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ABSTRACT

This report gives a simple analysis of the tubular pinch discharge as a circuit element based on the assumptions of cylindrical symmetry and small deviations from equilibrium. External observables, such as voltage and currents, may be interpreted under these assumptions in terms of plasma thickness and plasma motion, and therefore may be used for temperature and pressure estimates. The analysis is applied to an example of actual oscilloscopically observed signals.

A SIMPLE ANALYSIS OF THE TUBULAR PINCH DISCHARGE*

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INTRODUCTION

In the tubular pinch discharge[1] a hollow cylindrical plasma sleeve is created by passing a high current through a low-density gas along the annular space between two coaxial cylindrical return conductors, as indicated in Fig. 1. The configuration is analogous to a certain type of transmission line of particularly low inductance which was called "Triaxial Transmission Line," hence the device has been dubbed "Triax." The configuration thus formed is the simplest type of "hard-core" pinch[2] possible. Of course, without the help of any axial magnetic field the plasma is not completely stable.[3] However, the instabilities appear to grow sufficiently slowly, perhaps because of the stabilizing influence of the closely spaced conducting walls, so that observation of repeated and relatively regular and reproducible pinch oscillations are possible. An example of oscilloscope traces from which one can infer such oscillations is shown in Fig. 2, taken from Reference 1. It should be pointed out that the oscilloscope traces in Fig. 2 are superpositions made from three individual discharges for each trace in order to demonstrate the reproducibility of the signals. In this report the relationship between observable signals such as those reproduced in Fig. 2 and the behavior of the plasma is explored in some detail.

Some of the gross features of the tubular pinch may be adequately described in terms of suitably chosen simplified models. In all experiments to date, the idealized Triax discharge is characterized by two principal features. First, as in all conventional linear pinch configurations, complete cylindrical symmetry is assumed throughout i.e., the radial position r is considered to be the only variable in space. Second, since the entire discharge volume is completely enclosed by a copper return conductor, it is certain that the "net" magnetic flux inside the pinch chamber may be considered constant in time, at least during the brief time-span with which we are concerned here. In particular, in the absence of an externally produced magnetic field, the net flux inside the Triax chamber must remain zero at all times. It follows, in the case of the Triax discharge that the flux linked between the pinched plasma and the inner return conductor must be equal and opposite to the flux linked between the plasma and the outer return conductor at every instant of the discharge. This requirement will henceforth be termed "flux balance."

*Work done under the auspices of the U.S. Atomic Energy Commission.

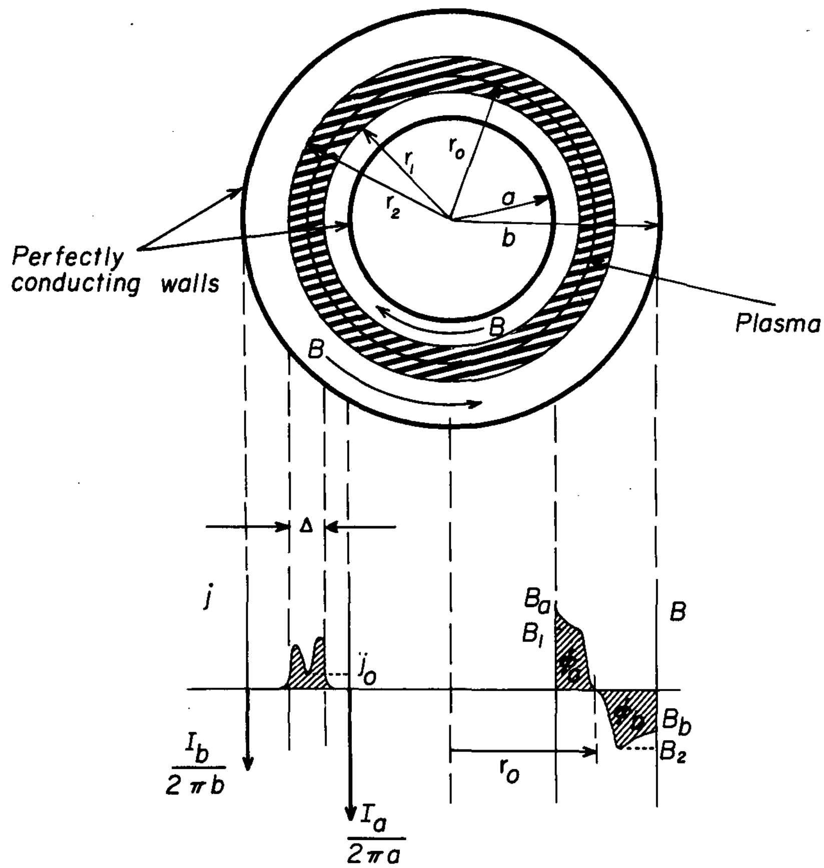

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Fig. 1. Tubular pinch geometry, current, and magnetic field.

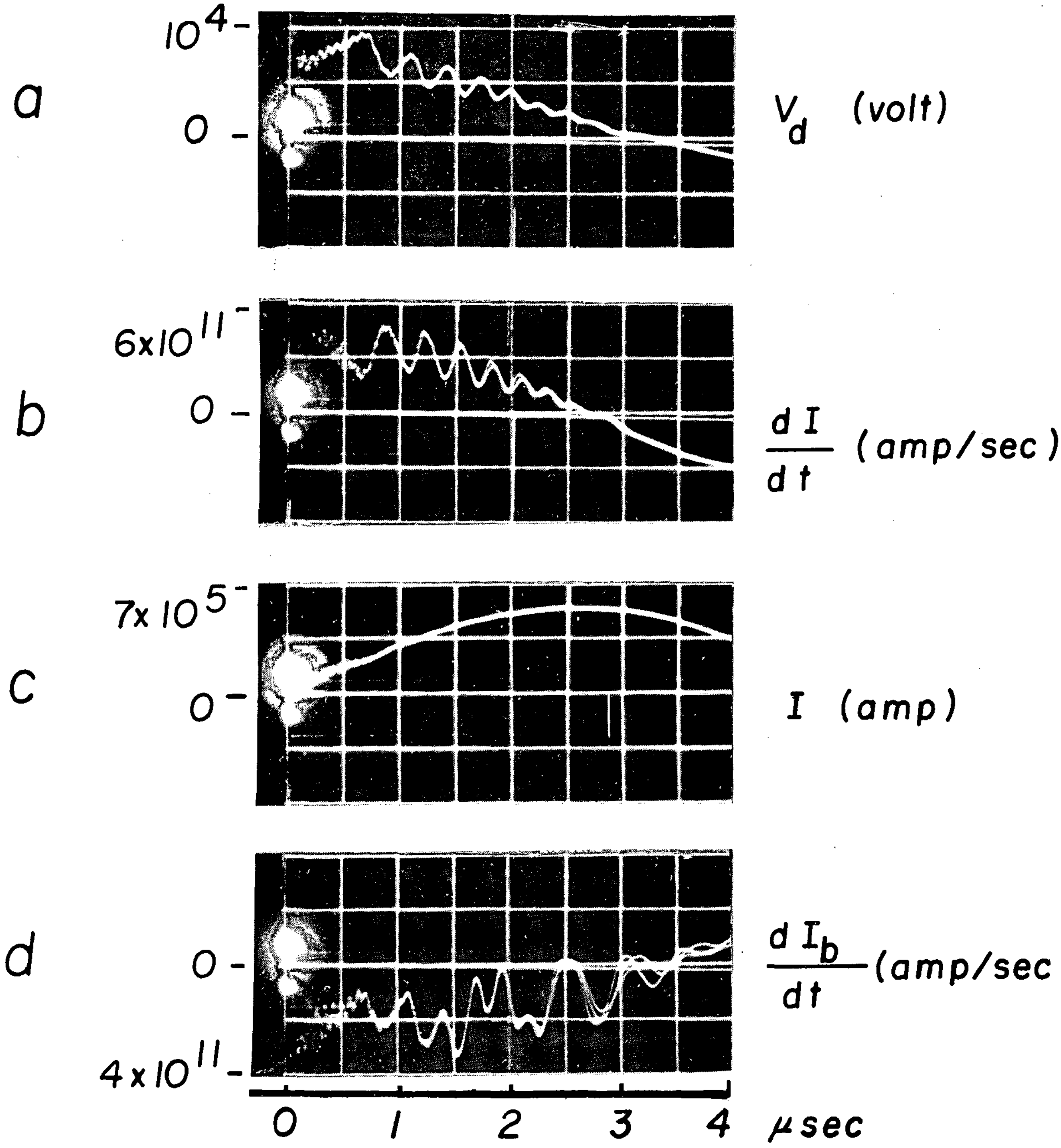

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Fig. 2. External observables in the tubular pinch discharge.

EQUILIBRIUM CONDITIONS

On the basis of the conditions given above, certain simple conclusions may be made immediately. For instance, if the plasma in equilibrium with the self-magnetic forces is pinched into a region $r_1 < r < r_2$ (see Fig. 1) such that the particle pressure at r_1 and r_2 is negligibly small, the following relations must hold:

$$4\pi \frac{dp}{dr} = - \frac{B}{r} \frac{d}{dr} (rB) = - \frac{1}{2r^2} \frac{d}{dr} (rB)^2;$$

therefore,

$$r_2^2 B_2^2 - r_1^2 B_1^2 = - 8\pi \int_{r=r_1}^{r=r_2} r^2 \, dp = 8 \int_{r=r_1}^{r=r_2} p \, d(\pi r^2). \quad (1)$$

This condition will be referred to as "pressure balance."

If we now use $p = nkT$, and note

$\frac{1}{2} r_1 B_1 = I_a$, the current on the inside conductor, whereas

$\frac{1}{2} r_2 B_2 = I_b$, the current on the outside conductor,

we obtain, using

$$I_a + I_b = I, \quad (2)$$

$$(I_b - I_a) I = I_b^2 - I_a^2 = 2k \int_{r=r_1}^{r=r_2} Tn d(\pi r^2) = 2kNT_{av}. \quad (3)$$

This is the Triax equivalent of the well-known conventional linear pinch relation. Here N is the total number of particles per unit length of the tube and, since no assumption was made concerning the temperature distribution, T_{av} is written to indicate an average over the plasma cross section.

It is worth noting that in principle at least the plasma temperature in the Triax can be determined directly from simple current measurements if equilibrium conditions can be reached. Even if the plasma is oscillating about its equilibrium configuration it is conceivable that suitable time averages can be used for temperature estimates. Quantitative inspection, however, reveals that the currents would have to be determined with extreme precision.

Equation (3) demonstrates also that in sheet pinches currents larger than in ordinary solid cylindrical pinches are needed to contain a plasma of a given energy content. The basic reason for this disadvantage is not only that a larger perimeter is needed to enclose the same number of particles but also that in such a double-surfaced plasma only half the current is used to produce magnetic pressure on one side.

Since the flux is in opposite directions on the two sides of the plasma it is clear that the magnetic field must be zero somewhere inside the plasma. Let the position of this surface of zero field be denoted by r_0 (see Fig. 1). Then the flux per unit length of tube linked by the plasma and the outer return conductor can be expressed in the form

$$\phi_b = \int_{r_0}^{b} |B|\,dr = 2I_b \ln \frac{b}{r_2} + \int_{r_0}^{r_2} |B|\,dr. \tag{4}$$

Similarly, the flux on the inside is given by

$$\phi_a = \int_{a}^{r_0} |B|\,dr = 2I_a \ln \frac{r_1}{a} + \int_{r_1}^{r_0} |B|\,dr. \tag{5}$$

And, since $\phi_a = \phi_b$, if we neglect the stray flux linked by the electrode structure, we can use the symmetric representation

$$\phi = I_a \ln \frac{r_1}{a} + I_b \ln \frac{b}{r_2} + \frac{1}{2}\int_{r_1}^{r_2} |B|\,dr, \tag{6}$$

which, presupposes as it were the condition of flux balance. Here $\int_{r_1}^{r_2} |B|\,dr$ represents the absolute magnitude of flux inside the current-carrying region. Equations (4) and (5) for the case of flux balance $\phi_a = \phi_b$, together with the condition $I_a + I_b = I = 2\pi \int_{r_1}^{r_2} j\; r dr$, determine the relative value of I_a and I_b when the current distribution is given as a function of r. In particular, when we may neglect the flux inside the plasma--i.e., in the skin current approximation--we have

$$I_a \ln \frac{r_1}{a} = I_b \ln \frac{b}{r_2}. \tag{7}$$

The condition of equilibrium--i.e., pressure balance--in this case requires $B_1 = B_2$ or $I_a/I_b = r_1/r_2$, so that we have

$$r_1 \ln \frac{r_1}{a} = r_2 \ln \frac{b}{r_2}. \tag{8}$$

If the plasma is very thin, so that we have $r_1 \approx r_2 \approx r_0$, we find $I_a \approx I_b \approx I/2$, regardless of the detailed current distribution, and therefore we may write

$$r_0^2 = ab. \tag{9}$$

If the stray flux around the electrode is taken into account, or if the current distribution has appreciable thickness, more complex expressions result. The above relations are useful, however, for semiquantitative interpretation of the observations.

ANALYSIS OF EXTERNAL OBSERVABLES IN THE TUBULAR PINCH

Since the discharge acts as a predominantly inductive circuit element in spite of the relatively low inductance of the configuration, it is useful to express the integrated flux distribution in terms of a lumped self-inductance of the entire Triax tube. The scheme is very simple, of course, if we retain the assumption of complete cylindrical symmetry. The self-inductance per unit length L is defined by

$$IL = \phi,$$

where ϕ is given by Eq. (6). I_a and I_b may be eliminated by means of Eqs.(4) and (5) for $\phi_b = \phi_a$ and the current continuity condition, Eq. (2).

The general result for the inductance per unit length is

$$L = \left(\ln \frac{b}{a}\,\frac{r_1}{r_2}\right)^{-1}\left(2\ \ln\frac{r_1}{a}\ \ln\frac{b}{r_2} + \frac{\phi_1}{I}\ \ln\frac{b}{r_2} + \frac{\phi_2}{I}\ \ln\frac{r_1}{a}\right), \tag{10}$$

where $\phi_1 = \int_{r_1}^{r_0} |B| dr$ and $\phi_2 = \int_{r_0}^{r_2} |B| dr$ are the flux contributions due to the magnetic field inside the current-carrying region.

For a thin sheet-plasma of thickness $r_2 - r_1 = \Delta \ll r_m$, where we define $r_m = (1/2)(r_1 + r_2)$, we may expand the logarithms and neglect all terms of order Δ^2 or smaller and we also may ignore ϕ_1 and ϕ_2 for $\Delta \ll b - a$.

We then obtain

$$L \approx 2\left(\ln\frac{b}{a}\right)^{-1} \ln\frac{r_m}{a}\ln\frac{b}{r_m} - \frac{\Delta}{r_m}\left[1 - 2\left(\ln\frac{b}{a}\right)^{-2} \ln\frac{r_m}{a}\ln\frac{b}{r_m}\right]. \tag{11}$$

It is readily shown in this approximation that for equilibrium, i.e., for pressure balance, we have $r_m \approx r_0 \approx \sqrt{ab}$ and L has its maximum value. If we now limit ourselves to small deviations δ from this equilibrium position of r_m -- that is, if we introduce $\delta = (r_m - \sqrt{ab}) \ll \sqrt{ab}$ --we obtain, for the inductance per unit length of the tube, the approximate expression

$$L \approx L_0 - \frac{\delta^2}{abL_0}\left(1 - \frac{\delta}{\sqrt{ab}}\right) - \frac{\Delta}{2\sqrt{ab}}\left(1 - \frac{\delta}{\sqrt{ab}}\right)\left(1 + \frac{\delta^2}{2abL_0^{\,2}}\right), \qquad (12)$$

where $L_0 = (1/2)\ln(b/a)$ represents the inductance per unit length of an infinitely thin sheet at $r = \sqrt{ab}$. Terms up to order $\delta^2/abL_0^{\,2}$ were not neglected, since L_0 for geometries with small spacing between return conductors, i.e., for $b - a \ll a$, has itself a small value. It is seen, however, that the inductance is very insensitive to displacements which are small compared with the annular spacing $b - a$, since then the term $\delta/\sqrt{ab}\,L_0$, which enters only quadratically, is definitely small compared with unity.

The conclusions concerning the variable inductance, as pointed out before,[1] are well borne out qualitatively by simple observation of the voltage between the driving electrode and the return ground (see **Fig. 2a**). Still better in this respect is the observation of the rate of change of current (**Fig. 2b**), because the voltage at the energy-supplying condenser cannot vary rapidly. As long as the entire circuit behaves like a series LCR-circuit we simply have

$$\frac{dL_t}{dt} = \frac{V_c}{I} - \frac{L_t}{I}\frac{dI}{dt} - R_t, \qquad (13)$$

where V_c is the condenser voltage, and L_t and R_t are the total inductance and resistance in the circuit. Since neither V_c nor I nor R_t can undergo very rapid fluctuations [the fluctuations in L_t and R_t are bound to be small compared with their absolute values (see **Fig. 2c**)], **Eq.** (13) shows that fluctuations in dI/dt are a good measure for oscillations in L_t. In particular, if we use Eq. (12) for the variable inductance we see that we have as a first approximation

$$\frac{d\Delta}{dt} \approx \frac{2\sqrt{ab}}{I\ell}\left(L_t\frac{dI}{dt} - V_c + IR_t\right), \qquad (14)$$

where ℓ is the length of the tubular plasma. This means that both the frequency and amplitude of the oscillations in dI/dt may be directly interpreted in terms of frequency and amplitude of the compression mode $d\Delta/dt$.

For example, Fig. 2b and 2c may be used for an estimate of $d\Delta/dt$. The conditions in this case were as follows: $a = 2.5$ cm; $b = 5$ cm; $\ell = 100$ cm; condenser bank $C = 45\ \mu f$, 20 kv; $L_t = 0.06\ \mu h = 60$ cm (emu). Approximating $|d\Delta/dt|_{max}$ by $[1/2]\,[(d\Delta/dt)_{max} - (d\Delta/dt)_{min}]$ for the purpose of this estimate, we find that the slowly varying quantities V_c and IR_t cancel out, that shortly after the first pinch we have $|d\Delta/dt|_{max} \approx 10^6$ cm/sec, and the period is about 0.4 μsec. In other words, the approximate magnitude of $|dL_t/dt|_{max}$ here is 30 milliohm. This impedance may be compared with R_d, the ohmic resistance of the discharge alone existing at the instant when dI/dt is 0, which follows from Eq. (13) when it is written for the discharge alone:

$$R_d = \frac{V_d}{I} - \frac{dL_d}{dt} - \frac{L_d}{I}\frac{dI}{dt}. \tag{15}$$

Since dL_d/dt is an oscillating term and is small at the time when dI/dt is 0, it can be averaged out. Using the data given in Fig. 2a, b and c we find $R_d \approx$ 2 milliohm. Unfortunately, for $dI/dt \neq 0$, R_d, L_d cannot be separately evaluated from the external observations and therefore the absolute value of Δ cannot be determined with any precision from the signals shown in Fig. 2 unless R_d is assumed negligible or at least constant. Equation (14) also clearly shows that the instants of "maximum pinching," defined by $d\Delta/dt = 0$, $d^2\Delta/dt^2 > 0$ are identified on Fig. 2b as the points where dI/dt is increasing and given by $dI/dt = (1/L_t)(V_c - IR_t)$, that is, somewhere halfway between the times where dI/dt has a minimum and a maximum. The time of the pinch is sometimes mistakenly identified with the instant when dI/dt has a minimum.

It has been customary in most instances to observe the "Triax" tube voltage V_d instead of the rate of change of current. Such measurement has the advantage of a direct calibration by displaying the tube voltage before breakdown. A relationship similar to that given by Eq. (14) involving V_d is, of course, easily derived. But it will not contain any additional information about $d\Delta/dt$ and it has the disadvantage that the discharge inductance L_d enters explicitly, making the approximations used in Eq. (14) less accurate.

It is clear that the displacement δ does not affect the observed total current or voltage of the tubular-pinch discharge unless its absolute value multiplied by its rate of change becomes comparable to the rate of change of the plasma thickness. For discharges at very high power level in tubes with a large ratio of radii of the return conductors such effects have indeed been noticed.

The displacement δ is detectable through its influence on the partial currents I_a and I_b. By neglecting the flux inside the plasma we deduce from Eq. (7) for the outside current

$$I_b = I\left(\ln \frac{b}{a}\frac{r_1}{r_2}\right)^{-1} \ln\frac{r_1}{a}. \tag{16}$$

For a thin plasma of thickness Δ and position r_m this becomes

$$I_b \approx \frac{I}{2L_0}\left[\ln\frac{r_m}{a} + \frac{\Delta}{2r_m}\left(\frac{\ln r_m/a}{L_0} - 1\right)\right], \tag{17}$$

where, as before, $L_0 = (1/2)\ln(b/a)$. It is interesting that for $r_m = \sqrt{ab}$ in this approximation $I_b^0 = (1/2)I$, irrespective of the thickness Δ. Finally, for small deflections from equilibrium this expression becomes

$$I_b \approx \frac{I}{2}\left[1 + \frac{\delta}{\sqrt{ab}\, L_0}\left(1 + \frac{\Delta}{\sqrt{ab}} - \frac{\delta}{4L_0}\right)\right], \tag{18}$$

which involves the deflection δ as a first-order correction.

It should be pointed out, however, that fluctuations of Δ also affect I_b directly through their influence on I according as shown in Eq. (14). The two modes of variation, $d\Delta/dt$ and $d\delta/dt$, are most easily distinguished by comparing I_b with I_a, since with $I_a = I - I_b$ the fluctuations due to δ are in opposite phase whereas those due to Δ must be in phase on the two return conductors. Again, as has been shown,[1] these features are very well demonstrated by the appropriate observations. Fortunately, in the experiments to date the two modes of fluctuations are also quite well separated in frequency, so that it becomes relatively easy with a little practice to distinguish them by simple inspection of oscilloscope traces, e.g., as shown in Fig. 2d.

The average temperature of the plasma in equilibrium may be very crudely estimated from Eq. (3), in conjunction with the flux-balance condition if the ohmic resistance of the discharge is known. Using the approximation for a thin plasma, we find from Eq. (3)

$$\Delta \approx \frac{4NkTr_0}{I^2} . \tag{19}$$

Assuming further that the current density is uniform in the channel of width Δ, one has an equation for the temperature,

$$\frac{\eta(T)}{T} \approx \frac{8\pi abNkR_d}{I^2 \ell} , \tag{20}$$

when $\eta(T)$ denotes the resistivity (expressed in the same units as the resistance R_d). If this is done, for instance, with our example in Fig. 2 - assuming that N (the number of particles per unit length of tube) is determined by the gas pressure before the discharge started (in this case 475 μ of deuterium), and using Spitzer's expression for the resistivity—we find that at peak current, when $R_d = 0.002\,\Omega$, $T \approx 25$ ev. If this value is inserted into Eq. (19), one finds $\Delta \approx 0.2$ cm at current peak, a rather reasonable number.

TUBULAR PINCH DYNAMICS

An explanation of the observed plasma compression and displacement oscillations in terms of some physical models still remains to be given. For completeness we shall include a brief derivation of the time of the first sheet pinch based on a simple snowplow model in an approximately plane geometry driven by a linearly rising current. The equations of motion for cylindrical geometry,

$$\frac{d}{dt}\left[\pi(r^2 - a^2)\rho\dot{r}\right] = \frac{I_a^2}{r} \qquad a \leq r < \frac{a+b}{2}$$

and

$$\frac{d}{dt}\left[\pi(b^2 - r^2)\rho\dot{r}\right] = -\frac{I_b^2}{r} \qquad \frac{a+b}{2} < r \leq b,$$

may be approximated in this case and averaged to give the single plane equation for a sinusoidally ringing circuit,

$$8\pi r_m^2 \rho \frac{d}{dt}(x\dot{x}) = I^2 = I_{max}^2 \sin^2 \omega t, \tag{21}$$

where $I = I_a + I_b$, $r_m = (a+b)/2$, and $x = r - a$ or $x = b - r$. The result is

$$x^2 = \frac{I_{max}^2}{4\pi r_m^2 \rho}\left(t^2 - \frac{\sin^2 \omega t}{\omega^2}\right) \approx \left(\frac{dI}{dt}\right)_0^2 \frac{t^4}{48\pi r_m^2 \rho}\left[1 - \left(\frac{dI}{dt}\right)_0^2 \frac{2t^2}{15 I_{max}^2}\right], \tag{22}$$

or, if the current layers move through a distance $x_1 = (b - a)/2$ to form the first pinch at time t_1, this time is given by

$$t_1 \approx \left(\frac{b^2 - a^2}{\left(\frac{dI}{dt}\right)_{av}}\right)^{1/2} (3\pi\rho)^{1/4}. \tag{23}$$

Experiments in which ρ was varied over a factor of 30 agree fairly well with Eq. (23).[4] Conversely, if the discharge is seen to pinch distinctly, Eq. (23) may be used to check on the effective gas density involved in the compression, since all other variables are determined within fairly narrow limits. There is some question concerning the magnitude of x_1, since the snowplow model does not really allow for an appreciable thickness of the gas or current layer. If the pinch, however, is defined as the time at which the inductance has reached its maximum value, then it is not unreasonable to use the quantity $x_1 = (b - a)/2$, since the pressure waves must actually have passed each other to be able to reverse the mass motion of the opposite snowplow. Although a gas dynamic-shock model would probably be a more correct picture, much refinement in this direction is not warranted, since in these present experiments the skin depth of the pinch currents is just as large as the compressed plasma.

One should discuss next the compression oscillations which normally are seen to follow the first pinch. A proper treatment of these oscillations is rather complicated since the plasma is known to carry current throughout the volume so that it represents a region of very nonuniform field.

As a very crude attempt, however, it might be argued that the duration of one compression cycle should be about equal to the time taken by a compression wave to traverse the plasma. We propose a model according to which

$$\Delta = \bar{v}_c \, \tau, \tag{24}$$

where τ is the period of the compression oscillations and

$$v_c = \left[(kT/m)(\gamma + 2/\beta) \right]^{1/2},$$

is the speed of compression waves transverse to the magnetic field. Here we define $\beta = 8\pi nkT/B^2$. If we assume that the deuterium used is completely dissociated, and collisions are sufficiently frequent to warrant the use of $\gamma = 5/3$, we find agreement, for the oscillations observed in low-power-level discharges, where $\tau \approx 0.3\,\mu$sec and $\Delta \approx 0.7$ cm, assuming $T \approx 20$ ev. This figure seems reasonable inasmuch as it is consistent with estimates from resistivity and pressure balance. But considering the simplicity of the model this may be fortuitous. It is reassuring, on the other hand, that the frequency of these oscillations is seen to increase somewhat because the plasma is expected to heat up through ohmic dissipation, whereas probe measurement showed that Δ did not decrease appreciably when the magnetic pressure increased.

Finally, we seek a simple model for the observed displacement oscillations. Clearly, these are due to an initial unbalance when the plasma is first pinched. The magnetic forces are driving the plasma towards the position of pressure balance but the momentum of the gas keeps the plasma swinging about the equilibrium position. These movements are usually much slower than the compression waves. As a matter of fact, only the first half or full cycle is usually regular and reproducible, indicating that thereafter the plasma can no longer be properly described as a true cylinder.

To obtain a quantitative expression for the frequency of small-amplitude oscillations in this mode it is merely necessary to estimate the magnetic restoring force on the cylindrical plasma. This is most readily done by starting from the expression for the inductance, since it can be shown that the restoring force at constant current is given by the expression

$$F = \frac{1}{2} I^2 \frac{dL}{dr_m} . \tag{25}$$

Substituting L from Eq. (12), and denoting the mass per unit length of the plasma sleeve by $N_0 m$, we obtain the equation

$$N_0 m \ddot{\delta} + \frac{I^2}{abL_0} \delta = 0, \tag{26}$$

so that the frequency is given by

$$\omega = \frac{I}{(N_0 mabL_0)^{1/2}} , \qquad (27)$$

When actual experimental numbers are inserted in Eq. (27) the frequency calculated is somewhat larger, perhaps by a factor of 2, than is actually observed. The reason for the discrepancy is not yet clear. It does not seem likely that the mass N_0m is actually much larger than assumed, because of the good agreement with pinch dynamics. It seems more likely that the radial oscillations do not follow the assumed cylindrical symmetry for a time long enough to permit a comparison with the analysis discussed in this report.

A proper analysis of the hydromagnetic plasma oscillations in sheet pinch devices requires numerical integrations and will be the subject of a separate report.[5]

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