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ORBIT DYNAMICS IN THE SPIRAL-RIDGED CYCLOTRON

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Lloyd Smith and Alper A. Garren

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ABSTRACT

Formulas are derived for the equilibrium orbit, isochronous condition, vertical and horizontal betatron frequencies, and for the effects of the 3/3 radial resonance in a three-fold geometry. The magnetic field is represented by a Fourier series in azimuth with amplitudes expanded in a Taylor series about the reference radius. The form is such that the various parameters may be deduced from an arbitrary set of field measurements in the median plane and the results obtained by direct substitution in algebraic formulas.

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INTRODUCTION

There exist in the literature numerous reports on the analytic description of particle motion in magnetic accelerators with azimuthally varying fields. However, there seems to be none in which the specific case of a medium-energy cyclotron is treated in enough detail to yield quantitative conclusions starting, say, from a set of magnetic-field measurements. In this report are developed explicit formulas for isochronism, betatron frequencies, and the 3/3 resonance in the radial motion. The magnetic field in the median plane is represented by a Fourier decomposition in angle with amplitudes expressed as power series in radius. The results are carried only to terms quadratic in the azimuthal variation, but exact in the number of sectors except for the vertical betatron frequency, which is treated by the smooth approximation.

MEDIAN PLANE EQUATIONS OF MOTION

We start from the equations of motion in the median plane in which θ appears as the independent variable:

$$\frac{dr}{d\theta} = r \tan \alpha \tag{1}$$

$$\frac{d\alpha}{d\theta} = 1 - r\,\mu(r, \theta) \sec \alpha \,. \tag{2}$$

Here α is the angle the orbit makes with the tangential direction, and $\mu(r, \theta)$ is the curvature of the orbit:

$$\mu(r, \theta) = \frac{e}{m\,c\,v} H_z(r, \theta) \quad . \tag{3}$$

For the variable, r, we substitute

$$r = r_0(1 + x) \ , \tag{4}$$

where r_0 is the radius at which the azimuthal average of μ is appropriate to the given particle momentum, i.e.,

$$r_0 \overline{\mu(r_0, \theta)} = 1 \quad . \tag{5}$$

Then $\mu(r, \theta)$ is expressed as a mixed Fourier and power series:

$$\mu = \overline{\mu(r_0, \theta)} \ \{ 1 + \mu' x + \mu'' \frac{x^2}{2} + \mu''' \frac{x^3}{6} + \sum_1^\infty [a_n \cos n\theta + b_n \sin n\theta]$$

$$+ x \sum_1^\infty [a'_n \cos n\theta + b'_n \sin n\theta] + \frac{x^2}{2} \sum_1^\infty [a''_n \cos n\theta + b''_n \sin n\theta]$$

$$+ \ \ldots \ \} \ , \tag{6}$$

where primes denote $\frac{d}{dx} = r_0 \frac{d}{dr}\Big|_{r=r_0}$, $\frac{d^2}{dx^2} = r_0^2 \frac{d^2}{dr^2}\Big|_{r=r_0}$, etc.

Although the derivatives of the average field should be small $(\sim \beta^2, \beta^4)$ for a low-energy cyclotron, they will be carried as zero-order quantities until the final steps. The expansion (6) is sufficient to include all contributions to frequencies quadratic in the azimuthal variations and all contributions to the nonlinear 3/3 resonance driving terms which are linear in the azimuthal variations.

To the same approximation, the equations of motion can be written:

$$\frac{dx}{d\theta} = \alpha + \alpha x + \frac{\alpha^3}{3} \tag{7}$$

$$-\frac{d\alpha}{d\theta} = (1 + \mu')x + \Sigma^o + (\mu' + \frac{\mu''}{2})x^2 + x(\Sigma^o + \Sigma') + \frac{\alpha^2}{2} + (1 + \mu')\frac{x\alpha^2}{2}$$

$$+ x^2(\frac{1}{2}\Sigma'' + \Sigma') + \frac{\alpha^2}{2}\Sigma^o + \frac{\mu''' x^3}{6} + \frac{\mu''}{2}x^3 , \tag{8}$$

where Σ^o, Σ', and Σ'' are abbreviations for the Fourier series and its first and second derivatives in x.

EQUILIBRIUM ORBIT; ISOCHRONOUS CONDITION

The equilibrium orbit can be obtained by successive approximations in the flutter. To first approximations, with $x = x_1$, and $\alpha = \alpha_1$, we have

$$\frac{dx_1}{d\theta} = \alpha_1 \tag{9}$$

and

$$-\frac{d\alpha_1}{d\theta} = (1 + \mu')x_1 + \Sigma^o , \tag{10}$$

with the periodic solutions:

$$x_1 = \sum_1^\infty \frac{1}{n^2 - (1 + \mu')} [a_n \cos n\theta + b_n \sin n\theta] \tag{11}$$

$$\alpha_1 = -\sum_1^\infty \frac{n}{n^2 - (1 + \mu')} [a_n \sin n\theta - b_n \cos n\theta] . \tag{12}$$

The differential equations for the second approximations are:

$$\frac{dx_2}{d\theta} = \alpha_2 + \alpha_1 x_1 \tag{13}$$

$$-\frac{d\alpha_2}{d\theta} = (1+\mu')x_2 + x_1^2(\mu' + \frac{\mu''}{2}) + \frac{\alpha_1^2}{2} + x_1(\Sigma^o + \Sigma'). \tag{14}$$

The solution of (13) and (14) is straightforward but cumbersome to write.[1] The only feature of significance to the present work is that x_2 (but not α_2) has a nonzero average value in θ :

$$\bar{x}_2 = -\frac{1}{2(1+\mu')} \sum_1^\infty \frac{1}{n^2 - (1+\mu')} \left\{ (a_n a'_n + b_n b'_n) + \frac{\frac{3}{2}n^2 - 1 + \frac{\mu''}{2}}{n^2 - (1+\mu')} (a_n^2 + b_n^2) \right\} . \tag{15}$$

We are now in a position to compute the length of the equilibrium orbit to second order. The path length is:

$$L = \int ds = \int_0^{2\pi} r\, d\theta \sqrt{1 + \tan^2\alpha} = r_0 \int_0^{2\pi} d\theta \left\{ 1 + x_1 + x_2 + \frac{\alpha_1^2}{2} \right\}$$

$$= 2\pi r_0 \left[1 + \bar{x}_2 + \frac{1}{2}\bar{\alpha}_1^2 \right] . \tag{16}$$

From Eqs. (12) and (15), we obtain

$$\frac{\Delta L}{2\pi r_0} = -\frac{1}{4(1+\mu')} \sum_{n=1}^\infty \frac{1}{n^2 - (1+\mu')} \left\{ \frac{n^2(2-\mu') - 2 + \mu''}{n^2 - (1+\mu')} [a_n^2 + b_n^2] + 2[a_n a'_n + b_n b'_n) \right\} . \tag{17}$$

---

[1] The complete second-order solution is given in Appendix 1.

The second term in the bracket is, of course, $\frac{d}{dx}\left[a_n^{\,2} + b_n^{\,2} \right]$, that is, roughly the rate of change of flutter with radius. The correction (17) tends to be small, but not negligible. For example, if there is only one Fourier component present, representing a number of sectors $n = N >> 1$, and of amplitude independent of radius,

$$\frac{\Delta L}{2\pi r_0} \sim -\frac{1}{2}\,\frac{a_N^{\,2} + b_N^{\,2}}{N^2} = -\frac{F^2}{N^2}\,, \tag{18}$$

where F^2 is the conventional "flutter factor" (mean square variation in field at $r = r_0$).

For $F^2 \sim 0.02$, $N = 3$, the correction is two parts in a thousand for isochronism.

To the extent that (17) is independent of small changes in the average magnetic field and can be regarded as a given function of r_0, the prescription for the variation of average field with radius for isochronism can be obtained as follows:

$$\omega = \frac{2\pi v}{L} = \frac{eH(r_0)}{mc(1 + \frac{\Delta L}{L})} = \frac{eH(r_0)}{m_0 c(1 + \frac{\Delta L}{L})}\left\{1 - \left[\frac{\omega r_0(1 + \frac{\Delta L}{L})}{c}\right]^2\right\}^{1/2}, \tag{19}$$

where m_0 is the particle rest mass. For ω independent of radius, Eq. (19) determines $H(r_0)$:

$$H(r_0) = \frac{m_0 c\,\omega}{e}\,(1 + \frac{\Delta L}{L})\left\{1 - \left[\frac{\omega r_0(1 + \frac{\Delta L}{L})}{c}\right]^2\right\}^{-1/2}, \tag{20}$$

or, approximately,

$$H(r_0) = \frac{m_0 c\,\omega}{e}\left[\,1 + \frac{\Delta L}{L} + \frac{1}{2}\beta^2\,\right]. \tag{21}$$

The coefficients in the expression of the average curvature in (6) may be obtained by differentiating (20). Neglecting $\frac{\Delta L}{L}$ for this purpose, we find:

$$\mu' = \frac{r}{H}\frac{dH}{dr} = p^2$$

$$\mu'' = p^2(1 + 3p^2) \tag{22}$$

$$\mu''' = 3p^4(3 + 5p^2) \quad ,$$

where p is the momentum in units of m_0c.

VERTICAL BETATRON FREQUENCY

Since the frequency of vertical oscillation will always be small compared to unity, the smooth approximation should be adequate. The linearized vertical equation of motion is[2]

$$\frac{d^2z}{ds^2} - \frac{\partial\mu}{\partial n} z = 0 \quad , \tag{23}$$

where s is distance measured along the equilibrium orbit, and $\frac{\partial\mu}{\partial n}$ is the derivative with respect to the outward normal of the curvature of the equilibrium orbit. To sufficient approximation, we have

$$\frac{\partial\mu}{\partial n} = \frac{1}{r_0}\frac{\partial\mu}{\partial x}\cos\alpha - \frac{\sin\alpha}{r}\frac{\partial\mu}{\partial\theta} = \mu'(1 - \frac{1}{2}\alpha_1^2) + \mu''(x_1 + x_2)$$

$$+ \Sigma' + \frac{\mu'''}{2}x_1^2 + x_1\Sigma'' - \alpha_1\frac{d}{d\theta}\Sigma^o \; . \tag{24}$$

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2 See, for example, E. D. Courant and H. S. Snyder, Annals of Physics 3, 1 (1958).

In the smooth approximation formulas for the frequency of a linear oscillator described by the equation

$$\frac{d^2 z}{ds^2} + g(s)\, z = 0 \quad , \tag{25}$$

we include the terms:[3]

$$\nu_z^2 = \langle g \rangle + \langle g_1^2 \rangle + 4 \langle g \rangle \langle g_2^2 \rangle \quad . \tag{26}$$

It must be remembered that the brackets denote averages with respect to s; the full result is

$$\begin{aligned} \nu_z^2 = -\mu' &+ \frac{1}{2} \sum_1^\infty \frac{n^4 - n^2\left(1 + \frac{\mu'}{2} - \frac{3}{2}\frac{\mu''}{1+\mu'}\right)}{[\, n^2 - (1+\mu')\,]^2} \left(a_n^2 + b_n^2\right) \\ &- \frac{1}{2} \sum_1^\infty \frac{\left[\, 1 - \frac{\mu''}{1+\mu'} \,\right]}{[\, n^2 - (1+\mu')\,]} \left(a_n a'_n + b_n b'_n\right) \\ &- \frac{1}{2} \sum_1^\infty \frac{a_n a''_n + b_n b''_n}{n^2 - (1+\mu')} + \frac{1}{2} \sum_1^\infty \frac{(n^2 - 4\mu')}{n^4} \left(a'^2_n + b'^2_n\right) , \end{aligned} \tag{27}$$

where some small terms in μ'^2, etc. have already been dropped.[4]

By the use of

$$a_n a'_n + b_n b'_n = \frac{1}{2} \frac{d}{dx} \left(a_n^2 + b_n^2\right) \tag{28}$$

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[3] For notation and derivation, see L. C. Teng, Rev. Sci. Instr. 27, 1051 (1956).

[4] The result without simplifications is given in Appendix 2.

and

$$a_n a''_n + b_n b''_n = \frac{1}{2} \frac{d^2}{dx^2} (a_n^2 + b_n^2) - (a'^2_n + b'^2_n) \qquad (29)$$

and if we neglect μ', μ'' compared to unity, Eq. (27) becomes

$$\nu_z^2 = -\mu' + F^2 + \sum_1^\infty \frac{a'^2_n + b'^2_n}{n^2} + \frac{1}{2} \sum_1^\infty \frac{a'^2_n + b'^2_n}{n^4}$$

$$+ \frac{1}{4} \left[2 - \frac{d}{dx} - \frac{d^2}{dx^2} \right] \sum_1^\infty \frac{a_n^2 + b_n^2}{n^2} , \qquad (30)$$

where $F^2 = \frac{1}{2} \sum_1^\infty (a_n^2 + b_n^2)$, the flutter factor.

The effect of a spiral angle is exhibited more explicitly by writing the Fourier components in the form

$$a_n(r) \cos n\,\theta + b_n(r) \sin n\,\theta = A_n(r) \cos n \left[\theta - \phi_n(r) \right] . \qquad (31)$$

Then we may write

$$a_n(r) = A_n(r) \cos n\,\phi_n(r) , \qquad b_n(r) = A_n(r) \sin n\,\phi(r), \qquad (32)$$

whence we have

$$a_n^2 + b_n^2 = A_n^2 \quad \text{and} \quad a'^2_n + b'^2_n = A'^2_n + A_n^2 n^2 \phi'^2_n , \qquad (33)$$

and for Eq. (30)

$$\nu_z^2 = -\mu' + F^2 + \sum_1^\infty A_n^2 \tan^2 \gamma_n + \frac{1}{2} \sum_1^\infty \frac{A_n^2 \tan^2 \gamma_n}{n^2} + \sum_1^\infty \left(\frac{1}{n^2} + \frac{1}{2n^4} \right) A'^2_n$$

$$+ \frac{1}{4} \left[2 - \frac{d}{dx} - \frac{d^2}{dx^2} \right] \sum_1^\infty \frac{A_n^2}{n^2} , \qquad (34)$$

where $\tan \gamma_n = r \frac{d}{dr} \bar{\phi}_n(r)$; i.e., γ_n is the spiral angle of the nth Fourier component. If there is only one harmonic, $n = N$, present, Eq. (34) reduces to:

$$\nu_z^2 = -\mu' + F^2(1 + 2\tan^2\gamma) + \frac{F^2}{N^2}\tan^2\gamma + \frac{A'^2}{N^2}\left(1 + \frac{1}{2N^2}\right)$$

$$+ \frac{1}{4N^2}\left[2 - \frac{d}{dx} - \frac{d^2}{dx^2}\right] A_n^2 \quad . \tag{35}$$

Any of the smaller terms in Eqs. (30) or (34) may be significant since one generally attempts a near cancellation of the dominant terms in order to minimize flutter and spiral requirements.

RADIAL BETATRON FREQUENCIES

The radial frequency will be close to unity, so that the phase advance of the oscillations per sector will be quite large in the cases of interest. Instead of the smooth approximation, we shall use a method designed to give the correction to the frequency from unity, exact in the number of sectors and quadratic in the field variations. The first step is to linearize Eqs (7) and (8), that is, replace x by $x_{eq} + \xi$ and α by $\alpha_{eq} + \eta$ and retain only terms of first order in ξ and η. Equations (7) and (8) become:

$$\frac{d\xi}{d\theta} = \eta + \alpha_{eq}\xi + x_{eq}\eta + \alpha_{eq}^2\eta \tag{36}$$

and

$$-\frac{d\eta}{d\theta} = (1 + \mu')\xi + 2(\mu' + \frac{\mu''}{2})x_{eq}\xi + (\Sigma^o + \Sigma')\xi + \alpha_{eq}\eta + \frac{(1 + \mu')}{2}[\alpha_{eq}^2\xi + 2\, x_{eq}\,\alpha_{eq}\eta]$$

$$+ 2\, x_{eq}\xi\,[\Sigma' + \frac{1}{2}\Sigma''] + \alpha_{eq}\eta\,\Sigma^o + \frac{\mu'''}{2}x_{eq}^2\xi + \frac{3\mu''}{2}x_{eq}^2\xi \;, \tag{37}$$

where $x_{eq}(\theta)$ and $\alpha_{eq}(\theta)$ describe the equilibrium orbit. If we change variables once more, replacing η by

$$\pi = \eta \left[1 - \frac{1}{2} \alpha_{eq}^{2} \right] , \tag{38}$$

the variable π is directly related to the radial component of momentum, and the equations take on a more symmetric form:

$$\frac{d\xi}{d\theta} = \pi \left[1 + x_{eq} + \frac{3}{2} \alpha_{eq}^{2} \right] + \xi \alpha_{eq} \tag{39}$$

$$-\frac{d\pi}{d\theta} = \pi \alpha_{eq} + \xi \left[(1 + \mu') + (2\mu' + \mu'')x_{eq} + \Sigma^{o} + \Sigma' + x_{eq}(2\Sigma' + \Sigma'') + \frac{1}{2}(\mu''' + 3\mu'')x_{eq}^{2} \right] . \tag{40}$$

These equations of motion follow from a Hamiltonian:

$$H = \frac{\pi^2}{2} \left[1 + x_1 + x_2 + \frac{3}{2} \alpha_1^{2} \right] + \frac{\xi^2}{2} \left[(1 + \mu') + (2\mu' + \mu'')(x_1 + x_2) + \Sigma^{o} + \Sigma' + x_1(2\Sigma' + \Sigma'') + \frac{1}{2}(\mu''' + 3\mu'')x_1^{2} \right] + \xi\pi \left[\alpha_1 + \alpha_2 \right] . \tag{41}$$

Here the equilibrium orbit has been replaced by the first- and second-order approximations of Eqs. (11), (12), and following, but retaining terms only up to quadratic in the field variations. The Hamiltonian can be further abbreviated by noting that oscillatory terms contribute to the frequency only as the square of their amplitude, so that it is legitimate to drop the oscillatory parts of terms already quadratic in the Fourier coefficients. Thus we have

$$H = \frac{\pi^2}{2} \left[1 + \bar{x}_2 + \frac{3}{2}\bar{\alpha}_1^{\,2} + x_1 \right] + \frac{\xi^2}{2} \left[1 + \mu' + (2\mu' + \mu'')\bar{x}_2 + \frac{1}{2}(\mu''' + 3\mu'')\bar{x}_1^{\,2} \right.$$

$$\left. + 2\,\overline{x_1\,\Sigma'} + \overline{x_1\,\Sigma''} + \Sigma^{o} + \Sigma' + (2\mu' + \mu'')x_1 \right] + \alpha_1 \xi \pi \,, \tag{42}$$

where a bar denotes, as before, an average in azimuth.

The problem, then, is to determine the additional effect of the oscillatory first-order terms on the frequency. To this end we proceed by using a generating function to establish a change of variable which will remove the first-order terms from the Hamiltonian. Using a function of the old coordinate and new momentum, we have

$$S(\xi, p, \Theta) = \xi\, p \left[1 + C(\Theta) \right] + \frac{A(\Theta)}{2}\, p^2 + \frac{B(\Theta)}{2}\, \xi^2 \tag{43}$$

and the relations

$$q = \frac{\partial S}{\partial p} \,, \qquad \pi = \frac{\partial S}{\partial \xi} \,. \tag{44}$$

The change of variables is

$$\begin{aligned} \xi &= q \left[1 - C + C^2 \right] - p \left[A - AC \right] \\ \pi &= p \left[1 + C - AB \right] + q \left[B - BC \right] \,. \end{aligned} \tag{45}$$

Equation (45) follows from Eq. (43) to second order if A, B, and C are first-order oscillatory functions of Θ. The transformed Hamiltonian is

$$\overline{H}(q, p, \Theta) = H \left[\xi(p, q), \pi(p, q), \Theta \right] + \frac{\partial S}{\partial \Theta} \,. \tag{46}$$

The algebraic manipulation required to write out Eq. (46) explicitly is rather formidable; what emerges is that if A, B, and C satisfy the relations

$$\frac{dA}{d\theta} = -x_1 - 2\,C \quad ,$$

$$\frac{dB}{d\theta} = -(2\mu' + \mu'')x_1 + 2(1 + \mu')C - \Sigma^{o} - \Sigma' \quad , \tag{47}$$

and

$$\frac{dC}{d\theta} = (1 + \mu')A - B - \alpha_1 \quad ,$$

then the transformed Hamiltonian is free of first-order oscillatory terms, and takes the explicit form, correct to second order,

$$\overline{H} = \frac{p^2}{2}\left[\, 1 + \overline{x_2} + \frac{3}{2}\overline{\alpha_1^{\,2}} + 2\,\overline{x_1 C} + \overline{C^2} - (1 + \mu')\overline{A^2}\,\right]$$

$$+ \frac{q^2}{2}\left[\, 1 + \mu' + (2\mu' + \mu'')\overline{x_2} + \frac{\mu''' + 3\mu''}{2}\,\overline{x_1^{\,2}} + 2\,\overline{x_1\Sigma'} + \overline{x_1\Sigma''}\right.$$

$$\left. + 2\,\overline{\alpha_1 B} + \overline{B^2} - (1 + \mu')\overline{C^2}\,\right] + q\,p\left[\,\overline{x_1 B} - (1 + \mu')\overline{AC} + \overline{BC} + \overline{\alpha_1 C}\,\right] . \tag{48}$$

Equation (48) does not depend explicitly on θ, and so represents a more or less familiar simple harmonic oscillator. The square of the frequency is equal, in the approximation considered, to the product of the coefficients of $\frac{p^2}{2}$ and $\frac{q^2}{2}$:

$$\nu_r^{\,2} = 1 + \mu' + (1 + 2\mu' + \mu'')\overline{x_2} + \frac{3}{2}\overline{\alpha_1^{\,2}} + \left(\frac{3\mu'' + \mu'''}{2}\right)\overline{x_1^{\,2}} + 2\,\overline{x_1\Sigma'}$$

$$+ \overline{x_1\Sigma''} + 2\,\overline{x_1 C} + 2\,\overline{\alpha_1 B} - \mu'\overline{C^2} - (1 + \mu')\overline{A^2} + \overline{B^2} \quad . \tag{49}$$

The coupled Eqs (47) can be solved readily enough (see Appendix 3). Finally, if we regard μ', etc., as small, Eq. (49) becomes

$$\nu_r^2 = 1 + \mu' + \frac{3}{2} \sum_1^\infty \frac{n^2}{(n^2 - 1)(n^2 - 4)} (a_n^2 + b_n^2)$$

$$+ \frac{1}{4} \frac{d}{dx} \sum_1^\infty \frac{5n^2 - 8}{(n^2 - 4)(n^2 - 1)} (a_n^2 + b_n^2) + \frac{1}{4} \frac{d^2}{dx^2} \sum_1^\infty \frac{a_n^2 + b_n^2}{n^2 - 1}$$

$$+ \frac{3}{2} \sum_1^\infty \frac{a'^2_n + b'^2_n}{(n^2 - 1)(n^2 - 4)} , \tag{50}$$

or in the form of Eq. (31),

$$\nu_r^2 = 1 + \mu' + \frac{3}{2} \sum_1^\infty \frac{n^2 A_n^2}{(n^2 - 1)(n^2 - 4)} + \frac{1}{4} \frac{d}{dx} \sum_1^\infty \frac{(5n^2 - 8)}{(n^2 - 4)(n^2 - 1)} A_n^2$$

$$+ \frac{1}{4} \frac{d^2}{dx^2} \sum_1^\infty \frac{A_n^2}{n^2 - 1} + \frac{3}{2} \sum_1^\infty \frac{n^2 \tan^2 \gamma_n}{(n^2 - 1)(n^2 - 4)} A_n^2$$

$$+ \frac{3}{2} \sum_1^\infty \frac{A'^2_n}{(n^2 - 1)(n^2 - 4)} . \tag{51}$$

The net gain over the smooth approximation lies in the explicit appearance of the factor $(n^2 - 4)$, introduced by the generating function. For three-fold symmetry, several iterations of the smooth approximation would be required to reproduce these terms as a power series in $1/N$.

Values computed from Eqs. (50) and (35) for the vertical frequencies have been compared for three-fold and four-fold measured field with the values obtained from the MURA "Ill-Tempered V" 704 code, indicating agreement satisfactory for practical purposes. (We hope to write a later report comparing such results in detail.)

RADIAL INSTABILITY

We shall consider specifically the radial, nonlinear 3/3 resonance, which arises in and is commonly believed to be a major objection to a three-fold geometry.

In order to treat this problem it is necessary to adjoin to Eqs. (36) and (37) the terms quadratic and cubic in ξ and η and attempt to solve the resulting equations. Since the linear problem alone yields a motion which contains the periodicity of the magnet structure as well as the slower oscillation of frequency given by Eq. (50), additional nonlinear driving terms arise from this self coupling. It is therefore more convenient to start from the transformed Hamiltonian, which describes a pure oscillation, and add to it the relevant non-linear terms. If the Hamiltonian (48) be denoted now by H_0, then the total Hamiltonian, H_T, turns out to be

$$H_T = H_0 + \frac{1}{2}(1 + C)q\,p^2 - \frac{1}{2}Ap^3 + [\,B - (\mu' + \frac{\mu''}{2})A\,]\,q^2 p + \frac{\alpha_1}{2}p^3$$

$$+ \frac{1}{8}p^4 + \frac{1}{8}(\mu'' + \frac{\mu'''}{3})q^4 + [\,(\mu' + \frac{\mu''}{2}) + \frac{3}{2}(\mu'' + \frac{\mu'''}{3})x_1 + \frac{1}{2}\Sigma''$$

$$+ \Sigma' - 3(\mu' + \frac{\mu''}{2})C\,]\,q^3/3 \quad , \tag{52}$$

where A, B, and C are the functions of θ in the linear generating function (43). For solving the nonlinear problem, we shall follow the method of "variable constants"' that is, let

$$q = \mathcal{A}(\theta)\sin[\,\nu_r\,\theta + \psi(\theta)\,] \tag{53}$$

and

$$p = \mathcal{A}(\theta)\cos[\,\nu_r\,\theta + \psi(\theta)\,] \quad , \tag{54}$$

setting $\nu_r = 1$ except where it appears in the argument of trigonometric functions. In the equations of motion, now for the amplitude, a, and the phase, ψ, only terms are kept which will contribute either to the nonlinear driving force or to a dependence on amplitude of the natural frequency. Thus, from Eq. (52), we obtain

$$\frac{da}{d\Theta} = -\frac{a^2}{8}\left[\cos(\nu_r\Theta + \psi) + 3\cos 3(\nu_r\Theta + \psi)\right]$$

$$+ \frac{a^2}{4}\left\{\left[\frac{1}{2}\Sigma'' + \Sigma' + \frac{3}{2}\left(\mu'' + \frac{\mu'''}{3}\right)x_1 - \frac{3}{2}C\right]\cos 3(\nu_r\Theta + \psi)\right.$$

$$\left. + \frac{3}{2}\left[\alpha_1 - A - 2B\right]\sin 3(\nu_r\Theta + \psi)\right\}, \tag{55}$$

and

$$\frac{d\psi}{d\Theta} = \frac{3}{8}a\left[\sin(\nu_r\Theta + \psi) + \sin 3(\nu_r\Theta + \psi)\right] - \frac{a}{4}\left\{\left[\frac{1}{2}\Sigma'' + \Sigma'\right.\right.$$

$$\left.\left. + \frac{3}{2}\left(\mu'' + \frac{\mu'''}{3}\right)x_1 - \frac{3}{2}C\right]\sin 3(\nu_r\Theta + \psi) - \frac{3}{2}\left[\alpha_1 - A - 2B\right]\cos 3(\nu_r\Theta + \psi)\right\}. \tag{56}$$

$$+ \frac{3}{16}a^2 .$$

The first term on the right in Eqs. (55) and (56) is the largest, but because it oscillates rapidly it contributes only in a higher order and only affects the frequency $\left[\frac{d\psi}{d\Theta}\right]$. In the present case, the higher-order contribution unfortunately serves to cancel the last term on the right side of (56), so that to the approximation used, there is no dependence of frequency on amplitude. The second members of Eqs. (55) and (56) contain the driving influence

of the resonance; the third harmonic component of the bracketed oscillatory functions combines with the sine and cosine of $3\,\nu_r\,\theta$ to produce trigonometric functions whose phase varies very slowly, giving rise to the possibility of phase locking and slowly accumulating changes in amplitude. If we keep only these secular terms, Eqs. (55) and (56) become:

$$\frac{da}{d\theta} = \frac{a^2}{16}\left\{[\,a''_3 + 5\,a'_3 + 3\,a_3\,]\cos[\,3(\nu_r - 1)\theta + 3\,\psi\,] - [\,b''_3 + 5\,b'_3 + 3\,b_3\,]\sin[\,3(\nu_r - 1)\theta + 3\,\psi\,]\right\} \tag{57}$$

and

$$\frac{d\psi}{d\theta} = -\frac{a}{16}\left\{[\,a''_3 + 5\,a'_3 + 3\,a_3\,]\sin[3(\nu_r - 1)\theta + 3\,\psi\,] + [\,b''_3 + 5\,b'_3 + 3\,b_3\,]\cos[\,3(\nu_r - 1)\theta + 3\,\psi\,]\right\} \tag{58}$$

where a_3, b_3, etc., are the field components first defined in Eq. (6). Recalling from Eqs. (31) to (33) that a'_3, a''_3, b'_3, and b''_3 include not only amplitude variations with radius but spiral angles as well, it is apparent that the magnitude of the driving force depends on an algebraic combination of almost all the magnet characteristics; i.e., flutter, spiral angle, rate of growth or decay of flutter, rate of change of spiral angle, and so on.

The system of Eqs. (57) and (58) can easily be integrated once. The result is:

$$\frac{a^3}{16}\left\{[\,a''_3 + 5\,a'_3 + 3\,a_3\,]\sin[\,3(\nu_r - 1)\theta + 3\,\psi\,] + [\,b''_3 + 5\,b'_3 + 3\,b_3\,]\cos[\,3(\nu_r - 1)\theta + 3\,\psi\,]\right\} - \frac{3}{2}(\nu_r - 1)\,a^2 = \text{const.} \tag{59}$$

Equation (59) is of the form

$$a^3 \; D \sin X = 24(\nu_r - 1)a^2 + \text{const.} , \tag{60}$$

where D is the square root of the sum of the squares of the bracketed coefficients and X the phase shifted argument of the trigonometric functions. The problem is really characterized by a single parameter, namely $\frac{\nu_r - 1}{D}$, the ratio of the deviation of the linear frequency from unity to the amplitude of the driving force. It is convenient to rewrite Eq. (60) as:

$$\sin X = \frac{24(\nu_r - 1)}{D} \frac{1}{a} + \frac{\text{const}}{a^3} . \tag{61}$$

Figure 1 shows a sketch of Eq. (61) for various values of the constant. Since Eq. (61) is an integral of the motion, the succession of values of a and X which an individual particle takes on are constrained to follow the curve on which it starts. It can be seen that there are two classes of orbits, those with initial conditions to the left of the heavy curve are bounded in amplitude, while all others lock into phase X = 0 with indefinitely growing amplitude. The maximum amplitude of the bounded region is, from Eq. (61),

$$a_{\max +} = \frac{16(\nu_r - 1)}{D} , \tag{62}$$

and the maximum amplitude for which the motion is bounded regardless of initial phase is

$$a_{\max -} = \frac{1}{2} a_{\max +} . \tag{63}$$

Amplitudes less than $a_{max\,-}$ are bounded, but might as much as double under the action of the driving force. With the help of Eq. (62), Eq. (61) may be put in a neater form:

$$\sin \chi = \frac{3}{2} \left[\frac{a}{a_{max\,+}} \right]^{-1} + \text{const.} \left[\frac{a}{a_{max\,+}} \right]^{-3} . \tag{64}$$

Figure 2 shows a plot of Eq. (64) for a few values of the constant, plotted this time in terms of particle displacement and slope by re-introducing Eqs. (53) and (54). The angular orientation of the figure is not fixed; it depends on the relative size of the bracketed terms in Eq. (59).

For a given magnetic field, $a_{max\,-}$, Eq. (63), can be computed from field measurements. If it is comfortably large, one may feel safe from resonance troubles. If the amplitudes expected in the machine fall in the unstable range, all is not lost, for the rate of growth of amplitude may be sufficiently slow that in the finite time for acceleration no substantial harm is done. The rate of growth may be overestimated from Eq. (57) by choosing the most unfavorable phase of the trigonometric functions, which is the one into which the phase will eventually lock (see Fig. 1). Equation (57) becomes

$$\frac{da}{dn} = \frac{2\pi\, da}{d\theta} \sim \frac{\pi}{8} a^2 D , \tag{65}$$

where D is the driving amplitude of Eq. (60), and n is the number of turns in the cyclotron. The fractional amplitude increase per turn is then

$$\frac{\Delta a}{a} = \frac{\pi}{8} a D . \tag{66}$$

For example, an amplitude of one-fourth inch at a radius of 10 inches means $a = \frac{1}{40}$. For $D \sim 2$, Eq. (66) gives

$$\frac{\Delta a}{a} \sim 0.03 \ .$$

A more precise evaluation of the seriousness of the effect could be obtained by returning to Eqs (57) and (58) and letting the machine parameters, and ν_r, vary with θ to represent the variation of the driving force with changing equilibrium radius. This would require numerical integration of Eqs. (57) and (58); for the effort involved it might be almost as simple to compute exact orbits including energy gain. We would prefer this approach, relying on the analytic expressions of this section as a guide in understanding the results of numerical work.

Finally, we give the parameters in the driving terms in the alternate form (32):

$$a_3 = A_3 \cos 3\phi \ ,$$

$$a'_3 = A'_3 \cos 3\phi - 3 A_3 \tan\gamma \sin 3\phi$$

$$a''_3 = A''_3 \cos 3\phi - 6 A'_3 \tan\gamma \sin 3\phi - 9 A_3 \tan^2\gamma \cos 3\phi$$
$$+ 3 A_3 [\tan\gamma - (\tan\gamma)'] \sin 3\phi$$

$$b_3 = A_3 \sin 3\phi$$

$$b'_3 = A'_3 \sin 3\phi + 3 A_3 \tan\gamma \cos 3\phi$$

$$b''_3 = A''_3 \sin 3\phi + 6 A'_3 \tan\gamma \cos 3\phi - 9 A_3 \tan^2\gamma \sin 3\phi$$
$$- 3 A_3 [\tan\gamma - (\tan\gamma)'] \cos 3\phi \ .$$

The term $9 A_3 \tan^2 \gamma$ in a''_3 and b''_3 is usually considered to be the most important, but this may not be the case if, for instance, an increasing or decreasing flutter has a large second derivative.

APPENDICES

Appendix 1: Equilibrium Orbit

The complete first- and second-order formulas for the equilibrium orbit are:

$$x_{eq} = x_1 + x_2 + \dots$$

$$\alpha_{eq} = \alpha_1 + \alpha_2 + \dots$$

$$x_1 = \sum_1^\infty \frac{a_n \cos\theta + b_n \sin n\theta}{n^2 - (1+\mu')}$$

$$x_2 = -\frac{1}{4(1+\mu')} \sum_1^\infty \frac{1}{n^2 - (1+\mu')} \Big[2(a_n a'_n + b_n b'_n)$$

$$+ \frac{3n^2 - 2 + \mu''}{n^2 - (1+\mu')} (a_n^2 + b_n^2) \Big] + \frac{1}{4} \sum_{n,m=1}^\infty \frac{1}{[n^2 - (1+\mu')][(n+m)^2 - (1+\mu')]}$$

$$\times \Big\{ [2(a_n a'_m - b_n b'_m) + \frac{4m^2 + nm - 2 + \mu''}{m^2 - (1+\mu')} (a_n a_m - b_n b_m)]\cos(n+m)\theta$$

$$+ [2(a_n b'_m + b_n a'_m) + \frac{4m^2 + nm - 2 + \mu''}{m^2 - (1+\mu')} (a_n b_m + b_n a_m)]\sin(n+m)\theta \Big\}$$

$$+ \frac{1}{4} \sum_{n \neq m}^\infty \frac{1}{[n^2 - (1+\mu')][(n-m)^2 - (1+\mu')]} \Big\{ [2(a_n a'_m + b_n b'_m)$$

$$+ \frac{4m^2 - nm - 2 + \mu''}{m^2 - (1+\mu')} (a_n a_m + b_n b_m)] \cos(n-m)\theta$$

$$+ [2(b_n a'_m - a_n b'_m) + \frac{4m^2 - nm - 2 + \mu''}{m^2 - (1+\mu')} (b_n a_m - a_n b_m)]\sin(n-m)\theta \Big\}$$

$$\alpha_1 = \sum_{1}^{\infty} \frac{n}{n^2 - (1 + \mu')} (b_n \cos n\theta - a_n \sin n\theta)$$

$$\alpha_2 = \frac{1}{2} \sum_{n,m}^{\infty} \frac{n + m}{[n^2 - (1 + \mu')][(n + m)^2 - (1 + \mu')]} \Bigg\{ \Bigg[(a_n b'_m + b_n a'_m)$$

$$+ \frac{1}{m^2 - (1 + \mu')} \left(\frac{1}{2} (4m^2 + nm - 2 + \mu'') - \frac{m[(n + m)^2 - (1 + \mu')]}{n + m} \right)$$

$$\times (a_n b_m + b_n a_m) \Bigg] \cos(n + m)\theta$$

$$+ \Bigg[(b_n b'_m - a_n a'_m) + \frac{1}{m^2 - (1 + \mu')} \Bigg(\frac{1}{2} (4m^2 + nm - 2 + \mu'')$$

$$- \frac{m[(n + m)^2 - (1 + \mu')]}{n + m} \Bigg) (b_n b_m - a_n a_m) \Bigg] \sin(n + m)\theta \Bigg\}$$

$$+ \frac{1}{2} \sum_{n \neq m}^{\infty} \frac{n - m}{[n^2 - (1 + \mu')][(n - m)^2 - (1 + \mu')]} \Bigg\{ \Bigg[(b_n a'_m - a_n b'_m)$$

$$+ \frac{1}{m^2 - (1 + \mu')} \left(\frac{1}{2} (4m^2 - nm - 2 + \mu'') - \frac{m[(n - m)^2 - (1 + \mu')]}{n - m} \right)$$

$$\times (b_n b_m - a_n a_m) \Bigg] \cos(n - m)\theta$$

$$+ \Bigg[(a_n a'_m + b_n b'_m) - \frac{1}{m^2 - (1 + \mu')} \Bigg(\frac{1}{2} (4m^2 - nm - 2 + \mu'')$$

$$- \frac{m[(n - m)^2 - (1 + \mu')]}{n - m} \Bigg) (a_n a_m + b_n b_m) \Bigg] \sin(n - m)\theta \Bigg\} .$$

Appendix 2: Vertical Betatron Frequency

The complete expression for ν_z^2 is:

$$
\begin{aligned}
\nu_z^2 = & -\mu' + \frac{1}{2}\sum_1^\infty \frac{n^2}{n^2 - (1+\mu')}\left(a_n^2 + b_n^2\right) \\
& + \frac{1}{4}\left(\mu' + \frac{3\mu''}{1+\mu'}\right)\sum_1^\infty \left(\frac{n}{n^2 - (1+\mu')}\right)^2 \left(a_n^2 + b_n^2\right) \\
& - \frac{1}{2}\left[\mu''\frac{2+\mu'}{1+\mu'} + \frac{\mu'''}{2} - \frac{(\mu'')^2}{2(1+\mu')}\right]\sum_1^\infty \frac{a_n^2 + b_n^2}{[n^2 - (1+\mu')]^2} \\
& - \frac{1+\mu'-\mu''}{2(1+\mu')}\sum_1^\infty \frac{a_n a'_n + b_n b'_n}{n^2 - (1+\mu')} + \mu''\sum_1^\infty \frac{a_n a'_n + b_n b'_n}{n^2[n^2 - (1+\mu')]} \\
& - 4\mu'\mu''\sum_1^\infty \frac{a_n a'_n + b_n b'_n}{n^4[n^2 - (1+\mu')]} - \frac{1}{2}\sum_1^\infty \frac{a_n a''_n + b_n b''_n}{n^2 - (1+\mu')} \\
& + \frac{(\mu'')^2}{2}\sum_1^\infty \frac{a_n^2 + b_n^2}{n^2[n^2 - (1+\mu')]^2} - 2\mu'(\mu'')^2\sum_1^\infty \frac{a_n^2 + b_n^2}{n^4[n^2 - (1+\mu')]^2} \\
& + \frac{1}{2}\sum_1^\infty \frac{1}{n^2}\left(a'^2_n + b'^2_n\right) - 2\mu'\sum_1^\infty \frac{1}{n^4}\left(a'^2_n + b'^2_n\right) .
\end{aligned}
$$

Appendix 3: Details of Calculation of ν_r

The explicit solution of coupled Eq. (47) is:

$$A = 2 \sum_1^\infty \frac{1}{n} \frac{1}{n^2 - 4(1+\mu')} [\, a'_n \sin n\theta - b'_n \cos n\theta \,]$$

$$+ \sum_1^\infty \frac{3n^2 + 2(2\mu' + \mu'')}{n[n^2 - (1+\mu')][n^2 - 4(1+\mu')]} [\, a_n \sin n\theta - b_n \cos n\theta \,]$$

$$B = \sum_1^\infty \frac{n^2 - 2(1+\mu')}{n[n^2 - 4(1+\mu')]} [\, b'_n \cos n\theta - a'_n \sin n\theta \,]$$

$$+ \sum_1^\infty \frac{n}{n^2 - 4(1+\mu')} \left[1 + \frac{2\mu' + \mu''}{n^2} \frac{n^2 - 2(1+\mu')}{n^2 - (1+\mu')} \right] \left[b_n \cos n\theta - a_n \sin n\theta \right]$$

$$C = - \sum_1^\infty \frac{1}{n^2 - 4(1+\mu')} [\, a'_n \cos n\theta + b'_n \sin n\theta \,]$$

$$- \sum_1^\infty \frac{1}{n^2 - 4(1+\mu')} \left[\frac{2n^2 - 2 + \mu''}{n^2 - (1+\mu')} \right] [\, a_n \cos n\theta + b_n \sin n\theta \,] \; .$$

With these equations together with Eqs. (6), (11), (12), and (15) for Σ', Σ'', x_1, α_1, and $\bar{x}_2$, respectively, the average values entering into Eq. (49) may be computed. One then obtains for ν_r^2 the complete second-order expression (we use the abbreviations $\sigma^2 = 1 + \mu'$, $\lambda^2 = 2\mu' + \mu''$):

$$
\begin{aligned}
\nu_r^2 = {} & 1 + \mu' + \sum_1^\infty (a_n^2 + b_n^2) \left\{ \frac{n^2}{2(n^2 - 4\sigma^2)^2} \left(1 + \frac{\lambda^2}{n^2} \frac{n^2 - 2\sigma^2}{n^2 - \sigma^2} \right)^2 \right. \\
& + \frac{n^2}{(n^2 - \sigma^2)(n^2 - 4\sigma^2)} \left(1 + \frac{\lambda^2}{n^2} \frac{n^2 - 2\sigma^2}{n^2 - \sigma^2} \right) - \frac{1 + 3\mu' + \mu''}{4\sigma^2} \frac{3n^2 - 2 + \mu''}{(n^2 - \sigma^2)^2} \\
& + \frac{3\mu'' + \mu'''}{4(n^2 - \sigma^2)^2} - \frac{2\sigma^2(n^2 - 1 + \frac{\mu''}{2})}{(n^2 - \sigma^2)^2(n^2 - 4\sigma^2)} - \frac{\sigma^2}{2} \left(\frac{3n^2 + 2\lambda^2}{n(n^2 - 4\sigma^2)(n^2 - \sigma^2)} \right)^2 \\
& \left. + \frac{3}{4}\sigma^2 \left(\frac{n}{n^2 - \sigma^2} \right)^2 \right\} + \sum_1^\infty (a_n a'_n + b_n b'_n) \left\{ \frac{n^2 - 2\sigma^2}{(n^2 - 4\sigma^2)} \left(1 + \frac{\lambda^2}{n^2} \frac{n^2 - 2\sigma^2}{n^2 - \sigma^2} \right) \right. \\
& + \frac{n^2 - 2\sigma^2}{(n^2 - \sigma^2)(n^2 - 4\sigma^2)} - \frac{1 + 3\mu' + \mu''}{2\sigma^2(n^2 - \sigma^2)} + \frac{1}{n^2 - \sigma^2} \left(1 - \frac{\lambda^2}{n^2} \frac{n^2 - 2\sigma^2}{n^2 - \sigma^2} \right) \\
& \left. - \frac{2\sigma^2(3n^2 + 2\lambda^2)}{n^2(n^2 - \sigma^2)(n^2 - 4\sigma^2)^2} \right\} + \frac{1}{2} \sum_1^\infty \frac{a'^2_n + b'^2_n}{(n^2 - 4\sigma^2)} + \frac{1}{2} \sum_1^\infty \frac{a_n a''_n + b_n b''_n}{(n^2 - \sigma^2)} .
\end{aligned}
$$

FIGURE CAPTIONS

Figure 1: Plot of the function $\sin \chi = \frac{3}{2}\left(\frac{a}{a_{max\,+}}\right)^{-1} - K\left(\frac{a}{a_{max\,+}}\right)^{-3}$ for various values of K. The left-hand portion of the curve for $K = \frac{1}{2}$ (heavy line) divides the stable from the unstable region.

Figure 2: Plot of $\frac{x}{a_{max\,+}} = \frac{a}{a_{max\,+}} \sin \frac{\chi}{3}$ vs. $\frac{1}{a_{max\,+}} \frac{dx}{d\theta} = \frac{a}{a_{max\,+}} \cos \frac{\chi}{3}$ for the integral curves of Fig. 1. If these curves are to represent the progression of x and $\frac{dx}{d\theta}$ at $\theta = 0^\circ$, 120°, and 240°, the curves should be rotated clockwise by an angle $\frac{1}{3} \tan^{-1} \left[\frac{b''_3 + 5b'_3 + 3b_3}{a''_3 + 5a'_3 + 3a_3} \right]$.

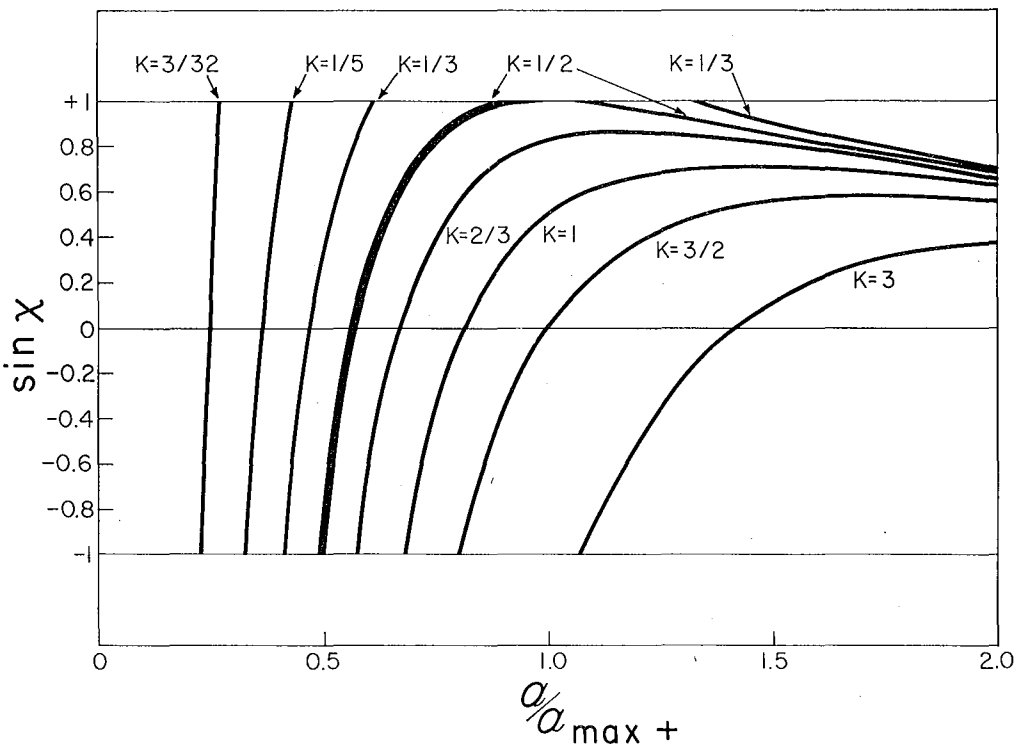
K=3/32
K=1/5
K=1/3
K=1/2
K=1/3
K=2/3
K=1
K=3/2
K=3
+1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
sin χ
0
0.5
1.0
1.5
2.0
a/a max +

Fig. 1

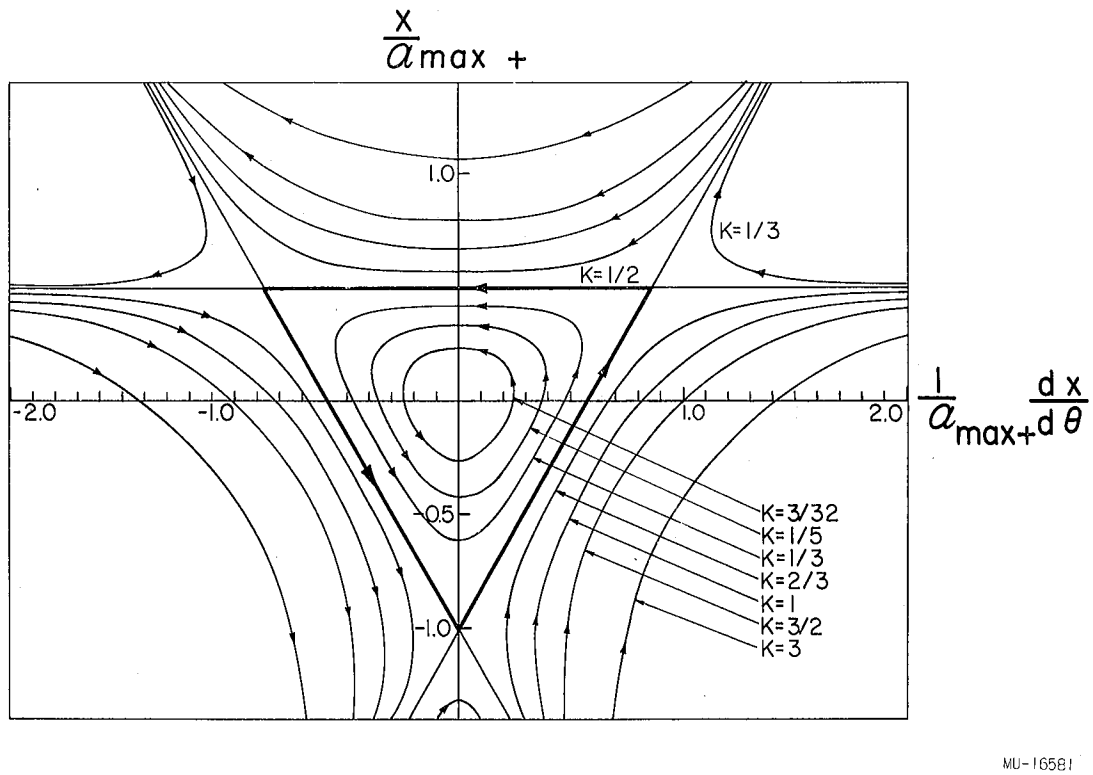
$\frac{x}{a_{max+}}$
1.0
-0.5
-1.0
-2.0
-1.0
1.0
2.0
$\frac{1}{a_{max+}}\frac{dx}{d\theta}$
K=1/3
K=1/2
K=3/32
K=1/5
K=1/3
K=2/3
K=1
K=3/2
K=3
MU-16581

Fig. 2.

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