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UCRL-3727

UNIVERSITY OF CALIFORNIA

Radiation Laboratory
Berkeley, California

Contract No. W-7405-eng-48

UNIQUENESS OF SOLUTIONS TO DISPERSION RELATIONS FOR POTENTIAL SCATTERING

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March 29, 1957

Printed for the U.S. Atomic Energy Commission

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ABSTRACT

Some properties of the dispersion relations for potential scattering are examined. It is shown that even for a potential of finite extent, the dispersion and unitarity relations do not define a unique scattering amplitude, so that they do not contain all the information that is derivable from the Schroedinger equation.

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INTRODUCTION

Dispersion relations connecting the real and imaginary parts of scattering amplitudes have been deduced from the assumptions of Lorentz invariance, microscopic causality, and certain symmetry properties quite independently of any particular Hamiltonian.[1] The

[1] M. L. Goldberger, Phys. Rev. 99, 979 (1955). For further references see, e.g., Capps and Takeda, Phys. Rev. 103, 1877 (1956).

connection with a model is made in the assignment of the mass spectrum, the postulated threshold behavior of the amplitudes, and in the assumed behavior of the amplitudes for infinite momenta. It has been conjectured that a quantum field theory might be completely defined by such dispersion relations together with the unitarity condition.[2] In such an approach it

[2] M. Gell-Mann, Proceedings of the Sixth Annual Rochester Conference on High Energy Nuclear Physics, Rochester, 1956.

is assumed that the dispersion relations, together with the unitarity condition, contain all the relevant information contained in the Hamiltonian, and that they yield a unique solution if the Hamiltonian does. One would then have a formulation of a field theory essentially in terms of observables, which would require none of the renormalization prescriptions of the canonical Hamiltonian theories.

However, this happy situation is not realized for certain simple models for which all solutions of the dispersion relations can be exhibited. Thus the dispersion-type equations that describe the "one-meson approximation" to a static-source meson theory have been solved by Castillejo, Dalitz, and Dyson,[3] who find that these equations have an infinite number of solutions. It is not known whether this ambiguity

[3] Castillejo, Dalitz, and Dyson, Phys. Rev. 101, 453 (1956).

remains if the static-source model is treated without the "one-meson approximation." This leads to an infinite number of coupled dispersion relations for which no solution has yet been exhibited. Similarly the two dispersion relations obtained for the Lee model[4] do not possess a

[4] T. D. Lee, Phys. Rev. 95, 1329 (1954). See also K. W. Ford, Phys. Rev. 105, 320 (1957).

unique solution, although the Hamiltonian formulation of this model does. Again, however, as in the "one-meson" model, the theory is a mutilation of a canonical field theory, so that the significance of the nonuniqueness is obscure.

It is therefore of interest to investigate the uniqueness of the solution to the exact dispersion relations which describe a reasonable, complete, and causal model, namely the scattering of a particle by a static scalar local potential of finite range. The Schroedinger (or Klein-Gordon) equations can be solved to give the scattering amplitude.

Conversely the scattering amplitude determines the potential[5] and so

[5] Jost and Kohn, Kgl. Danske Videnskab. Selskab. Mat.-fys. Medd. 27, No. 9 (1953).

contains all the information that is in the Hamiltonian. This scattering amplitude is a function of two variables, the momentum and the momentum transfer, and therefore is described by a much more complicated dispersion relation than that of the Lee model or the "one-meson approximation." Indeed, if the scattering amplitude is expandable into a convergent series of partial waves, this dispersion relation (plus the unitarity condition) decomposes into an infinite number of coupled nonlinear integral equations. In addition to their multiplicity, these integral equations differ in an important way from those previously studied: the inhomogeneous term need no longer be a rational function of its argument, but may have an essential singularity at infinity. Those mathematical procedures which produce the spectrum of solutions for the Lee model and the "one-meson approximation" are not then applicable. Nevertheless it is possible to show by specific examples that even for a square-well potential the solutions of the dispersion relation are not unique. Although these extra solutions cannot correspond to the scattering from any "reasonable" potential, they do demonstrate that the dispersion relations and the unitarity condition do not exhaust the content of the Hamiltonian theory.

THE DISPERSION RELATIONS

The assumed dispersion and unitarity relations for the scattering of a particle by a spherically symmetric static potential (with no bound states) are

$$f(k,\Delta) = V(\Delta) + \frac{1}{\pi}\int_{-\infty}^{\infty} dk' \quad \frac{\mathrm{Im}\, f(k',\Delta)}{k'-k-i\varepsilon} \tag{1}$$

where

$$\mathrm{Im}\, f(k,\Delta) = \frac{1}{4\pi m}\int d^3\Delta'\, \delta(\Delta'^2 + 2\underset{\sim}{k}\cdot\underset{\sim}{\Delta}')\, f^*(k, |\underset{\sim}{\Delta}-\underset{\sim}{\Delta}'|)\, f(k,\Delta'), \tag{2}$$

$$(k \text{ real})$$

and

$$f^*(k,\Delta) = f(-k,\Delta), \quad k \text{ real}. \tag{3}$$

The scattering amplitude $f(k,\Delta)$ is a function of the momentum k of the incident particle, and of its momentum transfer $\underset{\sim}{\Delta}$. The inhomogeneous term of Eq. (1), $V(\Delta)$, is the Born approximation for the scattering with momentum transfer $\underset{\sim}{\Delta}$, and of course depends only upon Δ for a static local potential. The unitarity relation (2) and the symmetry condition (3) follow directly from the reality of the potential, but the dispersion relation (1) holds only for certain classes of potentials. It is a consequence of Cauchy's theorem provided that

(a) $f(k,\Delta)$ is a regular function of k for fixed Δ, Δ real and $\mathrm{Im}\, k > 0$,

(b) $f(k,\Delta) - V(\Delta) \longrightarrow 0$ as $|k| \to \infty$, $\mathrm{Im}\, k \geq \delta > 0$,

(c) $k^{-1} f(k,\Delta)$ is bounded on the real axis.

Klein and Zemach have shown that for a potential of finite extent a, $f(k, \Delta)\exp(2ia\Delta)$ is a regular bounded function of k.[6] A similar result has been obtained by Wong for each term in the perturbation expansion of the scattering amplitude.[7] Klein and Zemach write the scattering amplitude

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6 Klein and Zemach, Dispersion Relations for Scattering by Nonrelativistic Particles (to be published).

7 D. Y. Wong, Physics Dept. Tech. Report No. 62, University of Maryland, Jan. 1957.

$$f(k,\Delta) = V(\Delta) + \frac{1}{4\pi}\int d^3r\, d^3r'\, e^{-i\underset{\sim}{\Delta}\cdot\underset{\sim}{r}}\, e^{-ik(\underset{\sim}{r}\cdot\hat{k}_1)}\, V(r) \times \langle r|\,(H - k^2/2m - i\varepsilon)^{-1}\,|r'\rangle\, V(r')\, e^{ik(\underset{\sim}{r}'\cdot\hat{k}_1)} \tag{4}$$

and show that

$$\left| f(k,\Delta) - V(\Delta)\right| \leq M/|k|\,\mathrm{Im}\,k \tag{5}$$

whenever

$$\int_0^\infty V^2(r)\,r^2\,dr < \infty \;;\quad \frac{1}{\Delta}\int_0^\infty \sin 2\Delta r\; V^2(r)\,r\,dr < \infty \tag{A}$$

The condition (5) and assumption (c) above insure the vanishing of $\int dk' \left[f(k',\Delta) - V(\Delta)\right]/(k' - k)$ over the semicircle from $+\infty$ to $-\infty$ in the upper half plane. Moreover, for potentials of finite extent, the integrations in Eq. (4) are over a finite region and the r.h.s. defines a regular function of k if $\mathrm{Im}\,k > 0$ and H has no negative eigenvalues

(bound states). If the integral in Eq. (4) is uniformly convergent, then $f(k, \Delta)$ will be a regular function of k even if the potential is not of finite extent.[8] In any region $\mathrm{Im}\, k \geq \delta > 0$ the uniform

[8] This result does not contradict the fact that for a large class of potentials which are not of finite extent there can appear redundant poles in the upper half plane in the s-wave scattering amplitude. (See for example V. Bargmann, Revs. Modern Phys. 21, 488 (1949)). The redundant poles appear in the individual partial-wave amplitudes, but cancel when these are combined to give the total scattering amplitude. This is similar to the result thateven though the partial-wave scattering amplitudes have essential singularities at infinity, the total amplitude approaches the constant $V(\Delta)$.

convergence will follow if for any ϵ there exists an $\bar{R}$, such that for any $R \geq \bar{R}$, the following conditions hold:

$$\int_R^\infty V^2(r)\, r^2 dr < \epsilon \quad ; \quad \frac{1}{\Delta}\int_R^\infty \sin 2\Delta r\, V^2(r)\, r\, dr < \epsilon \qquad [B]$$

UNIQUENESS

Clearly one solution of the dispersion and unitarity relations is that solution which results from solving the Schroedinger equation with the potential that appears in Eq. (1). If there is another solution of Eqs. (1) - (3), then it cannot correspond to the scattering by another potential which satisfies conditions (A) and (B), for if it did, then it would satisfy Eq. (1) with a different inhomogeneous term, which is impossible. Nevertheless alternate solutions can be constructed. Let us consider that solution of Eqs. (1) - (3) which is also the scattering amplitude calculated from the Schroedinger equation: $f_0(k, \Delta)$. We shall show that even for a square-well potential there exist an infinite number of other solutions identical to $f_0(k, \Delta)$ in all phase shifts except one, which we shall take to be the s wave.

If the s-wave part of $f_0(k, \Delta)$ has a phase shift $\delta_0(k)$, then

$$f_0(k,\Delta) - (2ik)^{-1}\left[\exp(2i\delta_0(k)) - 1\right] + (2ik)^{-1}\left[\exp(2i\delta(k)) - 1\right] \tag{6}$$

is unitary and a solution of Eq. (1) if

$k^{-1}\left[\exp(2i\delta_0(k)) - \exp(2i\delta(k))\right]$ is a regular function of k, which approaches zero as $|k| \longrightarrow \infty$ for $\operatorname{Im} k > 0$. [C]

We note first that if $V \equiv 0$ then $\exp(2i\delta_0(k)) \equiv S_0(k) = 1$, and

$$\exp(2i\delta(k)) = \prod_{i=1}^{N} \frac{(k-a_i)(k+a_i^*)}{(k+a_i)(k-a_i^*)} \equiv s(k) \; ; \quad \operatorname{Im} a_i > 0 \tag{7}$$

satisfies condition (C), so that there exist an infinite number of solutions in addition to $f_0(k, \Delta) \equiv 0$.

There is a class of potentials discussed by Bargmann[9] for which $S(k)$

[9] V. Bargmann, Revs. Modern Phys. 21, 488 (1949).

is a rational function of k, which approaches unity as $k \longrightarrow \infty$ (for example $V(r) = -V_0\, e^{-\lambda r}(1 + \beta\, e^{-\lambda r})^{-2}$). Here $S_0(k)$ has a finite number of poles in the upper half plane. In this case again

$$\exp\left(2i\,\delta(k)\right) = s(k)\, S_0(k) \tag{8}$$

is a satisfactory alternate solution, provided the a_i are chosen so that $s(k) = 1$ at the poles. This is the kind of situation which occurs in the "one-meson approximation" and in the Lee model, for which an infinite number of solutions could be exhibited in the form of Eq. (8).

In general, however, the situation is a great deal more complicated by the fact that $S_0(k)$ may have an essential singularity at infinity.[10]

[10] If $V(r)$ satisfies $\int_0^\infty r^n\, |V(r)|\, dr < \infty$ $(n = 1,2)$, then if $S_0(k)$ has no redundant poles (poles which do not correspond to bound states), it must have an essential singularity at infinity. The conditions on $V(r)$ imply a theorem of N. Levinson (Kgl. Danske Videnskab Salskab, Mat.-fys. Medd. 25, No. 9 (1949)) that $\int_{-\infty}^{\infty} dk\, S'(k)/S(k) = 0$ if there are no bound states. If $S(k) = 1 + O(1/k)$ at infinity in the upper half plane, then the absence of poles for $S(k)$ in the upper half plane implies an absence of zeros. But as we have $S(k)\, S(-k) = 1$, this implies $S(k) \equiv 1$. If, however, $S(k)$ is not identically equal to unity, and

there are no redundant poles, then $S(k)$ does not approach unity at infinity. Moreover, since we have $S(k)\,S(-k) = 1$, its behavior at infinity depends upon the direction of approach.

For example, for a potential of finite extent a, we have, quite generally,

$$S_0(k) = \exp(-2ika)\,\frac{1 + ikR_0(k^2)}{1 - ikR_0(k^2)} \tag{9}$$

where $R_0(k^2)$ is a Wigner R function,[11] and we cannot generate another

[11] E. P. Wigner, Annals of Math. 53, 36 (1950).

solution of the type $s(k)\,S_0(k)$ since $k^{-1}(s(k) - 1)\,S_0(k)$ is still singular at infinity. In the case of the square-well potential, for which

$$R_0(k^2) = K^{-1} \tan a K \quad , \tag{10}$$

where $K = k(1 + \lambda/k^2)^{\frac{1}{2}}$ and $\lambda/2m$ is the depth of the potential, we can exhibit several representative examples of alternate solutions, which satisfy condition (C) and thus establish non-uniqueness, even though they have some obvious nonphysical properties.

Example 1: Consider

$$\exp\left(2i\,\delta(k)\right) = \exp(-2ika)\,\frac{1 + ik\,R(k^2)}{1 - ik\,R(k^2)} \quad , \tag{11}$$

and choose

$$R(k^2) = K^{-1} \tan b K \quad . \tag{12}$$

This solution satisfies condition (C), provided $b > a$, and it is thus an acceptable alternate solution. It has the physically undesirable feature that the phase shift for (real) infinite momenta oscillates instead of vanishing, but this "unreasonable" behavior cannot be ruled out without imposing an additional condition to supplement Eqs. (1) - (3). Equation (12) can be immediately generalized to

$$R(k^2) = K^{-1} \sum_i \rho_i \tan b_i K / \sum_i \rho_i \quad ; \qquad b_i \geqslant a \quad . \tag{12'}$$

Example 2: Let us write

$$R(k^2) = -K^{-1} \cot a K \tag{13}$$

in Eq. (11). Condition (c) can again be shown to be satisfied. This solution has the property that for real infinite momenta $\exp\left(2i\,\delta(k)\right) \rightarrow -1$, i.e., the phase shift approaches $\pi/2$ for infinite momenta.

Example 3: Consider the form (11) with

$$R(k^2) = K^{-1} \tan(a + \frac{b}{k^2})K \quad . \tag{14}$$

This satisfies condition (C), provided $b < 0$. As we approach the real axis at $k = 0$ from above, $R(k^2)$ does not become real, but this does not violate unitarity because it is $kR(k^2)$, which vanishes as k approaches the origin, that enters into $\exp\left(2i\,\delta(k)\right)$ and so this is an acceptable solution.[12] It is clearly unphysical because the s-wave scattering cross

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[12] It is the unitarity condition which rules out the otherwise acceptable solution $R(k^2) = K^{-1} \tan(a + \sum_n \alpha_n^2/(\beta_n^2 - K^2))K$. Such a solution is not unitary along the discrete set of points $K = \pm\,\beta_n$.

section oscillates violently near $k = 0$; the "pathological" region may be made arbitrarily small by choosing b small enough. A variety of other types of solutions have been found but they also have the property that $\delta(\infty) - \delta(0) \neq m\pi$ (m an integer).

The difference between the "Bargmann potentials" and the square well may be restated by writing the dispersion relation for the s-wave scattering amplitude $F(k)$ ($= F^*(-k)$ for real k),

$$F(k) = G(k^2) + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{dk'}{k' - k - i\varepsilon}\, k' \left|F(k')\right|^2 , \tag{15}$$

which may be derived from Eqs. (1) - (3) by separating the scattering amplitude

$$\begin{aligned} f(k, \Delta) &= F(k) + \phi(k, \Delta) , \\ \frac{1}{2k^2}\int_0^{2k} \Delta\, \phi(k, \Delta)\, d\Delta &= 0 , \end{aligned} \tag{16}$$

and noting that

$$\begin{aligned} F(k) - \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{dk'}{k' - k - i\varepsilon}\, \mathrm{Im}\, F(k') &= V(\Delta) - \phi(k,\Delta) \\ &\quad + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{dk'}{k' - k - i\varepsilon}\, \mathrm{Im}\, \phi(k',\Delta) \\ &\equiv G(k^2) , \end{aligned}$$

because the left-hand side is independent of Δ and even in k. (The unitarity condition yields the relation $\mathrm{Im}\, F(k) = k\, |F(k)|^2$.) For the

cases of no potential and the "Bargmann potential", $G(k^2)$ is zero and a rational function of k^2 respectively, and the equation may be solved by the method used by Castillejo, Dalitz, and Dyson. On the other hand, for the square well $G(k^2)$ has an essential singularity at infinity, and non-uniqueness of the solutions of Eq. (15) cannot be established by this means.

If bound states are present, the inhomogeneous term of Eq. (1) must be appropriately altered to exhibit the poles of the scattering amplitude, but the conclusions of this paper are unchanged.

This work was supported in part by the National Science Foundation and in part by the U.S. Atomic Energy Commission.