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UNSTABLE PARTICLES AS TARGETS IN SCATTERING EXPERIMENTS

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ABSTRACT

A general method is suggested for analyzing the scattering of particle A by particle B, leading to three or more final particles, in order to obtain the cross section for the interaction of A with a particle which is virtually contained in B. Binding complications are absent if a plausible assumption about the location and residues of poles in the S-matrix is accepted. The method is useful for unstable particles from which free targets cannot be made; the special examples of pion and neutron targets are discussed in detail.

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I. INTRODUCTION AND RESULTS

A. The importance of measuring cross sections for such interactions as pion-pion, neutron-neutron, pion-neutron, electron-pion, etc. has long been recognized but no feasible way has been found for making targets from pions or neutrons. Deuteron targets have often been used with various subtraction procedures to give rough values for neutron cross sections but complications due to the presence of the unwanted proton have made precise interpretation impossible. Similarly it has been recognized that virtual pions in the cloud associated with physical nucleons might in some approximation be considered as targets, but here, even more than in the deuteron case, binding effects have obscured the desired two-body interactions. The purpose of this paper is to present a scheme for analyzing experiments with "complex" targets so as to obtain the elementary cross sections of target constituents, free from binding corrections.

The essential physical principle employed relates to the location and residue of poles in the scattering matrix. The existence of these

poles can be proved in local field theory, and the connection of the residues to physically measurable quantities may be made very plausible, although proofs have not yet been given for all interesting cases. Well known examples are the pole in the forward angular distribution for Coulomb scattering and the Weiszacker-Williams pole in the electron momentum transfer for processes induced by high energy electrons.[1] The location of other singularities of the S matrix, such as branch points, is of indirect importance to our scheme and here we resort to guesswork. In Section II of this paper the nonrelativistic deuteron problem is analyzed, to illustrate with a concrete example various essential aspects of our program.

B. From a practical standpoint the problem turns out to be largely one of choosing the right variables to represent the experimental results. To ensure that experimenters are not led by unfamiliarity with S-matrix theory to overlook the utility of the scheme, we present at once our prescription for analyzing experiments of the type,

$$A + B \rightarrow \text{three or more particles},$$

so as to obtain the cross section for the interaction of A with some constituent of B. An example might be the process

$$\text{(a)} \qquad \pi^+ + p \rightarrow p + \text{at least two mesons},$$

with the object of determining the total (π^+, π^0 cross section), or possibly

$$\text{(b)} \qquad n + d \rightarrow p + n + n,$$

with the object of determining the cross section for

$$n + n \rightarrow n + n.$$

One must in general deal with four masses. First there is the mass of the incident particle, which we shall call μ_1; then there is the mass of the "complex" target particle, M_1, and finally the two masses into which the target can virtually decompose. We shall call μ_2 the mass of the particle whose cross section is of interest, while M_2 refers to the recoil or "spectator" particle. For example, in case (a) above, we have

$$\mu_1 = \mu_2 = m_\pi,$$

and

$$M_1 = M_2 = M_p,$$

while in case (b) we have

$$\mu_1 = \mu_2 = M_n,$$

and

$$M_1 = M_d, \qquad M_2 = M_p.$$

The first experimental variable of interest will be called Δ^2 and is the invariant square of the difference of four momenta for the target (M_1) and spectator (M_2) particles. The laboratory kinetic energy of the recoiling spectator particle we call T_{2L}. Then we see that a linear relation holds between Δ^2 and T_{2L} (we use units in which $c = 1$):

$$\Delta^2 = 2M_1 T_{2L} - (M_1 - M_2)^2. \tag{1.1}$$

It is in fact convenient to use rather than Δ^2 a quantity $p^2 = 2M_2 T_{2L}$ which nonrelativistically is the square of the laboratory recoil momentum. Evidently the following relation is true:

$$p^2 = \frac{M_2}{M_1}\left[\Delta^2 + (M_1 - M_2)^2\right] . \tag{1.1'}$$

The second variable of prime interest will be called w^2 and is the square of the total energy of all the outgoing particles--excluding the spectator--in their barycentric system. If the angle as well as the momentum of the recoiling spectator is measured, w^2 can be calculated directly from energy-momentum conservation:

$$w^2 = (\omega_{1L} + M_1 - M_2 - T_{2L})^2 - (q_{1L}^2 - 2q_{1L}\, p_{2L} \cos \theta_L + p_{2L}^2). \qquad (1.2)$$

Here ω_{1L} and q_{1L} respectively are the total laboratory energy and momentum of the incident particle, while p_{2L} is the laboratory momentum of the recoiling spectator and θ_L its angle with respect to the incident-beam direction. Thus we write

$$M_2 + T_{2L} = \sqrt{p_{2L}^2 + M_2^2}$$

and

$$\omega_{1L} = \sqrt{q_{1L}^2 + \mu_1^2} .$$

Our method of analyzing the scattering experiment so as to obtain the total cross section* for the interaction of μ_1 with μ_2 requires a determination of the two-dimensional distribution,

$$\frac{\partial^2 \sigma(w^2, p^2)}{\partial w^2 \, \partial p^2} ,$$

which can be obtained through (1.1) and (1.2) if one measures the energy and angle distribution of the recoiling spectator in the laboratory system.

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* The procedure for determining differential cross sections will be described below.

To calculate the limits on the possible values of the variables w^2 and p^2 it is best to consider the over-all barycentric system, where we designate the total energy by W. The relation between W and the laboratory energy of the incident particle is

$$W = \sqrt{2M_1 \omega_{1L} + M_1^2 + \mu_1^2} , \tag{1.3}$$

and the upper limit of the variable w is $W - M_2$. The lower limit on w is the sum of the two smallest masses which can occur in the final state in addition to the spectator particle.

The upper and lower limits on p^2 depend on both W and w and require a slightly involved but straightforward calculation. In the over-all barycentric system let the recoil spectator energy be designated by E_2. One may easily show, then, that this relationship holds true

$$E_2 = \frac{W^2 + M_2^2 - w^2}{2W} . \tag{1.4}$$

Similarly in this same system, we designate the energy of the original target particle by E_1, so that

$$E_1 = \frac{W^2 + M_1^2 - \mu_1^2}{2W} . \tag{1.5}$$

Let the corresponding momenta be called $\vec{P}_2$ and $\vec{P}_1$. Then by definition,

$$\begin{aligned} \frac{M_1}{M_2} p^2 &= \Delta^2 + (M_1 - M_2)^2 \\ &= (\vec{P}_1 - \vec{P}_2)^2 - (E_1 - E_2)^2 + (M_1 - M_2)^2 \\ &= 2E_1E_2 - 2M_1M_2 + 2 \cos \theta \sqrt{(E_1^2 - M_1^2)(E_2^2 - M_2^2)} , \end{aligned} \tag{1.6}$$

where θ is the recoil angle in the barycentric system and ranges from 0 to 180°. Formula (1.6), together with (1.4) and (1.5), gives the range of p^2 for fixed values of W and w.

As an example of the above kinematical considerations, Fig. 1 shows the allowed regions of the (w^2, p^2) plane for case (a) with $W = 1.5\ M_p$ and $2.5\ M_p$. These total barycentric energies correspond to laboratory kinetic energies for the incident pion of about 0.45 and 2.6 Bev, respectively. A phase-space diagram for case (b) above is shown in Fig. 2. The importance of these phase-space diagrams to our scheme is discussed below.

C. Let us assume that for some range of w^2 and p^2 at a fixed total energy the differential cross section in these variables has been determined. Our method then prescribes that the following function be constructed:

$$F(w^2, p^2) = 2\pi \left(\frac{M_1}{M_2}\right)^2 \frac{q_{1L}^{\ 2}(p^2 - p_0^{\ 2})^2}{\sqrt{\frac{w^4}{4} - \frac{w^2}{2}(\mu_1^{\ 2} + \mu_2^{\ 2}) + \frac{1}{4}(\mu_1^{\ 2} - \mu_2^{\ 2})^2}} \frac{\partial^2 \sigma}{\partial p^2\, \partial w^2}, \tag{1.7}$$

where

$$p_0^{\ 2} = -\frac{M_2}{M_1}\left[\mu_2^{\ 2} - (M_2 - M_1)^2\right].$$

This formula will be motivated below in Section III, where it will be shown that for fixed w^2, if $\frac{\partial^2 \sigma}{\partial p^2\, \partial w^2}$ is extended to negative values of p^2, it has a second order pole at $\Delta^2 = -\mu_2^{\ 2}$ or $p^2 = p_0^{\ 2}$ and that the residue of this pole is directly related to the total cross section for the scattering of the incident particle by the particle of mass μ_2, at

total energy w in the barycentric system of these two particles. From the way we have constructed $F(w^2, p^2)$, it is clear that its value at $p^2 = p_0^2$ is essentially the residue in question. Because of final-state interactions involving the spectator it is expected that other singularities of $F(w^2, p^2)$ will appear in the neighborhood of $p^2 = p_0^2$, but nothing as important as the pole of interest. Therefore we believe that by extrapolation from the physical region it should be possible to determine $F(w^2, p_0^2)$.

From formula (1.1') it may be seen that we are speaking of an extrapolation in the recoil-spectator kinetic energy to the point

$$T_{2L}^{o} = -\frac{1}{2M_1}\left[\mu_2^2 - (M_1 - M_2)^2\right] , \tag{1.8}$$

which is always negative if the original target particle and its two virtual components are stable, that is, if we have

$$M_1 < M_2 + \mu_2, \quad M_2 < M_1 + \mu_2 \quad \text{and} \quad \mu_2 < M_1 + M_2.$$

In the physical region, T_{2L} is of course always positive, so an extrapolation over an interval at least equal to T_{2L}^{o} is required. However for case (a), which measures pion-pion scattering, this interval is only 10 Mev, while for case (b), which measures neutron-neutron scattering, it is only 1 Mev.

According to (1.6) the physical phase-space lower limit on p^2 approaches zero at

$$w^2 = w_0^2 = W^2 + M_2^2 - \frac{M_2}{M_1}(W^2 + M_1^2 - \mu_1^2) \tag{1.9}$$

and in the neighborhood of this point behaves quadratically:

$$p_{min}^2 \approx \frac{M_1^2}{(W^2 + M_1^2 - \mu_1^2)^2 - 4W^2 M_1^2}\,(w^2 - w_0^2)^2 . \tag{1.10}$$

Clearly our proposed extrapolation procedure is most feasible in the neighborhood of $w^2 = w_0^2$. In case (a) and in general when proton targets are used as a source of pions, w_0^2 equals m_π^2; therefore the point w_0 lies outside the physical region. In the physical region the lower limit on p^2 is always greater than zero, but (1.10) shows that for

$$(w^2 - m_\pi^2) \lesssim \frac{m_\pi}{M_p}\left[(W^2 + M_p^2 - m_\pi^2)^2 - 4W^2 M_p^2\right]^{1/2} ,$$

the lower limit is no larger in order of magnitude than the extrapolation distance m_π^2. Therefore for values of w^2 in this range one may still hope to be able to carry out our prescription. It is easy to show that a scattering experiment with a <u>free</u> target pion at rest, and with the same incident-pion laboratory energy as with the proton target, would correspond to a value of w^2 in the above allowed range. Thus our method permits a study of the same energy region that could be reached if real pion targets were available.

With a deuteron target and the proton as a spectator, the point w_0 occurs in the physical range for all but the lowest bombarding energies and closely corresponds to the unique value of w that would occur with a free-neutron target at rest. The possibility of reaching w_0 in the case of a deuteron but not in that of the proton perhaps reflects the fact that the neutron contained in the deuteron is closer to being a real particle than is the pion contained in the proton.

In Section II it will be shown that for any experiment designed to measure a neutron cross section with a deuteron target, including our example (b), the value of the function $F(w^2, p^2)$ at the position of the pole is to a very good approximation

$$F(w^2, p_0^{\,2}) = \frac{4}{M_p} \left(\frac{\alpha}{1 - \alpha r_0} \right) \sigma_{12}(w) , \tag{1.11}$$

where σ_{12} is the two-body total cross section of interest, α is the inverse deuteron "radius" and r_0 is the neutron-proton triplet effective range. The position of the pole is at $p_0^{\,2} = -\alpha^2$.

For experiments designed to measure neutral-pion cross sections with a proton target, the corresponding formula is

$$F(w^2, p_0^{\,2}) = - f_{0p}^{\,2} \, \sigma_{12}(w) , \tag{1.12}$$

where f_{0p} is the coupling constant for neutral pions to protons ($f_{0p}^{\,2} \approx 0.08$). The position of the pole in this case is at $p_0^{\,2} = - m_\pi^{\,2}$. If one wishes to measure a charged-pion cross section, with a neutron recoil, one uses the charged-pion coupling constant, $f_c^{\,2} \approx 2 f_0^{\,2}$.

Notice that the extrapolated value of $F(w^2, p^2)$ is negative in pion cross-section experiments but positive for neutron experiments. This circumstance results from the fact that a single virtual pion in the nucleon cloud must be in a P state, while the neutron in a deuteron is in a mixture of S and D states. Odd angular momentum in the complex target system in general gives rise to a negative residue for the pole in the cross section. This point will be elaborated in Section III below.

From a practical standpoint the negative residue in the pion problem is a severe disadvantage. It means that one must accurately determine not only the value of the function $F(w^2, p^2)$ in the neighborhood of $p^2 = 0$ but also at least its first derivative in order to perform the required extrapolation. There probably will be a peak in the cross section at low p^2 but this will be due to a first-order pole whose residue is not

unambiguously interpretable, since it may involve cross terms with parts of the amplitude that we cannot calculate. The effect on which we must depend is a tendency for the cross section to decrease at the last moment (as $p^2 \to 0$) as a result of the negative contribution from the second-order pole.

D. To conclude this prescription for the analysis of experiments, we generalize the foregoing to allow the determination of differential as well as total cross sections. First, when several outgoing channels are possible, there is an obvious correspondence between channels in the "elementary" reaction of interest and channels in the "complex" target reaction. For example, in our case (a) which involves the π-π interaction, if w is greater than $3m_\pi$ there may be both three-pion and two-pion final states. If one wishes to determine the purely elastic $\pi^+ - \pi^0$ cross section, the measurement should be restricted to processes of the type

$$\pi^+ + p \to p + \pi^+ + \pi^0 ,$$

excluding events in which three pions emerge, but otherwise the procedure stated above may be followed.

Should one wish to go further and measure the angular distribution for a two-body final state it is necessary to consider a variable corresponding to the barycentric angle of scattering for the two-body system of interest. The definition of this variable is not unique and will vary from problem to problem. In many cases, however, it seems natural to measure the energy ω_{3L} and momentum $\vec{q}_{3L}$ of one of the outgoing particles (say the π^+ in case (b)) and to evaluate the invariant quantity

$$q_1 \cdot q_3 = -\omega_{3L}\,\omega_{1L} + q_{1L}\,q_{3L}\cos\theta_{13L} , \qquad (1.13)$$

where θ_{13L} is the angle of the outgoing particle with respect to the incident beam in the laboratory system.

One may then consider the same invariant in the required barycentric system for particles 1 and 2, where the energy of the outgoing particle 3 is

$$\omega_{3b} = \frac{w^2 - \mu_4^2 + \mu_3^2}{2\,w}, \tag{1.14}$$

if μ_4 is the mass of the "other" particle in the reaction, $1 + 2 \rightarrow 3 + 4$. The momentum q_{3b} is of course $\sqrt{\omega_{3b}^2 - \mu_3^2}$. The energy of the incident particle in this system may be calculated if Δ^2 as well as w is known. One finds

$$\omega_{1b} = \frac{w^2 + \Delta^2 + \mu_1^2}{2\,w}, \tag{1.15}$$

and a corresponding momentum $q_{1b} = \sqrt{\omega_{1b}^2 - \mu_1^2}$. The cosine of the scattering angle in this system is then related to the invariant $q_1 \cdot q_3$ by a formula analogous to (1.13), so that one finds

$$\cos\theta_b = \frac{q_{1L}\,q_{3L}\cos\theta_{13L} + \omega_{3b}\,\omega_{1b} - \omega_{3L}\,\omega_{1L}}{q_{3b}\,q_{1b}}. \tag{1.16}$$

Thus it is possible to subdivide the events observed according to $\cos\theta_b$ and to extrapolate in p^2 at fixed θ_b in order to obtain the desired angular distribution. In Section II it will be explained that when a final-state interaction involving the spectator is important it may be necessary to avoid certain regions of the scattering angle θ_b. Since these regions are generally small, the determination of the total cross section for a given channel should not be too strongly affected by final-state interactions.

II. A NONRELATIVISTIC EXAMPLE: $N + D \rightarrow N + N + P$.

A. We consider a neutron, of momentum $\vec{q}_1$, incident on a deuteron at rest. The deuteron disintegrates, leaving a final state with two neutrons of momenta $\vec{q}_3$ and $\vec{q}_4$, and a proton of momentum $\vec{p}$. The contribution to the amplitude from the process in which the incident neutron is scattered by the neutron in the deuteron, the proton standing by as a spectator, is given by

$$a = \left\langle \vec{q}_3, \vec{q}_4 \mid T \mid -\vec{p}, \vec{q}_1 \right\rangle \phi(\vec{p}) , \tag{2.1}$$

where T is the neutron-neutron T matrix, ϕ is the Fourier-transform of the internal wave function of the deuteron, and where explicit spin functions have been omitted.

The main point of our paper is contained in the remark that $\phi(\vec{p})$ has a simple pole at $p^2 = -\alpha^2$ ($\frac{1}{\alpha}$ is the deuteron radius) whose residue is simply the normalization of the asymptotic wave function of the deuteron. The rest of the amplitude has no pole at this point. Furthermore, at $p^2 = -\alpha^2$, the T matrix is on the energy shell, so that it can yield direct information on neutron-neutron scattering. This evidently follows from the energy-conservation equations for Eq. (2.1):

$$\frac{q_1^2}{2M} - \frac{\alpha^2}{M} = \frac{p^2}{2M} + \frac{q_3^2}{2M} + \frac{q_4^2}{2M} . \tag{2.2}$$

The energy difference between the final and initial states of the T matrix in Eq. (2.1) is

$$\Delta E = \frac{q_3^2}{2M} + \frac{q_4^2}{2M} - \frac{q_1^2}{2M} - \frac{p^2}{2M} = -\frac{(\alpha^2 + p^2)}{M} , \tag{2.3}$$

so that when $p^2 = -\alpha^2$ we have $\Delta E = 0$ and the T matrix in Eq. (2.1) becomes a multiple of the neutron-neutron scattering amplitude.

The contribution of a in (2.1) to the cross section is

$$\frac{\Delta\sigma}{\Delta\tau} = \frac{4C^2}{Mq_1\pi^2} \int_\tau d\vec{p}\, d\vec{q}_3\, d\vec{q}_2\, \delta(\vec{p} + \vec{q}_3 + \vec{q}_4 - \vec{q}_1)$$

$$\delta\left(\frac{p^2}{2M} + \frac{q_3^2}{2M} + \frac{q_4^2}{2M} - \frac{q_1^2}{2M} + \frac{\alpha^2}{M}\right) \frac{|f|^2}{(p^2 + \alpha^2)^2} \quad . \tag{2.4}$$

In Eq. (2.4) the integral is to be extended over the region of interest τ. The quantity f, at $p^2 = -\alpha^2$, is the neutron-neutron scattering amplitude, that is $|f|^2$ is the neutron-neutron differential, unpolarized cross section in the center-of-mass system. The normalization of the asymptotic deuteron wave function, C, is:

$$C = \frac{2\alpha}{1 - r_0\alpha} \quad , \tag{2.5}$$

with r_0 the triplet effective range.[2] Strictly speaking, C^2 is a function of p^2, as the deuteron is not in a pure S state. In fact, one can easily show that C^2 must be replaced by $C^2(1 + (\frac{p}{\alpha})^4 \cdot \epsilon^2)$. Here $\epsilon \approx \sqrt{2}\, Q\, \alpha^2$, where Q is the deuteron quadrupole moment. Since $\epsilon \approx 0.02$, however, the difference may be safely ignored.

We do not need to take the exclusion principle explicitly into account. It is clear that for the process under consideration (N-N scattering, spectator proton) it enters only into the quantity f. For the other process of interest (N-P scattering, spectator neutron), the situation is

slightly more complicated, but the coefficient of the pole will not be affected by the exchange of the spectator and scattered particle.

In order to carry out the integrations indicated by Eq. (2.4) we introduce the variable $\vec{q} = \frac{1}{2}(\vec{q}_4 - \vec{q}_3)$, which is the final relative momentum of the scattered particles. In the notation of Section I, $w^2 = 4(q^2 + M^2)$; then we have

$$\frac{\Delta\sigma}{\Delta\tau} = \frac{4}{Mq_1\pi^2} C^2 \int_\tau d\vec{p}\, d\vec{q}\, \frac{|f|^2}{(p^2+\alpha^2)^2}\, \delta\left(\frac{p^2}{2M} + \frac{q^2}{M} + \frac{(\vec{q}_1 - \vec{p})^2}{4M} - \frac{q_1^2}{2M} + \frac{\alpha^2}{M}\right) . \tag{2.6}$$

We wish now to do the remaining integrals holding p^2, q^2 and z fixed, where

$$z = \frac{\vec{q}\cdot(\vec{q}_1 + \vec{p})}{q\,|\vec{q}_1 + \vec{p}\,|} . \tag{2.7}$$

The δ function in Eq. (2.6) shows that a measurement of the recoil energy and angle is equivalent to a measurement of q^2. The amplitude f is a function of the final relative energy, $q_f^2 = q^2$, as well as of z and the initial relative energy, q_i^2, where

$$q_i^2 = \left|\frac{\vec{q}_1 + \vec{p}}{2}\right|^2 = (p^2 + q^2 + \alpha^2). \tag{2.8}$$

At $p^2 = -\alpha^2$, we have already seen that by energy conservation q_f equals q_i, so that z approaches the scattering angle in the center-of-mass system. Thus we have

$$\int d\vec{q} = \int q^2 \, dq \, d\phi_q \, d(\cos \theta_q)$$

$$= \pi \, d(q^2) \, q \, dz \tag{2.9}$$

and

$$\int d\vec{p} = \pi \frac{d(p^2)}{q_1} \, d(\vec{p}\cdot\vec{q}_1) \, , \tag{2.10}$$

so that now we write

$$\frac{\partial^3\sigma}{\partial p^2 \, \partial q^2 \, \partial z} = \frac{4}{q_1^{\,2}} \, C^2 \, q \, \frac{|f|^2}{(p^2 + \alpha^2)^2} \, . \tag{2.11}$$

As p^2 approaches $-\alpha^2$ Eq. (2.11) becomes

$$\frac{\partial^3\sigma}{\partial p^2 \, \partial q^2 \, dz} \rightarrow \frac{4}{q_1^{\,2}} \, \frac{C^2}{(p^2 + \alpha^2)^2} \, \frac{d\sigma_{NN}}{d\Omega} \, (q, \theta_b) \, , \tag{2.12}$$

where $z = \cos \theta_b$. Integration over the variable z gives the total cross section:

$$\frac{\partial^2\sigma}{\partial p^2 \, \partial q^2} \rightarrow \frac{2q}{\pi q_1^{\,2}} \, \frac{C^2}{(p^2 + \alpha^2)^2} \, \sigma_{NN}^{\,T}(q) \, . \tag{2.13}$$

Formulas (2.12) and (2.13) yield the extrapolation procedure suggested in the first section.

We shall see in the next section that it is generally true that the distribution in energy and angle of the spectator particle extrapolates, via Eq. (2.13), to the total cross section of the other two particles at the appropriate energy even when multiple-production processes are involved.

B. We turn next to the important question of the limits on the variables p^2 and q^2. The limits on z are of course ± 1. Let us choose q^2 first. Clearly, in the center-of-mass system, we may have q^2 take all the available energy, or none of it. Therefore we write

$$0 \leq \frac{q^2}{M} \leq \frac{1}{3}\,\frac{q_1^{\,2}}{M} - \frac{\alpha^2}{M} . \tag{2.14}$$

In order to calculate the limits on p^2, we note that $p^2 = M^2(\Delta\vec{V})^2$, where $\Delta\vec{V}$ is the velocity transfer from the deuteron to the spectator proton. Since $\Delta\vec{V}$ is a Galilean invariant, we may calculate it in the over-all center-of-mass system. Let $\vec{P}_2$ be the proton recoil in this system. Then, by energy conservation, we have

$$\frac{P_2^{\,2}}{2M} + \frac{P_2^{\,2}}{4M} + \frac{q^2}{M} = \frac{1}{3}\,\frac{q_1^{\,2}}{M} - \frac{\alpha^2}{M} \tag{2.15}$$

or

$$\frac{3}{4}\,P_2^{\,2} = \frac{1}{3}\,q_1^{\,2} - \alpha^2 - q^2 .$$

The velocity transfer is $\Delta\vec{V} = \left(-\frac{\vec{P}_2}{M} + \frac{1}{3}\frac{\vec{q}}{M}\right)$ so that we obtain

$$p^2 = \left(\vec{P}_2 - \frac{1}{3}\,\vec{q}\,\right)^2 . \tag{2.16}$$

The upper and lower limits on p^2 are therefore given by $(P_2 \pm \frac{1}{3}\,q)^2$, where we have

$$P_2 = \left[\frac{4}{9}\,q_1^{\,2} - \frac{4}{3}\,(\alpha^2 + q^2)\right]^{1/2} .$$

This result is a special case of the general formula (1.6), taken in the nonrelativistic limit.

It is convenient to put these results on a plot of p^2/q_1^2 versus $(q^2 + \alpha^2)/q_1^2$ such as is shown in Fig. 2. Here the allowed region of q^2 and p^2 is included between the two lines. The point to which we must extrapolate is $p^2 = -\alpha^2$. Clearly the optimum q^2 is the one for which p^2 can take on the value zero. This occurs at $q_0^2 = \frac{q_1^2}{4} - \alpha^2$, a final center-of-mass energy which corresponds, neglecting the binding shift α^2, to the collision of the incident neutron with a neutron at rest in the laboratory. This is a second feature generally true for a deuteron target, irrespective of the particle striking the bound neutron: the minimum in the extrapolation distance, at least in the limit $\alpha^2 \to 0$, always occurs at that final energy corresponding to the fictitious two-particle collision in the laboratory.

C. The contribution of the pole at $p^2 = -\alpha^2$ to the total inelastic cross section is of the same order of magnitude as the total neutron-scattering cross section, so its effect is certain to be comparable to that of more complex processes. It is therefore probable that a successful extrapolation can be carried out in the deuteron case.

This order of magnitude may be estimated most simply by integrating Eq. (2.13) over p^2 and q^2. We have

$$\int \left(\frac{\partial^2 \sigma}{\partial q^2 \, \partial p^2} \right) dp^2 = \frac{2q}{\pi q_1^2} C^2 \left[\frac{1}{p_{min}^2 + \alpha^2} - \frac{1}{p_{max}^2 + \alpha^2} \right] \sigma_{NN}^T(q) \, . \tag{2.17}$$

If we neglect $\alpha^2/p_{max}^2 \sim \alpha^2/q_1^2$, the upper limit in Eq. (2.17) may be dropped compared to the lower. Further, if we expand $p_{min}^2(q^2)$ about q_0^2, we find, if we call $q^2 - q_0^2 = u$,

$$p_{min}^{\ 2} = (P_2 - \tfrac{1}{3} q_1)^2 \cong \frac{4u^2}{q_1^{\ 2}} . \tag{2.18}$$

The denominator of $\dfrac{1}{p_{min}^{\ 2} + \alpha^2}$ therefore limits the integral over q to values of $u^2 \lesssim \alpha^2 q_1^{\ 2}$ or $(\Delta q) \lesssim \alpha$. Assuming no violent q dependence in $\sigma_{NN}^{\ T}(q)$, we may neglect higher powers of $u/q_1^{\ 2}$ in the denominator of Eq. (2.17) and replace q by q_0 everywhere else. The remaining integral is

$$\sigma_{inel}^{\ T} \cong \int_{-\infty}^{\infty} du \frac{d\sigma}{dq^2} = \frac{2q_0\ \sigma_{NN}^{\ T}(q_0)}{\pi\ q_1^{\ 2}}\ c^2 \int_{-\infty}^{\infty} \frac{du}{(4u^2\ p_1^{\ 2}) + \alpha^2}$$

$$= \frac{c^2}{\alpha}\ \frac{q_0}{q_1}\ \sigma_{NN}^{\ T}(q_0) \approx \sigma_{NN}^{\ T}(q_0) . \tag{2.19}$$

D. We shall here discuss the residual dependence of the various terms in the production amplitude on the extrapolation variable p^2 once the pole has been removed. Of course the practicality of our scheme depends most critically on this dependence. Roughly stated, if the dependence on p^2 is too strong in the neighborhood of $p^2 \approx 0$ we will be unable to extrapolate to $p^2 = -\alpha^2$. More precisely, if there are singularities in the cross section which are closer to the physical region than the one at $p^2 = -\alpha^2$ then a polynomial extrapolation may fail.

The p^2 dependence may be divided into a part associated with those terms present in Eq. (2.1) and a part associated with other terms, such as final-state interactions. The first type is harmless, being given by the

characteristic momentum associated with the range of the nuclear potential. Thus the deuteron wave function satisfies the Schroedinger equation

$$\begin{aligned} \phi(\vec{p}) &= \frac{1}{p^2 + \alpha^2} \int V(\vec{p} - \vec{p}\,') \, \phi(\vec{p}\,') \, d\vec{p}\,' \\ &= \frac{1}{p^2 + \alpha^2} \int e^{i\vec{p}\cdot\vec{r}} \, V(\vec{r}) \, \psi(\vec{r}) d\vec{r} \, , \end{aligned} \tag{2.20}$$

and hence the singularities of the second factor are determined by the range of the potential. The proof for the dependence of the T matrix on its initial momentum is identical. Thus the dependence of (2.1) on p^2 is quite accurately given by the pole and its residue for a range of p^2 which is large compared to the extrapolation distance α^2.

The p^2 dependence of the rest of the amplitude is considerably more involved and much less favorable. One can qualitatively understand the difficulty by considering the final relative energies of the spectator and one or the other of the neutrons. These energies are

$$E_r^{\pm} = \frac{1}{4M} \left(\frac{\vec{q}_1}{2} - \frac{3\vec{p}}{2} \pm \vec{q} \right)^2 . \tag{2.21}$$

Clearly the scattering amplitude is brimming with singularities in the variables $E_r^{\pm}$, particularly in the neighborhood of $E_r = 0$. One need only recall the branch point at $E_r = 0$, the bound n-p state and the virtual-singlet state. (There are also other less obvious singularities associated with scattering by the spectator particle rather than with final-state interactions.) Since we have $\frac{q_1}{2} \approx q_0 \approx q$, we see from Eq. (2.21) that for small p the forward and backward directions will be

dangerous so that the extrapolation to the forward and backward differential cross section can probably not be carried out. As far as we have been able to determine, as long as $|z|$ is substantially smaller than 1, however, the nearest singularity in p^2 is the pole at $p^2 = -\alpha^2$, so that the extrapolation is possible in principle. Furthermore, the singularities in the forward and backward directions appear to be sufficiently weak so that, although they make an extrapolation to the differential cross section impossible at those points, they will not cause any practical difficulty in the total cross section. For example, a term in the total cross section of the form

$$\log \left[\frac{\alpha + \sqrt{p^2 + \alpha^2}}{\alpha} \right] ,$$

although it has a branch point at $p^2 + \alpha^2 = 0$, would show almost no trace of this singularity in the physical region compared to the rapidly varying term of interest, $\frac{1}{(p^2 + \alpha^2)^2}$. Calculations are being carried out on a special model to investigate these problems in more detail.

III. THE GENERAL PROBLEM

A. The central physical principle employed in this paper is the existence of poles in the S-matrix corresponding to single-particle "intermediate states." In the elastic scattering problem, $\pi + N \rightarrow \pi + N$, the fact that such poles exist in the energy variable has been rigorously proved; and it has recently been argued that for nucleon-nucleon scattering there are poles in the momentum-transfer variable. In both cases the residues of the poles are given by the renormalized pion-nucleon coupling constant. A generalization is required for the present application, and the following conjecture seems to us extremely plausible.

1. Consider an element of the S matrix corresponding to a definite total number of particles N (incoming plus outgoing) as a function of the $\frac{1}{2}N(N-3)$ independent invariants which remain after all particles are put on their mass shells and energy-momentum conservation is considered. Then, if it is possible to divide the particles involved into two groups, each of which has all the same quantum numbers (spin, charge, parity, etc.) as some single-particle state, we conjecture that there exists a pole in the S matrix at a point related to the mass of this particle. (In forming these two groups, if a particle is switched from incoming to outgoing or vice versa it is to be considered as the antiparticle with the opposite energy-momentum.) More precisely, if we choose one of the independent invariants to be P^2, the square of the total energy-momentum four vector for either group of particles, then the pole occurs at $P^2 = -m^2$, where m is the mass in question.

Consider for example pion-nucleon scattering, $\pi_1(q_1) + N_1(p_1) \rightarrow \pi_2(q_2) + N_2(p_2)$. Here one may form two groupings which lead to poles. First, the two incident particles (π_1, N_1) and the two

final particles (π_2, N_2) both can connect to a single-nucleon state, giving rise to a pole in the barycentric energy at $(p_1 + q_1)^2 = -M^2$. An alternative grouping is $(\bar{\pi}_1, N_2)$ and $(\bar{\pi}_2, N_1)$ which gives rise to a pole at $(p_1 - q_2)^2 = -M^2$. This latter variable is a combination of the conventional energy and momentum transfer. The last possible grouping, $(\pi_1, \bar{\pi}_2)$ and $(N_1, \bar{N}_2)$, has no pole associated with it if one ignores electromagnetic effects.

In nucleon-nucleon scattering we have,
$N_1(p_1) + N'_1(p'_1) \rightarrow N_2(p_2) + N'_2(p'_2)$, in which are three poles:
at $(p_1 + p'_1)^2 = -M_D^2$, corresponding to the deuteron; at $(p_1 - p_2)^2 = -m_\pi^2$ and at $(p_1 - p'_2)^2 = -m_\pi^2$, both corresponding to the pion. In pion-pion scattering there are no poles.

2. The residue of a particular pole in the S-matrix is conjectured to be given by the product of the (smaller dimensional) S-matrix elements which connect the two groups of particles to the intermediate particle on its mass shell. In the above elastic-scattering examples one is always considering groups containing two particles. The S-matrix element connecting such a group to a single particle, even though all three particles are on the mass shell, does not correspond to a physically realizable transition for stable particles. Nevertheless the matrix element may be defined by a process of analytic continuation and can be experimentally determined. It is well known for instance that for the transitions $\pi + N \rightleftarrows N$ or $N + \bar{N} \rightleftarrows \pi$ the value of the S-matrix element is essentially the pion-nucleon coupling constant. It is also known that for the transition $n + p \leftrightarrows d$ the S-matrix element is directly related to the normalization of the asymptotic wave function of the deuteron.

In this paper we are concerned with a problem where one of the groups in question contains two particles and the other three or more. As shown in Fig. 3 the smaller group consists of the complex target particle (M_1) and the spectator (M_2); the larger includes the incident particle (μ_1) and all outgoing particles except for the spectator. (We designate these outgoing particles by the symbol F.) The intermediate particle here is of mass μ_2.

Our basic conjecture is that the matrix element connecting the larger group (F + 1) to the intermediate particle on its mass shell is equal to the physical matrix element for the process $1 + 2 \rightarrow F$. A basis for this conjecture has been given above in Section II by considering a nonrelativistic deuteron problem in the impulse approximation; it can also be verified in relativistic-perturbation theory for the pion problem. We are, however, not able to give a general proof, although a proof for the case of real four-momenta has been given by Zimmermann.[3] For our purposes we require also complex four-momenta.

When we have a deuteron target $(M_1 = M_d)$ with a proton recoil $(M_2 = M_p)$ and wish to measure the neutron cross section $(\mu_2 = M_n)$, the residue of the pole in the S matrix at $\Delta^2 = - M_n^{\ 2}$ is the product of the matrix element for the process $d \rightleftarrows n + p$ with the amplitude for the incident particle (μ_1) to be scattered by the neutron. Correspondingly in the deuteron cross section there will be a second-order pole whose residue is a known multiple of the neutron cross section. Similar statements apply to the proton target when the object is the π^0 cross section. Let us now consider the calculation of explicit formulas for these residues.

B. We designate the total energy-momentum four vector of the F outgoing particles by the symbol Q while the "internal" state of these particles is labeled by the index n. The matrix element of essential interest is then

$$\left\langle Q, n \mid j_2 \mid q_1 \right\rangle , \tag{3.1}$$

where j_2 is the "current" operator associated with the particle of mass μ_2, and q_1 designates the incident particle of mass μ_1. When $(Q - q_1)^2$ equals $-\mu_2^2$ this matrix element describes the physical transition $1 + 2 \rightarrow F$. To establish a normalization, let us say that the total cross section for the scattering of particle 1 by particle 2 is

$$\sigma_{12}(w) = \frac{2\pi}{q'_1 \mu_2} \sum_n \left| \left\langle Q, n \mid j_2 \mid q_1 \right\rangle \right|^2 , \tag{3.2}$$

where $w = \sqrt{-Q^2}$, and q'_1 is the magnitude of the momentum of particle 1 in a frame where particle 2 is at rest. One may easily calculate that

$$q'_1 \mu_2 = \sqrt{\frac{w^4}{4} - \frac{w^2}{2}(\mu_1^2 + \mu_2^2) + \frac{1}{4}(\mu_1^2 - \mu_2^2)} . \tag{3.3}$$

The other matrix element that is required is

$$\left\langle p_2 \mid j_2 \mid p_1 \right\rangle , \tag{3.4}$$

where p_2 and p_1 designate the single-particle states of mass M_2 and M_1, respectively. We are interested in the case $(p_2 - p_1)^2 = -\mu_2^2$, where this matrix element is given by a single real number if all three

particles involved have spin zero. If nonzero spins occur, more than one number may be required, but we shall concern ourselves only with experiments where the initial state is unpolarized and no measurement is made of the spin of any final particle; in such a case only a spin average of the square of (3.4) need concern us. We shall call this average $4\pi\Gamma^2$ and normalize it so that for the process $p \rightarrow p + \pi^o$, we have

$$\Gamma^2 = g^2 \frac{(p_1 - p_2)^2}{4M_p^{\ 2}} = f^2 \frac{\Delta^2}{\mu^2}, \tag{3.5}$$

where $f^2 \approx 0.08$.

With the same normalization for the process $d \rightarrow n + p$, a very good approximation is given by

$$\Gamma^2 = \frac{4}{M_p} \frac{\alpha}{1 - \alpha r_0}, \tag{3.6}$$

as explained above in Section II.

The contribution to the cross section from the pole indicated in Fig. 3 may now be calculated. One finds

$$\frac{\Delta\sigma}{\Delta\tau} = \frac{2\pi}{q_{1L} M_1} \int_\tau \frac{4\pi\Gamma^2}{(\Delta_2 + \mu_2^{\ 2})} \left| \left\langle Q, n \mid j_2 \mid q_1 \right\rangle \right|^2 4M_1 M_2 \ \delta(p_2^{\ 2} + M_2^{\ 2}) \frac{d^4 p_2}{(2\pi)^3} \tag{3.7}$$

with the energy-momentum conservation condition

$$P = p_1 + q_1 = p_2 + Q \; . \tag{3.8}$$

We may now transform from p_2 to the variables of interest by observing that in the laboratory system we have

$$\Delta^2 = (p_2 - p_1)^2 = -M_2^{\,2} - M_1^{\,2} - 2E_{2L}\, M_1 \tag{3.9}$$

and

$$w^2 = -(P - p_1)^2 = W^2 + M_2^{\,2} + 2\, p_{2L}\, q_{1L} \cos \theta_L \; , \tag{3.10}$$

giving

$$\begin{aligned} \int d^4 p_2 \;\; \delta(p_2^{\,2} + M_2^{\,2}) &= \pi\, p_{2L}\, d \cos \theta_L\, dE_{2L} \\ &= \frac{\pi}{4M_1\, q_{1L}} \; d\Delta^2\, dw^2 \; . \end{aligned} \tag{3.11}$$

Remembering (3.2), we then get the final result for the limit as Δ^2 approaches $-\mu_2^{\,2}$:

$2w * k\{w; \mu_1, \mu_2\}$

$$\frac{\partial^2 \sigma}{\partial \Delta^2 dw^2} \rightarrow \frac{\Gamma^2}{2\pi} \left(\frac{M_2}{M_1}\right) \frac{\sqrt{\frac{w^4}{4} - \frac{w^2}{2}(\mu_1^{\,2} + \mu_2^{\,2}) + \frac{1}{4}(\mu_1^{\,2} - \mu_2^{\,2})}}{q_{1L}^{\,2}} \; \frac{\sigma_{12}(w)}{(\Delta^2 + \mu_2^{\,2})^2} \; , \tag{3.12}$$

which leads to the prescription given by (1.7), (1.11) and (1.12) when the relation (1.1") between p^2 and Δ^2 is used. It may easily be verified that (3.12) reduces to (2.13) in the nonrelativistic limit for a deuteron target.

C. We conclude by discussing the particular case of a proton target that is being used to determine the (π^+, π^o) cross section. Formula (3.12) here becomes

$$\frac{\partial^2\sigma}{\partial p^2\, dw^2} \xrightarrow[p^2 \to -\mu^2]{} \frac{f^2}{2\pi} \frac{p^2/\mu^2}{(p^2+\mu^2)^2} \frac{w\sqrt{\frac{w^2}{4} - \mu^2}}{q_{1L}^{\,2}} \sigma_{\pi^+\pi^o} , \tag{3.13}$$

where μ is the pion mass.

To establish the order of magnitude of the effect we may perform a rough integration of (3.13) over the allowed phase space (e.g. Fig. 1) assuming a constant value for $\sigma_{\pi\pi}$. The result for $W - M \gg \mu$ is a contribution to the total pion-nucleon cross section of the order of magnitude

$$\frac{f^2}{\pi} \left(\frac{M}{\mu}\right)^2 \left(\frac{W - M}{W + M}\right)^2 \sigma_{\pi^+\pi^o} . \tag{3.14}$$

Since $\frac{f^2}{\pi}\left(\frac{M}{\mu}\right)^2$ is of the order of magnitude unity, we see that at sufficiently high energies the full pion-pion cross section may be expected to contribute. Therefore, if $\sigma_{\pi^+\pi^o}$ is as large as 10 mb our pole should constitute an important part of the high-energy pion-nucleon interaction, since the observed total inelastic $\pi^+ - p$ cross section is only $\sim$ 20 mb., even though it includes also a (π^+, π^+) contribution, which occurs with twice the coefficient of $\sigma_{\pi^+\pi^o}$.

One may add here the qualitative remark that analyses of elastic pion-nucleon diffraction scattering in the Bev-energy range have shown a mean-square radius of the nucleon approximately equal to the charge- and magnetic-moment radii measured in the Stanford electron-scattering

experiments. This fact strongly suggests that the pion-pion interaction must be important since these large radii can only be understood in terms of a pion cloud. We expect, then, that a measurement of the type described in Section I will show a concentration of recoil protons at low kinetic energies, as predicted by formula (3.13).

Unfortunately, as stressed earlier, the magnitude of this concentration is not a quantitative measure of $\sigma_{\pi\pi}$. The difficulty is that in squaring the amplitude there will occur cross terms which lead to a first-order pole of unknown residue in the cross section. Only the second-order pole has a clearly interpretable coefficient, and in the physical region the second-order pole in (3.13) has a small and negative effect, since

$$\frac{p^2}{(p^2 + \mu^2)^2} = \frac{1}{p^2 + \mu^2} - \frac{\mu^2}{(p^2 + \mu^2)^2} . \tag{3.15}$$

In order to determine $\sigma_{\pi\pi}$ quantitatively the low-energy proton recoils must be measured with sufficient precision to determine the tendency of the cross section to decrease (or at least increase less rapidly) as p^2 approaches 0. Of course, as pointed out also by Goebel, the existence of a concentration at recoil-proton kinetic energies of the order of 10 Mev will constitute qualitative evidence for the $\pi - \pi$ interaction.[4]

In conclusion it should be emphasized that a negative experimental result would still be valuable it it gave an upper limit on the magnitude of $\sigma_{\pi\pi}$, since at present absolutely nothing is known about this cross section.

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FIGURE CAPTIONS

Fig. 1. Allowed region in p^2/M^2 vs w^2/M^2 for meson-nucleon collision at $W = 2.5$ M (2.6 Bev laboratory energy) and $W = 1.5$ M (0.45 Bev energy).

Fig. 2. Allowed region of $\frac{p^2}{q_1^2}$ and $\frac{q^2 + \alpha^2}{q_1^2}$ for neutron-deuteron collision. The lower limit on $\frac{q^2 + \alpha^2}{q_1^2}$ and the extrapolation distance below zero are both given by $\frac{\alpha^2}{q_1^2} \approx \frac{1}{E}$, where E is the neutron laboratory energy in Mev and $\vec{q}$ is the final relative momentum of the two neutrons.

Fig. 3. Diagram showing the particle groups corresponding to the pole of interest.

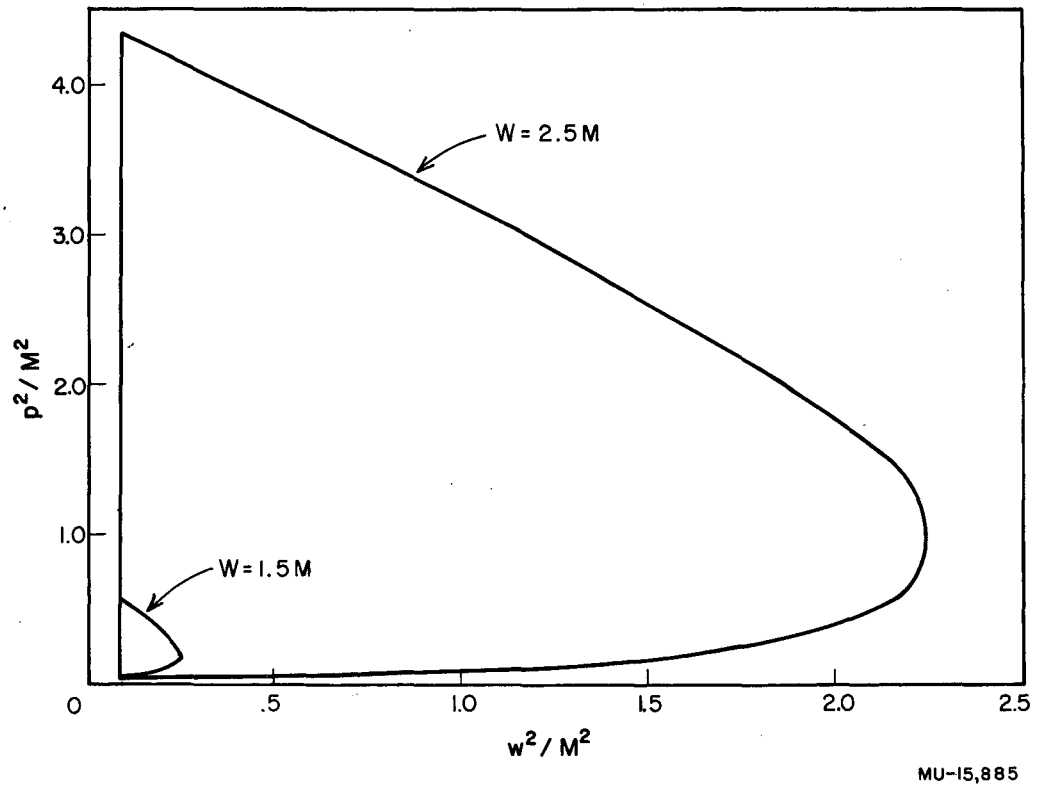

Fig. 1.

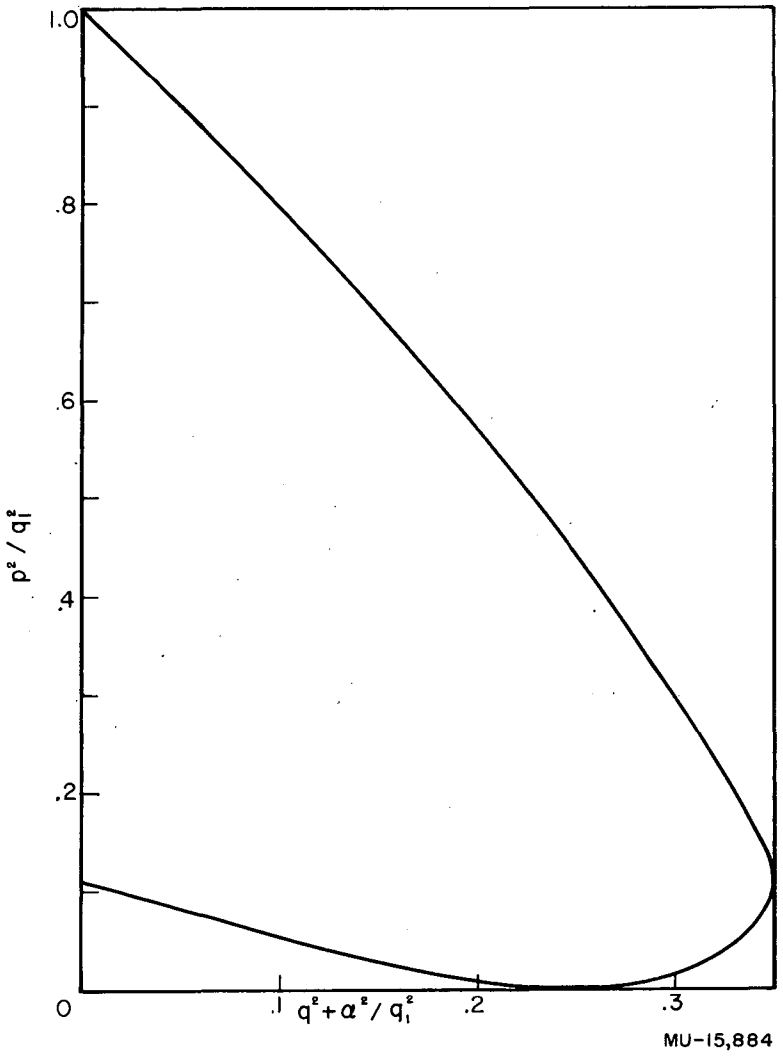

Fig. 2

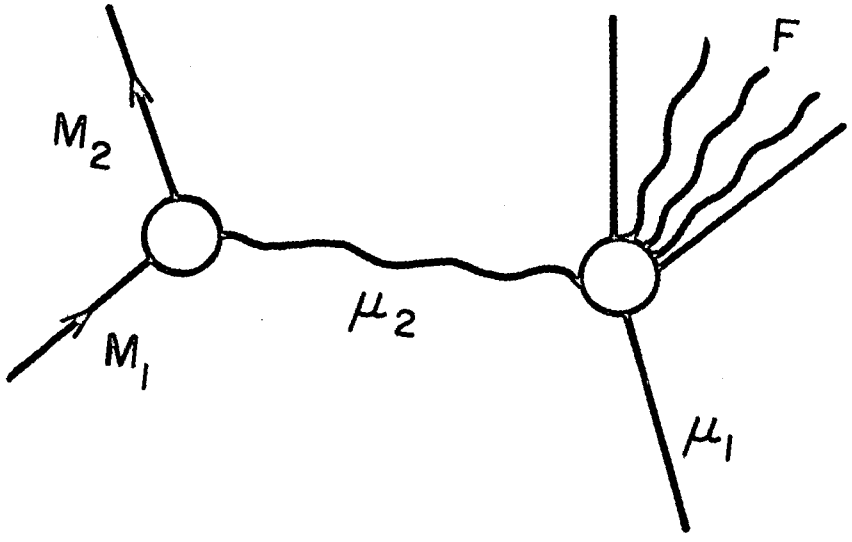

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Fig. 3