UCRL 8931 Rev.

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For Pub. Phys. Rev.

UCRL-8931-Rev.

UNIVERSITY OF CALIFORNIA

Lawrence Radiation Laboratory

Berkeley, California

Contract No. W-7405-eng-48

# CALCULATION OF $\alpha$-TRANSITION PROBABILITIES

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October 1959

Printed for the U. S. Atomic Energy Commission

# CALCULATION OF α-TRANSITION PROBABILITIES*

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## ABSTRACT

The decay rates for the ground state transitions of all polonium isotopes and the odd-even astatine isotopes are discussed on the basis of the nuclear shell model. Good agreement with experimental data is obtained. In particular the behavior of the reduced width as a function of the neutron number around the magic number N = 126 is well reproduced.

*This work was performed under the auspices of the United States Atomic Energy Commission.

**On leave from the University of Heidelberg, Heidelberg, Germany.

# CALCULATION OF $\alpha$-TRANSITION PROBABILITIES

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## INTRODUCTION

In a previous paper[1] a theory of alpha-decay was developed whose aim was to take into account the influence of nuclear structure on alpha-decay. The theory was found successful in explaining the fine structure of the $Po^{211}$ alpha-decay. The relative intensities of transitions leading to various states in the daughter nucleus and the coefficients of alpha-gamma angular correlations were obtained in good agreement with experimental values.

In this paper we shall first outline again briefly the derivation of the formulas which are then applied to discuss the ground state transitions of even-even, even-odd, and odd-even alpha emitters in the region of $Pb^{208}$.

The nuclear wave functions are approximated by shell model wave functions and good agreement is found with experimental data.[2]

## I. DERIVATION OF AN EXPRESSION FOR THE DECAY CONSTANT

We consider a system of A nucleons (Z protons and N neutrons) and describe it by means of the time dependent Schroedinger equation.

$$H \Phi(1 \ldots A; t) = i\hbar \dot{\Phi}(1 \ldots A; t) \qquad \text{I.1}$$

Assume for the Hamiltonian the following form:

$$H = -\sum_{i=1}^{A} \frac{\hbar^2}{2m} \Delta_i + V(1,2 \ldots A) \qquad \text{I.2}$$

V describes the interaction of all the particles. It is further assumed that the wave function $\Phi_0(1..A) = \Phi(1..A; t=0)$ is known and that this wave function is the wave function of the parent nucleus.

Next we separate off the trivial center of mass motion of the system and decompose the remaining Hamiltonian in several terms

$$H = H_\alpha(1234) + H_K(5..A) - \frac{\hbar^2}{2M} \Delta_R + W(\alpha, K) \qquad \text{I.3}$$

where 1, 2, are protons, 3, 4, are neutrons.

$H_\alpha$ describes the internal motion of a system consisting of two protons and two neutrons. $H_K$ describes the internal motion of the remaining A-4 nucleons. R is the relative distance between the center of mass of particles 1; 2; 3; 4 and the center of mass of the remaining particles 5; 6; ...A; and $-\frac{\hbar^2}{2M}\Delta_R$ is therefore the operator of the kinetic energy associated with the relative motion of the two groups of nucleons. W ($\alpha K$) accounts for the interaction between the two groups of particles. We should mention that the Hamiltonian H has not lost its symmetry properties.

For obvious reasons we shall refer from now on to these two groups of nucleons as $\alpha$-particle and daughter nucleus. Consequently we call the solutions of the equations

$$H_\alpha \chi_\alpha^\tau = \epsilon_\alpha \chi_\alpha^\tau \qquad \text{I.4}$$

$$H_K \Psi_K^\sigma = E_K \Psi_K^\sigma$$

the internal wave functions of the $\alpha$-particle and the daughter nucleus. In these equations $\tau$ and $\sigma$ are short hand notations for those sets of quantum numbers that are necessary to determine the solutions $\chi_\alpha^\tau$ and $\psi_K^\sigma$ completely.

Furthermore if $\chi_\alpha^\tau \; \chi_\alpha^{\tau'}$ and $\psi_K^\sigma \psi_K^{\sigma'}$ are bound state solutions of I.4 the following relation holds:

$$\langle \chi_\alpha^{\tau'} \psi_K^{\sigma'} | W(\alpha K) | \chi_\alpha^\tau \psi_K^\sigma \rangle = \delta_{\tau\tau'} \delta_{\sigma\sigma'} V(\mathcal{R}) \qquad \text{I.5}$$

for $R \geq R_0$ and $R_0 \approx 9.0 \cdot 10^{-13}$ cm

The meaning of this relation is that always when the $\alpha$-particle and the daughter nucleus are well separated in space the interaction between them is described by a simple potential. Furthermore $V$ approaches the Coulomb potential $\frac{2(Z-2)e^2}{R}$ rapidly. For most purposes therefore it will be sufficient to use $\frac{2(Z-2)e^2}{R}$ instead of $V(\mathcal{R})$.

These considerations indicate that it might be reasonable to write for the time-dependent wave function $\Phi(1\ldots A; t)$:

$$\Phi(1\ldots A; t) = a(t)\, \Phi_0(1\ldots A) + \sum_{\tau\sigma L m} \int d\epsilon\, A \cdot \left\{ \chi_\alpha^\tau \cdot \psi_K^\sigma \cdot \varphi_L(R,\epsilon) \cdot Y_L^m\left(\frac{\mathcal{R}}{R}\right) \right\} b_{\tau\sigma L m}(\epsilon, t) \qquad \text{I.6}$$

A is an antisymmetrisation and mornalization operator, while $\varphi_L(R,\epsilon)$ is a solution of the equation,

$$\left[ -\frac{\hbar^2}{2M} \frac{1}{R} \frac{d^2}{dR^2} R + \frac{\hbar^2}{2M} \frac{L(L+1)}{R^2} + \frac{2(Z-2)e^2}{R} - \epsilon \right] \varphi_L(R,\epsilon) = 0 \qquad \text{I.7}$$

and is normalized as follows:

$$\int \varphi_L(R,\epsilon)\,\varphi_L(R\,\epsilon')\,R^2 dR = \delta(\epsilon-\epsilon') \qquad \text{I.8}$$

To get unique solutions[3] for the coefficients $a(t)$ and $b_{\tau\sigma L m}(\epsilon t)$ it is necessary to impose the condition:

$$\left\langle \Phi_0(1\ldots A) \,\middle|\, \sum_{\tau\sigma L m} \int d\epsilon\, A\left\{\chi_\alpha^\tau\, \psi_K^\sigma\, \varphi_L(R\epsilon)\, Y_L^m\right\} \cdot b_{\tau\sigma L m}(\epsilon,t) \right\rangle = \sigma \qquad \text{I.9}$$

Of course $\Phi(1\ldots A;t)$ may always be expanded in the above form (Eq. 6) as long as we sum over a complete set of functions $\chi_\alpha^\tau$ and $\psi_K^\sigma$. But we know that for energetic reasons the terms in the sum over $\tau$ and $\sigma$ corresponding to unbound states cannot contribute to the alpha-decay. Therefore we split the sum into one over bound states only and one in which $\tau$ or $\sigma$ or both correspond to unbound states and neglect the latter sum. With this approximation and taking into account angular momentum conservation as well as the fact that there is only a single bound state of the alpha-particle we rewrite $\Phi(1\ldots A;t)$ in the following form:

$$\Phi(1\ldots A;t) = a(t)\,\Phi_{0\,J}^{M}(1\ldots A) + \sum_{j\sigma L}\int d\epsilon$$

$$b_{j\sigma L}(\epsilon,t)\, A\left\{\chi_\alpha\, \varphi_L(R\epsilon) \sum_m C(Lj J;\, m\, M-m)\, Y_L^m\left(\frac{r}{R}\right) \Psi_{K\,j\sigma}^{M-m}\right\} \qquad \text{I.10}$$

where is the angular momentum of the parent nucleus, $J$ the angular momentum of the daughter nucleus and $j$ the angular momentum of the $\alpha$-particle. The Clebsch-Gordan coefficient $C(Lj J;\, m\, M-m)$ couples $L$ and $j$ to give the resulting angular momentum $J$.

Introducing this expression for $\Phi$ into the Schroedinger equation (I.1) we get a system of coupled integrodifferential equations for $a(t)$ and $b_{J\sigma L}(\epsilon t)$. But this system of equations decouples and becomes rather simple if one more approximation is introduced. We neglect nondiagonal matrix elements of the type

$$\langle \Phi^{M}_{J J\sigma L \epsilon} | H - H_\alpha - H_K + \frac{\hbar^2}{2M}\Delta_R - \frac{2(Z-2)e^2}{R} | \Phi^{M}_{J J'\sigma' L' \epsilon'} \rangle$$

where

$$\Phi^{M}_{J J\sigma L \epsilon} = A \left\{ \chi_\alpha \varphi_L(R,\epsilon) \sum_m C(L J J; m\, M-m)\, Y_L^m\, \psi_{K J\sigma}^{M-m} \right\}$$

which means we neglect the interaction of the alpha-particle with the daughter nucleus via the nuclear forces* and take only into account the Coulomb interaction; a reasonable approximation at the alpha-energies involved in natural alpha-decays.

The simplified equations read:

$$i\hbar\, \dot{a}(t) = a(t) E_0 + \sum_{J\sigma L} \int d\epsilon\, b_{J\sigma L}(\epsilon t) \langle \Phi^{M}_{0J} | H - E_0 | \Phi^{M}_{J J\sigma L \epsilon} \rangle$$

$$i\hbar \left[ \dot{b}_{J\sigma L}(\epsilon t) + \dot{a}(t) \langle \Phi_{J J\sigma L \epsilon} | \Phi^{M}_{0J} \rangle \right] = (\epsilon_\alpha + E_K + \epsilon)\, b_{J\sigma L}(\epsilon, t) \qquad \text{I.11}$$

$$+ a(t) \langle \Phi^{M}_{J J\sigma L \epsilon} | H | \Phi^{M}_{0J} \rangle$$

where $E_0 = \langle \Phi^{M}_{0J} | H | \Phi^{M}_{0J} \rangle$

---

*If we use V (R) as defined in Eq.I.5 instead of the Coulomb potential we take into account a part of the nuclear interaction.

Using standard techniques[4] to solve the equations approximately we get:

$$a(t) = \exp\left(-\frac{i}{\hbar}(E_0 + \Gamma - i\gamma)t\right)$$

$$b_{J\sigma L}(\epsilon, t) = \frac{\exp\left(-\frac{i}{\hbar}(E_0+\Gamma-i\gamma)t\right) - \exp\left(-\frac{i}{\hbar}(E_K+\epsilon_\alpha+\epsilon)t\right)}{E_0 + \Gamma - E_K - \epsilon_\alpha - \epsilon - i\gamma} \langle \Phi^M_{0J} | H - E_0 | \Phi^M_{J J \sigma L \epsilon} \rangle$$

$$\gamma = \pi \sum_{JL\sigma} \left| \langle \Phi^M_{J J \sigma L \epsilon} | H - E_0 | \Phi^M_{0J} \rangle \right|^2_{\epsilon = E_0 - \epsilon_\alpha - E_K + \Gamma} \qquad \text{I.12}$$

$$\Gamma = \sum_{J\sigma L} \int d\epsilon \, \frac{1 - \exp\left(-\frac{i}{\hbar}(\epsilon_\alpha + E_K + \epsilon - \Gamma - E_0 + i\gamma)t\right)}{E_0 + \Gamma - E_K - \epsilon_\alpha - \epsilon - i\gamma} \left| \langle \Phi^M_{J J \sigma L \epsilon} | H - E_0 | \Phi^M_{0J} \rangle \right|^2$$

The conditions for the approximate solution to be a good one are

$$\gamma \ll | E_0 - E_K - \epsilon_\alpha | = \epsilon_0$$

$$\Gamma \ll \epsilon_0 \qquad \text{I.13}$$

and

$$\left| \langle \Phi^M_{J J \sigma L \epsilon} | H - E_0 | \Phi^M_{0J} \rangle \right|^2$$

has to vary slowly compared to $\left[ (E_0 + \Gamma - \epsilon_\alpha - E_K - \epsilon)^2 + \gamma^2 \right]^{-1}$ if $\epsilon$ varies between $\epsilon_0 - \gamma$ and $\epsilon_0 + \gamma$. For all natural alpha-emitters these conditions are well fulfilled.

The problem of determining the decay constant is now reduced to the calculation of the matrix elements $\langle \Phi^M_{0J} | H - E_0 | \Phi^M_{J J \sigma L \epsilon} \rangle$.

But those matrix elements cannot be calculated unless $\Phi_{0J}^{M}$ is defined in a rigorous way. The somewhat vague statement "the wave-function $\Phi_{0J}^{M}$ should describe the parent nucleus of the alpha-decay" cannot be considered as a definition of $\Phi_{0J}^{M}$.

A natural way to define $\Phi_{0J}^{M}$ would be to obtain it from a consideration of the formation process of the alpha-emitter. But on a nuclear time scale all alpha-emitters are nearly stable. If it were not for the repulsive Coulomb-interaction between the alpha-particle and the daughter nucleus there would be no alpha-decay at all. Therefore another definition of $\Phi_{0J}^{M}$ is suggested. It has to be a bound state solution of a Schroedinger equation.

$$H_0 \Phi_{0J}^{M} = E^{(0)} \Phi_{0J}^{M} \qquad \text{I.14}$$

$H_0$ is defined as follows:

$H_0 = H$ if all nucleons are confined in a spherical* volume $\Omega_0$ with radius $r_0$

$H_0 = H - V_c(r) + V_c(r_0)$ if one or more nucleons are outside the above defined volume $\Omega_0$. $V_c(r)$ is the Coulomb interaction between the nucleons inside and those outside $\Omega_0$.

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*Of course one may assume $\Omega_0$ to be non spherical if one deals with a deformed nucleus.

$r_0$ is so defined that the interaction between one selected proton (neutron) and all the others that are in $\Omega_0$ is repulsive (zero) if the selected proton (neutron) is outside $\Omega_0$ but mostly attractive if the proton (neutron) is inside $\Omega_0$. Of course there is a maximum value for $r_0$ because of the condition that the eigen value problem with $H_0$ (Eq.I.14) should have bound state solutions. This definition assures that $H_0$ is equal to the exact Hamiltonian $H$ in that part of the configuration space where the nuclear forces between the nucleons play an important role and therefore $\Phi_{0J}^{M}$ should be a very close approximation to an exact wave function as derived for instance from considering the formation process of an alpha-emitter. By means of these assumptions about $\Phi_{0J}^{M}$ the matrix element can be simplified considerably.

$$\langle \Phi_{0J}^{M} | H - E_0 | \Phi_{JJ\sigma L\epsilon}^{M} \rangle = \left[ \binom{Z}{2} \binom{N}{2} \right]^{\frac{1}{2}} \cdot$$

$$\langle \Phi_{0J}^{M} | H - H_0 | \chi_\alpha \varphi_L(R,\epsilon) \sum_m C(LJJ; m\, M\text{-}m)\, Y_L^m \Psi_{J\sigma}^{M-m} \rangle$$

$$= \int_{R_0}^{\infty} R^2 dR \int d\Omega_R \int d\xi_\alpha\, d\xi_K\, \Phi_{0J}^{*M} (H - H_0)\, \chi_\alpha \varphi_L(R\epsilon) \qquad \text{I.15}$$

$$\sum_m C(LJJ; m\, M\text{-}m)\, Y_L^m \left(\frac{\mathbf{R}}{R}\right) \Psi_{J\sigma}^{M-m} \left[ \binom{Z}{2} \binom{N}{2} \right]^{\frac{1}{2}}$$

The first equality holds because of $H-E_0$ being a symmetric operator and $\Phi_{0J}^{M}$ an antisymmetric wave function and the fact that $E_0 = E^{(0)}$ is a very good approximation. The second equality holds because of the definition of $\Phi_{0J}^{M}(1,\ldots A)$ (Eq. I.14).

$R_0$ is somewhat smaller than $r_0$ because of the finite size of the alpha-particle. The relation between $R_0$ and $r_0$ is schematically illustrated in Figure 1.

Because of the freedom in choosing $r_0$ and consequently $R_0$, the distance $r_0$ and hence $R_0$ is chosen so that for $R \geq R_0$:

$$H\, \chi_\alpha \varphi_L(R,\epsilon) \sum_m C(LJJ;\, m\, M-m)\, Y_L^m\left(\frac{\mathbf{R}}{R}\right) \Psi_{J\sigma}^{M-m} =$$

$$= \left(E_K + \epsilon_\alpha + \epsilon\right) \chi_\alpha \varphi_L(R,\epsilon) \sum_m C(LJJ;\, m\, M-m) \qquad \text{I.16}$$

$$Y_L^m\left(\frac{\mathbf{R}}{R}\right) \Psi_{J\sigma}^{M-m}$$

Using Eqs. I.15 and I.16 and integrating by parts on $R$ the following expression is obtained for the matrix element in question:

$$\left\langle \Phi_{0J}^{M} \middle| H-E_0 \middle| \Phi_{JJ\sigma L\, \epsilon_0}^{M} \right\rangle = \left[\binom{Z}{2}\binom{N}{2}\right]^{1/2} \frac{\hbar^2}{2M} \cdot$$

$$\int d\xi_\alpha\, d\xi_K\, R_0^2\, d\Omega_R \cdot \left\{ \Phi_{0J}^{*M} \left.\frac{\partial \varphi_L(R,\epsilon)}{\partial R}\right|_{R=R_0} - \right. \qquad \text{I.17}$$

$$\left. \left.\frac{\partial \Phi_{0J}^{*M}}{\partial R}\right|_{R=R_0} \varphi_L(R,\epsilon) \right\} \chi_\alpha \sum_m C(LJJ;\, m\, M-m)\, Y_L^m\, \Psi_{J\sigma}^{M-m}$$

Introducing for $\varphi_L(\epsilon_0 R)$ the WKB approximation

$$\varphi_L(R\epsilon_0) = \left(\frac{2M}{\pi\hbar^2}\right)^{1/2} \frac{1}{R_0 q_L^{1/2}} \cdot \frac{1}{2} \exp\left(-\int_R^{R_u} q_L \, dR\right)$$

$$q_L = \left[\frac{2M}{\hbar^2}\left(\frac{2(Z-2)e^2}{R} + \frac{\hbar^2}{2M}\frac{L(L+1)}{R^2} - \epsilon_0\right)\right]^{1/2} \qquad \text{I.18}$$

$$q_L(R_u) = \sigma$$

and defining a function $G_{JJ\sigma L}(R)$

$$G_{JJ\sigma L}(R) = R_0^{3/2} \int d\xi_n \, d\xi_\alpha \, d\Omega_r \cdot$$

$$\left(\Phi_{0J}^{*M} \chi_\alpha \sum_m C(LJJ; m\,M-m)\, Y_L^m\left(\frac{r}{R}\right)\right.$$

$$\left.\Psi_{J\sigma}^{M-m}\right) \left[\binom{Z}{2}\binom{N}{2}\right]^{1/2} \qquad \text{I.19}$$

the expression for the decay constant is brought into the following form:

$$\lambda = \frac{1}{\hbar} \sum_{JL\sigma} \exp\left(-2\int_{R_0}^{R_u} q_L \, dR\right) \cdot \frac{\hbar^2}{2MR_0^2}$$

$$\frac{(R_0 q_L - 1)^2}{2R_0 q_L} \left| G_{JJ\sigma L}(R_0) + \frac{R_0}{1-R_0 q_L} \left.\frac{\partial G_{JJ\sigma L}}{\partial R}\right|_{R=R_0} \right|^2 \qquad \text{I.20}$$

$$= \frac{1}{\hbar} \sum_{JL\sigma} P_L(\epsilon_0) \cdot \delta^2_{JJ\sigma L}$$

where,

$$P_L(\epsilon_0) = \exp\left(-2\int_{R_0}^{R_u} q_L \, dR\right)$$

$$\delta^2_{JJ\sigma L} = \frac{\hbar^2}{2MR_0^2} \frac{(R_0 q_L - 1)^2}{2R_0 q_L} \left| G_{JJ\sigma L} + \frac{R_0}{1-R_0 q_L} \left.\frac{\partial G}{\partial R}\right|_{R=R_0} \right|^2$$

$P_L$ is the well known barrier penetrability as already derived by Gamow in 1928 and $\delta^2_{J\nu\sigma L}$ is the reduced width that accounts for the influence of the nuclear structure on alpha decay.

As far as the connection of our treatment with the treatment of other authors is concerned we refer to reference 1.

We should also mention that there is no difficulty in using more sophisticated potentials $V(\mathcal{R})$ than the simple Coulomb potential. One has to replace the Coulomb potential $\frac{2(Z-2)e^2}{R}$ in Eq. I.17 by $V(\mathcal{R})$ to calculate now the somewhat different function $\varphi_L(R\epsilon_0)$. If necessary this replacement has been made in the application of the theory, although all formulas are given for the pure Coulomb potential.

## II. GROUND STATE TRANSITIONS IN THE REGION OF $Pb^{208}$

To apply the theory developed in the preceeding chapter we shall approximate the nuclear wave functions by some sort of shell model wave functions.

The following assumptions determine completely all the wave functions.

(1) The alpha-particle wave function is a Gaussian type wave function.

$$\chi_\alpha = \chi_0^0(12)\,\chi_0^0(34)\,\exp\left(-\frac{\beta}{2}(\xi_1^2+\xi_2^2+\xi_3^2)\right)\left(\frac{2\beta^{3/2}}{\frac{1}{2}!\,4\pi}\right)^{3/2} \qquad \text{II.1}$$

$$\xi_1 = \frac{1}{\sqrt{2}}(\kappa_1-\kappa_2)\,;\quad \xi_2 = \frac{1}{\sqrt{2}}(\kappa_3-\kappa_4)\,;\quad \xi_3 = \frac{1}{2}(\kappa_1+\kappa_2-\kappa_3-\kappa_4)$$

where $\chi_0^0(ik)$ is a singlet spin function of particles $i$ and $k$.

For the constant $\beta$ we are going to use the value

$$\beta = 0.625 \cdot 10^{+26}; \text{ cm}^{-2}$$

This value of $\beta$ was chosen so that the RMS radius of the charge density is equal to the measured value.[5]

(2) The wave functions of the parent and daughter nucleus are shell model wave functions with seniority 0 (even-even nuclei) or 1 (even-odd and odd-even nuclei). That means an even number of particles is coupled pairwise to angular momentum zero. In an odd mass nucleus the last odd particle is therefore responsible for the total angular momentum and the parity.

Furthermore we assume no configurational mixing and the single particle states involved are taken from the spectra of nuclei which have one particle more or less than $Pb^{208}$ [6,7] ($Tl^{207}$, $Pb^{207}$, $Bi^{209}$, $Pb^{209}$). For the radial part of the single-particle wave functions harmonic oscillator functions are used.

$$\Phi_{jln}^{m} = \alpha^{3/4} \left[\frac{2 \cdot n!}{(n+l+\frac{1}{2})!}\right]^{\frac{1}{2}} (\sqrt{\alpha}\, r)^{l} L_n^{l+\frac{1}{2}}(\alpha r^2) \qquad \text{II.2}$$

$$\exp\left(-\frac{\alpha}{2} r^2\right) \sum_{\mu} C(l\tfrac{1}{2}j;\, \mu\, m-\mu)\, Y_l^{\mu}\left(\frac{\mathbf{r}}{r}\right) \chi_{\frac{1}{2}}^{m-\mu}$$

where

$$L_n^{l+\frac{1}{2}} = \sum_{k=0}^{n} \binom{n+l+\frac{1}{2}}{n-k} \frac{(-1)^k}{k!} (\alpha r^2)^k$$

For the constant $\alpha$ we shall use $\alpha = 0.175 \cdot 10^{26}$ cm$^{-2}$, a value which gives RMS radii of the nuclei that are consistent with the measured values.[8]

For $R_0$ we shall use $R_0 = 9.0 \cdot 10^{-13}$ cm. But only the product $\alpha R_0^2$ enters into the reduced width and furthermore we shall only calculate relative transition probabilities. Therefore the choice of $\alpha$ or $R_0$ alone is not too critical.

With the above assumptions (1) and (2) we get for the reduced widths $\delta^2_{JJL}$ where $J$ is the angular momentum of the parent nucleus; $J$ the angular momentum of the daughter nucleus and $L$ the angular momentum of the outgoing alpha particle.

even-even nuclei :

$$\delta^2_{000} = \frac{1}{16} N_1 (2J_1 + 3 - N_1) N_3 \cdot (2J_3 + 3 - N_3) \cdot R^2_{\substack{n_1 \ell_1 \, n_1 \ell_1 \\ n_3 \ell_3 \, n_3 \ell_3}} (R_0) \qquad \text{II.3}$$

$N_1$ is the number of protons in the unfilled subshell with quantum numbers $n_1 \; \ell_1 \; J_1$ in the parent nucleus. $N_3$ is the number of neutrons in the unfilled subshell with quantum numbers $n_3 \; \ell_3 \; J_3$ in the parent nucleus. $R^2$ involves the radial parts of the wave functions and is given below.

even-odd nuclei:

$$\delta^2_{J_1 J_1 0} = \frac{1}{16}(N_1-1)(2J_1+2-N_1)\,N_3$$

$$(2J_3+3-N_3)\;R^2_{\substack{n_1 \ell_1 \; n_1 \ell_1 \\ n_3 \ell_3 \; n_3 \ell_3}}(R_0) \qquad \text{II.4}$$

$$\delta^2_{J_1 J_1 L} = \frac{1}{4}(N_1-1)(2J_1+2-N_1)\left[\frac{2J_1+1}{2J_1-1}\,C(J_1 J_1 L;\tfrac{1}{2}-\tfrac{1}{2})\right]^2 \cdot$$

$$N_3\,(2J_3+3-N_3)\;R^2_{\substack{n_1 \ell_1 \; n_1 \ell_1 \\ n_3 \ell_3 \; n_3 \ell_3}}(R_0)$$

$$\delta^2_{J_2 J_1 L} = \frac{1}{4}\,N_1\;C^2(J_1 J_2 L;\tfrac{1}{2}-\tfrac{1}{2})\;N_3 \cdot$$

$$(2J_3+3-N_3)\;R^2_{\substack{n_1 \ell_1 \; n_2 \ell_2 \\ n_3 \ell_3 \; n_3 \ell_3}}(R_0)$$

$N_1$ is again the number of nucleons in the subshell $n_1, \ell_1, J_1$. In the first two cases $(\delta^2_{J_1 J_1 0}\;\; \delta^2_{J_1 J_1 L})$ $N_1$ is odd. In the third case $(\delta^2_{J_2 J_1 L})$ $N_1$ is even and there is one more particle in the state $n_2\;\ell_2\;J_2$ in the parent nucleus.

$R^{n_1 l_1 n_2 l_2}_{n_3 l_3 n_4 l_4}(R_0)$ has the following form:

$$R^{n_1 l_1 n_2 l_2}_{n_3 l_3 n_4 l_4} = \Big[ n_1!\,(n_1+l_1+\tfrac{1}{2})!\; n_2!\,(n_2+l_2+\tfrac{1}{2})!\; n_3!\,(n_3+l_3+\tfrac{1}{2})!$$

$$n_4!\,(n_4+l_4+\tfrac{1}{2})!\,\Big]^{-\frac{1}{2}} \left(\frac{2\alpha}{\alpha+\beta}\right)^{N+\frac{L}{2}} \frac{N!}{2^{2N+L}} \Bigg[ \frac{\hbar^2}{MR_0^2} \cdot$$

$$\frac{(R_0 q_L - 1)^2}{R_0 q_L} \left(\sqrt{\alpha}\,R_0\right)^3 \left(\frac{2\sqrt{\alpha\beta}}{\alpha+\beta}\right)^9 \left(\tfrac{1}{2}!\right)^3 \Bigg]^{\frac{1}{2}} \exp(-2\alpha R_0^2)$$

$$\sum_{\rho} B_\rho(n_1 l_1 n_2 l_2 n_3 l_3 n_4 l_4) \left(\frac{\beta-\alpha}{2\alpha}\right)^{N-\rho} \frac{(-1)^\rho \rho!}{N!}$$

$$4^{N-\rho} \left[2(\alpha+\beta)R_0^2\right]^{\frac{L}{2}} K_\rho^{L+\frac{1}{2}}\left(2(\alpha+\beta)R_0^2\right)$$

where II.5

$$K_\rho^{L+\frac{1}{2}} = \left(1 + \frac{4\alpha R_0^2 - L}{R_0 q_L - 1}\right) L_\rho^{L+\frac{1}{2}}\left(2(\alpha+\beta)R_0^2\right)$$

$$- \frac{4(\alpha+\beta)R_0^2}{R_0 q_L - 1} L_\rho^{L+\frac{1}{2}\,\prime}\left(2(\alpha+\beta)R_0^2\right)$$

and the prime means differentiation with respect to the argument $2(\alpha+\beta)R_0^2$ . The coefficients $B_\rho(n_1 l_1 n_2 l_2 n_3 l_3 n_4 l_4)$ are defined by the following equation:

$$n_1!\, n_2!\, n_3!\, n_4!\, L_{n_1}^{l_1+\frac{1}{2}}(x')\, L_{n_2}^{l_2+\frac{1}{2}}(x')\, L_{n_3}^{l_3+\frac{1}{2}}(x)$$

$$L_{n_4}^{l_4+\frac{1}{2}}(x') = \sum_{\rho} B_\rho(n_1 l_1 n_2 l_2 n_3 l_3 n_4 l_4)\, x'^{\,\rho+\frac{L-l_1-l_2-l_3-l_4}{2}} \qquad \text{II.6}$$

or

$$B_\rho(n_1 l_1 n_2 l_2 n_3 l_3 n_4 l_4) = n_1!\, n_2!\, n_3!\, n_4!$$

$$\sum_{\nu_1 \nu_2 \nu_3 \nu_4} \binom{n_1+l_1+\frac{1}{2}}{n_1-\nu_1}\binom{n_2+l_2+\frac{1}{2}}{n_2-\nu_2}\binom{n_3+l_3+\frac{1}{2}}{n_3-\nu_3}\binom{n_4+l_4+\frac{1}{2}}{n_4-\nu_4}\frac{(-1)^{\nu_1+\nu_2+\nu_3+\nu_4}}{\nu_1!\,\nu_2!\,\nu_3!\,\nu_4!}$$

where the summation is restricted by

$$2\rho + L = 2(\nu_1+\nu_2+\nu_3+\nu_4) + l_1 + l_2 + l_3 + l_4$$

With the help of these formulas and using the configurations listed in Table I, we have calculated the reduced widths that are compared to the experimental data[2] in Figs. 2, 3, and 4. In these diagrams the experimental and calculated reduced widths are plotted versus the mass number for the even-even polonium isotopes, the odd-even astatine isotopes and the even-odd polonium isotopes seperately.

The reduced width of $Po^{210}$ is taken as a standard and set equal to the experimental reduced width. We hope that in doing this, we minimize the ambiguities introduced by the choice of the radical wave functions and the parameters $\alpha$ and $R_0$.

It may be shown that varying the parameters within reasonable limits affects only the absolute magnitude of the radial part of the reduced width $R^2$, but leaves nearly unaffected the relative magnitudes we are mainly interested in. This result indicates also that it is sometimes allowed to use a very handy approximation for the radial part $R_{\substack{n_1 l_1 n_2 l_2 \\ n_3 l_3 n_4 l_4}}(R_0)$ which arises if one sets $\beta \gg \alpha$. Then

$$R_{\substack{n_1 l_1 n_2 l_2 \\ n_3 l_3 n_4 l_4}}(R_0) \sim e^{-2\alpha R_0^2} (\sqrt{\alpha} R_0)^{l_1+l_2+l_3+l_4} \left[\frac{n_1!\, n_2!\, n_3!\, n_4!}{(n_1+l_1+\frac{1}{2})!\,(n_2+l_2+\frac{1}{2})!\,(n_3+l_3+\frac{1}{2})!\,(n_4+l_4+\frac{1}{2})!}\right]^{\frac{1}{2}} \cdot L_{n_1}^{l_1+\frac{1}{2}}(\alpha R_0^2)\, L_{n_2}^{l_2+\frac{1}{2}}(\alpha R_0^2)\, L_{n_3}^{l_3+\frac{1}{2}}(\alpha R_0^2)\, L_{n_4}^{l_4+\frac{1}{2}}(\alpha R_0^2) \qquad \text{II.7}$$

a very useful expression if one wants results quickly.

Table I

Shell Model Configurations for Nuclei in the Region of $Pb^{208}$

| Element | Proton Config. | Neutron Config. | Element | Proton Config. | Neutron Config. |
|---|---|---|---|---|---|
| $Po^{202}$ | $(1h_{9/2})^2_o$ | $(p_{1/2})^{-2}_o(f_{5/2})^{-4}_o(p_{3/2})^{+2}_o$ | $Pb^{198}$ | Closed Shell | $(p_{1/2})^{-2}(f_{5/2})^{-6}_o(p_{3/2})^{-2}_o$ |
| $Po^{202}$ | " | $(p_{1/2})^{-2}_o(f_{5/2})^{-4}_o$ | $Pb^{200}$ | " | $(p_{1/2})^{-2}_o(f_{5/2})^{-4}_o(p_{3/2})^{-2}_o$ |
| $Po^{206}$ | " | $(p_{1/2})^{-2}_o(f_{5/2})^{-2}_o$ | $Pb^{202}$ | " | $(p_{1/2})^{-2}_o(f_{5/2})^{-4}_o$ |
| $Po^{208}$ | " | $(p_{1/2})^{-2}_o$ | $Pb^{204}$ | " | $(p_{1/2})^{-2}_o(f_{5/2})^{-2}_o$ |
| $Po^{210}$ | " | Closed Shell | $Pb^{206}$ | " | $(p_{1/2})^{-2}_o$ |
| $Po^{212}$ | " | $(2g_{9/2})^2_o$ | $Pb^{208}$ | " | Closed Shell |
| $Po^{214}$ | " | $(2g_{9/2})^4_o$ | $Pb^{210}$ | " | $(2g_{9/2})^2_o$ |
| $Po^{216}$ | " | $(2g_{9/2})^6_o$ | $Pb^{212}$ | " | $(2g_{9/2})^4_o$ |
| $Po^{218}$ | " | $(2g_{9/2})^8_o$ | $Pb^{214}$ | " | $(2g_{9/2})^6_o$ |
| $At^{203}$ | $(h_{9/2})^3_{9/2}$ | $(p_{1/2})^{-2}_o(f_{5/2})^{-4}_o(p_{3/2})^{-2}_o$ | $Bi^{199}$ | $1h_{9/2}$ | $(p_{1/2})^{-2}(f_{5/2})^{-6}_o(p_{3/2})^{-2}_o$ |
| $At^{205}$ | " | $(p_{1/2})^{-2}_o(f_{5/2})^{-4}_o$ | $Bi^{201}$ | " | $(p_{1/2})^{-2}_o(f_{5/2})^{-4}_o(p_{3/2})^{-2}_o$ |
| $At^{207}$ | " | $(p_{1/2})^{-2}_o(f_{5/2})^{-2}_o$ | $Bi^{203}$ | " | $(p_{1/2})^{-2}_o(f_{5/2})^{-4}_o$ |
| $At^{209}$ | " | $(p_{1/2})^{-2}_o$ | $Bi^{205}$ | " | $(p_{1/2})^{-2}_o(f_{5/2})^{-2}_o$ |
| $At^{211}$ | " | Closed Shell | $Bi^{207}$ | " | $(p_{1/2})^{-2}_o$ |
| $At^{213}$ | " | $(2g_{9/2})^2_o$ | $Bi^{209}$ | " | Closed Shell |
| $At^{215}$ | " | $(2g_{9/2})^4_o$ | $Bi^{211}$ | " | $(2g_{9/2})^2_o$ |

Table I (cont'd.)

| Element | Proton Config. | Neutron Config. | Element | Proton Config. | Neutron Config. |
|---|---|---|---|---|---|
| $At^{217}$ | $(1h_{9/2})^{3}_{9/2}$ | $(2g_{9/2})^{6}_{o}$ | $Bi^{213}$ | $1h_{9/2}$ | $(2g_{9/2})^{4}_{o}$ |
| $At^{219}$ | " | $(2g_{9/2})^{8}_{o}$ | $Bi^{215}$ | " | $(2g_{9/2})^{6}_{o}$ |
| $Po^{203}$ | $(1h_{9/2})^{2}_{o}$ | $(3p_{1/2})^{-2}_{o}(f_{5/2})^{-3}_{5/2}(p_{3/2})^{-2}_{o}$ | $Pb^{199}$ | Closed Shell | $(3p_{1/2})^{-2}_{o}(2f_{5/2})^{-5}_{5/2}(p_{3/2})^{-2}_{o}$ |
| $Po^{205}$ | " | $(3p_{1/2})^{-2}_{o}(f_{5/2})^{-3}_{5/2}$ | $Pb^{201}$ | " | $(3p_{1/2})^{-2}_{o}(2f_{5/2})^{-3}_{5/2}(p_{3/2})^{-2}_{o}$ |
| $Po^{207}$ | " | $(3p_{1/2})^{-2}_{o}(f_{5/2})^{-1}_{5/2}$ | $Pb^{203}$ | " | $(3p_{1/2})^{-2}_{o}(2f_{5/2})^{-3}_{5/2}$ |
| $Po^{209}$ | " | $(3p_{1/2})^{-1}_{1/2}$ | $Pb^{205}$ | " | $(3p_{1/2})^{-2}_{o}(2f_{5/2})^{-1}_{5/2}$ |
| $Po^{211}$ | " | $2g_{9/2}$ | $Pb^{207}$ | " | $(3p_{1/2})^{-1}_{1/2}$ |
| $Po^{213}$ | " | $(2g_{9/2})^{3}_{9/2}$ | $Pb^{209}$ | " | $2g_{9/2}$ |
| $Po^{215}$ | " | $(2g_{9/2})^{5}_{9/2}$ | $Pb^{211}$ | " | $(2g_{9/2})^{3}_{9/2}$ |
| $Po^{217}$ | " | $(2g_{9/2})^{7}_{9/2}$ | $Pb^{213}$ | " | $(2g_{9/2})^{5}_{9/2}$ |

The expression so obtained is identical with what one would get using the formula proposed by Tolhoek and Brissaard.[8] But one has to be very careful in using this expression because it favors too much the formation of alpha-particles from single particle states with high angular momenta.

## III. DISCUSSION

The comparison of experimental and calculated reduced widths in Figs. 2, 3, and 4 show clearly that taking into account the nuclear structure, even in a very crude approximation, gives results that agree well with the general features of the experimental data. Especially the behavior of the reduced width when crossing the neutron number 126 is well reproduced. It seems to us that one no longer needs to introduce a sudden jump of the nuclear radius at the double magic nucleus $Pb^{208}$ to explain the difference in the reduced width of $Po^{210}$ and $Po^{212}$. The increase of the reduced width when going from N = 126 to N = 128 is quite naturally brought by the change in the single particle states involved in the alpha-decay. Particles in the $2g_{9/2}$ subshell are favored over particles in the $3p_{1/2}$ and $2f_{5/2}$ subshell in forming an alpha-particle, first because of the higher angular momentum (compare Eq. II.3) and second, because the $2g_{9/2}$ level belongs to a higher major shell and therefore the radial part of the wave function is larger too at the edge of the nucleus. This effects the quantity $R^{n_1 l_1 n_2 l_2}_{n_3 l_3 n_4 l_4}(R_0)$ (compare Eq. II.5).

The remaining discrepancies may have several reasons. First we have neglected configurational mixing. Configurational mixing under the influence of a short range attractive force however tends to bring the particles closer

together which would increase the overlap with the alpha-particle wave functions. This increase would be, roughly speaking, proportional to the number of states which are available. The number of states however increases with an increasing number of particles outside or holes in closed shells. To give a more quantitative background to these considerations the reduced widths for the decays $Po^{211} \xrightarrow{\alpha} Pb^{207}$ and $Po^{210} \xrightarrow{\alpha} Pb^{206}$ have been computed with the wave functions given by True and Ford[9] and Newby.[10] The main difference between the two decays is that there is no configurational mixing in the $Pb^{207}$ wave function (one hole in a closed shell) if the shell model has any justification at all. But there is an appreciable amount of mixing in the $Pb^{206}$ wave function. The influence of the mixing in the $Po^{210}$ and $Po^{211}$ wave function (protons) drops out in first approximation if one forms the ratio $\delta^2(Po^{211})/\delta^2(Po^{210})$. The result of the calculation is, if we set $\delta^2(Po^{210}) = 0.676\ 10^{-2}$ Mev, $\delta^2(Po^{211}) = 0.230\ 10^{-2}$ Mev and experimentally we have $\delta^2(Po^{211}) = 0.286\ 10^{-2}$ Mev.

A second reason for some discrepancies could be that even by forming ratios of transition probabilities we have not eliminated the errors introduced by the harmonic oscillator wave functions which might differ considerably from more exact single-particle wave functions.

A further reason for discrepancies could be that there is a sort of "clustering" of particles in the nuclear surface which is not taken into account by the conventional configurational mixing. But we feel that this "clustering" would mainly effect the absolute value of the reduced width, which comes out too small with the wave functions we have used, unless rather unreasonable values for the parameters $\alpha$ and $\beta$ are chosen.[1] A reason for this

"feeling" is that if the shell model gives an adequate description of the nucleus at all, the nuclear wave functions should be approximated reasonably by products of single-particle wave functions inside the nucleus. "Clustering" should be effective in the low density region of the nuclear surface. But the "cluster wave function" has to join smoothly to the shell model wave function. Therefore shell model wave functions should give the relative amplitudes of different "clusters" (for instance $\alpha$-particles with different angular momenta) to a good approximation. Together with the fact that using oscillator wave functions we have greatly underestimated the magnitude of the single particle wave functions in the surface region, these considerations would explain why shell model wave functions give good agreement between theoretical and experimental values for the relative transition probabilities but fail to do so for the absolute values.

We may conclude with a remark about the alpha particle wave function. Concerning this wave function we feel quite sure, that Eq. II.1 provides a good approximation to the actual wave function. There are experimental[5] as well as some theoretical[11] reasons for this feeling. High energy electron scattering experiments[5] show clearly that a gaussian type charge density is a good approximation to the real charge density.

Acknowledgments

I am grateful to Professor I. Perlman and Professor J. O. Rasmussen for the hospitality extended to me at the Lawrence Radiation Laboratory and for many helpful discussions.

I also wish to thank the Deutschen Forschungsgemeinschaft for a grant.

REFERENCES

1. H. J. Mang, Zs. f. Phys. 148, 572 (1957), for details also see H. J. Mang, Sitzungsberichte der Heidelberger Akademie der Wissenschaften (in press - 1958) and H. J. Mang, University of California Lawrence Radiation Laboratory Report, UCRL-8931, October 1959.

2. J. O. Rasmussen, Phys. Rev. 113, 1593 (1959) and University of California Lawrence Radiation Laboratory Report, UCRL-8517, March 1959.

3. H. Casimir, Physica 1, 193 (1934).

4. W. Heitler, Quantum Theory of Radiation, 3rd edition, Clarendon Press, p. 181-189 and the references quoted there.

5. R. Hofstadter, Rev. Modern Phys. 28, 3, 214 (1956).

6. D. Strominger and J. M. Hollander, Decay Schemes, University of California Radiation Laboratory Report, UCRL-8289, June 1958.

7. D. Strominger, J. M. Hollander, and G. T. Seaborg, Rev. Modern Phys. 30, 2, Part II, 585-904, April 1958.

8. P. J. Brussaard and H. A. Tolhoek, Physica 24, 238 (1958).

9. W. True and K. Ford, Phys. Rev. 109, 1675 (1958).

10. N. Newby, Thesis, Indiana University (1958); and N. Newby and E. J. Konopinsley, Phys. Rev. 115, 434 (1959).

11. H. J. Mang and W. Wild, Zs. f. Phys. 154, 182 (1959).

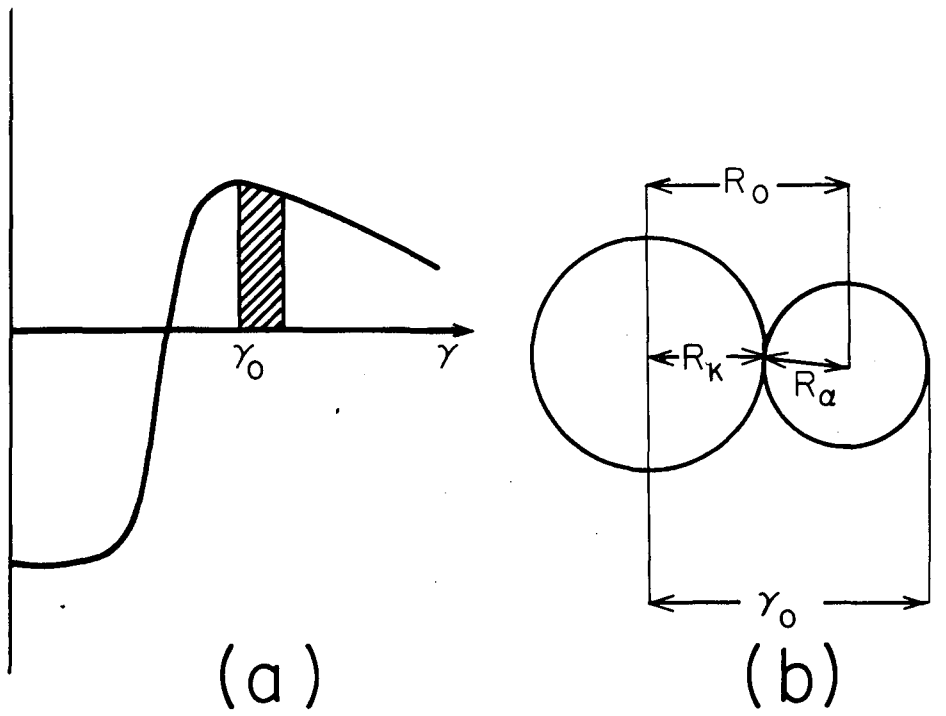


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Fig. 1.

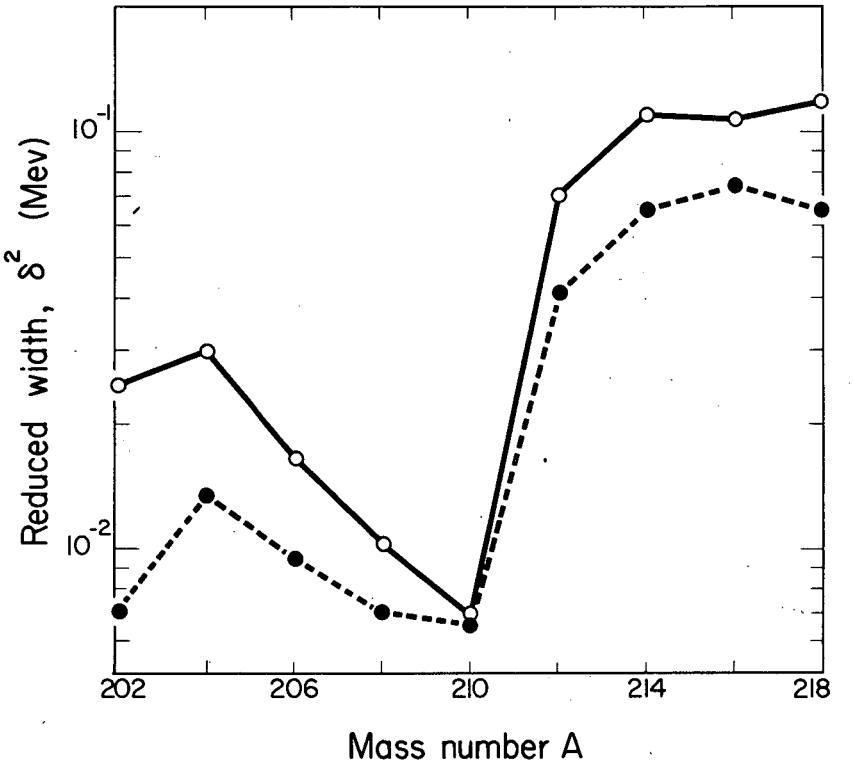


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Fig. 2.

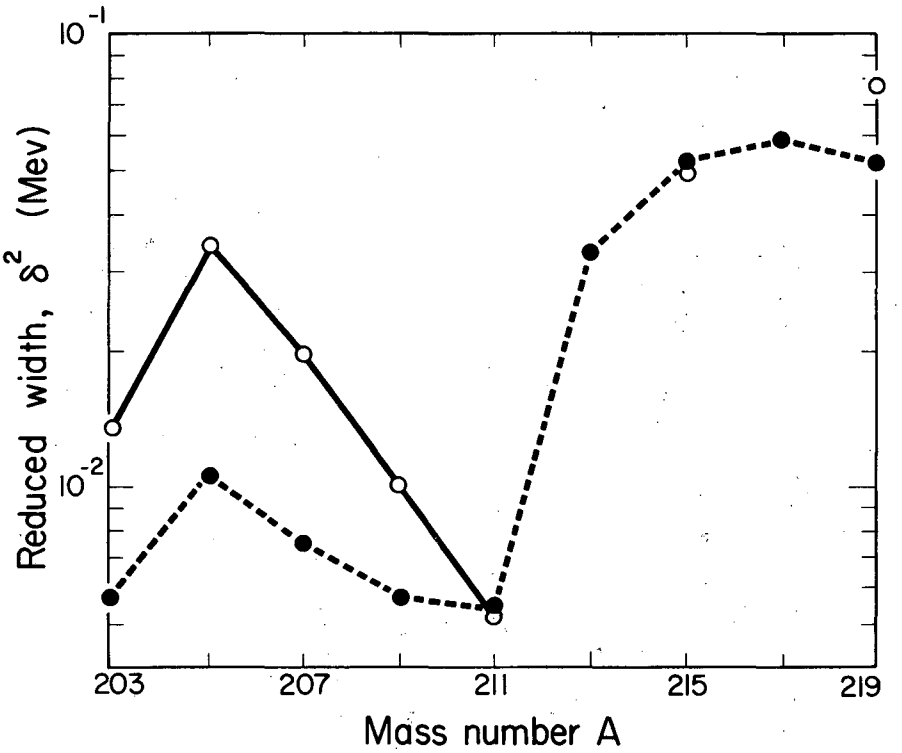


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Fig. 3.

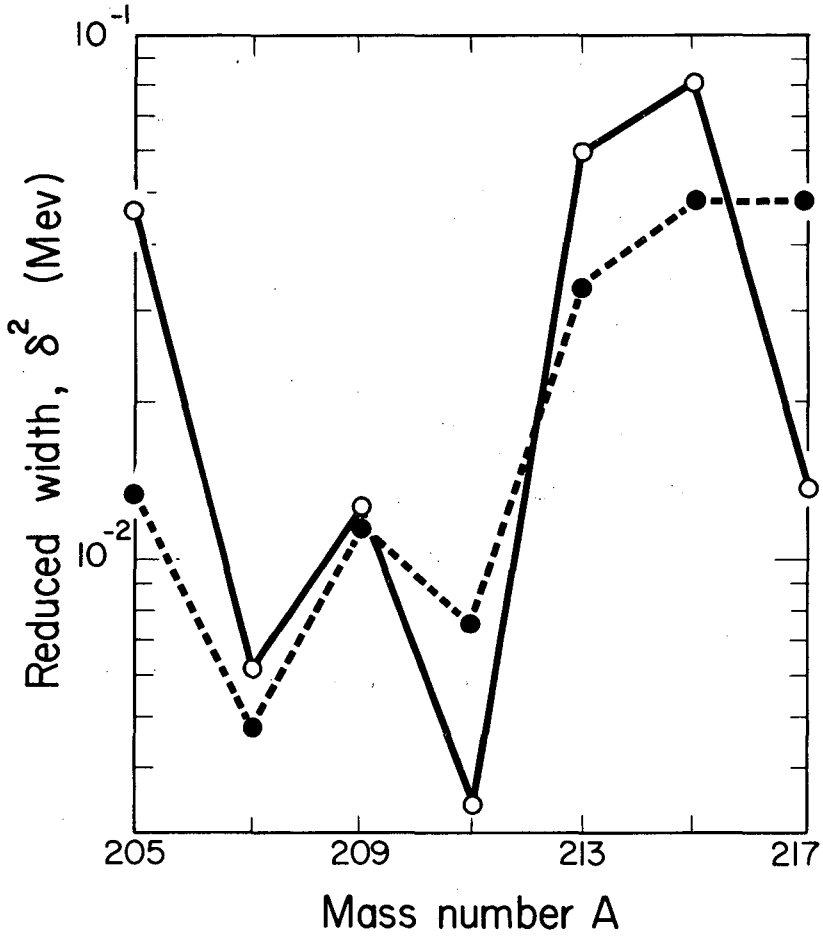


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Fig. 4.

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