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# COHERENT ELECTROMAGNETIC EFFECTS IN HIGH-CURRENT PARTICLE ACCELERATORS: II. ELECTROMAGNETIC FIELDS AND RESISTIVE LOSSES

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## ABSTRACT

Coherent electromagnetic fields arising from an azimuthally modulated beam are considered. The beam is completely enclosed in a toroidal vacuum tank of rectangular cross section and highly conducting walls. Expressions are given for the image currents arising from low harmonics of the beam circulation-frequency. These expressions are then used to evaluate resistive losses in the walls of the chamber. Expressions are given for fields arising from harmonics of the revolution frequency high enough that the beam may be in resonance with a characteristic mode of the vacuum chamber. The results are generalized to provide a description of the electric field in the neighborhood of a resonance. Numerical examples of resistive losses are given, indicating that these effects will not be serious for circulating currents of the order of 1 amp. Some properties of high-order Bessel functions, required for a description of the resonant chamber modes and the energy lost in their excitation, are developed in an Appendix.

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## I. INTRODUCTION

In most particle accelerators currently in use, the total number of particles is not sufficiently large to produce coherent effects that warrant special consideration. As the number of particles and thus the circulating current in the machine is increased, some of these accompanying phenomena may become troublesome.

In this paper we investigate the electromagnetic fields arising from the current and charge distributions of a beam of particles in an accelerator vacuum tank.[1] In general, such a beam of high-velocity particles will have an azimuthal variation in density which will give rise to large coherent electromagnetic fields. It is noted that these fields contain "resonant" and "nonresonant" parts, the former arising from a resonant excitation of the cavity modes at a multiple of the particle circulation frequency.[2] These

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resonant fields are of particular interest because of the forces they exert on coasting beams, which may produce instabilities.[3] This problem will be treated in Part III of the series, where use will be made of the results presented here.

The electromagnetic fields associated with the particles provide a mechanism for loss of energy from the beam. These losses are of two types. The first is the resistive loss arising from image currents in the walls of the vacuum chamber and is largely due to the low harmonics of the beam circulation frequency. This loss may be calculated to good approximation by neglecting the curvature of the vacuum tank. The second loss is due to wall currents specifically associated with resonant modes which may be excited by a high harmonic of the orbital frequency. Expressions are given for the power dissipated by each of these effects, and numerical examples are given which indicate that such losses are negligible in many practical instances.

Sections II, III, and IV are devoted to determining the nonresonant fields, resonant fields, and fields near resonance, respectively. Section V contains numerical examples of energy loss, while the Appendix is devoted to a discussion of the properties of the resonant modes.

## II. NONRESONANT FIELDS

For the lower-order harmonics, the wall currents are substantially divergence-free image currents (i.e., uninfluenced appreciably by time-dependent induced charges), distributed in such a manner that the normal component of the magnetic field vanishes at the boundaries. Since the field configuration will be substantially that found in a straight pipe of rectangular cross sections and transverse dimensions small in comparison to a wavelength, the distribution of image currents can be found readily by methods analogous to those employed in corresponding two-dimensional electrostatic problems.[1] Therefore, we employ a coordinate system ($x$, $y$, $z \equiv R\theta$) in which the toroid is straightened.

The current distribution

$$I = \sum_n I_n \cos n(\theta - \omega_0 t), \tag{2.1}$$

centrally located within a metallic chamber enclosing the region

$$-w/2 \leqslant x \leqslant w/2, \quad -h/2 \leqslant y \leqslant h/2,$$

gives rise to an image-current distribution such that $H_n = 0$; this distribution is as follows: On the top and bottom we have

$$I_{surf} = -\frac{1}{w}\sum_n I_n \left[\sum_m \operatorname{sech}(2m+1)\frac{\pi}{2}\frac{h}{w}\cos(2m+1)\pi\frac{x}{w}\right]\cos n(\theta-\omega_0 t); \tag{2.2a}$$

and on the sides,

$$I_{surf} = -\frac{1}{h}\sum_n I_n \left[\sum_m \operatorname{sech}(2m+1)\frac{\pi}{2}\frac{w}{h}\cos(2m+1)\pi\frac{y}{h}\right]\cos n(\theta-\omega_0 t), \tag{2.2b}$$

directed azimuthally. For $h \ll w$, the expression for the surface current in the top and bottom boundaries may be simplified by writing it in the approximate form[4, 5]

$$I_{surf} \cong -\frac{1}{w}\sum_n I_n \left[\int_0^\infty \operatorname{sech}(\pi h/w)t\cos(2\pi x/w)t\,dt\right]\cos n(\theta-\omega_0 t)$$

$$= -\frac{1}{2h}\sum_n I_n \operatorname{sech}\frac{\pi x}{h}\cos n(\theta-\omega_0 t). \tag{2.3}$$

The nonresonant contribution to the resistive loss is immediately obtained from Eqs. (2.2a and b) in terms of the surface resistances appropriate to the frequencies of the individual harmonics,[6] as

$$P = \pi \mathcal{R}_1 \left[ \frac{R}{w} \sum_m \operatorname{sech}^2 (2m+1) \frac{\pi}{2} \frac{h}{w} + \frac{R}{h} \sum_m \operatorname{sech}^2 (2m+1) \frac{\pi}{2} \frac{w}{h} \right] \left[ \sum_n n^{1/2} I_n^2 \right], \tag{2.4}$$

in which $R$ denotes the radius of the accelerator. If $h \ll w$, the first of the two sums over $m$ distinctly dominates, and one may write

$$P \cong \pi \mathcal{R}_1 \frac{R}{w} \left[ \int_0^\infty \operatorname{sech}^2 (\pi h/w) t \, dt \right] \cdot \sum_n n^{1/2} I_n^2 = \mathcal{R}_1 \frac{R}{h} \sum_n n^{1/2} I_n^2. \tag{2.5}$$

Equation (2.5) could have been obtained directly from the approximate expression, Eq. (2.3), which in this limit was given for the surface-current density in the upper and lower surfaces.

If desired the expressions just derived for the resistive loss may alternatively be expressed in terms of the Fourier coefficients of the linear charge density or of the number of particles per radian at the orbit radius, $R_B$. Thus we may write

$$\lambda = \sum_n \lambda_n \cos n(\theta - \omega_0 t) \text{ charge per unit length} \tag{2.6a}$$

and

$$N = \sum_n N_n \cos n(\theta - \omega_0 t) \text{ particles per radian,} \tag{2.6b}$$

by use of the relations

$$I_n = \omega_0 R_B \lambda_n \tag{2.7a}$$

and

$$\lambda_n = eN_n / R_B. \tag{2.7b}$$

The emf per turn associated with the resistive loss, furthermore, is given by

$$V = \frac{2\pi}{e\omega_0} \frac{P}{N_t}, \tag{2.8}$$

where $N_t$ denotes the total number of particles in the beam.

Thus the nonresonant resistive loss alternatively may be expressed conveniently in the forms

$$V = 2\pi^2 e\omega_o \mathcal{R}_1 \left[\frac{R}{w} \sum_m \operatorname{sech}^2(2m+1)\frac{\pi}{2}\frac{h}{w} + \frac{R}{h} \sum_m \operatorname{sech}^2(2m+1)\frac{\pi}{2}\frac{w}{h}\right]\left[\sum_n n^{1/2} \frac{N_n^2}{N_t}\right], \tag{2.9}$$

or for $h \ll w$,

$$V \cong 2\pi e\omega_0 \mathcal{R}_1 \frac{R}{h} \sum_n n^{1/2} \frac{N_n^2}{N_t}. \tag{2.10}$$

## III. FIELDS ASSOCIATED WITH A RESONANT MODE

It is well known that in a straight wave guide, all electromagnetic modes have phase velocities greater than the velocity of light, c. As shown in Appendix I, at any radius within a toroidal cavity it is possible to find modes that have, at that radius, azimuthal phase velocities less than c. Such modes have eigenfrequencies that are very high harmonics of the beam-circulation frequency. It is therefore possible for an azimuthally modulated beam of relativistic particles to excite one or more electromagnetic modes of the chamber.[2] The fields of such high-order modes may be large. The concomitant resistive losses then warrant separate evaluation, despite the relatively low magnitudes of the Fourier components responsible for the excitation of these modes. The curvature of the chamber is essential for the excitation of the resonant modes, and these high-order solutions may well show a radial dependence that differs materially from that of a simple circular function. It is expedient, therefore, to use cylindrical polar coordinates $(r, \theta, z)$ and to consider the fields expressed in terms of solutions (Z) of Bessel's equation, with the imposition of boundary conditions at $r = a, b$ appropriate to the type of mode under consideration.

Rather than commencing with a general solution for the electromagnetic fields excited by the beam and then extracting a particular resonant term, it is convenient to employ from the start only the field components that are associated with the resonant mode of interest. Power will be supplied to such a mode by the work that the beam current performs against the longitudinal electric field $E_\theta$. Excitation will be strongest if $E_\theta$ is precisely out-of-phase with the beam current. In the steady state, this power may be equated to the resistive losses in the chamber walls. Both the level of the electromagnetic excitation and the power loss are thereby determined in terms of the appropriate Fourier component of the beam current. In what follows, we employ this procedure to obtain expressions for the power loss associated with a resonant TE mode

and, independently, for the loss arising from a resonant TM mode. In each case the results are expressed in terms of the loss factor (Q) of the chamber for the particular mode under consideration.

We assume that the beam has negligible cross-sectional area and is located at $r = R_B$, $z = 0$. For a resonant mode of angular frequency $\omega_r$, the power is given by

$$P = \int_{\text{circumference}} I(-E_\theta)ds = 2\pi R_B \left\langle -E_\theta I \right\rangle_{av} \quad . \tag{3.1}$$

The loss factor is defined by

$$Q = \omega_r \frac{[\text{stored energy}]}{P} , \tag{3.2}$$

so that the power may be written as

$$P = \frac{(2\pi R_B)^2 \left\langle -E_\theta I \right\rangle_{av}^2}{\omega_r[\text{stored energy}]} \quad Q. \tag{3.3}$$

For a resonant TE mode within a chamber of inner and outer radii a and b, one may employ a field configuration of the form (MKS units):

$$B_\theta = -Ank \frac{Z}{r} \sin kz \cos n(\theta - \omega_0 t) \tag{3.4a}$$

$$B_r = -Ak \frac{dZ}{dr} \sin kz \sin n(\theta - \omega_0 t) \tag{3.4b}$$

$$B_z = Aq^2 Z \cos kz \sin n(\theta - \omega_0 t) \tag{3.4c}$$

$$E_\theta = -A\omega_r \frac{dZ}{dr} \cos kz \cos n(\theta - \omega_0 t) \tag{3.4d}$$

$$E_r = -An\omega_r \frac{Z}{r} \cos kz \sin n(\theta - \omega_0 t) \tag{3.4e}$$

$$E_z = 0. \tag{3.4f}$$

Here Z represents a solution of Bessel's equation, $\frac{d}{dr}(r\frac{dZ}{dr}) + (q^2 r - \frac{n^2}{r}) Z = 0$, subject to the Neumann boundary conditions $[dZ/dr]_a = [dZ/dr]_b = 0$; $q^2 + k^2 = \omega_r^2/c^2$; k is an odd multiple of $\pi/h$; $\omega_r = n\omega_0$; and the phase intentionally has been chosen so that $-E_\theta$ is in phase with the current $I_n \cos n(\theta - \omega_0 t)$.

With these fields, then, we have

$$\langle -E_\theta I \rangle_{av} = \frac{I_n}{2} A \omega_r \left[\frac{dZ}{dr}\right]_B , \tag{3.5}$$

the subscript B denoting that the derivative is to be evaluated at $r = R_B$. The stored energy is

$$\frac{\epsilon_0}{2} \iiint E^2 dv + \frac{1}{2\mu_0} \iiint B^2 dv = \frac{\pi A^2 h}{2\mu_0} q^2 \left(\frac{\omega_r}{c}\right)^2 \int_a^b r Z^2 dr. \tag{3.6}$$

Accordingly, we have

$$P_{TE} = 2\pi \mu_0 c^2 I_n^2 \frac{R_B^2}{\omega_r q^2 h} \frac{[dZ/dr]_B^2}{\int_a^b r Z^2 dr} Q_{TE}$$

$$= 2\pi \mathcal{Z} I_n^2 \frac{R_B^2}{(\omega_r/c) q^2 h} \frac{[dZ/dr]_B^2}{\int_a^b r Z^2 dr} Q_{TE}, \tag{3.7}$$

where $\mathcal{Z}$ denotes $(\mu_0/\epsilon_0)^{1/2} = \mu_0 c = 120\pi = 377$ ohms.

In cases for which the annular width of the chamber is small in comparison to the diameter ($w \ll 2R$), this last result may be written conveniently in the approximate form

$$P_{TE} \cong 16\pi \mathcal{Z} I_n^2 \frac{R}{(\omega_r/c) q^2 w^3 h} \frac{[dZ/du]_B^2}{\int_{-1}^{1} Z^2 du} Q_{TE}, \tag{3.8}$$

where the dimensionless variable u is such that

$$r \equiv \frac{b+a}{2} + \frac{w}{2} u, \tag{3.9}$$

with

$$w \equiv b - a.$$

The loss factor, $Q_{TE}$, may also be evaluated[8] in the conventional way from these fields and expressed in terms of the relevant properties of the characteristic solution Z:

$$Q_{TE} = \frac{\zeta}{4R} \frac{(\omega_r/c)^3 h}{k^2} \left\{ 1 + \frac{h}{4q^2} \left[ \frac{n^2}{b} \left( 1 + \frac{b}{a} \frac{[Z(a)]^2}{[Z(b)]^2} \right) + \right.\right.$$

$$\left.\left. \frac{q^4 b}{k^2} \left( 1 + \frac{a}{b} \frac{[Z(a)]^2}{[Z(b)]^2} \right) \right] \frac{[Z(b)]^2}{\int_a^b r Z^2 dr} \right\}^{-1} . \tag{3.10}$$

Again some simplification results for $w \ll 2R$, for which we have

$$Q_{TE} \cong \frac{\zeta}{4R} \frac{(\omega_r/c)^3 h}{k^2} \left\{ 1 + \frac{h}{2q^2 w} \left( \frac{n^2}{R^2} + \frac{q^4}{k^2} \right) \left( 1 + \frac{[Z(-1)]^2}{[Z(1)]^2} \frac{[Z(1)]^2}{\int_{-1}^{1} Z^2 du} \right) \right\}^{-1} , \tag{3.11}$$

where the arguments of Z are now understood to represent values of the dimensionless variable u.

Under potentially-resonant conditions, the required properties of the characteristic function Z can depend in a fairly sensitive way on the parameters of the structure and are best determined by computation. Typical values (cf. Ref. 8, Table IX) in a resonant situation are

$$\frac{[Z(-1)]^2}{[Z(1)]^2} \leq 0.85, \quad \frac{[Z(1)]^2}{\int_{-1}^{1} Z^2 du} \cong 0.52,$$

and, for a beam centrally located within the aperture (at u = 0).

$$\frac{[dZ/du]_B^2}{\int_{-1}^{1} Z^2 du} \cong 0.42.$$

For a resonant TM mode, similarly, one may employ a field configuration of the form (MKS units):

$$B_\theta = -A \frac{\omega_r}{c^2} \frac{dZ}{dr} \sin kz \cos n(\theta - \omega_0 t) \tag{3.12a}$$

$$B_r = -A \frac{n\omega_r}{c^2} \frac{Z}{r} \sin kz \sin n(\theta - \omega_0 t) \tag{3.12b}$$

$$B_z = 0 \tag{3.12c}$$

$$E_\theta = -Ank \frac{Z}{r} \cos kz \cos n(\theta - \omega_0 t) \tag{3.12d}$$

$$E_r = -Ak \frac{dZ}{dr} \cos kz \sin n(\theta - \omega_0 t) \tag{3.12e}$$

$$E_z = -Aq^2 Z \sin kz \sin n(\theta - \omega_0 t) \tag{3.12f}$$

in which the solution $Z$ of Bessel's equation now must conform to the Dirichlet boundary conditions

$$Z(a) = Z(b) = 0. \tag{3.13}$$

With these fields we have

$$\left\langle -E_\theta I \right\rangle_{av} = \frac{I_n}{2} Ank \left[\frac{Z}{R}\right]_B, \tag{3.14}$$

and the stored energy is

$$\frac{\epsilon_0}{2} \iiint E^2 dv + \frac{1}{2\mu_0} \iiint B^2 dv = \frac{\pi A^2 h q^2}{2\mu_0 c^2} \left(\frac{\omega_r}{c}\right)^2 \int_a^b r Z^2 dr. \tag{3.15}$$

Accordingly, we may write

$$P_{TM} = 2\pi I_n^2 \frac{n^2 k^2}{(\omega_r/c)^3 q^2 h} \frac{Z_B^2}{\int_a^b r Z^2 dr} Q_{TM}, \tag{3.16}$$

and for $w << 2R$,

$$P_{TM} \cong 4\pi \} I_n^2 \frac{n^2 k^2}{(\omega_r/c)^3 q^2 Rwh} \frac{Z_B^2}{\int_{-1}^1 Z^2 du} Q_{TM}. \tag{3.17}$$

Finally, the loss factor $Q_{TM}$ may be evaluated[8] for this mode, with the result

$$Q_{TM} = \frac{\zeta}{4R} \frac{\omega_r}{c} h \left\{ 1 + \frac{bh}{4q^2} \left[ 1 + \frac{a}{b} \frac{(dZ/dr)_a^2}{(dZ/dr)_b^2} \right] \frac{(dZ/dr)_b^2}{\int_a^b r\, Z^2\, dr} \right\}^{-1}, \tag{3.18}$$

which, for $w << 2R$, may be written

$$Q_{TM} \cong \frac{\zeta}{4R} \frac{\omega_r}{c} h \left\{ 1 + \frac{2h}{q^2 w^3} \left[ 1 + \frac{(dZ/du)_{-1}^2}{(dZ/du)_1^2} \right] \frac{(dZ/du)_1^2}{\int_{-1}^{1} Z^2\, du} \right\}^{-1}. \tag{3.19}$$

The required properties of the characteristic function $Z$ are again best determined by computation. Illustrative values (Ref. 8, Table VIII) are

$$\frac{(dZ/du)_{-1}^2}{(dZ/du)_1^2} \doteq 0.04, \qquad \frac{(dZ/du)_1^2}{\int_{-1}^{1} Z^2\, du} \doteq 8.4,$$

and, for a centrally located beam,

$$\frac{Z_B^2}{\int_{-1}^{1} Z^2\, du} \doteq 0.79.$$

Because $q$ is of the order of $n/b$ (or $n/R$, for $w << 2R$), and $n^2(w/2R)^3$ is normally of the order of unity under resonant circumstances, it may be seen that the second term in the denominator of $Q_{TM}$ will be very much smaller than unity. In contrast, the second term in the denominator of $Q_{TE}$ could play a strong or even dominating role. This situation may be regarded as arising in the following way. In the TE case, the $B_z$ field component is (for $n$ sufficiently large to attain resonance) by far the largest of the three components of $\underline{B}$. The associated current, which is in the side walls only, consequently dominates. For a TM mode, on the other hand, component $B_z$ vanishes and there is no such dominance as occurs in the TE case. For the TM resonance, the factor $\int_{-1}^{1} Z^2\, du$ enters in estimating energy stored and the resistive loss in the upper and lower surfaces.

It thus effectively cancels, in the evaluation of $Q_{TM}$. With the TE fields, the energy involves this integral and the loss is determined by the quantities $[Z(-1)]^2$ and $[Z(1)]^2$ which serve to specify the current density $I_\theta$ associated with $B_z$ at $r = a, b$.

It is appropriate, therefore, to simplify Eqs. (3.11) and (3.19), which were applicable only for $w \ll 2R$, as

$$Q_{TE} \cong \frac{\mathcal{Z}}{16\mathcal{R}} \frac{(\omega_r/c)^3 w^4}{R\,\eta^3 n^2} \frac{\int_{-1}^{1} Z^2 du}{[Z(-1)]^2 + [Z(1)]^2} , \tag{3.20}$$

and

$$Q_{TM} \cong \frac{\mathcal{Z}}{4\mathcal{R}} \frac{\omega_r}{c} h , \tag{3.21}$$

Here $\eta$ denotes $\frac{w}{2R}$ and the product $\eta^3 n^2$ is a convenient quantity to employ in estimating the location of the resonances that may be excited in a chamber of small transverse dimensions. Finally, if Eqs. (3.20) and (3.21) are respectively combined with Eqs. (3.8) and (3.17), the following resistive losses result:

$$P_{TE} = \pi \frac{\mathcal{Z}^2}{\mathcal{R}} I_n^2 \frac{(\omega_0 R/c)^2}{\eta^3 n^2} \frac{w}{h} \frac{[dZ/du]_B^2}{[Z(-1)]^2 + [Z(1)]^2} , \tag{3.22}$$

$$P_{TM} = (2m+1)^2 \left|\frac{\pi}{2}\right|^3 \frac{\mathcal{Z}^2}{\mathcal{R}} I_n^2 \frac{(w/h)^2}{(\omega_0 R/c)^2 \eta^3 n^2} \frac{Z_B^2}{\int_{-1}^{1} Z^2 du} . \tag{3.23}$$

## IV. FIELDS NEAR RESONANCE

The stability of an intense beam will be influenced by the self-generated electric fields which are enhanced by proximity to resonance. For the purposes of Part III of this series, we extend the results of Section II to obtain required expressions for the longitudinal electric field. Under resonant conditions, the longitudinal electric field of a TE mode is of the form of Eq. (3.4d) in which coefficient A is expressible through use of Eqs. (3.1) and (3.5) as

$$A = \frac{P}{\pi I_n R_B \omega_r [dZ/dr]_B} . \tag{4.1}$$

By use of Eqs. (3.7) and (4.1), the longitudinal electric field at resonance is found to be

$$E_\theta = -2 \mathfrak{Z} I_n \frac{Q_{TE} R_B}{(\omega_r/c) q^2 h} \frac{[dZ/dr]_B^2}{\int_a^b r Z^2 dr} \cos(n\theta - \omega_r t). \tag{4.2}$$

Equation (4.2) may be generalized for frequencies near the resonant frequency by replacing

$$Q_{TE} \cos(n\theta - \omega_r t)$$

by

$$\omega^2 \frac{(\omega_r^2/Q_{TE})\cos(n\theta - \omega t) + (\omega_r^2 - \omega^2)\sin(n\theta - \omega t)}{(\omega_r^2 - \omega^2)^2 + (\omega_r^2/Q_{TE})^2} ,$$

wherein we have not distinguished between $\omega$ and $\omega_r$, except in the arguments of the circular functions and in the resonant term $(\omega_r^2 - \omega^2)$. With this substitution, Eq. (4.2) may be written

$$E_\theta = \mathfrak{Z} I_n \frac{R_B}{(\omega_r/c) q^2 h} \frac{[dZ/dr]_B^2}{\int_a^b r Z^2 dr}$$

$$\times \left[ \frac{i\omega^2 e^{i(n\theta - \omega t)}}{(\omega_r^2 - \omega^2) - i(\omega_r^2/Q_{TE})} - \frac{i\omega^2 e^{-i(n\theta - \omega t)}}{(\omega_r^2 - \omega^2) + i(\omega_r^2/Q_{TE})} \right] \tag{4.3}$$

and gives the field generated by a current $I_n \cos(n\theta - \omega t)$, or by $\frac{1}{2} I_n [e^{i(n\theta-\omega t)} + e^{-i(n\theta-\omega t)}]$. For the perturbation analysis of Part III, it is convenient to employ specifically the complex field associated with a perturbation of the number of particles per radian, expressed in the form of a complex number. A perturbation

$$\delta N = N_n e^{i(n\theta - \omega t)} ,$$

or an associated perturbed current

$$\delta I = e\omega_0 N_n e^{i(n\theta - \omega t)} ,$$

should thus, from Eq. (4.3), have associated with it the longitudinal field

$$E_\theta = 2i\mathcal{Z} e N_n \frac{cR_B}{nq^2 h} \frac{[dZ/dr]_B^2}{\int_a^b r Z^2 dr} \omega^2 \times \frac{e^{i(n\theta - \omega t)}}{(\omega_r^2 - \omega^2) - i(\omega_r^2/Q_{TE})} . \tag{4.4}$$

When $w \ll 2R$, $q \approx n/R$ and Eq. (4.4) may be written in the somewhat simpler approximate form

$$E_\theta \cong 16 i \mathcal{Z} e N_n \frac{cR^2}{n^3 w^3 h} \frac{[dZ/du]_B^2}{\int_{-1}^{1} Z^2 du} \omega^2 \times \frac{e^{i(n\theta - \omega t)}}{(\omega_r^2 - \omega^2) - i(\omega_r^2/Q_{TE})} . \tag{4.5}$$

To proceed in a similar way to evaluate the longitudinal electric field of a TM mode near resonance, we have

$$A = \frac{P}{\pi I_n nk Z_B} \tag{4.6}$$

from Eqs. (4.1) and (3.14). By use of Eqs. (3.16) and (4.6), the resonance longitudinal electric field, Eq. (3.12d) is

$$E_\theta = -2\mathcal{Z} I_n \frac{Q_{TM} n^2 k^2}{(\omega_r/c)^3 q^2 R_B h} \frac{Z_B^2}{\int_a^b r Z^2 dr} \cos(n\theta - \omega_r t) . \tag{4.7}$$

For frequencies near the resonant frequency, Eq. (4.7) is generalized in the same manner as employed in connection with the TE resonance to read

$$E_\theta = \mathfrak{z}\, I_n \frac{n^2 k^2}{(\omega_r/c)^3 q^2 R_B h} \frac{Z_B^2}{\int_a^b r\, Z^2\, dr}$$

$$\times \left[ \frac{i\omega^2 e^{i(n\theta - \omega t)}}{(\omega_r^2 - \omega^2) - i(\omega_r^2/Q_{TM})} - \frac{i\omega^2 e^{-i(n\theta - \omega t)}}{(\omega_r^2 - \omega^2) + i(\omega_r^2/Q_{TM})} \right] . \tag{4.8}$$

For a perturbation

$$\delta N = N_n e^{i(n\theta - \omega t)} ,$$

Eq. (4.8) gives the associated field

$$E_\theta = 2i\mathfrak{z}\, eN_n \frac{k^2 c}{n(\omega_0/c)^2 q^2 R_B h} \frac{Z_B^2}{\int_a^b r\, Z^2\, dr} \omega^2$$

$$\times \frac{e^{i(n\theta - \omega t)}}{(\omega_r^2 - \omega^2) - i(\omega_r^2/Q_{TM})} ; \tag{4.9}$$

for $w << 2\,R$, with $q \approx n/R$ and $k = (2\,m + 1)\,\pi/h$, we have

$$E_\theta \cong 4(2m+1)^2 \pi^2 i\, \mathfrak{z}\, eN_n \frac{cR^2}{n^3(\omega_0 R/c)^2 wh^3} \frac{Z_B^2}{\int_{-1}^{1} Z^2\, du} \omega^2$$

$$\times \frac{e^{i(n\theta - \omega t)}}{(\omega_r^2 - \omega^2) - i(\omega_r^2/Q_{TM})} . \tag{4.10}$$

## V. NUMERICAL EXAMPLES

We have calculated the nonresonant power loss using the parameters of the Berkeley Bevatron, a typical strong-focusing machine such as the CERN proton synchrotron, and the Stanford electron-storage rings. Results are given in Table I. For a proton machine in which the radio frequency operates on a harmonic m, (m = 1 for the Bevatron, m = 10 for AGS) we take the azimuthal distribution of particles to be

$$N(\theta) = \frac{2N}{\pi m a} \left[1 - (\theta/a)^2\right]^{1/2},$$

in which N is the total number of particles. This distribution leads to Fourier coefficients of the current given by

$$I_n = \frac{2e\omega_0 N}{\pi m a} J_1(na) = \frac{4I_C}{na} J_1(na),$$

for n an integral multiple of m. Other Fourier coefficients are zero. The total circulating current is $I_C$. The azimuthal distribution of particles in the Stanford storage rings will be taken as Gaussian. We thus have

$$N(\theta) = (2\pi \langle\theta^2\rangle_{av})^{-1/2} e^{-1/2\,\theta^2/\theta^2_{av}},$$

from which it follows that

$$I_n = I_C e^{-n^2 \langle\theta^2\rangle_{av}/2}.$$

The resistivity $\rho$ of the conducting walls is taken somewhat arbitrarily to be $10^{-4}$ ohm-cm for all numerical examples. If the true resistivity $\rho_t$ of the walls is known, the results in Table I should be altered by a factor $10^2 \rho_t^{1/2}$, with $\rho_t$ in ohm-cm.

We have calculated the resonant power loss for the Bevatron only, using Eq. (3.22). Inserting values $R_B$ = 50 ft, b = 52 ft, h = 1 ft, and $\gamma$ = 6 into Eq. (A-18), we find that resonance can occur for $n \approx 650$. The ratio w/h is 4 for this machine, and the resonant energy loss $\delta E$ is of the order of 0.1 ev. For the strong-focusing machine, we insert

Table I. Parameters of three accelerators and the nonresonant energy loss per particle per turn. The circulating current assumed in the calculation is $I_C$, and $(f_0 = \omega_0/2\pi)$ is the particle circulation frequency. The parameters $\alpha$ and $\langle\theta^2\rangle_{av}$ characterize the extent of particles in the rf phase.

| Machine | $\mathcal{R}_1$ (ohms) | R/h | $I_c$ (amp) | $f_0$ (cps) | $\alpha$ | $\langle\theta^2\rangle_{av}$ | $\delta E$ (ev) |
|---|---|---|---|---|---|---|---|
| Bevatron | $3.14\times10^{-3}$ | 50 | 4 | $2.5\times10^{6}$ | 1 | | 3.56 |
| CERN | $1.37\times10^{-3}$ | $10^{3}$ | 1 | $4.8\times10^{5}$ | 0.1 | | 24.6 |
| Stanford | $10\times10^{-3}$ | 36 | 1 | $2.5\times10^{7}$ | | 0.014 | 21.8 |

$R_B = 100$ m, $b = 100.15$ m, $h = 0.1$ m, and $\gamma = 25$ into Eq. (A18), and find that resonance is possible at values of $n \sim 8 \times 10^5$. This is sufficiently high that the resonant energy loss is negligible.

For the electron-storage rings, we use $R_B = 142$ cm, $b = 150$ cm, $h = 5$ cm, and $\gamma = 10^3$. Resonance is found to be possible with the 275th harmonic, but the 275th Fourier component of the Gaussian distribution is so small that resonant power losses do not warrant consideration.

# APPENDICES

## Appendix I. Azimuthal Phase Velocities and Possible Resonance

The eigenfrequencies $\omega_\ell$ of the cavity modes which can be excited by the beam are given by $(\omega_\ell/c)^2 = q^2 + (p\pi/h)^2$ with $q$ the characteristic value of Bessel's equation and $p$ an odd integer. The angular phase velocity, is simply $\omega_\ell/n$, and the azimuthal phase velocity, $v_\theta$, is $\omega_\ell r/n$. We thus have

$$(v_\theta/c)^2 = (qr/n)^2 + (p\pi r/nh)^2.$$

Obviously the second term may be made negligibly small by choosing $p = 1$ and $n \gg p\pi r/h$. For $p > 1$, this term may still be made small, but only for much larger values of n. With this term negligibly small, we may have $v_\theta < c$ at any radius $r$ within the vacuum tank for which $qr/n < 1$. It is then possible for a relativistic beam of particles to be circulating with a velocity coinciding with the phase velocity of the mode. This is the resonant condition referred to in this work. We now show that for sufficiently large $n$ it is possible to satisfy $qr/n < 1$ at any radius within a toroidal cavity.

For TE modes the appropriate solution of Bessel's equation is

$$Z_n(r) = Y'_n(qa)\, J_n(qr) - J'_n(qa)\, Y_n(qr),$$

with the values of $q$ determined by the boundary condition $Z'_n(b) = 0$. In Fig. 1 we have plotted qualitatively the function $J'_n(x)/Y'_n(x)$ vs $x$ for large values of n. The maximum of the curve occurs at $x = n$, and the half-width is of the order of $n^{1/3}$. The lowest characteristic value $q_0$ may be found approximately by selecting $q_0 a < n$ and $n < q_0 b < j'_{n1}$ such that

$$J'_n(q_0 a)/Y'_n(q_0 a) = J'_n(q_0 b)/Y'_n(q_0 b).$$

The first zero of $J'_n$ is designated by $j'_{n1}$, and occurs approximately at $x = n + 0.81\, n^{1/3}$ for large n. The ratio b/a is fixed and determines how far down the curve we must place our values. Thus in Fig. 1, the portion of the vacuum tank for which $q_0 r/n < 1$ holds is represented by the region of the abscissa between $q_0 a$ and n.

For fixed b/a, the portion of the vacuum tank for which $q_0 r/n < 1$ does not hold diminishes to zero as n approaches infinity. This can be seen by noting that $q_0$ is of the order of n/b and thus $q_0 b - q_0 a \sim n(1-a/b)$. For any b/a, it is possible to choose a value of n such that this quantity is very much greater than the half-width of the curve, which is of order $n^{1/3}$. We must then place $q_0 b$ very close to its maximum value $j'_{n1}$, while $q_0 a$ is located far to the left of n. The portion of the vacuum vessel represented by the region of the abscissa between n and $j'_{n1}$ therefore becomes negligibly small, as n increases without limit, compared to the portion between $q_0 a$ and n. In the latter portion, $q_0 r/n < 1$ holds. More accurate values of $q_0$ will be found in Appendix II.

For TM modes, the appropriate solution of Bessel's equation is

$$Z_n(r) = Y_n(qa)\, J_n(qr) - J_n(qa)\, Y_n(qr),$$

with the values of q determined by the boundary condition $Z_n(b) = 0$. The lowest characteristic value may be found approximately by a graphical technique analogous to that used above. In Fig. 2 we have plotted qualitatively the function $J_n(x)/Y_n(x)$ vs x. For large n, the first zero $j_{n1}$ of $J_n$, occurs approximately at $n + 1.86\, n^{1/3}$, while $y_{n2}$, the second zero of $Y_n$, occurs approximately at $n + 2.54\, n^{1/3}$. Hence for large n the first characteristic value for TM modes always has the limits $n + 1.86\, n^{1/3} < qb < n + 2.54 n^{1/3}$. Again, more quantitative evaluation will be found in Appendix II. We merely wish to point out here that, for this first TM solution, the condition $qr/n < 1$ holds for some portion of the vacuum-tank aperture.

## Appendix II. High-Order Solutions of Bessel's Equation for a Narrow Annulus

### A. Introduction

The lowest characteristic values, $q$ and the associated characteristic functions $Z(r)$, of interest here are those which arise from Bessel's equation when $n$ is large and when $(b-a)/(b+a) \ll 1$. As shown in Appendix I, the lowest characteristic values will be in the neighborhood of $n/b$. To find whether a resonant electromagnetic mode will be excited by a modulated beam moving within the vacuum chamber, however, the characteristic values must be determined with some accuracy, because of the strong cancellation involved in computing the quantity $k = [(n\omega_0/c)^2 - q^2]^{1/2}$. The quantity assumes values which are odd multiples of $\pi/h$ in resonant modes. It is accordingly appropriate to examine directly[8] the characteristic solutions of Bessel's equation, subject to our particular boundary conditions, without reference to the customary Bessel and Neumann functions $J_n$ and $Y_n$.

### B. Analysis

It is convenient to introduce the quantity $\eta \equiv \frac{b-a}{b+a} = \frac{w}{2R_0}$ and, because of the strong cancellation mentioned above, to define

$$\delta \equiv \eta^2 [(qR_0)^2 - n^2] \tag{A-1a}$$

and

$$u \equiv 2 \frac{r - R_0}{w} . \tag{A-1b}$$

In terms of these quantities, we have $r = R_0(1+\eta u)$, with $-1 \leq u \leq 1$, and Bessel's equation assumes the form

$$\frac{d}{du}\left[(1+\eta u)\frac{dZ}{du}\right] + \left[\delta(1+\eta u) + \frac{(2+\eta u)}{(1+\eta u)} \cdot \eta^3 n^2 \cdot u\right] Z = 0. \tag{A-2}$$

For $\eta \ll 1$, the characteristic values, $\delta$, and the characteristic functions for this equation may be obtained[8] by a perturbation method provided $n$ is not too large. In this way we find for the first Neumann solution (TE mode):

$$\delta \cong \frac{1}{3}\eta^4 n^2 - \frac{8}{15}\eta^6 n^4 \tag{A-3a}$$

$$Z \propto 1 + \eta^3 n^2 \left(u - \frac{u^3}{3}\right); \tag{A-3b}$$

For the first Dirichlet solution (TM mode):

$$\delta \cong \left(\frac{\pi}{2}\right)^2 - \frac{\eta^2}{4} + \left(1 - \frac{6}{\pi^2}\right)\left(n^2 - \frac{1}{4}\right)\eta^4 \tag{A-4a}$$

$$Z \propto \left\{\cos\frac{\pi}{2}u + \frac{1}{\pi}\left(n^2 - \frac{1}{4}\right)\eta^3\left[(1-u^2)\sin\frac{\pi}{2}u - \frac{2}{\pi}u\cos\frac{\pi}{2}u\right]\right\}(1+\eta u)^{-1/2}; \tag{A-4b}$$

and for the second Neumann solution (TE mode):

$$\delta \cong \left(\frac{\pi}{2}\right)^2 + \frac{3}{4}\eta^2 + \left(1 + \frac{10}{\pi^2}\right)\eta^4 n^2 \tag{A-5a}$$

$$Z \propto \sin\frac{\pi}{2}u - \frac{1}{2}\eta\left(u\sin\frac{\pi}{2}u + \frac{2}{\pi}\cos\frac{\pi}{2}u\right) - \frac{1}{\pi}\eta^3 n^2\left[\left(1 + \frac{4}{\pi^2} - u^2\right)\cos\frac{\pi}{2}u + \frac{2}{\pi}u\sin\frac{\pi}{2}u\right]. \tag{A-5b}$$

The region of applicability of the foregoing expressions is that for which $\eta^3 n^2 \ll 1$. Of greater significance for our present purposes, however, are the results for the case $\eta^3 n^2 > 1$, which we discuss below.

Since our interest here is confined to the case $\eta \ll 1$, it is convenient to approximate the differential equation for $Z$ by

$$\frac{d^2 Z}{du^2} + [\delta + 2\eta^3 n^2 u]\, Z = 0. \tag{A-6}$$

Solutions of this approximate equation may then be written explicitly in terms of Bessel functions of order 1/3. Specifically, we take

$$Z \propto \xi^{1/2}\left[J_{1/3}\left(\frac{\xi^{3/2}}{3\eta^3 n^2}\right) + J_{-1/3}\left(\frac{\xi^{3/2}}{3\eta^3 n^2}\right)\right], \tag{A-7}$$

where $\xi$ denotes $\delta + 2\eta^3 n^2 u$. The particular ratio of the coefficients of $J_{1/3}$ and $J_{-1/3}$ is selected to insure a decreasing exponential solution to the left of the "classical turning point," $u_c = -\frac{\delta}{2\eta^3 n^2}$. When

$\eta^3 n^2$ is fairly large in comparison to unity, such a solution will drop sufficiently rapidly in that region to satisfy the boundary condition required at $u = -1$ (i.e., at $r = a$).

Asymptotic forms for the characteristic values of $\delta$ may then be found immediately by application of the desired boundary conditions at $u = 1$, with the aid of published tables.[9] The following estimates of $\delta$, applicable in cases in which $\eta^3 n^2$ is at least somewhat larger than unity, are obtained.

For the first Neumann solution (TE mode):

$$\delta \cong -2\eta^3 n^2 + 1.61724\, \eta^2 n^{4/3}, \tag{A-8}$$

for the first Dirichlet solution (TM mode):

$$\delta \cong -2\eta^3 n^2 + 3.71151\, \eta^2 n^{4/3}, \tag{A-9}$$

for the second Neumann solution (TE mode):

$$\delta \cong -2\eta^3 n^2 + 5.15619\, \eta^2 n^{4/3}. \tag{A-10}$$

The nature of the characteristic functions can be seen conveniently from a graph (Fig. 3) of

$$Z \propto v^{1/2} \left[ J_{1/3}(v^{3/2}) + J_{-1/3}(v^{3/2}) \right] \quad \underline{\text{vs}} \quad v, \tag{A-11}$$

with $v$ defined by the relation

$$v \equiv \frac{\delta + 2\eta^3 n^2 u}{3^{2/3} \eta^2 n^{4/3}} . \tag{A-12}$$

The various characteristic solutions of interest are then depicted by this curve, with the $u = 1$ boundary appropriately located at the maximum, zero, or minimum of the function plotted. When $\eta^3 n^2$ is large, the solutions are highly localized near $u = 1$. Their values exceed $\frac{1}{e} Z_{max}$ only in an interval $\Delta u$ of width $1.35\, \eta^{-1} n^{-2/3}$, $2.39\, \eta^{-1} n^{-2/3}$, or $3.12\, \eta^{-1} n^{-2/3}$, respectively, for the three characteristic solutions discussed here. This property, and others useful in the application of the characteristic solutions, depend only

upon the value of $\eta^3 n^2$ and may be estimated from the graph or evaluated computationally.[8]

## C. The Possibility of Resonance

The possibility that an azimuthally modulated beam may excite a resonant electromagnetic mode of a toroidal vacuum chamber may be examined by reference to the equation

$$k = [(n\omega_0/c)^2 - q^2]^{1/2}, \tag{A-13}$$

where $k = (2m+1)\pi/h$. In terms of the average radius of the chamber, $R_0$ and the radius of the particle orbit $R_B$, this relation may be written

$$(qR_0)^2 = n^2(\beta R_0/R_B)^2 - (2m+1)^2(\pi R_0/h)^2 \tag{A-14a}$$

or

$$\delta = \eta^2 n^2 [(\beta R_0/R_B)^2 - 1] - [(m+\tfrac{1}{2})\pi w/h]^2 . \tag{A-14b}$$

For a relativistic beam moving close to the center of the aperture, $\beta R_0/R_B$ will be close to unity. The ratio $w/h$ is normally greater than unity. For resonance to occur, therefore, $\delta$ must be somewhat negative and hence, $\eta^3 n^2$ would be roughly of order unity for the lower-order resonant modes. Somewhat lower values of $n$ could give rise to resonant excitation if $R_B < R_0$, while $\beta$ materially less than unity will require larger values of n. There is, in fact, a limiting value for the particle energy below which resonance will not occur, even with $R_B = a$, as can be seen from the following argument. If we have

$$R_B = R_0 - w/2, \tag{A-15}$$

we can write

$$\delta = \eta^2 n^2 [\beta^2/(1-\eta)^2 - 1] - [(m+\tfrac{1}{2})\pi w/h]^2, \tag{A-16}$$

and resonance certainly cannot occur in any mode if we have

$$\eta^2 n^2 [\beta^2/(1-\eta)^2 - 1] < -2\eta^3 n^2,$$

i.e., for

$$\beta^2/(1-\eta)^2 - 1 + 2\eta < 0,$$

$$\beta^2 < (1 - 2\eta)(1-\eta)^2,$$

or

$$\gamma^2 < \frac{1}{1 - (1-2\eta)(1-\eta)^2} \cong \frac{1}{4\eta} . \qquad \text{(A-17)}$$

For $R_B = R_0$, however, significant resonances may arise for values of n sufficiently great that $\eta^3 n^2$ is in the range 4 to 30. A convenient general expression is obtained from Eq. (A-14b) by neglecting terms proportional to $(\eta^3 n^2)^{2/3}$ which appear in Eqs. (A-8) through (A-10). The first resonance then is seen to occur for harmonic numbers such that:

$$\eta^2 n^2 \left[ \left( \frac{\beta\, b/R_B}{1+\eta} \right)^2 - 1 \right] = -2\eta^3 n^2 + \left( \frac{\pi w}{2h} \right)^2 ,$$

$$n = (\pi R_B/h)[1 - 1/\gamma^2 - (1-2\eta)(1+\eta)^2 (R_B/b)^2]^{-1/2} \qquad \text{(A-18)}$$

$$\cong (\pi R_B/h)[1 - 1/\gamma^2 - (R_B/b)^2]^{-1/2}. \qquad \text{(A-18a)}$$

## D. Salient Properties of the Characteristic Solution

With $\eta^3 n^2 > 1$, the characteristic solutions differ considerably from simple circular functions. This fact affects the coupling between the beam and the electromagnetic fields and modifies the numerical values of the loss factor Q. For purposes of this paper it may suffice to state that computational results[8] indicate $[dZ/du]_0^2 / \{[Z(-1)]^2 + [Z(1)]^2\}$ does not appreciably exceed 0.40 for the first Neumann solution (for $\eta^3 n^2 \sim 3$). For the second Neumann solution, this quantity assumes the value $\pi^2/8$ for $\eta^3 n^2$ small, vanishes for $\eta^3 n^2 \sim 6$, attains a maximum value of approximately 4.0 for $\eta^3 n^2 \sim 20$, and decreases thereafter. The quantity $[dZ/du]_0^2 / \int_{-1}^{1} Z^2\, du$ for the first Neumann solution has a maximum value of approximately

0.71 at $\eta^3 n^2 \sim 4$, drops to 0.41 at $\eta^3 n^2 \sim 10$, and becomes less than 0.13 for $\eta^3 n^2 \geqslant 20$. For the second Neumann solution it is $\pi^2/4$ for $\eta^3 n^2$ small, vanishes for $\eta^3 n^2 \sim 6$, attains a maximum value of approximately 4.5 for $\eta^3 n^2 \sim 20$, and decreases thereafter. Finally, for the first Dirichlet solution, the quantity $[Z(0)]^2/\int_1^1 Z^2\, du$ drops steadily from a value unity, for $\eta^3 n^2 \sim 4$, 0.37 at $\eta^3 n^2 \sim 10$, and 0.10 at $\eta^3 n^2 \sim 20$.

## References

1. The treatment given in this paper is somewhat intuitive (and consequently simple) in its approach, and therefore not as rigorous as might be desired. A more extensive, rigorous treatment may be found in V. Kelvin Neil, "A Study of Some Coherent Electromagnetic Effects in High-Current Particle Accelerators," (Thesis) Lawrence Radiation Laboratory Report UCRL-9124, April 26, 1960.

2. The possible existence of resonance is discussed in the Appendix; see also reference 3.

3. C. E. Nielsen, A. M. Sessler, and K. R. Symon, Proceedings of the International Conference on High-Energy Accelerators and Instrumentation (CERN; Geneva, Switzerland; September, 1959), pp. 239-252, esp. p. 246.

4. W. Gröbner and N. Hofreiter, Integraltafel, (Springer, Vienna, 1950), Part II, Sec. 335, Eq. (11a), p. 136.

5. The result for the case $w/h \to \infty$ may also be obtained directly by reference to a corresponding electrostatic problem treated by William R. Smythe in his Static and Dynamic Electricity, Second edition, (McGraw-Hill Book Co.; New York, 1950), Sec. 4.20, p. 85, for a line charge centrally located between a pair of parallel conducting plates. From Smythe's result, the current density in the boundary surfaces becomes

$$I_{surf} = -\frac{1}{\pi}\sum_{n} I_n \left\{\frac{d}{dx}\left[\tan^{-1}\left(\tanh\frac{\pi x}{2h}\right)\right]\right\}\cos n(\theta - \omega_0 t)$$

$$= -\frac{1}{2h}\sum_{n} I_n \operatorname{sech}\frac{\pi x}{h}\cos n(\theta - \omega_0 t),$$

as found in our Eq. (2.3). This result also follows from the analysis by W. K. H. Panofsky and M. Phillips in their Classical Electricity and Magnetism (Addison Wesley Publishing Co., Cambridge, Mass., 1955), Chap. 3, Sec. 6, p. 45 ff.

6. The surface resistance is defined as the resistivity, $\rho$, divided by the skin depth, $\delta$. It may be written $\mathcal{R} = n^{1/2}\mathcal{R}_1$, where the surface resistance for the fundamental frequency is in MKS units,

$$\mathcal{R}_1 = \frac{\rho}{\delta_1} = \left(\frac{\mu_0 \omega_0 \rho}{2}\right)^{1/2} = \frac{\mu_0 \omega_0 \delta_1}{2} .$$

Correspondingly, the skin depth for the fundamental frequency is

$$\delta_1 = \left(\frac{2\rho}{\mu_0 \omega_0}\right)^{1/2} .$$

7. It will be noted that the convenience of this form lies in the fact that it may be used to evaluate P in terms of I (or its Fourier component $I_n$) and Q <u>without any special normalization</u> of the fields which describe the resonant mode of interest.

8. L. Jackson Laslett and William Lewish, <u>Evaluation of the Zeros of Cross-Product Bessel Functions</u>, (Iowa State University, Ames, Iowa, 1960), unpublished work.

9. National Bureau of Standards Computation Laboratory, <u>Tables of Bessel Functions of Fractional Order</u>, Vols. I and II, (Columbia University Press, New York, 1948-49).

## Figure Legends

Fig. 1. Qualitative graph of $J_n'(x)/Y_n'(x)$ for large n. The radial aperture of the vacuum tank is represented by the region of the abscissa between $q_0a$ and $q_0b$. The ith zero of $J_n'$ is $j_{n'i}$, and the ith zero of $Y_n'$ is $y_{n'i}$.

Fig. 2. Qualitative graph of $J_n(x)/Y_n(x)$ for large n. The radial aperture of the vacuum tank is represented by the region of the abscissa between qa and qb. The ith zero of $J_n$ is $j_{ni}$ and the ith zero of $Y_n$ is $Y_{ni}$.

Fig. 3. Graph of the universal radial function Z(v), as defined by Eqs. (A-11) and (A-12).

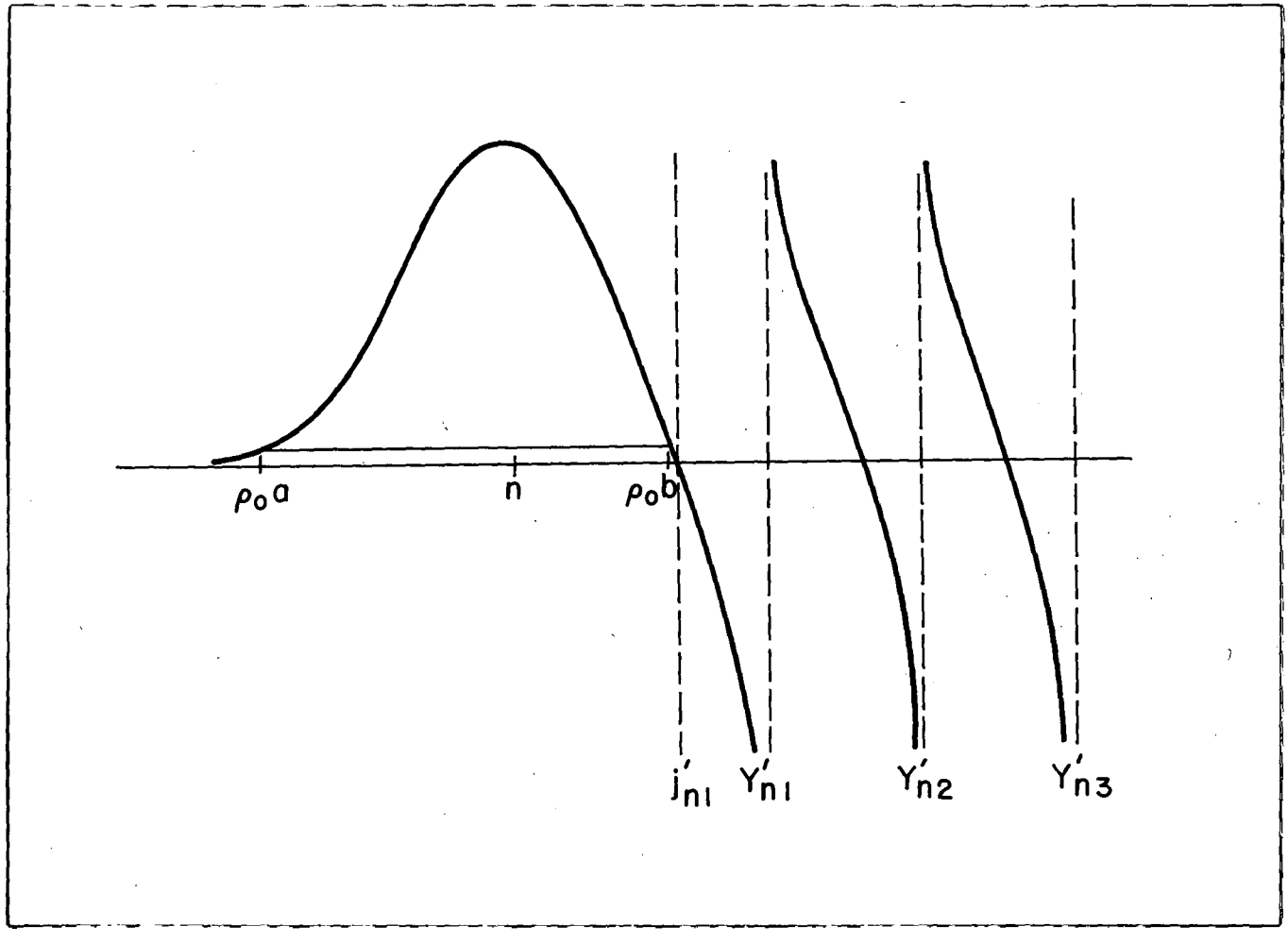


Fig. 1.

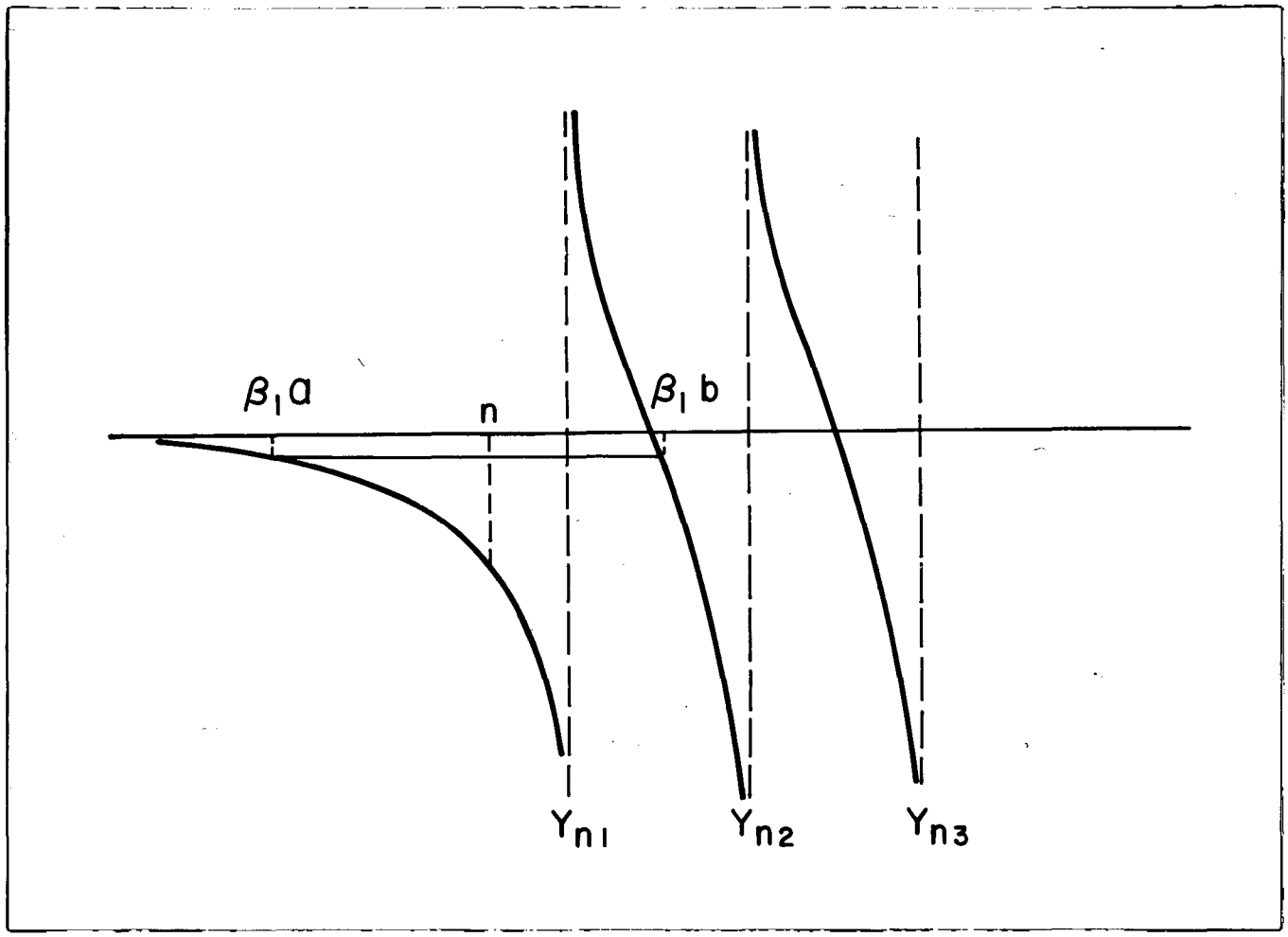


Fig. 2.

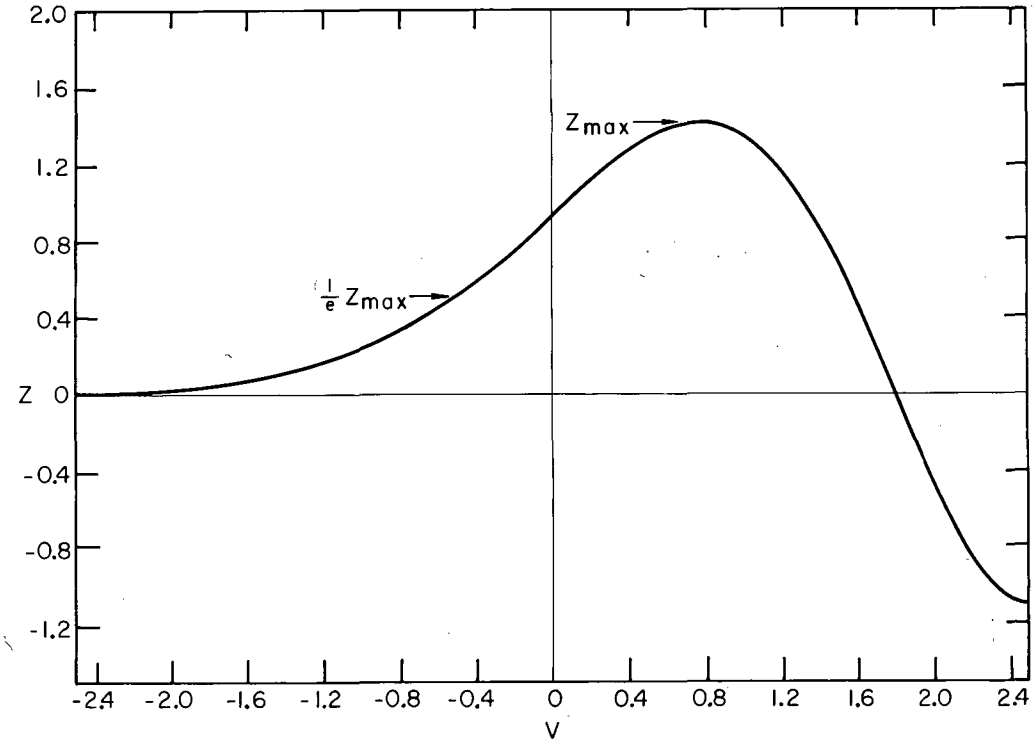
2.0
1.6
1.2
0.8
0.4
Z 0
-0.4
-0.8
-1.2
Zmax
1/e Zmax
-2.4 -2.0 -1.6 -1.2 -0.8 -0.4 0 0.4 0.8 1.2 1.6 2.0 2.4
V

Fig. 3.