct$ and $x'^i = r, \theta, z, ct$, where

$$\begin{aligned} R &= r, \\ \Theta &= \theta + \omega t, \\ Z &= z, \\ cT &= ct, \end{aligned} \tag{A16}$$

then $dx^i = \breve{\alpha}^i_k \, dx'^k$, where

$$\breve{\alpha}^{\text{row}}_{\text{col}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \omega/c \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{A17}$$

and

$$\alpha^{\text{row}}_{\text{col}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -\omega/c \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{A18}$$

For the transformation of a covariant tensor from the rotating system into the laboratory frame, we have

$$t_{ik} = \alpha_i^l \, \alpha_k^m \, t'_{lm} \, , \tag{A19}$$

and for the first electromagnetic stress tensor F_{ik}, using (16) for F'_{lm} (in the primed system) we get, in the laboratory system,

$$F_{ik} = \begin{pmatrix} 0 & crB'_z & -cB'_\theta & E'_r - \omega r B'_z \\ & 0 & crB'_r & rE'_\theta \\ & & 0 & E'_z + \omega r B'_r \\ & & & 0 \end{pmatrix} \tag{A20}$$

where the primed quantities are those measured in the rotating system. By the same reasoning used in Part IV of the text to show that $\vec{E}$ and $\vec{B}$ are actually the usual electromagnetic forces on a test charge, we see that the components of the electric and magnetic field measured in the fixed system are

$$\vec{B} = \vec{B}' , \tag{A21}$$

$$E_\theta = E'_\theta \, ,$$

$$E_r = E'_r - \omega r B'_z \, , \text{ and} \tag{A22}$$

$$E_z = E'_z + \omega r B'_r \, .$$

The second set of these equations looks very much like the transformation equations for electric field strength in special relativity,

$$\vec{E}_\perp = \gamma(\vec{E}'_\perp - \vec{v} \times \vec{B}_\perp) \text{ and } E_{\parallel} = E'_{\parallel} \, , \tag{A23}$$

where $\parallel$ and $\perp$ refer to the components of E and B parallel and perpendicular to v. The first set, however, is quite different. The magnetic field vector is unchanged by the rotation, whereas in special relativity the transformation would be

$$B_{\parallel} = B'_{\parallel}$$

and

$$\vec{B}_\perp = \gamma(\vec{B}'_\perp - \vec{v} \times \vec{E}_\perp / c^2) . \text{*} \tag{A24}$$

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* In connection with these transformation equations for $\vec{E}$ and $\vec{B}$ and for most practical applications involving plasma physics, it should be noticed that, because of the good conductivity, $\vec{E}$ will be considerably less than $\vec{B}$, and because of the factor of c^2 in the denominator of (A24), it is generally found that, to a very good approximation, $\vec{B}$ is equal to $\vec{B}'$, even for uniform motion.

We have already seen other consequences of the nonrelativity of rotary motion. An apparent paradox that the equation (A21) above leads to is afforded by considering a ring of charge at rest in the laboratory frame and an observer whirling around the axially symmetric distribution with an angular velocity ω. We know that if the ring is rotated an observer in the laboratory frame will see a magnetic field, but our equations tell us that the observer rotating around the fixed-charge ring will see no magnetic field, because in the laboratory frame the only field is electrostatic, and $\vec{B}$ is unchanged by rotation. Thus the two cases of rotating observer and rotating charge distribution are not symmetric. We have neglected, however, the effect of the charges of opposite sign at infinity, which are, of course, in violent motion with respect to the (rotating) observer. This point will be discussed more fully in a subsequent paper. A similar problem that avoids the question of charges at infinity, has been considered by Schiff,[9] who considers two concentric oppositely charged spheres with an observer rotating around them. Because, in the rest frame, all components of the stress tensor vanish here, they must similarly vanish in all systems.

APPENDIX III

I have been unable to find in the literature an explicit derivation of this Eq. (51), but at one time a great deal of importance was attached to the results of an experiment performed by H. A. Wilson[10] to check whether the correct factor in the brackets was $(1 - 1/\mu K\epsilon_0 c^2)$ or $(1 - 1/K)$. A derivation of the correct factor is given below.

Consider two parallel conducting plates in the laboratory system, parallel to the y-z plane, and of infinite extent. Let these be separated by a distance L, and consider a slab of insulating magnetic material, infinite in the y and z directions and of thickness (L - 2a) in the x direction, to be moving in the positive y direction with velocity v. Let a magnetic field B that is directed in the positive z direction exist in the laboratory system, and let the dielectric constant and permeability of the slab be k and μ respectively (i.e., if primed quantities refer to a system of coordinates in which the slab is at rest, then

$$D' = K\epsilon_0 E' \text{ and}$$

$$B' = \mu H' = K_m \mu_0 H',$$

where K_m is the <u>relative</u> permeability).

We shall let the subscripts "0" and "i" refer to quantities measured in the space between the plates and the dielectric slab, and inside the slab, respectively, and unprimed quantities refer to the laboratory system. We then attach an electrostatic voltmeter (of very large capacitance, so that we don't have to concern ourselves with charge division between the conducting plates and the external voltmeter) to the two plates (clearly, this will be a

"Gedanken" experiment, if the leads are to be brought around the infinite slab). We then have

(a) $\vec{E}_\perp' = \gamma(\vec{E}_\perp + \vec{v} \times \vec{B}_\perp)$,

(b) $\vec{B}_\perp' = \gamma(\vec{B}_\perp - \vec{v} \times \vec{E}_\perp/c^2)$, $\gamma = 1/\sqrt{1-v^2/c^2} = 1/\sqrt{1-\beta^2}$,

(c) $\vec{B}' = \mu\vec{H}'$, and

(d) $\vec{D}' = K\epsilon_0\vec{E}'$.

In our case these equations become

(a) $E_x' = \gamma(E_x + vB)$,

(b) $B' = \gamma(B + vE_x/c^2)$,

(c) $B' = \mu H'$, and

(d) $D' = K\epsilon_0 E'$.

If the capacitance of the measuring instrument is large enough we shall have no charge density on the plates, so that if $\underline{\sigma}$ denotes the charge density measured in the laboratory system, then

$$\sigma = \sigma' = E_0 = 0.$$

From (A27a) and (A27d), we find

$$E_0' = \gamma v B_0 = D_0'/\epsilon_0 = D_i'/\epsilon_0 = KE_i' ,$$

and thus

$$\begin{aligned} E_i &= \gamma(E_i' - vB_i') = (\gamma^3 v/K)(B_0' - vE_0'/c^2) - \gamma^2 v(B_i + vE_i/c^2) \\ &= vB_0/K - vB_i \\ &= vB_i(\mu_0/K\mu - 1)/(1 - \beta^2/K\mu). \end{aligned}$$

As $\epsilon_0\mu_0 = 1/c^2$ in our units, we see that, neglecting terms in second order of (v/c), this is just Eq. (51).

APPENDIX IV

It is evident that Eqs. (30) depend upon the definitions of F_{ij} and H_{ij} with which we started, i.e., Eqs. (16) and (24). We could equally well have started with the contravariant tensors F^{ij} and H^{ij}, letting, for example, F^{ij} be as defined in Eq. (16). In this case, calculating F_{ij} from F^{ij} in the usual way and substituting into Eq. (5), we would have obtained

$$\operatorname{div} \vec{B} = \frac{\omega r}{c^2}(\operatorname{curl} \vec{E})_\theta - \frac{\omega E_z}{c^2},$$

$$(\operatorname{curl} \vec{E})_z = 0,$$

$$(\operatorname{curl} \vec{E})_r = 0, \text{ and}$$

$$(\operatorname{curl} \vec{E})_\theta = -\frac{2\omega^2 r\gamma^2}{c^2} E_z - \omega r\gamma^2 \operatorname{div} \vec{B},$$

and if H^{ij} is similarly defined as in (16) (with $\vec{H}/c$ and $\vec{D}$ replacing $c\vec{B}$ and $\vec{E}$), the second two of Maxwell's equations would have been exactly as given in (2c) and (2d). In this case, however, the field vectors would not have had the usual meaning we assigned them in Part IV.

Or, in a similar way, we could have started with either F_{ij} or F^{ij} defined as in (16) in the laboratory system, and calculated the corresponding tensors in the rotating system from (A19). In this case, assuming no θ dependence for the fields, Eqs. (2) will be found to be preserved exactly, but the simple relation (36) no longer obtains, but is replaced by a relation between $\vec{D}$ and $\vec{H}$ on the one hand and $\vec{B}$ and $\vec{E}$ on the other, similar to (33).

Thus it is seen that there are many ways to attack this problem, and indeed this flexibility is present in all general relativistic transformations. They must all lead to the same physical results, however, although the nomenclature may be different.

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