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ELECTRIC FIELD CORRELATION AND PLASMA DYNAMICS

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ABSTRACT

The correlation function for the electric field in a fully ionized plasma is derived and its application to the problem of finding the force on a charge moving slowly through a plasma is described. The correlation function is evaluated for a plasma with and without a magnetic field and this function is then converted to the spectrum of the mean-square fluctuation in the random electric field. The application of the generalized fluctuation-dissipation theorem then gives the mean force on a slowly moving test charge. The relation of this to other treatments is briefly discussed.

ELECTRIC FIELD CORRELATION AND PLASMA DYNAMICS*

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I. INTRODUCTION

In a fully ionized plasma the correlation coefficients are important fundamental quantities from which a number of plasma properties can be obtained. In this paper we consider the evaluation, and an application, of the time correlation of the electric field

$$H_{\alpha\beta}(\tau) = \langle E_\alpha(t)\, E_\beta(t+\tau) \rangle ,$$

where E_α is a component of the electric field at a point and the average is over an ensemble. The more general coefficient

$$\langle E(\underset{\sim}{x}, t)\; E(\underset{\sim}{x} + \underset{\sim}{y}, t + \tau) \rangle$$

can be discussed along similar lines.

The application of this correlation function is illustrated by the calculation of the coefficient of dynamical friction (the force on a slowly moving charge in a plasma in thermal equilibrium). The calculation is performed both with and without a magnetic field and this coefficient is

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used to find an approximate value of the resistivity. The method adopted to obtain the coefficient of dynamical friction is first to transform the correlation function to give the spectrum of the mean square (thermal) fluctuations in electric field within the plasma. This is achieved by the use of the Wiener-Khintchine theorem.[1] The generalized fluctuation-dissipation,[2] or Nyquist, relation is then used to relate this spectrum (the analogue of the 'power spectrum' in noise analysis) to the dynamical friction in somewhat the same way that the noise power spectrum of a resistor is related to its resistance.

II. THE CORRELATION COEFFICIENT

Since the field is the sum of contributions of individual particles the correlation coefficient can be written

$$H_{\alpha\beta}(\tau) = e^2 \left\langle \sum_{ij} \frac{(\underset{\sim}{r}_i)_\alpha \, (\underset{\sim}{r}'_j)_\beta}{r_i^{\,3} \; r'^{\,3}_j} \right\rangle , \qquad (1)$$

where $\underset{\sim}{r}_i$ is the position of the ith particle at t and $\underset{\sim}{r}'_i$ its position at $t + \tau$, and e is the charge on the particles. Only one species of particle is considered here but the generalization is obvious.

We now introduce the approximation, discussed below, that in this expression the particles can be regarded as independent, then only terms with $i = j$ contribute to the ensemble average and this average can be performed over each particle separately since the ensemble distribution function is then the product of individual particle distribution functions. These are

themselves products of the velocity distribution function and the (uniform) spatial distribution function. The average over coordinate space merely involves an integration over r and

$$H_{\alpha\beta}(\tau) = \left\langle ne^2 \int \frac{(\underset{\sim}{r}_i)_\alpha \, (\underset{\sim}{r}'_j)_\beta \, d^3\underset{\sim}{r}}{r_i^3 \; r'^3_j} \right\rangle_V ,$$

where now the average is to be taken only over the particle distribution in velocity space.

It is convenient to introduce a function which plays the role of a potential,[3]

$$K_{(\underset{\sim}{b}-\underset{\sim}{a})} = ne^2 \int \frac{1}{|\underset{\sim}{r} + \underset{\sim}{a}|} \cdot \frac{1}{|\underset{\sim}{r} + \underset{\sim}{b}|} \, d^3\underset{\sim}{r} \; . \qquad (2)$$

Then if we write $\underset{\sim}{r}' = \underset{\sim}{r} + \underset{\sim}{s}$ so that $\underset{\sim}{s}$ is a function only of the velocity and the time separation τ, (e.g., for completely uncorrelated motion in zero magnetic field $\underset{\sim}{s} = \underset{\sim}{v}\,\tau$); then

$$H_{\alpha\beta}(\tau) = \left\langle - \frac{\partial^2}{\partial s_\alpha \; \partial s_\beta} \; K(\underset{\sim}{s}) \right\rangle_V \; .$$

The integration in K is easily evaluated and yields

$$H_{\alpha\beta}(\tau) = 2\pi ne^2 \left\langle \left\{ \frac{\delta_{\alpha\beta}}{s} - \frac{s_\alpha \; s_\beta}{s^3} \right\} \right\rangle_V \; . \qquad (3)$$

In this formula it must be remembered that $\underset{\sim}{s}$ is a function only of $\underset{\sim}{v}$ and τ.

The assumption of independent particles means that the correlation coefficient (3) becomes incorrect for very short or very long τ and we must accordingly modify $H(\tau)$. To discuss it exactly in these limits would involve calculations well beyond the scope of this paper. However, a simple modification of (3), sufficient for our present purposes, can be obtained in the following way.

In taking the spatial average it was assumed that the ensemble distribution was uniform in r, whereas it should have been weighted by

$$\exp\left(-\sum_{i<j} V_{ij}/kT\right)$$

where V_{ij} is the interaction energy. Now this factor is ≈ 1 unless the configuration is such that any two particles are within some distance of order

$$b \simeq e^2/kT$$

at which their interaction energy is approximately equal to the thermal energy. Now it is clear that the divergence in $H(\tau)$ as $\tau \to 0$ comes from the contribution to the integral (2) from small r, and it is in this region that the neglect of the Boltzman factor is most serious. However here it can be allowed for by introducing a cutoff in the integration at $r = b$ so that

$$H_{\alpha\beta}(\tau) = 0 \quad \text{if} \quad s < b \,. \tag{4}$$

At long times the assumption of independent particle trajectories (embodied in the neglect of terms with $i \neq j$) is incorrect. A measure of the time for which the independent approximation is realistic is given

by the period of collective oscillations, which in the absence of magnetic fields is the plasma period τ_p, beyond which collective effects are certainly important; therefore we further impose

$$H_{\alpha\beta}(\tau) = 0 , \quad \tau > \tau_p \simeq \sqrt{\frac{m}{4\pi ne^2}} . \tag{5}$$

The usefulness of these approximations depends on the fact that interesting results are insensitive to the values of b and τ_p.

III. DYNAMICAL FRICTION, $\underset{\sim}{B} = 0$

To illustrate the use of the correlation function, consider first the case of plasma without a magnetic field. Then from isotropy $H_{\alpha\beta} = 0$ if $\alpha = \beta$ and the other components are equal and are denoted by $H(\tau)$. Then

$$H(\tau) = 2\pi ne^2 \left\langle \frac{\sin^2 \psi}{s} \right\rangle_V ,$$

where ψ is the angle between $\underset{\sim}{s}$ and the axis along which we are computing H. We now insert the values $s = v\tau$, and perform the averaging for a Maxwellian distribution, then

$$H(\tau) = \frac{8\pi^{1/2}\, ne^2}{3\,\tau} \sqrt{\frac{m}{2kT}} . \tag{6}$$

The generalized Nyquist theorem[2] can now be used to obtain the dynamical friction on a slowly moving charge. If the mean force on a

test charge e_1 oscillating in the plasma (assumed again to be in thermal equilibrium) with frequency p is $F(p)\, e^{ipt}$ and the velocity is $v(p)\, e^{ipt}$, then a frictional coefficient can be defined by

$$e_1 E(p) = F(p) = \mu_{(p)}\, v(p) \quad ,$$

and the Nyquist formula states that the mean square fluctuating force on the same test particle at rest is

$$\langle F^2 \rangle = \int_{-\infty}^{\infty} G(p)\, dp \quad ,$$

where

$$G(p) = \frac{kT}{\pi}\, \mu'(p)$$

and where μ' is the real part of μ.

The spectral function $G(p)$ is related to the time-correlation coefficient by the Wiener-Khintchine theorem[1]

$$e_1^2\, H(\tau) = \int_{-\infty}^{\infty} G(p)\, e^{ip\tau}$$

and the corresponding inverse relation, so that we can express $\mu'(p)$ in terms of the correlation coefficient of the fluctuating force, namely

$$\mu'(p) = \frac{e_1^2}{kT} \int_0^{\infty} H(\tau) \cos(p\tau) d\tau \quad . \tag{7}$$

The dynamical friction, as usually defined, refers to the quantity $\mu'(0)$ $(= \mu)$,

$$\mu = \frac{e_1^2}{kT} \int_0^\infty H(\tau)\, d\tau \quad . \tag{8}$$

Inserting the value for $H(\tau)$, we find that the friction on a slowly moving charge e_1 is

$$F = \mu v = \frac{8\pi^{1/2}\, ne^2\, e_1^2}{3\, kT} \sqrt{\frac{m}{2kT}}\; v \log \Lambda \quad ; \tag{9}$$

here Λ is the ratio λ/b , where λ is the Debye length. Strictly we should defer taking the velocity average until after the τ integration, but the error is negligible provided $\lambda \gg b$ as is usually the case.

IV. PLASMA RESISTIVITY

The result obtained above can be used to derive an approximate value for plasma resistivity. To do this consider a model of the plasma which is the opposite extreme to the Lorentz model (in the Lorentz model electron-electron interactions are neglected), namely one in which electron-electron interactions are sufficiently strong to maintain the electrons in thermal equilibrium even under the influence of the electric field, which merely produces a relative velocity w between ions and electrons. Then the ion-electron drag produces a rate of momentum transfer

$$P = N \mu w$$

(N = ion density = n/Z), while the current density is

$$j = new \; ,$$

so that the resistivity, as defined by Spitzer,[4] is

$$\eta_1 = \frac{16}{3} \frac{\pi^{1/2} m^{1/2} e^2 Z \log \Lambda}{(2\ kT)^{3/2}} \quad . \tag{10}$$

The resistivity of the Lorentz model is[4]

$$\eta_L = \frac{1}{2} \frac{\pi^{3/2} m^{1/2} e^2 Z \log \Lambda}{(2\ kT)^{3/2}} \quad ,$$

while the actual value, for ions of $Z = 1$, as calculated by numerical integration[5] of the Fokker-Planck equation, is

$$\eta_s = \frac{\eta_L}{0.58} = 0.51\ \eta_1 \quad ,$$

so that the true value lies near the geometric mean of the Lorentz value, which is correct as $Z \to \infty$, and the value derived here, which is correct as $Z \to 0$.

V. DYNAMICAL FRICTION IN MAGNETIC FIELD

In a magnetic field with the z axis in the direction of $\underset{\approx}{B}$ the off-diagonal element H_{xy} is nonzero and can be calculated in the same way as the diagonal components H_{zz} (denoted here by $H_{||}$) and $H_{xx} = H_{yy}$ ($= H_{\perp}$), which are discussed in detail below.

In the presence of a magnetic field the particle orbits are no longer straight lines, but it is still true that for the diagonal components

$$H_{\alpha\alpha} = 2\pi n e^2 \left\langle \frac{\sin^2 \psi}{s} \right\rangle \quad ,$$

and if $s = (x, y, z)$,

$$H_{||}(\tau) = 2\pi ne^2 \left\langle \frac{x^2 + y^2}{s^3} \right\rangle ,$$

$$H_{\perp}(\tau) = 2\pi ne^2 \left\langle \frac{y^2 + z^2}{s^3} \right\rangle .$$

A. Parallel Component $H_{||}$

It is convenient to express the velocity in polar coordinates v, θ, ϕ with the θ axis along the direction of B. Then

$$x = \frac{2v}{\omega} \sin\theta \sin\frac{\omega\tau}{2} \cos\left(\phi + \frac{\omega\tau}{2}\right) ,$$

$$y = \frac{2v}{\omega} \sin\theta \sin\frac{\omega\tau}{2} \sin\left(\phi + \frac{\omega\tau}{2}\right) ,$$

$$z = \frac{2v}{\omega} \cdot \frac{\omega\tau}{2} \cos\theta ,$$

where $\omega = eB/mc$, and if we write $q = \frac{\omega\tau}{2}$,

$$H_{||}(\tau) = 2\pi ne^2 \left\langle \frac{\frac{4v^2}{\omega^2} \sin^2\theta \sin^2 q}{\left[\frac{4v^2}{\omega^2} (q^2 \cos^2\theta + \sin^2 q \sin^2\theta)\right]^{3/2}} \right\rangle . \tag{13}$$

The average over θ can be separated from that over v, and one finds

$$H_{||}(\tau) = \pi ne^2 \left\langle \frac{1}{v} \right\rangle \frac{1}{2} \int_{-1}^{+1} \frac{(1 - \mu^2) \sin^2 q}{[\sin^2 q + (q^2 - \sin^2 q)\mu^2]^{3/2}} \, d\mu ,$$

leading to a coefficient of friction, from Eq. (8),

$$\mu_{||} = \frac{2\pi n e^2 e_1^2}{kT} \left\langle \frac{1}{v} \right\rangle \int dq \int_0^{+1} d\mu \frac{(1-\mu^2)\sin^2 q}{[\sin^2 q + (q^2 - \sin^2 q)\mu^2]^{3/2}} . \tag{14}$$

The singularities of the integrand at the points where the denominator vanishes --i.e., at $\mu = 0$, $q = n\pi$ --are excluded from the range of integration by the condition (4), since this implies $H = 0$ if

$$(\sin^2 q + (q^2 - \sin^2 q)\mu^2) < \frac{b^2}{4a^2} , \tag{15}$$

where a is the Larmor radius.[6]

Only situations with $b^2 \ll 4a^2$ are of interest and in this case it can be shown, by expanding the integrand about $\mu = 0$, $q = n\pi$, $(n \geqslant 1)$ that the additional contribution from each singularity obtained by neglecting the subsidiary condition (15) is of order

$$\frac{b}{4a} \cdot \frac{1}{n} ;$$

so that the total contribution from all singularities is of the order

$$\frac{b}{4a} \sum_1^N \frac{1}{n} ,$$

where N is limited so that πN lies within the range of the q integration. It will become apparent that this contribution forms a negligible fraction of the total integral over the range $0 \leqslant \mu \leqslant 1$, $\frac{\omega b}{2v} < q < q_{max}$.

After the μ integration one finds

$$\mu_{\parallel} = \frac{2\pi n e^2 e_1^2}{kT} \left\langle \frac{1}{v} \right\rangle \int \frac{1}{q} \left[\frac{1}{(1 - Q^2)} - \frac{Q^2}{(1 - Q^2)^{3/2}} \log\left(\frac{1 + \sqrt{1 - Q^2}}{|Q|} \right) \right] dq \tag{16}$$

where $Q = \sin q/q$.

Now if $q \ll 1$ then the integrand $\approx \frac{2}{3q}$, while if $q \gg 1$ the integrand $\approx 1/q$, so that the effect of the magnetic field on the integration is very small. In fact if $\frac{\lambda}{a} \ll 1$ then $q \ll 1$ over the whole range, and the upper limit on the τ, or q , integration is again set by the plasma period so that the effect of the field is completely negligible and

$$\mu_{\parallel}(B) = \mu , \qquad \text{if} \quad \lambda \ll a .$$

On the other hand, if $a \ll \lambda$, then we can estimate the value of the integral (16) by dividing the integration into two regions in each of which the asymptotic form may be used, that is, we put the integrand equal $1/q$ if $q > 1$ and equal $\frac{2}{3q}$ if $q < 1$. Then

$$\mu_{\parallel} = \frac{8\pi^{1/2} n e^2 e_1^2}{3kT} \sqrt{\frac{m}{2kT}} \left[\log \frac{2a}{b} + \frac{3}{2} \log \frac{\omega\tau_m}{2} \right] , \tag{17}$$

where τ_m is the cutoff introduced by collective effects. In the absence of magnetic field $\tau_m \simeq 1/\omega_p$, but in the presence of magnetic field the frequency of collective oscillations depends on direction, being ω_p along the lines of force and $\sqrt{\omega_p^2 + \omega^2}$ across them. It is therefore not immediately apparent what τ_m should be, but the work of Rostoker and Rosenbluth[7]

indicates that it should be $(\omega_p\,\omega)^{-1/2}$ so that finally

$$\mu_{\parallel} = \frac{8\pi^{1/2}\, ne^2\, e_1^{\,2}}{3kT} \sqrt{\frac{m}{2kT}} \left[\log \frac{2a}{b} + \frac{3}{4} \log \frac{\lambda}{4a} \right] . \tag{18}$$

It will be noted that, even disregarding the uncertainty in the cutoff, the effect of the magnetic field is extremely small.

B. Transverse Component $H_{\perp}(\tau)$

Proceeding as in the case of the parallel component one finds

$$H_{\perp}(\tau) = \pi ne^2 \omega \left\langle \frac{1}{v} \right\rangle \int_0^1 \frac{q^2\,\mu^2 + \frac{1}{2}\sin^2 q(1-\mu^2)}{(\sin^2 q + (q^2 - \sin^2 q)\mu^2)^{3/2}}\, d\mu \tag{19}$$

and

$$\mu_{\perp} = \frac{2\pi\, ne^2\, e_1^{\,2}}{kT} \left\langle \frac{1}{v} \right\rangle \int_0^1\!\!\int \frac{q^2\,\mu^2 + \frac{1}{2}\sin^2 q(1-\mu^2)}{(\sin^2 q + (q^2 - \sin^2 q)\mu^2)^{3/2}}\, d\mu\, dq . \tag{20}$$

As in the case of $\mu_{\parallel}$ it can be shown that the contribution from the singularities is negligible and the μ integration gives

$$\mu_{\perp} = \frac{2\pi\, ne^2\, e_1^{\,2}}{kT} \left\langle \frac{1}{v} \right\rangle \int \frac{1}{q} \left[\frac{(1 - \frac{1}{2}Q^2)}{(1-Q^2)^{3/2}} \log \frac{(1 + \sqrt{1-Q^2})}{|Q|} - \frac{1}{2(1-Q^2)} \right] dq . \tag{21}$$

If $q \ll 1$ the integrand again reduces to $\frac{2}{3q}$, so that

$$\mu_{\perp}(B) = \mu \quad \text{if} \quad \lambda \ll a \ .$$

In the regime $\lambda \gg a$ the region of integration can again be divided according as $q \gtrless 1$. The contribution from $q < 1$ gives

$$\frac{8\pi^{1/2}\, ne^2\, e_1^{\,2}}{3kT} \sqrt{\frac{m}{2kT}} \log \frac{2a}{b} \ .$$

In the region $q > 1$ an approximation which is sufficient in a search for gross effects is to write the integral as approximately

$$\int_1^{\omega\tau_m/2} \frac{1}{q} \left(\log 2q - \log |\sin q| - \frac{1}{2}\right) dq$$

and then to replace the middle term by its mean value, so that finally

$$\mu_{\perp} = \frac{8\pi^{1/2}\, ne^2\, e_1^{\,2}}{3kT} \sqrt{\frac{m}{2kT}} \left(\log \frac{2a}{b} + \frac{3}{4} \log^2 \frac{\omega\tau_m}{2} + \frac{3}{2}\left(\log 4 - \frac{1}{2}\right) \log \frac{\omega\tau_m}{2}\right) \ . \tag{22}$$

CONCLUSION

A general expression for the electric field correlation in a plasma has been derived according to which the correlation falls off as $1/\tau$ over a wide range of τ. From the correlation an equivalent expression for the spectrum of the mean-square fluctuating electric field can be derived and

by the use of Nyquist's theorem one then finds a value for the coefficient of friction. In a plasma without magnetic field this is

$$\mu = \frac{8\pi^{1/2}\, ne^2\, e_1^{\,2}}{3kT} \sqrt{\frac{m}{2kT}} \log \Lambda \quad . \tag{23}$$

Our treatment so far refers to the drag on a test particle of small charge but effectively infinite mass (so that its velocity is unaffected by the drag). One usually considers a test particle of finite mass, in which case one must correct for its recoil. For a field particle of given velocity this correction involves replacing the real field particle mass by its reduced mass, which means that (23) must be multiplied by

$$\left(1 + \frac{m}{m_{test}}\right) \quad .$$

The result then agrees with that obtained by Gasiorowicz et al.[8] and by Spitzer.[4] It appears to differ from that given by Rostoker and Rosenbluth[7] by a factor $\sqrt{2}$.

In the case that the 'test' particle is an ion and the field particles are electrons the correction is negligible, so that we do not need to modify the estimate (10) of the resistivity on this account.

In the presence of an applied magnetic field an explicit expression has been given for the diagonal components of the friction tensor $\mu_{\alpha\beta}$, and these have the approximate forms given in the text. Apart from the factor $\sqrt{2}$, mentioned above, the result for $\mu_{\parallel}$ also agrees with that given by Rostoker and Rosenbluth, but the result for $\mu_{\perp}$ appears to differ both numerically and qualitatively by the appearance of a term of the form $\log^2 \omega\tau_m$.

Since it is really only the electric field which influences a particle in a plasma it may be more realistic to deal with the field directly, as in the present paper, rather than introduce the concept of 'collisions'.

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REFERENCES

1. M. C. Wang and G. E. Uhlenbeck, Revs. Modern Phys. 17, 326 (1945).

2. H. B. Callen and T. A. Welton, Phys. Rev. 83, 34 (1951); also L. D. Landau E. M. Lifshitz, Statistical Physics (Pergamon Press, London, 1958).

3. Strictly this integral is divergent but the divergent part disappears on taking the derivative, which is all that is required.

4. Lyman Spitzer, Physics of Fully Ionized Gases (Interscience Publishers, Inc., New York, 1956).

5. L. Spitzer and R. Harm, Phys. Rev. 89, 977 (1953).

6. We again refer to an average velocity in discussing the cutoff.

7. N. Rostoker and M. N. Rosenbluth, Phys. Fluids 3, 1 (1960).

8. S. Gasiorowicz, M. Neuman, and R. J. Riddell, Phys. Rev. 101, 922 (1956).

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