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ALPHA DECAY OF EVEN-EVEN DEFORMED NUCLEI

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March 25, 1958

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ABSTRACT

Approximate analytic methods are developed for calculating the amplitudes of alpha partial waves at the surface of deformed even-even nuclei. A two-term expansion modifying the ordinary Coulomb function to account for nuclear quadrupole coupling is applied. The amplitudes of alpha partial waves at the nuclear surface are tabulated for eight choices of phase and three values of the intrinsic nuclear quadrupole moment. A detailed comparison is made between this treatment, that of Fröman, and the numerical integration of Rasmussen and Hansen for Cm^{242}.

An approximate method of calculating phase shifts induced by the nuclear quadrupole moment is developed and compared with the results of Rasmussen and Hansen for Cm^{242}.

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I INTRODUCTION

When we consider the interaction of an emitted alpha particle with the daughter nucleus in the heavy-element region, the nonsphericity of the nucleus plays an important role. Because the emitted alpha particle may interact with the quadrupole field of the nucleus, the experimental intensities that are observed for decay to various states of the daughter are not the same as the intensities at the nuclear surface. It is this quadrupole interaction which complicates the solution of the differential equations describing the alpha-decay process, as it couples the alpha partial waves which differ from each other in angular momentum by two units. From a numerical integration of these equations, based on experimental alpha-group intensities, one may obtain the amplitudes of the alpha partial waves at the nuclear surface, as well as the amounts of the phase shifts due to the quadrupole terms. In this paper, we describe an approximate analytical method of obtaining the partial-wave amplitudes at the nuclear surface for favored alpha decay to members of a rotational group.

II MATHEMATICAL FORMULATION

Making the usual multipole expansion of the potential V, we obtain[1]

$$V = 2e^2 \sum_{p=1}^{Z} \sum_{\lambda=0}^{\infty} r_p^{\lambda} \frac{P_\lambda(\cos\theta')}{r^{\lambda+1}}, \tag{1}$$

where $\underline{e}$ is the electrostatic unit of charge; $\underline{p}$ is the index labeling the protons in the daughter nucleus; $\underline{P_\lambda(\cos\theta')}$ is the Legendre polynomial of order $\underline{\lambda}$;

*This work was done under the auspices of the U.S. Atomic Energy Commission.

r is the distance of the alpha particle from the center of mass of the recoil nucleus; θ' is the angle between the vectors from the center of mass to the alpha particle and the proton. In this treatment, we include the quadrupole and monopole terms of the Coulomb field.

We construct a solution to the Schroedinger equation of this problem by expanding ψ, the wave function, in spherical polar coordinates:

$$\psi(r, \theta, \phi) = \sum_{\ell} 1/r\, u_{\ell, I_f}(r) \quad Y^{I,M}_{\ell, I_f, T}(\theta, \phi, \chi i) , \tag{2}$$

in which ℓ is the angular momentum of the alpha particle; I, the spin of the parent, and M, its projection, are both 0; I_f, the spin of the final state, is the same as ℓ:

$$Y^{IM}_{\ell, I_f, T}(\theta, \phi, \chi i) = Y^{00}_{\ell, \ell, T}(\theta, \phi, \chi i) = \sum_{\ell} \frac{(-1)^m}{\sqrt{2\ell+1}} Y^m_{\ell}(\theta,) \Phi^T_{\ell, -m}(\chi i), \tag{3}$$

where m is the projection of ℓ on a space-fixed axis and $\Phi^T_{\ell, -m}(\chi i)$ represents the recoil nucleus. We then substitute the expansion (2) into Schroedinger's equation, multiply by $Y^{00*}_{\ell', \ell', T}$, and integrate over all variables except r, to obtain a set of ordinary coupled differential equations in r. The equations for favored alpha decay of even-even nuclei have been derived previously[2] and they are

$$u_0''(r) - \left(\frac{4m Z e^2}{\hbar^2 r} - \frac{2m E_0}{\hbar^2}\right) u_0(r) = \frac{2m Q_0 e^2}{\hbar^2 r^3} \frac{u_2(r)}{\sqrt{5}} , \tag{4a}$$

$$u_2''(r) - \left(\frac{4m Z e^2}{\hbar^2 r} - \frac{2m E_2}{\hbar^2} + \frac{6}{r^2}\right) u_2(r) = \frac{2mQ_0 e^2}{\hbar^2 r^3} \left(\frac{u_0(r)}{\sqrt{5}} + \frac{2u_2(r)}{7} + \frac{6u_4(r)}{7\sqrt{5}}\right), \tag{4b}$$

$$u_4''(r) - \left(\frac{4m Z e^2}{\hbar^2 r} - \frac{2m E_4}{\hbar^2} + \frac{20}{r^2}\right) u_4(r) = \frac{2m Q_0 e^2}{\hbar^2 r^3} \left(\frac{6u_2(r)}{7\sqrt{5}} + \frac{20u_4(r)}{77} + \frac{15u_6(r)}{11\sqrt{13}}\right), \tag{4c}$$

$$u_6''(r) - \left(\frac{4m Z e^2}{\hbar^2 r} - \frac{2m E_6}{\hbar^2} + \frac{42}{r^2}\right) u_6(r) = \frac{2m Q_0 e^2}{\hbar^2 r^3} \left(\frac{15}{11\sqrt{13}} u_4(r) + \frac{14u_6(r)}{55}\right), \tag{4d}$$

including only the alpha partial waves having angular momenta 0, 2, 4, and 6. Z is the charge of the daughter nucleus, m is the reduced mass of the system, E_ℓ is the total decay energy of the ℓth-wave alpha, u_ℓ is defined in Eq. (2), and Q_0 is the intrinsic nuclear quadrupole moment.

III ANALYTIC APPROXIMATION

We can see that the solutions of the set of Eqs.(4) would be regular or irregular Coulomb functions, were it not for the quadrupole moment, i.e., the right-hand side of the Eqs.(4) would vanish if Q_0 were zero. From an examination of the series expansion of the W-K-B integrand, we surmised that the radial wave functions of the alpha partial waves might be well represented by functions of the form $(\alpha_\ell + \frac{\beta_\ell}{r^{3/2}} G_\ell(r)$ in the region of the nuclear surface; α_ℓ and β_ℓ are parameters fixed over all "r" and G_ℓ is the irregular Coulomb function. It is clear that $\beta_\ell / r^{3/2} \to 0$ as $r \to \infty$, so that one may identify α_ℓ as the square root of the quotient of the alpha partial-wave intensity and its velocity. We determine β_ℓ by substituting the approximate solution into the differential equations and demanding that they be exactly satisfied at some arbitrary intermediate distance (2.0×10^{-12} cm seemed to be optimum). The equations that one obtains are algebraic, and their right-hand sides are the same as those of Eqs. (4), simply substituting $(\alpha_\ell + \frac{\beta_\ell}{r^{3/2}} G_\ell, (r)$ for $u_\ell(r)$. The equations are

$$\beta_0\left[-\frac{3\,G'_0(r)}{r^{5/2}} + \frac{15}{4}\,\frac{G_0(r)}{r^{7/2}}\right] = \text{right-hand side of Eq. (4a)} \tag{5a}$$

$$\beta_2\left[-\frac{3\,G'_2(r)}{r^{5/2}} + \frac{15}{4}\,\frac{G_2(r)}{r^{7/2}}\right] = \text{right-hand side of Eq. (4b)} \tag{5b}$$

$$\beta_4\left[-\frac{3\,G'_4(r)}{r^{5/2}} + \frac{15}{4}\,\frac{G_4(r)}{r^{7/2}}\right] = \text{right-hand side of Eq. (4c)} \tag{5c}$$

$$\beta_6\left[-\frac{3G'_6(r)}{r^{5/2}} + \frac{15}{4}\,\frac{G_6(r)}{r^{7/2}}\right] = \text{right-hand side of Eq. (4d)} \tag{5d}$$

When we obtain the values of α_ℓ and β_ℓ, we may readily compute the alpha partial-wave amplitudes at the nuclear surface by simply setting $r=r_0$ in the approximate solution.

The distance at which we demand that the approximate solutions satisfy the differential equations is somewhat arbitrary; the reasons for choosing 2.0×10^{-12} cm. are completely pragmatic, i.e., using this value, we found that we could get best agreement with results of detailed numerical integrations for both Cm^{242} and U^{233},[3,4] If we choose 1.5 or 2.5×10^{-12} cm the variation in surface amplitudes over this range is about 10%. It seems quite reasonable that if we were to add another term to the approximate solutions, i.e., $(\alpha_\ell + \frac{\beta_\ell}{r^{3/2}} + \frac{\gamma_\ell}{r^3}) G_\ell(r)$, these variations could be minimized.

As quadrupole moments are not known very well in the heavy-element region, we have calculated the surface amplitudes for several values of the nuclear quadrupole moment. As α_ℓ is the square root of the intensity of the ℓ-wave divided by its velocity, there is a sign ambiguity. We have included all eight phase choices here. Angular correlation work on Am^{241} seems to indicate[5] that the D wave is in phase with the S wave within the potential barrier, but the interpretation of the U^{233} alignment experiment is somewhat ambiguous.[6]

The data used in this analysis, which came from two summary compilations,[1,7] are given in Table I, and the results in Table II.

Table I

Data used in analysis of α-decay data of even-even heavy elements

Isotope	Energy (Mev)	Abundance (%)	$\sqrt{I}$[a]
	5.421	71	8.426
$_{90}Th^{228}$	5.338	28	5.292
	5.173	0.2	0.448
	4.883	0	0[b]
	4.682	76	8.718
$_{90}Th^{230}$	4.615	24	4.899
	4.467	0.2	0.4472
	4.206	0	0[b]
	5.884	67.2	8.197
$_{92}U^{230}$	5.813	32	5.656
	5.685	0.3	0.550
	5.380	0	0
	5.318	68	8.246
$_{92}U^{232}$	5.261	32	5.656
	5.134	0.32	0.566
	4.919	0	0[b]
	4.768	72	8.485
$_{92}U^{234}$	4.717	28	5.292
	4.594	0.3	0.550
	4.397	0	0[b]
	4.499	73	8.544
$_{92}U^{236}$	4.449	27	5.196
	4.339	0.5	0.71
energy	-----	0	0[b]
	5.763	69	8.306
$_{94}Pu^{236}$	5.716	31	5.567
	5.610	0.2	0.4472
	5.442	0.002	0.0447
	5.495	72	8.485
$_{94}Pu^{238}$	5.452	28	5.292
	5.352	0.09	0.300
	5.204	0.004	0.0632
	5.162	76	8.718
$_{94}Pu^{240}$	5.118	24	4.899
	5.014	0.1	0.316
	4.851	0.003	0.055
$_{94}Pu^{242}$	4.898	76	8.718
	4.858	24	4.899
	6.110	73.7	8.585
$_{96}Cm^{242}$	6.066	26.3	5.130
	5.965	0.035	0.187
	5.806	0.006	0.0775
	5.802	76.7	8.758
$_{96}Cm^{244}$	5.760	23.3	4.83
	5.662	0.017	0.13
	5.510	0.004	0.0632
	6.753	78	8.832
$_{98}Cf^{246}$	6.711	22	4.690
	6.615	0.16	0.4
	6.469	0.015	0.123
	6.024	83	9.11
$_{98}Cf^{250}$	5.980	17	4.120
	6.112	82	9.055
$_{98}Cf^{252}$	6.069	15	3.872
	5.969	0.2	0.447
	5.811	0	0[b]
	7.20	85	9.22
$_{100}Fm^{254}$	7.16	15	3.872
	7.064	0.4	0.632
	6.96	0	0[b]

[a]The difference in velocity of the various waves was ignored in calculating these boundary conditions.

[b]No α-intensity given.

Table II

Results

The first two columns of the data compilation give the charge Z and mass A of the parent nucleus. The next four columns give the relative amplitudes of the alpha partial waves on the spherical surface given by $R=R_0A^{1/3}$(where R_0 is 1.45×10^{-13} cm). The amplitudes are given for all eight choices of phase, and three values of the intrinsic quadrupole moment, Q_0, are used in the calculation for each phase choice.

		Relative phase			
		Plus	Plus	Plus	Plus
Z	A	$\ell = 0$	$\ell = 2$	$\ell = 4$	$\ell = 6$
$Q_0 = 8$					
90	228	1.0000	1.5541	0.8884	0.2156
90	230	1.0000	1.4577	0.9354	0.2215
92	230	1.0000	1.5048	0.7491	0.1779
92	232	1.0000	1.4742	0.7948	0.1773
92	234	1.0000	1.4113	0.8359	0.1838
92	236	1.0000	1.4378	1.0522	0.2368
94	236	1.0000	1.3350	0.5678	0.3233
94	238	1.0000	1.2568	0.4598	0.3757
94	240	1.0000	1.2045	0.4874	0.4456
94	242	1.0000	1.1671	0.2095	0.0183
96	242	1.0000	1.1400	0.3409	0.3160
96	244	1.0000	1.1241	0.3053	0.2867
98	246	1.0000	1.0819	0.4245	0.3763
98	250	1.0000	0.9750	0.1693	0.0151
98	252	1.0000	0.9640	0.4155	0.0820
100	254	1.0000	0.9282	0.4403	0.0913
$Q_0 = 11$					
90	228	1.0000	1.6016	0.9661	0.2932
90	230	1.0000	1.5162	1.0028	0.2994
92	230	1.0000	1.5527	0.8291	0.2451
92	232	1.0000	1.5242	0.8676	0.2423
92	234	1.0000	1.4688	0.9021	0.2498
92	236	1.0000	1.5002	1.1090	0.3175
94	236	1.0000	1.3923	0.6512	0.3605
94	238	1.0000	1.3169	0.5443	0.4014
94	240	1.0000	1.2702	0.5694	0.4723
94	242	1.0000	1.2227	0.2833	0.0341
96	242	1.0000	1.2078	0.4234	0.3360
96	244	1.0000	1.1910	0.3851	0.3041
98	246	1.0000	1.1622	0.5057	0.4048
98	250	1.0000	1.0489	0.2338	0.0288
98	252	1.0000	1.0504	0.4770	0.1175
100	254	1.0000	1.0236	0.5037	0.1303
$Q_0 = 14$					
90	228	1.0000	1.6446	1.0410	0.3716
90	230	1.0000	1.5691	1.0685	0.3770
92	230	1.0000	1.5965	0.9064	0.3146
92	232	1.0000	1.5696	0.9382	0.3084
92	234	1.0000	1.5210	0.9668	0.3162
92	236	1.0000	1.5563	1.1647	0.3964
94	236	1.0000	1.4454	0.7323	0.4017
94	238	1.0000	1.3729	0.6268	0.4314
94	240	1.0000	1.3315	0.6499	0.5028
94	242	1.0000	1.2749	0.3564	0.0551
96	242	1.0000	1.2712	0.5052	0.3610
96	244	1.0000	1.2539	0.4644	0.3263
98	246	1.0000	1.2374	0.5871	0.4374
98	250	1.0000	1.1188	0.2999	0.0475
98	252	1.0000	1.1318	0.5405	0.1564
100	254	1.0000	1.1134	0.5698	0.1731

		Relative phase			
		Plus	Plus	Minus	Plus
Z	A	$\ell = 0$	$\ell = 2$	$\ell = 4$	$\ell = 6$
$Q_0 = 8$					
90	228	1.0000	1.4181	0.2691 -	0.1530 -
90	230	1.0000	1.3030	0.4027 -	0.1721 -
92	230	1.0000	1.3947	0.1543 -	0.1158 -
92	232	1.0000	1.3511	0.2420 -	0.1256 -
92	234	1.0000	1.2748	0.3363 -	0.1398 -
92	236	1.0000	1.2510	0.5682 -	0.1962 -
94	236	1.0000	1.2630	0.0232 -	0.1562
94	238	1.0000	1.2088	0.0623	0.2674
94	240	1.0000	1.1480	0.0120	0.3194
94	242	1.0000	1.1671	0.2095	0.0183
96	242	1.0000	1.1147	0.1351	0.2590
96	244	1.0000	1.1052	0.1496	0.2450
98	246	1.0000	1.0327	0.0335	0.2671
98	250	1.0000	0.9750	0.1693	0.0151
98	252	1.0000	0.9020	0.0954 -	0.0536 -
100	254	1.0000	0.8565	0.1241 -	0.0622 -
$Q_0 = 11$					
90	228	1.0000	1.4321	0.1511 -	0.1796 -
90	230	1.0000	1.3215	0.2971 -	0.2095 -
92	230	1.0000	1.4154	0.0442 -	0.1320 -
92	232	1.0000	1.3699	0.1367 -	0.1481 -
92	234	1.0000	1.2965	0.2386 -	0.1695 -
92	236	1.0000	1.2641	0.4679 -	0.2440 -
94	236	1.0000	1.3015	0.0749	0.1442
94	238	1.0000	1.2558	0.1551	0.2606
94	240	1.0000	1.1979	0.1022	0.3076
94	242	1.0000	1.2227	0.2833	0.0341
96	242	1.0000	1.1754	0.2206	0.2614
96	244	1.0000	1.1668	0.2315	0.2493
98	246	1.0000	1.0989	0.1184	0.2613
98	250	1.0000	1.0489	0.2338	0.0288
98	252	1.0000	0.9695	0.0335 -	0.0630 -
100	254	1.0000	0.9299	0.0620 -	0.0743 -
$Q_0 = 14$					
90	228	1.0000	1.4481	0.0405 -	0.1929 -
90	230	1.0000	1.3412	0.1973 -	0.2352 -
92	230	1.0000	1.4372	0.0599	0.1361 -
92	232	1.0000	1.3899	0.0373 -	0.1597 -
92	234	1.0000	1.3188	0.1456 -	0.1892 -
92	236	1.0000	1.2790	0.3733 -	0.2809 -
94	236	1.0000	1.3388	0.1694	0.1412
94	238	1.0000	1.3009	0.2453	0.2617
94	240	1.0000	1.2458	0.1903	0.3030
94	242	1.0000	1.2749	0.3564	0.0551
96	242	1.0000	1.2328	0.3052	0.2703
96	244	1.0000	1.2251	0.3127	0.2596
98	246	1.0000	1.1619	0.2034	0.2625
98	250	1.0000	1.1188	0.2999	0.0475
98	252	1.0000	1.0341	0.0308	0.0664 -
100	254	1.0000	1.0003	0.0029	0.0799 -

		Relative phase			
		Plus	Minus	Plus	Minus
Z	A	$\ell = 0$	$\ell = 2$	$\ell = 4$	$\ell = 6$
$Q_0 = 8$					
90	228	1.0000	1.7168	0.5373	0.2771
90	230	1.0000	1.4269	0.7083	0.2861
92	230	1.0000	1.6578	0.3310	0.2095
92	232	1.0000	1.5605	0.4628	0.2179
92	234	1.0000	1.3735	0.5858	0.2284
92	236	1.0000	1.3348	0.9460	0.3154
94	236	1.0000	1.3202	0.0876	0.2452
94	238	1.0000	1.1906	0.0490	-0.4062
94	240	1.0000	1.0631	0.0256	0.4675
94	242	1.0000	1.1017	0.2661	-0.0232
96	242	1.0000	0.9800	0.1509	-0.3747
96	244	1.0000	0.9612	0.1714	-0.3491
98	246	1.0000	0.8024	0.0009	0.3720
98	250	1.0000	0.7132	0.1859	-0.0166
98	252	1.0000	0.5943	0.1662	0.0739
100	254	1.0000	0.4914	0.2069	0.0849
$Q_0 = 11$					
90	228	1.0000	1.9144	0.4715	0.4187
90	230	1.0000	1.5204	0.6958	0.4286
92	230	1.0000	1.8500	0.2304	0.3098
92	232	1.0000	1.7097	0.4007	0.3243
92	234	1.0000	1.4591	0.5589	0.3388
92	236	1.0000	1.3917	0.9798	0.4734
94	236	1.0000	1.4205	0.0357	-0.2636
94	238	1.0000	1.2637	0.1826	-0.4607
94	240	1.0000	1.1025	0.0847	-0.5167
94	242	1.0000	1.1633	0.3968	-0.0478
96	242	1.0000	1.0134	0.2765	-0.4313
96	244	1.0000	0.9906	0.2925	-0.4028
98	246	1.0000	0.7935	0.0907	-0.4073
98	250	1.0000	0.7008	0.2671	-0.0329
98	252	1.0000	0.5479	0.1315	0.1017
100	254	1.0000	0.4267	0.1817	0.1180
$Q_0 = 14$					
90	228	1.0000	2.1911	0.3815	0.5985
90	230	1.0000	1.6419	0.6813	0.6030
92	230	1.0000	2.1195	0.0930	0.4314
92	232	1.0000	1.9120	0.3193	0.4542
92	234	1.0000	1.5697	0.5262	0.4714
92	236	1.0000	1.4623	1.0236	0.6660
94	236	1.0000	1.5545	0.1925	-0.2970
94	238	1.0000	1.3604	0.3482	-0.5369
94	240	1.0000	1.1548	0.2172	-0.5827
94	242	1.0000	1.2433	0.5547	-0.0858
96	242	1.0000	1.0589	0.4268	-0.5070
96	244	1.0000	1.0306	0.4361	-0.4740
98	246	1.0000	0.7867	0.1957	-0.4539
98	250	1.0000	0.6894	0.3576	-0.0566
98	252	1.0000	0.4965	0.0964	0.1301
100	254	1.0000	0.3547	0.1583	0.1527

		Relative Phase							
		Plus	Minus		Minus		Minus	
Z	A	$\ell=0$	$\ell=2$		$\ell=4$		$\ell=6$	
$Q_0=8$								
90	228	1.0000	2.0118	-	1.5490	-	0.3846	-
90	230	1.0000	1.7284	-	1.5163	-	0.3663	-
92	230	1.0000	1.8933	-	1.2802	-	0.3120	-
92	232	1.0000	1.8119	-	1.3218	-	0.3018	-
92	234	1.0000	1.6326	-	1.3229	-	0.2969	-
92	236	1.0000	1.6865	-	1.6749	-	0.3830	-
94	236	1.0000	1.4557	-	0.8685	-	0.5186	-
94	238	1.0000	1.2760	-	0.6641	-	0.5768	-
94	240	1.0000	1.1584	-	0.6819	-	0.6594	-
94	242	1.0000	1.1017	-	0.2661	-	0.0232	-
96	242	1.0000	1.0221	-	0.4540	-	0.4600	-
96	244	1.0000	0.9919	-	0.3970	-	0.4104	-
98	246	1.0000	0.8806	-	0.5546	-	0.5300	-
98	250	1.0000	0.7132	-	0.1859	-	0.0166	-
98	252	1.0000	0.6839	-	0.5084	-	0.1048	-
100	254	1.0000	0.5936	-	0.5306	-	0.1153	-
$Q_0=11$								
90	228	1.0000	2.4274	-	2.1179	-	0.6653	-
90	230	1.0000	2.0221	-	1.9731	-	0.6078	-
92	230	1.0000	2.2571	-	1.7621	-	0.5414	-
92	232	1.0000	2.1360	-	1.7709	-	0.5121	-
92	234	1.0000	1.8859	-	1.7122	-	0.4898	-
92	236	1.0000	1.9695	-	2.1290	-	0.6256	-
94	236	1.0000	1.6429	-	1.1736	-	0.6966	-
94	238	1.0000	1.4009	-	0.9030	-	0.7272	-
94	240	1.0000	1.2532	-	0.9019	-	0.8130	-
94	242	1.0000	1.1633	-	0.3968	-	0.0478	-
96	242	1.0000	1.0797	-	0.6261	-	0.5624	-
96	244	1.0000	1.0386	-	0.5510	-	0.4966	-
98	246	1.0000	0.9146	-	0.7251	-	0.6478	-
98	250	1.0000	0.7008	-	0.2671	-	0.0329	-
98	252	1.0000	0.6829	-	0.6204	-	0.1631	-
100	254	1.0000	0.5800	-	0.6401	-	0.1779	-
$Q_0=14$								
90	228	1.0000	3.0635	-	2.9673	-	1.1087	-
90	230	1.0000	2.4425	-	2.6080	-	0.9602	-
92	230	1.0000	2.8055	-	2.4689	-	0.9011	-
92	232	1.0000	2.6105	-	2.4103	-	0.8300	-
92	234	1.0000	2.2416	-	2.2417	-	0.7663	-
92	236	1.0000	2.3687	-	2.7510	-	0.9711	-
94	236	1.0000	1.9029	-	1.5796	-	0.9471	-
94	238	1.0000	1.5694	-	1.2080	-	0.9300	-
94	240	1.0000	1.3791	-	1.1754	-	1.0132	-
94	242	1.0000	1.2433	-	0.5547	-	0.0858	-
96	242	1.0000	1.1570	-	0.8357	-	0.6963	-
96	244	1.0000	1.1012	-	0.7362	-	0.6085	-
98	246	1.0000	0.9625	-	0.9285	-	0.7971	-
98	250	1.0000	0.6894	-	0.3576	-	0.0566	-
98	252	1.0000	0.6861	-	0.7471	-	0.2356	-
100	254	1.0000	0.5690	-	0.7629	-	0.2550	-

		Relative phase			
		Plus	Plus	Plus	Minus
Z	A	$\ell = 0$	$\ell = 2$	$\ell = 4$	$\ell = 6$
$Q_0 = 8$					
94	236	1.0000	1.3331	0.5217	0.1090 -
94	238	1.0000	1.2543	0.3958	0.2254 -
94	240	1.0000	1.2015	0.4117	0.2810 -
96	242	1.0000	1.1377	0.2834	0.2198 -
96	244	1.0000	1.1219	0.2527	0.2086 -
98	246	1.0000	1.0791	0.3579	0.2294 -
$Q_0 = 11$					
94	236	1.0000	1.3891	0.5915	0.0572 -
94	238	1.0000	1.3125	0.4610	0.1825 -
94	240	1.0000	1.2650	0.4705	0.2361 -
96	242	1.0000	1.2036	0.3480	0.1880 -
96	244	1.0000	1.1873	0.3159	0.1810 -
98	246	1.0000	1.1572	0.4180	0.1900 -
$Q_0 = 14$					
94	236	1.0000	1.4405	0.6604	0.0024 -
94	238	1.0000	1.3662	0.5260	0.1360 -
94	240	1.0000	1.3236	0.5298	0.1879 -
96	242	1.0000	1.2649	0.4134	0.1513 -
96	244	1.0000	1.2482	0.3800	0.1487 -
98	246	1.0000	1.2298	0.4801	0.1460 -

		Relative phase			
		Plus	Plus	Minus	Minus
Z	A	$\ell = 0$	$\ell = 2$	$\ell = 4$	$\ell = 6$
$Q_0 = 8$					
94	236	1.0000	1.2612	0.0696 -	0.2781 -
94	238	1.0000	1.2062	0.0019 -	0.3355 -
94	240	1.0000	1.1450	0.0640 -	0.4097 -
96	242	1.0000	1.1123	0.0774	0.2777 -
96	244	1.0000	1.1031	0.0969	0.2510 -
98	246	1.0000	1.0299	0.0334 -	0.3405 -
$Q_0 = 11$					
94	236	1.0000	1.2982	0.0145	0.2770 -
94	238	1.0000	1.2513	0.0712	0.3265 -
94	240	1.0000	1.1927	0.0025	0.4053 -
96	242	1.0000	1.1712	0.1449	0.2641 -
96	244	1.0000	1.1630	0.1621	0.2368 -
98	246	1.0000	1.0939	0.0300	0.3369 -
$Q_0 = 14$					
94	236	1.0000	1.3338	0.0963	0.2681 -
94	238	1.0000	1.2940	0.1434	0.3110 -
94	240	1.0000	1.2377	0.0687	0.3945 -
96	242	1.0000	1.2264	0.2128	0.2443 -
96	244	1.0000	1.2193	0.2280	0.2170 -
98	246	1.0000	1.1541	0.0950	0.3264 -

		Relative phase			
		Plus	Minus	Plus	Plus
Z	A	$\ell = 0$	$\ell = 2$	$\ell = 4$	$\ell = 6$
$Q_0 = 8$					
94	236	1.0000	1.3168	- 0.1618	0.4495
94	238	1.0000	1.1861	- 0.0496	0.5189
94	240	1.0000	1.0581	- 0.1377	0.6082
96	242	1.0000	0.9761	- 0.0664	- 0.4107
96	244	1.0000	0.9577	- 0.0954	- 0.3655
98	246	1.0000	0.7980	- 0.0952	0.4844
$Q_0 = 11$					
94	236	1.0000	1.4127	- 0.0806	0.5469
94	238	1.0000	1.2537	- 0.0302	- 0.6043
94	240	1.0000	1.0917	- 0.0863	0.7055
96	242	1.0000	1.0049	- 0.1477	- 0.4606
96	244	1.0000	0.9831	- 0.1771	- 0.4045
98	246	1.0000	0.7841	- 0.0514	0.5543
$Q_0 = 14$					
94	236	1.0000	1.5391	- 0.0211	- 0.6609
94	238	1.0000	1.3411	- 0.1269	- 0.7016
94	240	1.0000	1.1343	- 0.0271	0.8160
96	242	1.0000	1.0429	- 0.2428	- 0.5130
96	244	1.0000	1.0166	- 0.2721	- 0.4440
98	246	1.0000	0.7691	- 0.0042	0.6302

		Relative Phase			
		Plus	Minus	Minus	Plus
Z	A	L=0	L=2	L=4	L=6
$Q_0 = 8$					
94	236	1.0000	1.4522	- 0.7935	- 0.1812
94	238	1.0000	1.2714	- 0.5649	- 0.3526
94	240	1.0000	1.1534	- 0.5690	- 0.4218
96	242	1.0000	1.0182	- 0.3693	- 0.3273
96	244	1.0000	0.9885	- 0.3208	- 0.3054
98	246	1.0000	0.8761	- 0.4598	- 0.3303
$Q_0 = 11$					
94	236	1.0000	1.6349	- 1.0549	- 0.1270
94	238	1.0000	1.3908	- 0.7486	- 0.3484
94	240	1.0000	1.2421	- 0.7285	- 0.4225
96	242	1.0000	1.0711	- 0.4964	- 0.3340
96	244	1.0000	1.0311	- 0.4351	- 0.3136
98	246	1.0000	0.9050	- 0.5812	- 0.3230
$Q_0 = 14$					
94	236	1.0000	1.8865	- 1.4011	- 0.0407
94	238	1.0000	1.5494	- 0.9812	- 0.3318
94	240	1.0000	1.3579	- 0.9245	- 0.4137
96	242	1.0000	1.1407	- 0.6494	- 0.3332
96	244	1.0000	1.0870	- 0.5707	- 0.3156
98	246	1.0000	0.9444	- 0.7240	- 0.3060

IV COMPARISON WITH OTHER TREATMENTS

If we let $a_{\ell'}$ be the expansion coefficients of the Legendre functions on our spherical nuclear surface and b_ℓ be the reciprocal of the product of the hindrance factor and the centrifugal barrier reduction factor, there exists a matrix such that $b_\ell = \sum_{\ell'} k_{\ell\ell'}(B)\ a_{\ell'}$ for all phase choices of b_ℓ. Fröman has derived the elements of such matrices in his treatment of the alpha decay of spheroidal nuclei.[8] To calculate the value of B, we use Fröman's Eq. (VI-9), leaving out the term unity in the final factor and using (VI-2) to obtain his q_0. This matrix has been calculated for Cm^{242} alpha decay as well as an equivalent matrix derived by Rasmussen and Hansen from numerical integration[3] of the wave equation. We compare these two matrices to the one obtained from this treatment of the problem.

Matrix derived from numerical integration by Rasmussen and Hansen

$\ell = 0$	$\ell = 2$	$\ell = 4$	$\ell = 6$
1.015+0.0 116i	-0.1674-0.0176i	0.01166+0.00217i	-0.0005093-0.000130i
-0.2107-0.0456i	0.9542-0.00158i	-0.1195-0.00592i	0.007260+0.000679i
0.02114+0.0135i	-0.1899-0.0595i	0.9191-0.00360i	-0.1008-0.00187i
-0.001089-0.0 0216i	0.01885+0.0187i	-0.2052-0.0893i	0.9086-0.0241i

Matrix derived from Fröman's treatment

$\ell = 0$	$\ell = 2$	$\ell = 4$	$\ell = 6$
1.019	-0.193	0.014	-0.0005
-0.193	0.908	-0.158	0.014
0.014	-0.158	0.917	-0.155
-0.0005	0.014	-0.155	0.917

Matrix derived from analytic treatment of this paper

$\ell = 0$	$\ell = 2$	$\ell = 4$	$\ell = 6$
1.032	-0.1855	0.0166	-0.00121
-0.2405	0.9181	-0.1213	0.00879
0.0358	-0.2010	0.9197	-0.0979
-0.00684	0.0383	-0.2579	0.9070

It is interesting to note that the Fröman treatment gives a matrix which corresponds rather closely to the real elements of the numerical integration treatment, although Fröman's treatment necessitates a symmetric matrix, which is not the case in the numerical treatment. The present treatment corresponds more nearly to the sum of the real and imaginary parts of the matrix elements of the numerical treatment, and the matrix is not symmetric.

V APPROXIMATE METHOD FOR OBTAINING PHASE SHIFTS

Although the phase shifting due to the quadrupole interaction does not affect any experimental observables in alpha decay of even-even nuclei, it will enter into such things as alpha-gamma angular correlations and alpha angular distributions for aligned nuclei in odd-mass nuclei.

We may obtain approximate relations for the phase shifting due to the quadrupole interaction by noting a few things. The main contribution to the quadrupole phase shifting comes when $u_i'' = 0$, (i.e., near the classical turning point). When this is the case, the functions u_i may be represented by straight lines. On the other hand, the u_i's are also representable as Coulomb functions in this region. Making use of this information, we may estimate the quadrupole phase shifting. We set $u_i = \alpha_i (r - r_i)$, where α_i is the square root of the intensity and r_i is the point at which the Coulomb functions of this E and ℓ would have its node. As the formulae are well known for the phase shifting due to angular momentum, we treat this

interaction as a pseudo angular momentum. We then set

$$\ell'(\ell' + 1) = \ell\ (\ell + 1) + 2m/\hbar^2\ r_i^0\, Q_0\ e^2 \sum_j a_{ij}(u_j/u_i)\ ;$$

evaluating the second term at the point where $u_i'' = 0$, i.e., r_i^0, we obtain

$$2m\ Q_0/\hbar^2 r_i^0\ \ e^2 \sum_j a_{ij}(a_j/a_i)(r_i^0 - r_j)/(r_i^0 - r_i)\ .$$

To determine the points r_i and r_i^0, we make use of the relations given for irregular Coulomb functions[9] at the turning point,

$$\frac{G'_\ell}{G_\ell} = -\ \frac{\Gamma\ (2/3)}{\Gamma\ (1/3)}\ \left(\frac{\rho_\ell}{3}\right)^{-1/3} \left[1 + \frac{\ell(\ell + 1)}{\rho_\ell^{\ 2}}\right]^{1/3}\ , \qquad (6)$$

and since the functions u_i are also straight lines in this region, $G'_i = a_i/k_i$. We calculate $\rho_{\ell'}$ (the point at which $u''_i = 0$) by the relation[9]

$$\rho_\ell = \eta + \sqrt{\eta^2 + \ell(\ell + 1)}\ . \qquad (7)$$

We use an iterative procedure. First we calculate ρ_ℓ neglecting the quadrupole interaction; we then put in the quadrupole interaction to calculate $\ell'(\ell' + 1)$ and then calculate $\rho_{\ell'}$. We then calculate another value for $\ell'(\ell' + 1)$ and continue until $\rho_{\ell'}$ does not change. $\rho_{\ell'} - \rho_{\ell'}$ = quadrupole phase shift.

We may compare these phase shifts with those obtained through the detailed calculations by Rasmussen and Hansen (R and H)[3] for Cm^{242}, assuming all alpha waves in phase.

Quadrupole phase shifts

	R and H	Approximation		R and H	Approximation
θ_0	-0.009	-0.03	$\theta_0 - \theta_2$	0.051	0.07
θ_2	-0.060	-0.10	$\theta_0 - \theta_4$	0.67	0.67
θ_4	-0.68	-0.70	$\theta_0 - \theta_6$	0.066	+0.06
θ_6	-0.075	-0.09			

We note that the agreement is fairly good for the differences in phase shift. It is the cosine of the difference in phase shift which enters into angular-correlation experiments and the approximation seems useful for calculations of correlations for odd-mass nuclei.

VI SUMMARY

Through approximate methods, we are able to obtain information concerning the alpha decay of deformed nuclei which was heretofore obtainable only through detailed numerical integrations. We have developed approximate methods for calculating both alpha partial-wave amplitudes at the nuclear surface and δ phase shifting caused by nuclear quadrupole deformation.

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