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KINEMATICS AND DISPERSION RELATIONS FOR GENERAL PRODUCTION PROCESSES

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ABSTRACT

On the basis of heuristic arguments it is shown that the amplitude for the reaction $a_1 + a_2 \rightarrow b_1 + b_2 + b_3$ and the channels associated with it allows a dispersion representation analogous to that given by Mandelstam for processes of type $a_1 + a_2 \rightarrow b_1 + b_2$, provided that the singularities of the amplitude can be assumed to be restricted, in complete analogy to Mandelstam's case, to certain parts of real hyperplanes in the (complex) space of the invariant variables $s_{ik} = (q_i + q_k)^2$. (Here the $q_i (i=1,2,\cdots,5)$ are the particle four-momenta.) The question whether or not this assumption is actually fulfilled and to what extent it may be violated is not discussed in this paper.

A Lorentz-invariant description due to Kibble for the boundary of the physical region of the process $a_1 + a_2 \rightarrow b_1 + b_2$ is generalized for arbitrary reactions and discussed in terms of scattering angles for some special cases. After suitable generalization of the Breit frame a set of ten one-dimensional dispersion relations analogous to the three one-dimensional relations of Mandelstam is obtained by using a method due to Polkinghorne. Each relation apart from pole terms consists of six dispersion integrals, each of which corresponds to a certain reaction channel. The absorptive parts are obtained from analytic continuation of the unitarity condition in the respective channel. For obtaining such a result it is essential to keep fixed not four variables of type s_{ik} but three such variables and a fourth variable ν, which was formerly introduced by Polkinghorne and which is a general linear

function of those s_{ik}, which have not been kept fixed. Provided that there are no complex singularities each of these one-dimensional dispersion relations can--in a formal way--be derived from a two-dimensional representation, in which certain three variables s_{ik} are fixed and which consists of twelve double integrals. It is suggested that if we had analyticity with regard to all variables and only real singularities a possible representation in terms of fivefold dispersion integrals would be of considerable complexity and consist of at least 162 terms.

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I. INTRODUCTION

The method of dispersion relations has in recent years found a wide application for the study of elementary particle reactions. Most of the work, however, deals with reactions of the type $a_1 + a_2 \rightarrow b_1 + b_2$, while the theory of those with more than two particles in the final state is still in a very preliminary stage. One reason for this is that even with only three particles in the final state the theory is already much more complicated. Nevertheless, a further development of the theory seemed to us very desirable.

As is well known, the theory at present is being developed on various levels simultaneously.

(a) From a heuristic point of view, approaches of a more or less formal character are being carried out with the aim of suggesting plausible formulations of equations and theorems that interconnect the various amplitudes.

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(b) Attempts are being made to correlate these equations and theorems with experimental data, eventually after appropriate approximations.

(c) The heuristic approaches have to be put on a mathematically rigorous basis, starting from the "axioms" of quantum field theory. This part is by far the most difficult and has so far been carried through successfully for only a small number of problems.

This paper clearly falls into the first category, but we hope to contribute in subsequent publications to the two others also. Generally speaking, the aim of this paper is to put the theory in a form as closely as possible analogous to Mandelstam's formulation (1) of the theory of reactions of type $a_1 + a_2 \rightarrow b_1 + b_2$. In the later sections we specialize on reactions $a_1 + a_2 \rightarrow b_1 + b_2 + b_3$, but as much as possible the formulation is in more general terms. A discussion of specific physical reactions, in particular pion production in pion-nucleon collisions, has also been carried out, but is to be presented in a later publication. Here we restrict ourselves to those aspects of the theory which can be formulated quite generally.

The first problem of course, is the definition of appropriate variables. As one might expect, we will introduce quantities $s_{ik} = (q_i + q_k)^2$, where q_i, q_k are particle four-momenta, and will call these s variables. They are related to one another by various linear equations, which will be discussed in detail. As for the description of the physical region, we will generalize the Lorentz-invariant, elegant formulation due to Kibble (2). All these kinematical questions can, of course, be treated on a mathematically rigorous basis. The next point will be to write down dispersion relations, not only for the reaction $a_1 + a_2 \rightarrow b_1 + b_2 + b_3$, but also for the associated "channels" $a_1 + \bar{b}_1 \rightarrow \bar{a}_2 + b_2 + b_3$, $a_1 + \bar{b}_2 \rightarrow \bar{a}_2 + b_1 + b_3$, etc. In order to do this in a rigorous theory one would have to prove the possibility of various analytic

continuations. In the present context, however, we are content with an application of the heuristic and purely formal techniques developed by Polkinghorne (3). All objections that could be raised against his work would likewise apply to this one. Keeping an arbitrary incoming and an arbitrary outgoing particle in the state vectors, and after generalizing the Breit system in an appropriate way (such that the vector of momentum transfer between these two particles has a vanishing timelike component, which, of course, is possible only for negative momentum transfers), we can readily apply Polkinghorne's techniques and obtain in this way a set of 10 one-dimensional dispersion relations, each connecting a set of six "reaction channels" to one another. Since the amplitudes are determined by five independent variables, the important question arises how to choose four variables which are kept fixed. For obvious reasons we cannot for our purposes simply choose four s variables, but we have to take three s variables and a quantity ν, which is a general linear function of s variables. If those s variables which represent the total center-of-mass energy in the respective "reaction channels" are introduced into the dispersion integrals as integration variables, the dispersion relations assume a very neat form, which is completely analogous to that of the three one-dimensional dispersion relations given by Mandelstam (1).

This is an interesting starting point for speculations on the form of possible multidimensional dispersion relations. The first thing to do here is to remove the quantities ν from the picture. We will show that, at least on a purely formal basis, for each of the one-dimensional dispersion relations one can write down a two-dimensional dispersion relation in which three s-variables are kept fixed and from which the corresponding one-dimensional relation can be derived. It would be interesting to see if these two-dimensional representations could be proved in perturbation theory. It is to be expected that, if their

validity can be established at all, it will be restricted as in Mandelstam's case (1) to certain combinations of external masses with normal or with only certain types of anomalous thresholds. Possibly there will also be restrictions on the fixed variables. Up to now these questions are completely open.

Once the two-dimensional representations formulated solely in terms of s-variables have been established, we might ask the further question: If the amplitudes were analytic functions in all five variables and if there were a five-dimensional dispersion representation, what would it look like? Mandelstam's result might suggest that we have to write a fivefold dispersion integral for each combination of five independent s variables. If this were the case, the representation would consist, besides the pole terms, of 162 fivefold integrals. It is well realized that within the framework of a rigorous field theoretical approach the present calculations do not prove anything. It is hoped, however, that they will not be quite useless, but may provide us with some definite suggestions on what the results of a more complete theory might be. G. F. Chew (4), in his recent outline of the possible framework of a complete dynamical theory for strong interactions, points out why investigations of this type are desirable and what use can be made of them. He even raises doubts that correct final answers can be obtained from quantum field theory in its present form.

II. DEFINITION OF VARIABLES

We begin with the consideration of a general reaction, for which the total number of ingoing and outgoing particles is n. Each particle is characterized by a four-momentum q_i $(i = 1, 2, \cdots, n)$, which has a positive timelike component for an incoming particle, and a negative timelike component for an outgoing particle. With this convention all momenta in Fig. 1 are pointing inward and our formulae maintain a maximum of symmetry. The mass of the ith particle is given by[1] $q_i^2 = m_i^2$, and energy and momentum conservation are expressed by

$$\sum_{i=1}^{n} q_i = 0. \tag{1}$$

Now denote by σ_0 the set of indices $i = 1, 2, \cdots, n$, let σ be a subset of σ_0, and $\bar{\sigma}$ that subset of σ_0 which is complementary to σ (i.e., $\sigma \cap \bar{\sigma} = 0$ and $\sigma \cup \bar{\sigma} = \sigma_0$). Each such σ defines a "reaction channel," i.e., it may be associated with the reaction in which the particles characterized by $i \epsilon \sigma$ are incoming (and $i \epsilon \bar{\sigma}$ outgoing). The square of the total center-of-mass energy in this reaction channel is given by

$$s_\sigma = \left(\sum_{i \epsilon \sigma} q_i\right)^2 . \tag{2}$$

This Lorentz-invariant expression will subsequently be called an "s variable," and we shall try to make as wide a use of these variables as possible. For obvious reasons we are particularly interested in reaction channels with only two incoming particles, and we introduce a special notation for the associated s variables,

$$s_{ik} = s_{ki} = (q_i + q_k)^2 = m_i^2 + m_k^2 + 2q_i q_k . \tag{3}$$

From energy and momentum conservation we obtain a set of $\frac{1}{2}\sum_{r=0}^{n}\binom{n}{r} = 2^{n-1}$ identities for the s variables,

$$s_\sigma = s_{\bar{\sigma}} \quad . \tag{4}$$

All the s_σ can be expressed in terms of s_{ik} and m_i^2 :

$$s_\sigma = \sum_{i\epsilon\sigma}\sum_{k\epsilon\sigma} q_i q_k = \sum_{\substack{i,k\epsilon\sigma\\ i<k}} s_{ik} - (r-2)\sum_{i\epsilon\sigma} m_i^2 \,, \tag{5}$$

where r denotes the number of indices contained in σ. Therefore we can deduce from the identities (4) a number of (not necessarily independent) relations for the s_{ik} ,

$$\sum_{\substack{i,k\epsilon\sigma\\ i<k}} s_{ik} - (r-2)\sum_{i\epsilon\sigma} m_i^2 = \sum_{\substack{i,k\epsilon\bar{\sigma}\\ i<k}} s_{ik} - (n-r-2)\sum_{i\epsilon\bar{\sigma}} m_i^2 \,. \tag{6}$$

For later purposes it will be convenient to have some of these equations written explicitly:

for $r = 0$, σ empty: $$0 = \sum_{\substack{i,k=1\\ i<k}}^{n} s_{ik} - (n-2)\sum_{i=1}^{n} m_i^2 \,; \tag{7a}$$

for $r = 1$, $\sigma = j$: $$m_j^2 = \sum_{\substack{i,k\neq j\\ i<k}} s_{ik} - (n-3)\sum_{i\neq j} m_i^2 \,; \tag{7b}$$

for $r = 2$, $\sigma = j,\ell$: $$s_{j\ell} = \sum_{\substack{i,k\neq j\\ i,k\neq \ell\\ i<k}} s_{ik} - (n-4)\sum_{i\neq j,\ell} m_i^2 \,. \tag{7c}$$

By subtracting (7b) from (7a) we obtain another useful equation,

$$\sum_{\substack{k=1 \\ k\neq j}}^{n} s_{jk} = (n - 4)\, m_j^2 + \sum_{i=1}^{n} m_i^2 \quad . \tag{8}$$

Equations (7c) and (8) will be the most frequently used relations. In the case $n = 4$, (7c) reduces to $s_{12} = s_{34}$, $s_{13} = s_{24}$, $s_{14} = s_{23}$ and each of the relations (7a), (7b), (8) reduces to the well-known equation

$$s_{12} + s_{13} + s_{14} = m_1^2 + m_2^2 + m_3^2 + m_4^2 \quad .$$

III. CONDITIONS FOR THE PHYSICAL REGION

In the physical region of any reaction the four-momenta q_i of the particles involved $(i = 1,2,\cdots,n,$ where n is the total number of incoming and outgoing particles) are real timelike vectors (since $q_i^2 = m_i^2 > 0$). This fact will be used in this section to derive inequalities that tell us where in the space of s variables the physical regions of the various reaction channels are situated.

<u>Theorem I</u>: In the physical region any two of the n four-momenta q_i satisfy the inequality

$$\Delta_{i_1 i_2} \equiv \begin{vmatrix} m_{i_1}^2 & q_{i_1} q_{i_2} \\ q_{i_1} q_{i_2} & m_{i_2}^2 \end{vmatrix} \leq 0 , \tag{9}$$

any three of the n four-momenta q_i satisfy the inequality

$$\Delta_{i_1 i_2 i_3} \equiv \begin{vmatrix} m_{i_1}^2 & q_{i_1} q_{i_2} & q_{i_1} q_{i_3} \\ q_{i_2} q_{i_1} & m_{i_2}^2 & q_{i_2} q_{i_3} \\ q_{i_3} q_{i_1} & q_{i_3} q_{i_2} & m_{i_3}^2 \end{vmatrix} \geq 0, \tag{10}$$

any four of the n four-momenta q_i satisfy the inequality

$$\Delta_{i_1 i_2 i_3 i_4} \equiv \begin{vmatrix} m_{i_1}^2 & q_{i_1} q_{i_2} & q_{i_1} q_{i_3} & q_{i_1} q_{i_4} \\ q_{i_2} q_{i_1} & m_{i_2}^2 & q_{i_2} q_{i_3} & q_{i_2} q_{i_4} \\ q_{i_3} q_{i_1} & q_{i_3} q_{i_2} & m_{i_3}^2 & q_{i_3} q_{i_4} \\ q_{i_4} q_{i_1} & q_{i_4} q_{i_2} & q_{i_4} q_{i_3} & m_{i_4}^2 \end{vmatrix} \equiv 0. \tag{11}$$

Analogous determinants, constructed from more than four four-momenta q_i vanish.

Remark: This theorem can be formulated in terms of s variables by using Eq. (3). Also we could, following Kibble (2), express $\Delta_{i_1 i_2}$, $\Delta_{i_1 i_2 i_3}$, $\Delta_{i_1 i_2 i_3 i_4}$ as homogeneous polynomials in s variables. The resulting expressions, however, would in general (i.e., for $n \geq 5$) not be uniquely determined, since we have many relations (not only one) among the s variables.

Proof: First we prove (11). Denoting the metric tensor by $g^{\mu\nu}$ ($g^{00} = -g^{11} = -g^{22} = -g^{33} = 1$, $g^{ik} = 0$ for $i \neq k$), we can write the left-hand side of (11) in the form

$$\Delta_{i_1 i_2 i_3 i_4} = \begin{vmatrix} \sum_{\mu\nu} q_{i_1\mu} g^{\mu\nu} q_{i_1\nu} & \sum_{\mu\nu} q_{i_1\mu} g^{\mu\nu} q_{i_2\nu} & \cdots \\ \sum_{\mu\nu} q_{i_2\mu} g^{\mu\nu} q_{i_1\nu} & \sum_{\mu\nu} q_{i_2\mu} g^{\mu\nu} q_{i_2\nu} & \cdots \\ \cdots & \cdots & \cdots \end{vmatrix} \tag{12}$$

$$= \det\,(q_{i_1} q_{i_2} q_{i_3} q_{i_4}) \cdot \det\,(g^{\mu\nu}) \cdot \det\,(q_{i_1} q_{i_2} q_{i_3} q_{i_4}).$$

Here $\det(q_{i_1} q_{i_2} q_{i_3} q_{i_4})$ denotes the determinant, whose rth row $(r = 1, 2, 3, 4)$ consists of the four components of the vector q_{i_r}. Equation(11) follows from the fact that $\det(g^{\mu\nu}) = -1$ and $(\det(q_{i_1} q_{i_2} q_{i_3} q_{i_4}))^2 > 0$, if all the four-momenta $q_{i_1}, q_{i_2}, q_{i_3}, q_{i_4}$ are real. To prove (10), we first find a Lorentz frame in which the fourth component of each of the three vectors $q_{i_1}, q_{i_2}, q_{i_3}$ vanishes. The argument then proceeds in the same way as before; we have, however, instead of $\det(g^{\mu\nu})$, the three-rowed determinant

$$\begin{vmatrix} g^{00} & g^{01} & g^{02} \\ g^{10} & g^{11} & g^{12} \\ g^{20} & g^{21} & g^{22} \end{vmatrix} = +1$$

Similarly we proceed for the proof of (9). The last statement follows from the fact that any five four-vectors are linearly dependent, and therefore by combining rows and columns in an appropriate way the elements of a whole column can be made to vanish. Introducing into (9), by means of (3), the variable $s_{i_1 i_2}$, we find

$$-\Delta_{i_1 i_2} = \frac{1}{4}\left(s_{i_1 i_2} - (m_{i_1} + m_{i_2})^2\right)\left(s_{i_1 i_2} - (m_{i_1} - m_{i_2})^2\right) \geq 0. \tag{13}$$

Thus the hypersurface in the space of the s_{ik} $(i, k = 1, \cdots, n, i \neq k)$ defined by $\Delta_{i_1 i_2} = 0$ consists of two hyperplanes. In the physical region we have either $s_{i_1 i_2} < (m_{i_1} - m_{i_2})^2$ or $s_{i_1 i_2} > (m_{i_1} + m_{i_2})^2$. The first inequality obviously holds when particle i_1 is in the initial and i_2 in the final state, or vice versa; the second inequality holds, when both particles are in either the initial or final state. Thus we have

<u>Definition:</u> $s_{i_1 i_2}$ is called an "energy-type variable" if in the physical region $s_{i_1 i_2} > (m_{i_1} + m_{i_2})^2$. It is called a "momentum-transfer type variable" if in the physical region $s_{i_1 i_2} < (m_{i_1} - m_{i_2})^2$.

<u>Theorem II:</u> The six hyperplanes, defined by $\Delta_{i_1 i_2} = 0$, $\Delta_{i_1 i_3} = 0$, and $\Delta_{i_2 i_3} = 0$ are tangent to the hypersurface defined by $\Delta_{i_1 i_2 i_3} = 0$. Similarly the four hypersurfaces defined by $\Delta_{i_1 i_2 i_3} = 0$, $\Delta_{i_1 i_2 i_4} = 0$, $\Delta_{i_1 i_3 i_4} = 0$, and $\Delta_{i_2 i_3 i_4} = 0$ are tangent to the hypersurface defined by $\Delta_{i_1 i_2 i_3 i_4} = 0$.

<u>Proof:</u> In order to show that $\Delta_{i_1 i_2} = 0$ is tangent to $\Delta_{i_1 i_2 i_3} = 0$, we need only prove that all points common to both hypersurfaces are double points. Writing the equation $\Delta_{i_1 i_2 i_3} = 0$ in terms of s variables, we obtain

$$0 = \Delta_{i_1 i_2 i_3} = m_{i_1}^2 m_{i_2}^2 m_{i_3}^2 + \frac{1}{4}\,(s_{i_1 i_2} - m_{i_1}^2 - m_{i_2}^2)(s_{i_1 i_3} - m_{i_1}^2 - m_{i_3}^2)$$

$$(s_{i_2 i_3} - m_{i_2}^2 - m_{i_3}^2)$$

$$- \frac{1}{4}\left[m_{i_1}^2 (s_{i_2 i_3} - m_{i_2}^2 - m_{i_3}^2)^2 + m_{i_2}^2 (s_{i_1 i_3} - m_{i_1}^2 - m_{i_3}^2)^2\right.$$

$$\left. + m_{i_3}^2 (s_{i_1 i_2} - m_{i_1}^2 - m_{i_2}^2)^2\right] . \qquad (14)$$

Inserting $s_{i_1 i_2} = (m_{i_1} \pm m_{i_2})^2$, we find

$$0 = \Delta_{i_1 i_2 i_3} = - \frac{1}{4}\left[m_{i_1} (s_{i_2 i_3} - m_{i_2}^2 - m_{i_3}^2)\right.$$

$$\left. \mp m_{i_2} (s_{i_1 i_3} - m_{i_1}^2 - m_{i_3}^2)\right]^2 , \qquad (15)$$

which proves our statement on $\Delta_{i_1 i_2} = 0$. In a similar fashion we show, that $\Delta_{i_1 i_3} = 0$ and $\Delta_{i_2 i_3} = 0$ are tangent to $\Delta_{i_1 i_2 i_3} = 0$. Now we prove the second part of our theorem. With $x_1 = q_{i_1} q_{i_4}$, $x_2 = q_{i_2} q_{i_4}$, $x_3 = q_{i_3} q_{i_4}$ we can write

$$\Delta_{i_1 i_2 i_3 i_4} = m_{i_4}^{\ 2} \Delta_{i_1 i_2 i_3} + x_1^2 a_{11} + x_2^2 a_{22} + x_3^2 a_{33} + 2 x_1 x_2 a_{12}$$

$$+ 2 x_1 x_3 a_{13} + 2 x_2 x_3 a_{23} , \qquad (16)$$

with coefficients a_{ik} depending on $s_{i_1 i_2}$, $s_{i_1 i_3}$ and $s_{i_2 i_3}$. By straightforward calculation one can show

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{13} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{22} & a_{23} \\ a_{23} & a_{33} \end{vmatrix} = (m_{i_1}^{\ 2} + m_{i_2}^{\ 2} + m_{i_3}^{\ 2}) \Delta_{i_1 i_2 i_3} \qquad (17)$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{vmatrix} = (\Delta_{i_1 i_2 i_3})^2 . \qquad (18)$$

From this we conclude: For $\Delta_{i_1 i_2 i_3} = 0$, then $\Delta_{i_1 i_2 i_3 i_4}$ is a quadratic form $\sum_{i,k=1}^{3} x_i x_k a_{ik}$ with eigenvalues $\lambda_1 \neq 0$, $\lambda_2 = \lambda_3 = 0$. Therefore,

three coefficients b_1, b_2, b_3 (depending on $s_{i_1 i_2}$, $s_{i_1 i_3}$, $s_{i_2 i_3}$) exist, such that $\Delta_{i_1 i_2 i_3 i_4} = \epsilon\,(b_1 x_1 + b_2 x_2 + b_3 x_3)^2$, ϵ being the sign of λ_1. This proves that the two hypersurfaces $\Delta_{i_1 i_2 i_3} = 0$ and $\Delta_{i_1 i_2 i_3 i_4} = 0$ have only double points in common and are thus tangent to each other.

The meaning of this theorem can best be seen from a discussion of the case $n = 4$, for which we refer the reader to Kibble's paper (2). Roughly speaking we may say: The set of inequalities (11) is more restrictive than the set of inequalities (10); these in turn are stronger than the inequalities (9). In saying this we have considered all reaction channels simultaneously. When we wish to pick out a specific channel, we have in addition to specify which s variables are of the "energy type" and which are of the "momentum-transfer type".

In special cases some of the inequalities (9), (10), (11) admit a simple interpretation in terms of scattering angles. To show this let us consider (10). Denoting the timelike and spatial part of q_i $(i = 1, 2, \cdots, n)$ by q_{i0} and $\vec{q}_i$ respectively, and defining the angle θ_{ik} by $|\vec{q}_i| \cdot |\vec{q}_k| \cos\theta_{ik} = \vec{q}_i \cdot \vec{q}_k$, we find the general expression

$$\Delta_{i_1 i_2 i_3} = (q_{i_1 0} \left|\vec{q}_{i_2}\right| \cdot \left|\vec{q}_{i_3}\right| \cdot \sin\theta_{i_2 i_3} + q_{i_2 0} \left|\vec{q}_{i_1}\right| \cdot \left|\vec{q}_{i_3}\right| \cdot \sin\theta_{i_1 i_3}$$

$$+ q_{i_3 0} \left|\vec{q}_{i_1}\right| \cdot \left|\vec{q}_{i_2}\right| \cdot \sin\theta_{i_1 i_2})^2$$

$$+ \vec{q}_{i_1}^{\,2}\, \vec{q}_{i_2}^{\,2}\, \vec{q}_{i_3}^{\,2} \; (\cos^2\theta_{i_1 i_2} + \cos^2\theta_{i_1 i_3} + \cos^2\theta_{i_2 i_3}$$

$$- 2\cos\theta_{i_1 i_2} \cos\theta_{i_1 i_3} \cos\theta_{i_2 i_3} - 1)$$

$$+ 2 q_{i_1 0}\, q_{i_2 0} \left|\vec{q}_{i_1}\right| \cdot \left|\vec{q}_{i_2}\right| \cdot \vec{q}_{i_3}^{\,2} (\cos(\theta_{i_1 i_3} + \theta_{i_2 i_3}) - \cos\theta_{i_1 i_2})$$

$$+ 2 q_{i_1 0}\, q_{i_3 0} \left|\vec{q}_{i_1}\right| \cdot \left|\vec{q}_{i_3}\right| \vec{q}_{i_2}^{\,2} (\cos(\theta_{i_1 i_2} + \theta_{i_3 i_2}) - \cos\theta_{i_1 i_3})$$

$$+ 2 q_{i_2 0}\, q_{i_3 0} \left|\vec{q}_{i_2}\right| \cdot \left|\vec{q}_{i_3}\right| \vec{q}_{i_1}^{\,2} (\cos(\theta_{i_2 i_1} + \theta_{i_3 i_1}) - \cos\theta_{i_2 i_3}) \,. \tag{19}$$

It is easily seen that the last four terms vanish if $\vec{q}_{i_1}$, $\vec{q}_{i_2}$, $\vec{q}_{i_3}$ are coplanar. This is the case, for example, when we consider a reaction with three particles in the final (or initial) state in its center-of-mass system and when q_{i_1}, q_{i_2}, q_{i_3} are the four-momenta of these three particles. Another simple case is a reaction with only two particles in the initial (or final) state. When q_1 and q_2 are the four-momenta of the two incoming particles, then in the center-of-mass system we can put $\vec{q}_1^{\,2} = \vec{q}_2^{\,2} = \vec{q}^{\,2}$. The last four terms in (19) vanish, and the first one reduces to

$$\Delta_{12i_3} = s_{12} \cdot \vec{q}^{\,2} \cdot \vec{q}_{i_3}^{\,2} (1 - \cos^2 \theta_{1i_3}) \geq 0 \quad . \tag{20}$$

The expression for (11), which is quite complicated in the general case, reduces now to

$$\Delta_{12i_3i_4} = - s_{12} \vec{q}^{\,2} \vec{q}_{i_3}^{\,2} \vec{q}_{i_4}^{\,2} (1 - \cos^2 \theta_{1i_3} - \cos^2 \theta_{1i_4} - \cos^2 \theta_{i_3i_4}$$

$$+ 2 \cos \theta_{1i_3} \cos \theta_{1i_4} \cos \theta_{i_3i_4}) \leq 0. \tag{21}$$

Obviously to fulfill (20) and (21) we have to require

$$-1 \leq \cos \theta_{1i_3} \leq +1 \quad \text{and} \quad \cos (\theta_{1i_3} + \theta_{1i_4}) \leq \cos \theta_{i_3i_4} \leq \cos (\theta_{1i_3} - \theta_{1i_4}).$$

IV. DEFINITION OF AMPLITUDES

In this section we collect some definitions that are useful for the discussion of dispersion relations. In doing so we restrict ourselves to the consideration of reactions with only two incoming and an arbitrary number of outgoing particles. (Polkinghorne (3) has given a heuristic derivation of dispersion relations for a reaction with one nucleon and m mesons in the initial state and one nucleon and n mesons in the final state ($m \geq 1$, $n \geq 1$). For this derivation a particular causality requirement had to be used. To fulfill this condition a "causal product" was defined, and it has been explicitly assumed that the amplitude possesses a representation in terms of these "causal products." Subsequent investigations (5), however, showed that in the S-matrix formalism using reduction formulae (5, 6, 7) and local commutativity such representations would be obtained only for $m = 1$ or $n = 1$. Thus, if we have both $m \geq 2$ and $n \geq 2$, the connection of Polkinghorne's generalized dispersion relations with the S-matrix formalism is not clear.) For reasons of simplicity we consider only scalar hermitean fields. In the following we assume that the four-momenta of the two incoming particles are given by q_1 and q_2, those of the n-2 outgoing particles by $q_3, q_4, \cdots q_n$. Using the LSZ-formalism (6) and denoting the time-like component of $q_i (i = 1, 2, \cdots n)$ by q_{i0}, we can express the matrix elements of the S matrix and its adjoint $S^\dagger$ as follows:

$$S(q_n, q_{n-1}, \cdots, q_3; q_2, q_1) = \frac{(-i)\cdot(2\pi)^4\cdot\delta^4(q_1+q_2+\cdots+q_n)}{\sqrt{(2\pi)^{3n}\cdot 2|q_{10}|\cdot 2|q_{20}|\cdot\ldots\cdot 2|q_{n0}|}}\; T_R(q_n, q_{n-1}, \cdots, q_3; q_2, q_1), \tag{22a}$$

$$S^{\dagger}(q_n, q_{n-1}, \cdots, q_3; q_2, q_1) = \frac{(-i)\cdot(2\pi)^4 \cdot \delta^4(q_1+q_2+\cdots+q_n)}{\sqrt{(2\pi)^{3n} \cdot 2|q_{10}| \cdot 2|q_{20}| \cdot \ldots \cdot 2|q_{n0}|}}$$

$$T_A(q_n, q_{n-1}, \cdots, q_3; q_2, q_1) \tag{22b}$$

where

$$T_R(q_n, q_{n-1}, \cdots, q_3; q_2, q_1) = \Omega \int d^4x_3 d^4x_4 \cdot \ldots \cdot d^4x_{n-1}\, e^{-iq_3x_3 - iq_4x_4 - \ldots - iq_{n-1}x_{n-1}}$$

$$\langle -q_n | \theta \left[j_{n-1}(x_{n-1}), \cdots, j_3(x_3) : j_2(0) \right] | q_1 \rangle \tag{23a}$$

$$T_A(q_n, q_{n-1}, \cdots, q_3; q_2, q_1) = \Omega \int d^4x_3 d^4x_4 \cdot \ldots \cdot d^4x_{n-1}\, e^{-iq_3x_3 - iq_4x_4 - \ldots - iq_{n-1}x_{n-1}}$$

$$\langle -q_n | \theta \left[j_2(0) : j_3(x_3), \cdots, j_{n-1}(x_{n-1}) \right] | q_1 \rangle \,. \tag{23b}$$

Here we have used Screaton's notation (7); Ω is defined as

$$\Omega = (2\pi)^3 \cdot (-i)^{n-3} \sqrt{2|q_{10}| \cdot 2|q_{n0}|}\,, \tag{24}$$

and $j_k(x_k) = -(\Box + m_k^2)\, \phi_k(x_k)$ is the current associated with the field $\phi_k(x_k)$, which describes the particle of momentum q_k. The expressions

$$\theta\,[j_{n-1}(x_{n-1}),\ j_{n-2}(x_{n-2}),\cdots,j_3(x_3):\ j_2(0)] =$$

$$= \sum \theta\,(x_{n-1} - x_{n-2})\ \theta\,(x_{n-2} - x_{n-3})\cdots\ \theta\,(x_4 - x_3)\ \theta\,(x_3)$$

$$\left[j_{n-1}(x_{n-1})\ldots\left[j_4(x_4),\ [j_3(x_3),\ j_2(0)]\right]\cdots\right], \qquad (25a)$$

$$\theta\,[j_2(0):\ j_3(x_3),\ j_4(x_4),\cdots,j_{n-1}(x_{n-1})]$$

$$= \sum \theta\,(-x_3)\ \theta(x_3 - x_4)\cdots\ \theta(x_{n-3} - x_{n-2})\ \theta(x_{n-2} - x_{n-1})$$

$$\left[\ldots\left[\left[j_2(0),\ j_3(x_3)\right],\ j_4(x_4)\right]\cdots\ j_{n-1}(x_{n-1})\right], \qquad (25b)$$

where the summation goes over all permutations of 3, 4, $\cdots$, n-1, are apart from a factor $(-i)^{n-3}$ identical to the advanced and retarded commutators as defined in LSZ (6). The expressions (23) will be used to define dispersive and absorptive amplitudes D and A:

$$D(q_n, q_{n-1}, \cdots, q_3; q_2, q_1) = \frac{1}{2}\,[T_R\,(q_n,\ q_{n-1}, \cdots, q_3; q_2, q_1) + T_A\,(q_n, q_{n-1}, \cdots, q_3; q_2, q_1)] \qquad (26a)$$

$$A(q_n, q_{n-1}, \cdots, q_3; q_2, q_1) = \frac{1}{2i}\,[T_R(q_n, q_{n-1}, \cdots, q_3; q_2, q_1) - T_A(q_n, q_{n-1}, \cdots, q_3; q_2, q_1)], \qquad (26b)$$

such that

$$T_R = D + iA, \tag{27a}$$

$$T_A = D - iA. \tag{27b}$$

If we can (denoting by $*$ the complex conjugate) show

$$\left.\begin{aligned} T_R^*(q_n, q_{n-1}, \cdots, q_3; q_2, q_1) &= T_A(q_n, q_{n-1}, \cdots, q_3; q_2, q_1), \\ T_A^*(q_n, q_{n-1}, \cdots, q_3; q_2, q_1) &= T_R(q_n, q_{n-1}, \cdots, q_3; q_2, q_1), \end{aligned}\right\} \tag{28}$$

then we can conclude that the dispersive and absorptive amplitudes D and A are real functions. In fact, assuming the validity for our theory of the TCP theorem (8), Eq. (28) holds in the physical region, where all four-momenta $q_i (i=1,2,\cdots,n)$ are real. This is most easily seen by expressing T_R and T_A in terms of vacuum expectation values (cf. LSZ (6), Eq. (43)), taking the complex conjugate, substituting $x_i \to -x_i$ $(i=1,2,\cdots,n)$, and using the TCP theorem in Jost's formulation,

$$\langle 0 \left| j_{i_n}(-x_{i_n}) \cdot \ldots \cdot j_{i_2}(-x_{i_2}) \cdot j_{i_1}(-x_{i_1}) \right| 0 \rangle$$

$$= \langle 0 \left| j_{i_1}(x_1) \cdot j_{i_2}(x_{i_2}) \cdot \ldots \cdot j_{i_n}(x_{i_n}) \right| 0 \rangle . \tag{29}$$

The representations (23a) and (23b) are, of course, not the only representations of T_R and T_A in terms of expectation values between one-particle states. We could as well have kept the other incoming particle (with momentum q_2) in the state vector and "converted" Particle 1 into a field operator. Likewise we could have "converted" the outgoing particle with four-momentum q_n into a field operator and kept any one of the other outgoing particles $(3,4,\cdots,n-1)$ in the state vector. Indeed, these other representations will also be of importance for us later on.

V. GENERALIZATIONS OF BREIT'S FRAME OF REFERENCE

It is well known that in order to derive for elastic scattering of a Particle 1 by a Particle 2 the dispersion relation in which the momentum-transfer $(q_1 + q'_1)^2$ is kept constant,[2] it is most convenient to use the so-called Breit system. This is the frame of reference in which for the timelike components of the four-momenta q_1, q'_1 we have

$$q_{10} + q'_{10} = 0, \tag{30}$$

and for the spacelike components

$$\vec{q}_1 = \vec{q}'_1 \quad . \tag{31}$$

In discussing more general reactions we shall wish to keep the momentum transfer between particles of unequal mass constant. Then we cannot fulfill simultaneously the two equations corresponding to (30) and (31). Usually one chooses to satisfy the second equation. This, however, would not be of great use for our purposes, since we wish to generalize Polkinghorne's (3) heuristic proof of general dispersion relations for the case in which there are only two incoming particles but the masses of the particles involved may all be different from each other. For this proof it is essential that the vector of momentum transfer between the two particles, which are kept in the state vectors in a representation of type (23) of T_R and T_A (in the case of (23) the vector $q_1 + q_n$) have a vanishing timelike component. In addition we will require that in this system the timelike components q_{i0} of all four-momenta $q_i (i=1,2,\cdots,n)$ be equal to invariants that depend linearly on the total energy s_{12}. This leads uniquely to replacing (30) and (31) by

$$q_{10} + q_{n0} = 0, \tag{32}$$

$$(1 + c)\,\vec{q}_1 = (1 - c)\,\vec{q}_n, \tag{33}$$

and to the ansatz

$$q_{i0} = \omega_i \equiv \frac{(1 + c)\,q_i q_1 - (1 - c)\,q_i q_n}{\sqrt{((1 + c)\,q_1 - (1 - c)\,q_n)^2}} \quad \text{for } i=2,3,\cdots,n-1. \tag{34}$$

Evidently (32) cannot be satisfied unless $s_{1n} \leq 0$.[3] Therefore we will assume explicitly in the following that $s_{1n} \leq 0$. Equation (33) has the solution $\vec{q}_1 = (1 - c)\,\vec{p}$, $\vec{q}_n = (1 + c)\,\vec{p}$, and this inserted into (32) yields $m_1^2 - m_n^2 + \vec{q}_1^{\,2} - \vec{q}_n^{\,2} = m_1^2 - m_n^2 - 4\,c\vec{p}^{\,2} = 0$. On the other hand, $s_{1n} = (q_{10} + q_{n0})^2 - (\vec{q}_1 + \vec{q}_n)^2 = -4\,\vec{p}^{\,2}$. Thus c can be expressed in terms of s_{1n} as

$$c = \frac{m_n^2 - m_1^2}{s_{1n}} \quad . \tag{35}$$

The $\omega_i (i=2,3,\cdots,n-1)$ can now be expressed entirely in terms of s variables: we have, using (3),

$$2(q_i q_1 - q_i q_n + c\,(q_i q_1 + q_i q_n)) =$$

$$= s_{i1} - s_{in} + \frac{m_n^2 - m_1^2}{s_{1n}}\,(s_{i1} + s_{in} + s_{1n} - 2\,m_i^2 - m_1^2 - m_n^2). \tag{36}$$

From (8) we find

$$s_{i1} + s_{in} = - \sum_{\substack{k=2 \\ k\neq i}}^{n-1} s_{ik} + (n-4)\, m_i^{\,2} + \sum_{k=1}^{n} m_k^{\,2} \quad . \tag{37}$$

Therefore, with the abbreviation (note that $\Delta = 2q_{10} = -2q_{n0}$)

$$\Delta^2 \equiv ((1+c)\, q_1 - (1-c)\, q_n)^2 = 2(m_1^{\,2} + m_n^{\,2}) - s_{1n} - \frac{(m_n^{\,2} - m_1^{\,2})^2}{s_{1n}} \, , \tag{38}$$

we can write ω_i $(i=2,3,\cdots,n-1)$ as

$$\omega_i = \frac{1}{2\Delta} \left(s_{i1} - s_{in} + \frac{m_n^{\,2} - m_1^{\,2}}{s_{1n}} \left(s_{1n} - \sum_{\substack{k=2 \\ k\neq i}}^{n-1} s_{ik} + (n-6)\, m_i^{\,2} + \sum_{k=2}^{n-1} m_k^{\,2}\right)\right). \tag{39}$$

Furthermore it is useful for the following to introduce

$$\nu_i = - \frac{\omega_i}{\omega_2} \qquad \text{for } i=2,3,\cdots,n-1 \quad . \tag{40}$$

From (1) and (32) we have $q_{20} + q_{30} + \cdots + q_{n-1,0} = 0$, therefore

$$\nu_2 + \nu_3 + \cdots + \nu_{n-1} = 0. \tag{41}$$

We will also make use of quantities

$$\nu_{ik} = \nu_{ki}^{\,-1} = \frac{\nu_i}{\nu_k} = \frac{\omega_i}{\omega_k} \quad \text{for} \quad i,k=2,3,\cdots,n-1 \ . \tag{42}$$

In particular we have $\nu_{i2} = -\nu_i$. In the physical region ω_2 represents the energy of an incoming particle and is thus positive. All other ω_i belong to outgoing particles and are negative. Therefore all ν_i except $\nu_2 = -1$ are positive quantities. After these preparations we can proceed to discuss the dispersion relations.

VI. DISPERSION RELATIONS

An important point in the derivation of general dispersion relations is the selection of a set of variables, which are kept fixed at real values in the physical region of the considered process. Polkinghorne (3) suggested choosing the ν_i defined in (40) and in addition to this constructing a set of linear combinations δ^α of the four-momenta $q_1, q_2, \cdots, q_n$, such that the timelike component of each δ^α vanishes, and keeping the squares of the spatial parts $(\vec{\delta}^{\,\alpha})^2$ at fixed values. But since we have restricted ourselves to reactions with only two incoming particles, it is actually sufficient to have only one such vector, namely the momentum transfer between the two particles, which have been kept in the state vectors--for example, in representation (23) the vector $q_1 + q_n$. Thus in order to derive the dispersion relation for the amplitude as represented in (23) we propose to keep fixed the quantities $\nu_i (i=3, 4, \cdots, n-1)$, s_{1n}, and all s_{ik} with $2 \leq i \leq n-1$, $2 \leq k \leq n-1$ and $i \neq k$. Not all these variables, of course, are independent of one another. As shown in the preceeding section, for $s_{1n} < 0$, the timelike component of $q_1 + q_n$ can be made to vanish, and then an examination of Polkinghorne's heuristic derivation of the dispersion relations shows that his arguments can be applied even in our general mass case without any changes. We thus find that $T_R(\omega_2, \nu_3, \cdots, \nu_{n-1}, s_{23}, \cdots, s_{n-2,n-1})$ is an analytic function in the upper half of the complex ω_2 plane, while $T_A(\omega_2, \nu_3, \cdots, \nu_{n-1}, s_{23}, \cdots, s_{n-2,n-1})$ is an analytic function in the lower half plane (except for cuts along the real axis). We define $T(\omega_2, \nu_3, \cdots, \nu_{n-1}, s_{23}, \cdots, s_{n-2,n-1})$ to be the analytic function, which equals T_R in the upper half plane and T_A in the lower half plane.

We then can write the dispersion relation[4] (ignoring the eventual necessity of subtractions)

$$D(\omega_2;\ \nu_3,\cdots,\nu_{n-1},\ s_{23},\cdots,s_{n-2,n-1})$$

$$= \frac{1}{\pi}\ P \int_{-\infty}^{+\infty} d\omega'_2\ \frac{A(\omega'_2;\ \nu_3,\cdots,\nu_{n-1},\ s_{23},\cdots,s_{n-2,n-1}}{\omega'_2 - \omega_2} \tag{43}$$

or

$$T_R(\omega_2;\ \nu_3,\cdots,\nu_{n-1},\ s_{23},\cdots,s_{n-2,n-1})$$

$$= \lim_{\epsilon\to+0}\ \frac{1}{\pi} \int_{-\infty}^{+\infty} d\omega'_2\ \frac{A(\omega'_2;\ \nu_3,\cdots,\nu_{n-1},\ s_{23},\cdots,s_{n-2,n-1})}{\omega'_2 - (\omega_2 + i\epsilon)} \tag{44}$$

Since we have not necessarily $s_{1n} < 0$ throughout the physical region, we assume that the dispersion relations can be continued analytically to those parts where $s_{1n} \geq 0$. In this connection note that the quantities ν_i $(i=2,\cdots,n-1)$, on which the amplitudes depend, are not singular at $s_{1n} = 0$.

One might ask whether or not one could avoid introducing the variables $\nu_3,\cdots,\nu_{n-1}$ and replace them simply by s variables. In order to decide this question let us consider the simplest example, $n = 5$, which exhibits all essential features. As is easily demonstrated, there are five linearly independent s variables, and therefore in order to write one-dimensional dispersion relations four independent variables have to be kept fixed. As such we have proposed to choose

$$\nu_3 = -\frac{\omega_3}{\omega_2} = -\frac{s_{13} - s_{35} + \dfrac{m_5^2 - m_1^2}{s_{15}}(s_{15} - s_{23} - s_{34} + m_2^2 + m_4^2)}{s_{12} - s_{25} + \dfrac{m_5^2 - m_1^2}{s_{15}}(s_{15} - s_{23} - s_{24} + m_3^2 + m_4^2)} \tag{45}$$

and s_{23}, s_{24}, s_{34}. From (7c) we have

$$s_{15} = s_{23} + s_{24} + s_{34} - m_2^2 - m_3^2 - m_4^2, \tag{46}$$

so that s_{15} is also fixed. The remaining s variables are s_{12}, s_{13}, s_{14} and s_{25}, s_{35}, s_{45}. Now we shall wish to replace the integral with respect to ω'_2 in (44) by a sum of integrals with respect to s variables, so that we obtain a dispersion relation resembling Mandelstam's one-dimensional relations. For this one has to discuss in some detail the structure of the absorptive amplitudes. As will be shown in a moment, all six reaction channels, whose s variables have not yet been fixed, contribute to the dispersion relation and therefore the remaining six s variables appear in the denominators of the dispersion integrals. Hence none of these six s variables can be chosen as a fourth fixed variable, and we necessarily have to resort to certain functions of the s like the above ν_3.

Altogether the structure of the absorptive amplitude leads to the conclusion that in general (n arbitrary) all s variables s_σ, where σ contains either 1 or n, but not 1 and n simultaneously, will appear in the denominators of the dispersion integrals. Therefore no other s variables than s_{1n}, s_{23}, s_{24}, $\cdots$, $s_{n-2,n-1}$ or variables dependent on these can be chosen as fixed s variables. For all further needs we have to make use of the ν_i.

Since a complete discussion of the general case would be too involved, we shall in the following again restrict ourselves to $n = 5$.

We will now eliminate from (43) or (44) the integration with respect to ω'_2 in favor of integrations over s'_{1i} or s'_{i5} $(i=2,3,4)$. For this purpose we have to decompose the absorptive amplitude and introduce appropriate sums over intermediate states. First we observe that from $\theta(x) + \theta(-x) = 1$ we obtain

$$\theta(x_1 - x_2)\,\theta(x_2 - x_3) - \theta(x_3 - x_2)\,\theta(x_2 - x_1) = \theta(x_1 - x_2) - \theta(x_3 - x_2)\,, \tag{47a}$$

or, equivalently,

$$\theta(x_1 - x_2)\,\theta(x_2 - x_3) - \theta(x_3 - x_2)\,\theta(x_2 - x_1) = \theta(x_2 - x_3) - \theta(x_2 - x_1). \tag{47b}$$

Therefore, using the Jacobi identity, we find

$$\begin{aligned}
&\frac{1}{2i}\left(\theta[j_4(x_4),\ j_3(x_3):\ j_2(0)] - \theta\,[j_2(0):\ j_3(x_3),\ j_4(x_4)]\right) \\
&= \frac{1}{2i}\Big(-\theta\,(x_3 - x_4)\left[j_2(0),\ \left[j_3(x_3),\ j_4(x_4)\right]\right] + \theta(x_3)\left[j_4(x_4),\ \left[j_3(x_3),\ j_2(0)\right]\right] \\
&\qquad + \theta(-x_4)\left[j_3(x_3),\ \left[j_2(0),\ j_4(x_4)\right]\right]\Big).
\end{aligned} \tag{48}$$

For the right-hand side we can find other equivalent expressions, for example

$$\begin{aligned}
&\frac{1}{2i}\Big(-\theta(x_4 - x_3)\left[j_2(0),\ \left[j_4(x_4),\ j_3(x_3)\right]\right] + \theta\,(-x_3)\left[j_4(x_4),\ \left[j_2(0),\ j_3(x_3)\right]\right] \\
&\qquad + \theta(x_4)\left[j_3(x_3),\ \left[j_4(x_4),\ j_2(0)\right]\right]\Big),
\end{aligned} \tag{49}$$

and by taking the average of (48) and (49) we obtain

$$\begin{aligned}
&\frac{1}{4i}\Big(-\epsilon(x_3 - x_4)\,[j_2(0),\ \left[j_3(x_3),\ j_4(x_4)\right]] + \epsilon(x_3)\left[j_4(x_4),\ \left[j_3(x_3),\ j_2(0)\right]\right] \\
&\qquad + \epsilon(x_4)\left[j_3(x_3),\ \left[j_4(x_4),\ j_2(0)\right]\right]\Big),
\end{aligned} \tag{50}$$

where $\epsilon(x) = \theta(x) - \theta(-x)$.

Using any one of these formulae, we can define a decomposition of the absorptive amplitude into partial amplitudes; for example, using (48), we define, for $i = 2,3,4$ (let i, k, ℓ be $(2,3,4)$, $(3,2,4)$, or $(4,3,2)$),

$$A^{(1i)}(\omega_2, \nu_3, s_{23}, s_{34}, s_{42}) = \frac{\Omega}{2i} \int d^4x_2\, d^4x_3\, d^4x_4\, \delta^4(x_2)\, \theta(x_k - x_\ell)\, e^{-iq_3x_3 - iq_4x_4}$$

$$\langle -q_5 | \left[j_k(x_k),\ j_\ell(x_\ell) \right] j_i(x_i)\, | q_1 \rangle\ , \qquad (51a)$$

$$A^{(5i)}(\omega_2, \nu_3, s_{23}, s_{34}, s_{42}) = \frac{\Omega}{2i} \int d^4x_2\, d^4x_3\, d^4x_4 \delta^4\, (x_2)\, \theta(x_k - x_\ell)\, e^{-iq_3x_3 - iq_4x_4}$$

$$\langle -q_5 |\ j_i(x_i) \left[j_k\, (x_k),\ j_\ell(x_\ell) \right] | q_1 \rangle\ . \qquad (51b)$$

Thus we have

$$A(\omega_2, \nu_3, s_{23}, s_{34}, s_{42}) = A^{(12)} - A^{(13)} - A^{(14)} - A^{(52)} + A^{(53)} + A^{(54)}. \qquad (52)$$

The question arises whether analogous decompositions of the absorptive amplitude derived by using (49) or (50) instead of (48) are equivalent to (51). We shall come to this in a moment.

Let us remark that the indices on the right-hand side of (52) indicate with which s variable we wish to associate the respective partial amplitude. It will turn out that $A^{(1i)}$ resp. $A^{(5i)}$ can be obtained by analytic continuation via the unitarity condition from the imaginary part of the amplitude in the physical region of that reaction channel, in which $\sqrt{s_{1i}}$ resp. $\sqrt{s_{5i}}$ represents the total center-of-mass energy. To make this statement somewhat more explicit we now introduce a resolution of unity into (51).

Let the one-particle states of our theory be associated with masses $\mu_1, \mu_2, \cdots, \mu_m$, and the mass of the lowest two-particle state be μ_c; then the resolution of unity is given by

$$I = \sum_{\gamma} \int d^4p \int d\mu^2 \, \rho(\mu^2) \, \delta\left(p_0 - \sqrt{\mu^2 + \vec{p}^{\,2}}\right) |p\gamma\rangle \langle p\gamma| \tag{53}$$

with the weight function

$$\left.\begin{aligned} \rho(\mu^2) &= \sum_{\alpha=1}^{m} \delta(\mu^2 - \mu_\alpha^2) + \rho_c(\mu^2) \\ \rho_c(\mu^2) &= 0 \quad \text{for} \quad \mu^2 < \mu_c^2 \, . \end{aligned}\right\} \tag{54}$$

Introducing this into (51) and using

$$\delta\left(p_0 - \sqrt{\mu^2 + \vec{p}^{\,2}}\right) = 2p_0 \, \theta(p_0) \, \delta(p_0^2 - (\mu^2 + \vec{p}^{\,2})) = 2p_0 \, \theta(p_0) \, \delta(p^2 - \mu^2),$$

we find

$$A^{(1i)}(\omega_2, \nu_3, s_{23}, s_{34}, s_{42})$$

$$= -\pi \, \rho(s_{1i}) \, \theta(q_{10} + q_{i0}) \, T(q_5, q_k; q_\ell, (q_1 + q_i)\gamma) \, \Gamma(-(q_1 + q_i)\gamma | q_i | q_1), \tag{55}$$

$$A^{(5i)}(\omega_2, \nu_3, s_{23}, s_{34}, s_{42})$$

$$= -\pi\rho(s_{5i}) \, \theta(-q_{50} - q_{i0}) \, \Gamma(q_5 | q_i | - (q_5 + q_i)\gamma) \, T((q_5 + q_i)\gamma, q_k; q_\ell, q_1), \tag{56}$$

where we have used the abbreviations

$$\Gamma(p_1\alpha_1|q_i|p_2\alpha_2) = (2\pi)^3\sqrt{2|p_{10}|\cdot 2|p_{20}|}\ \langle -p_1\alpha_1|j_i(0)|p_2\alpha_2\rangle\ , \tag{57}$$

$$T(p_1\alpha_1,\ q_k:\ q_\ell,\ p_2\alpha_2) = -i\ (2\pi)^3\sqrt{2|p_{10}|\ \cdot\ 2|p_{20}|}\int d^4y\ e^{-iq_k y}$$

$$\langle -p_1\alpha_1\ |\theta(y)\ [j_k(y),\ j_\ell(0)]\ |p_2\alpha_2\ \rangle. \tag{58}$$

Here $\langle -p_1\alpha_1|$ and $|p_2\alpha_2\rangle$ are arbitrary state vectors, which satisfy $p_1 + q_i + p_2 = 0$ in (57) and $p_1 + q_k + q_\ell + p_2 = 0$ in (58). When $\langle -p_1\alpha_1|$ and $|p_2\alpha_2\rangle$ both are one-particle states and describe an outgoing particle α and an incoming particle β respectively, then $\Gamma(p_1\alpha_1|q_i|p_2\alpha_2)$ continued analytically to the point where q_i^2 equals m_i^2 is by definition the renormalized coupling constant $g_{\alpha i\beta}$. Similarly $T(p_1\alpha_1,\ q_k;\ q_\ell\ p_2\alpha_2)$, when continued analytically to a region where p_2 and q_ℓ are real and belong to the physical region of the reaction $\beta + \ell \to \alpha + k$, is just the amplitude of this reaction. In general, however, the intermediate states in (55), (56) are multiparticle states. Thus, when $\langle -(q_1 + q_i)\gamma|$ in (55) represents an outgoing state of particles $\alpha_1, \cdots, \alpha_r$ with total center-of-mass energy $\sqrt{s_{1i}}$, then $\Gamma(-(q_1 + q_i)\gamma|q_i|q_1)$ is essentially the amplitude of the reaction $1 + i \to \alpha_1 + \cdots + \alpha_r$ in its physical region, likewise $T(q_5,\ q_k;\ q_\ell,\ (q_1 + q_i)\gamma)$ is the amplitude of the reaction $5 + k + \ell \to \alpha_1 + \cdots + \alpha_r$ and is also in the physical region, provided that $(\omega_2, \nu_3, s_{23}, s_{34}, s_{42})$ lies in the physical region of $1 + i \to k + \ell + 5$. Therefore the partial amplitude $A^{(1i)}\ (\omega_2, \nu_3, s_{23}, s_{34}, s_{42})$, when continued analytically to the physical region of the reaction $1 + i \to k + \ell + 5$, represents the imaginary part of the reaction amplitude for $1 + i \to k + \ell + 5$, and the right-hand side of (55) is nothing else than the unitarity condition. In a similar fashion we can discuss (56).

We can now investigate the possible equivalence to (51) of partial amplitudes defined with help of (49) or (50). After intermediate states have been introduced, the only part that might be affected is the analog of (58). As long as we consider multiparticle intermediate states all three possible forms of the partial amplitudes $A^{(1i)}$ and $A^{(5i)}$ are equivalent, since then $T(p_1\alpha_1\ q_k;\ q_\ell,\ p_2\alpha_2)$ and its analogs are related to amplitudes, where particles k and ℓ are either both incoming or both outgoing. The order in which k and ℓ are taken out of the state vector and converted into current operators clearly does not matter. But considering one-particle intermediate states and using (49), we would now find the analog of (58) related by analytical continuation to the reaction amplitude $\beta + k \to \alpha + \ell$ instead of $\beta + \ell \to \alpha + k$. Since we know from the Mandelstam representation that both these reaction amplitudes are one and the same analytic function, the analytic continuation from the physical region of $\beta + k \to \alpha + \ell$ or from $\beta + \ell \to \alpha + k$ to the unphysical points needed in (55) and (56) will in either case lead to the same result. From this consideration it is also evident that a linear combination of (48) and (49) is not admissible. Previously several authors (9, 10) have used the decomposition (50) and were consequently led to the conclusion that the residues of the one-particle pole terms essentially are the product of a renormalized coupling constant with the dispersive part only of a reaction amplitude of type $\alpha + k \to \ell + \beta$. A perturbation calculation, however, contradicts this result and predicts instead of the dispersive part the full amplitude, and this comfirms our above conclusions.

Using (37), (39), and (40), we now can express ω_2 in terms of ν_3, s_{23}, s_{24}, s_{34} and any one of the six variables s_{1i} or s_{i5} (for i=2,3,4). From (39) and the remarks following this equation we see: When $\omega_2 \to +\infty$ $(-\infty)$, then each of the three "energy-type" variables s_{12}, s_{35}, s_{45} also tends to

$+\infty$ $(-\infty)$, while each of the three "momentum-transfer-type" variables s_{13}, s_{14}, s_{25} tends to $-\infty$ $(+\infty)$. Thus introducing these variables as integration variables into (44) and using (52) and the content of (55) and (56), we can write (the small imaginary parts in the denominators of the dispersion integrals have to be taken negative, when the integral is with respect to an "energy-type" variable, and positive when it is with respect to a "momentum-transfer type" variable):

$$T_R(\omega_2, \nu_3, s_{23}, s_{34}, s_{42}) = \sum_{\alpha=1}^{m} \sum_{i=2}^{4} \left\{ \frac{A_\alpha^{(1i)}(s_{1i} = \mu_\alpha^2, \nu_3, s_{23}, s_{34}, s_{42})}{\mu_\alpha^2 - s_{1i}} + \frac{A_\alpha^{(5i)}(s_{5i} = \mu_\alpha^2, \nu_3, s_{23}, s_{34}, s_{42})}{\mu_\alpha^2 - s_{5i}} \right\}$$

$$+ \lim_{\epsilon \to 0} \frac{1}{\pi} \sum_{i=2}^{4} \left\{ \int_{\mu_c^2}^{\infty} ds'_{1i} \frac{A^{(1i)}(s'_{1i}, \nu_3, s_{23}, s_{34}, s_{42})}{s'_{1i} - s_{1i} \mp i\epsilon} + \int_{\mu_c^2}^{\infty} ds'_{5i} \frac{A^{(5i)}(s'_{5i}, \nu_3, s_{23}, s_{34}, s_{42})}{s'_{5i} - s_{5i} \mp i\epsilon} \right\}. \qquad (59)$$

Here we have taken into account that $A^{(1i)}$ as a function of s_{1i} due to (54), (55), and (56) vanishes for $s_{1i} < \mu_c^2$ apart from isolated points μ_α^2. Likewise $A^{(5i)}$ behaves as a function of s_{5i}. $A_\alpha^{(1i)}$ and $A_\alpha^{(5i)}$ are the expressions (55) and (56) taken at $s_{1i} = \mu_\alpha^2$ and $s_{5i} = \mu_\alpha^2$ respectively, but with the factor $\pi \cdot \rho(s_{1i})$ resp. $\pi \cdot \rho(s_{5i})$ left out. If we specified the reaction so that we could use selection rules, we would eventually be able to show that the residues of some of the pole terms vanish and to find higher values for the lower limits of the integrations. This will be seen in detail when in subsequent papers we treat reactions involving pions, nucleons and antinucleons.

The dispersion relation (59) for the reaction $1 + 2 \rightarrow 3 + 4 + 5$ connects this reaction "channel" with five other channels, namely with the reactions $1 + 3 \rightarrow 2 + 4 + 5$, $1 + 4 \rightarrow 2 + 3 + 5$, $5 + 2 \rightarrow 1 + 3 + 4$, $5 + 3 \rightarrow 1 + 2 + 4$, $5 + 4 \rightarrow 1 + 2 + 3$. If we had derived a dispersion relation for any of these other five reactions, using the same methods as above and also keeping Particles 1 and 5 in the state vectors, then we would have found formally exactly the same result, the only difference being that now the variables ν_3, s_{23}, s_{34}, s_{42} would be fixed at certain values in the physical region of one of the other five reactions.

In all the foregoing we have assumed that Particles 1 and 5 have been kept in the state vectors. Obviously we could have chosen any other pair of particles instead. We would then have obtained a different dispersion relation connecting a different set of six channels to one another. All together, we can construct 10 different one-dimensional dispersion relations, each relating a set of six channels to one another. Focusing our attention on a specific reaction, say $1 + 2 \rightarrow 3 + 4 + 5$, we find that there are six dispersion relations in which a dispersion integral with respect to the square of the total energy s_{12} occurs, and that there are four other dispersion relations in which the total energy is a fixed variable.

It is quite informative to write down in form of a table the various sets of fixed variables and related channels.

TABLE I: The One-Dimensional Dispersion Relations For $n = 5$

Particles kept in the state vectors	13	14	15	23	24	25	34	35	45	12
Fixed s variables	s_{13}	s_{14}	s_{15}	s_{23}	s_{24}	s_{25}	s_{34}	s_{35}	s_{45}	s_{12}
	s_{24}	s_{23}	s_{23}	s_{14}	s_{13}	s_{13}	s_{12}	s_{12}	s_{12}	s_{34}
	s_{25}	s_{25}	s_{24}	s_{15}	s_{15}	s_{14}	s_{15}	s_{14}	s_{13}	s_{35}
	s_{45}	s_{35}	s_{34}	s_{45}	s_{35}	s_{34}	s_{25}	s_{24}	s_{23}	s_{45}
Integration variables, corresponding to related channels	s_{12}	s_{12}	s_{12}	s_{21}	s_{21}	s_{21}	s_{31}	s_{31}	s_{41}	s_{13}
	s_{14}	s_{13}	s_{13}	s_{24}	s_{23}	s_{23}	s_{32}	s_{32}	s_{42}	s_{14}
	s_{15}	s_{15}	s_{14}	s_{25}	s_{25}	s_{24}	s_{35}	s_{34}	s_{43}	s_{15}
	s_{32}	s_{42}	s_{52}	s_{31}	s_{41}	s_{51}	s_{41}	s_{51}	s_{51}	s_{23}
	s_{34}	s_{43}	s_{53}	s_{34}	s_{43}	s_{53}	s_{42}	s_{52}	s_{52}	s_{24}
	s_{35}	s_{45}	s_{54}	s_{35}	s_{45}	s_{54}	s_{45}	s_{54}	s_{53}	s_{25}

In addition to the fixed s variables one of the functions ν has to be kept fixed. The first six columns correspond to dispersion relations in which integrals with respect to s_{12} occur, the last four are for fixed s_{12}.

VII. SOME CONSIDERATIONS TOWARDS A GENERALIZATION OF THE MANDELSTAM REPRESENTATION

For $n=4$ one has a set of three one-dimensional dispersion relations, each relating two of the three reaction channels. These one-dimensional relations form the starting point for Mandelstam's (1) discussion of double dispersion relations. For $n=5$ the analogous set of dispersion relations is given symbolically in Table I. It is therefore tempting to search for possible forms of multiple dispersion relations, from which those of Table I can be deduced. Such considerations can, of course, be only of a purely formal character. What we wish to assert is this: The dispersion relation (59) can be derived from a two-dimensional dispersion relation with fixed s_{23}, s_{24}, s_{34}, if the singularities of the amplitude can be assumed to be restricted to the real plane of the remaining two variables and if they are located there in a way similar to that in Mandelstam's case $n=4$. If such a two-dimensional relation should exist, one would expect it to hold (analogously to $n=4$) only for certain combinations of external masses and possibly only for restricted values of the fixed variables.

At present the only conceivable method of investigating this question uses the framework of perturbation theory. For $n=4$ it has thus been possible to establish the validity of the double-dispersion relation (11), but for $n=5$ not even the analytic properties of the contribution from the simplest loop diagram are known in detail. There is evidence that in general there are complex singularities (12, 13), but their precise location is unknown. Among these singularities there are complex pole terms. Cook and Tarski (14) have investigated their position for some special cases and found that they may or may not lie on the physical sheet. It should be kept in mind that the singularity from a certain graph may in some cases cancel out singularities contributed by other graphs of the perturbation series (15). Also it has been

suggested by Eden that, even if there are complex singularities on the physical sheet, their occurence might be restricted to a finite number of graphs so that the rest of the amplitude (including all higher-order contributions) might satisfy a double-dispersion relation (16). From this point of view there is a certain interest in answering the question on what the double-dispersion relations for $n=5$ may possibly look like. Eventually the formulas given below will have to be modified to include contributions from complex contours.

The first problem, which presents itself quite naturally, is to remove the quantities ν from the picture, in other words to search for double-dispersion relations in which three s variables are kept fixed and which consist of a sum of double integrals with respect to s variables. Obviously every pair of integration variables together with the three fixed variables has to form a set of five independent s variables. Considering again the case in which Particles 1 and 5 have been kept in the state vector, and neglecting for the moment possible pole terms, we are thus led to the following ansatz, which is the most general one, if we assume that the singularities are restricted to certain parts of real hyperplanes in complete analogy to Mandelstam's case:

$$
\begin{aligned}
T(P) = \frac{1}{\pi^2} \int_{\mu_c^2}^{\infty} ds'_{12} \int_{\mu_c^2}^{\infty} ds'_{13} \, & \frac{A^{(12,13)}}{(s'_{12} - s_{12})(s'_{13} - s_{13})} \\
+ \frac{1}{\pi^2} \int_{\mu_c^2}^{\infty} ds'_{12} \int_{\mu_c^2}^{\infty} ds'_{14} \, & \frac{A^{(12,14)}}{(s'_{12} - s_{12})(s'_{14} - s_{14})} \\
+ \frac{1}{\pi^2} \int_{\mu_c^2}^{\infty} ds'_{13} \int_{\mu_c^2}^{\infty} ds'_{14} \, & \frac{A^{(13,14)}}{(s'_{13} - s_{13})(s'_{14} - s_{14})} \\
+ \frac{1}{\pi^2} \int_{\mu_c^2}^{\infty} ds'_{25} \int_{\mu_c^2}^{\infty} ds'_{35} \, & \frac{A^{(25,35)}}{(s'_{25} - s_{25})(s'_{35} - s_{35})} \\
+ \frac{1}{\pi^2} \int_{\mu_c^2}^{\infty} ds'_{25} \int_{\mu_c^2}^{\infty} ds'_{45} \, & \frac{A^{(25,45)}}{(s'_{25} - s_{25})(s'_{45} - s_{45})} \\
+ \frac{1}{\pi^2} \int_{\mu_c^2}^{\infty} ds'_{35} \int_{\mu_c^2}^{\infty} ds'_{45} \, & \frac{A^{(35,45)}}{(s'_{35} - s_{35})(s'_{45} - s_{45})} \\
+ \frac{1}{\pi^2} \int_{\mu_c^2}^{\infty} ds'_{12} \int_{\mu_c^2}^{\infty} ds'_{35} \, & \frac{A^{(12,35)}}{(s'_{12} - s_{12})(s'_{35} - s_{35})} \\
+ \frac{1}{\pi^2} \int_{\mu_c^2}^{\infty} ds'_{12} \int_{\mu_c^2}^{\infty} ds'_{45} \, & \frac{A^{(12,45)}}{(s'_{12} - s_{12})(s'_{45} - s_{45})} \\
+ \frac{1}{\pi^2} \int_{\mu_c^2}^{\infty} ds'_{13} \int_{\mu_c^2}^{\infty} ds'_{45} \, & \frac{A^{(13,45)}}{(s'_{13} - s_{13})(s'_{45} - s_{45})}
\end{aligned}
\tag{60}
$$

$$+ \frac{1}{\pi^2} \int_{\mu_c^2}^{\infty} ds'_{25} \int_{\mu_c^2}^{\infty} ds'_{13} \frac{A^{(13,25)}}{(s'_{25} - s_{25})(s'_{13} - s_{13})}$$

$$+ \frac{1}{\pi^2} \int_{\mu_c^2}^{\infty} ds'_{25} \int_{\mu_c^2}^{\infty} ds'_{14} \frac{A^{(14,25)}}{(s'_{25} - s_{25})(s'_{14} - s_{14})}$$

$$+ \frac{1}{\pi^2} \int_{\mu_c^2}^{\infty} ds'_{35} \int_{\mu_c^2}^{\infty} ds'_{14} \frac{A^{(14,35)}}{(s'_{35} - s_{35})(s'_{14} - s_{14})} \tag{60}$$

Here the "spectral functions" $A^{(ij,k\ell)}$ depend on s'_{ij}, $s'_{k\ell}$ and the fixed variables s_{23}, s_{24}, s_{34}. On the right-hand side, of course, all 12 terms have to be taken at the same point P in the five-dimensional space of s variables as on the left-hand side. P can be given either in terms of $(\omega_2, \nu_3, s_{23}, s_{34}, s_{42})$ or in terms of any of the 12 combinations $(s_{12}, s_{13}, s_{23}, s_{34}, s_{42})$, $(s_{12}, s_{14}, s_{23}, s_{34}, s_{42})$, $\cdots$. The denominators of all integrals have to be thought of as being furnished in the usual fashion with small imaginary parts. Likewise we could have given a formulation in terms of Cauchy principal values. As in Mandelstam's case we should add on the right-hand side of (60) certain one-dimensional dispersion integrals, but we left these out, being interested mainly in the two-dimensional integrals.

Let us now investigate how the one-dimensional dispersion relation (59) can be derived from (60). Let i, k, ℓ be a permutation of 2, 3, 4, and define

$$A_i = \frac{m_5^2 - m_1^2}{s_{15}} (s_{15} - s_{ik} - s_{i\ell} + m_k^2 + m_\ell^2), \tag{61}$$

$$B_i = s_{ik} + s_{i\ell} - m_i^2 - \sum_{k=1}^{5} m_k^2 , \tag{62}$$

$$C_i = B_i + A_i, \qquad D_i = B_i - A_i; \tag{63}$$

then we can write, using (37), (39), and (42),

$$\nu_{ik} = \frac{2s_{1i} + C_i}{2s_{1k} + C_k} = \frac{2s_{5i} + D_i}{2s_{5k} + D_k} = - \frac{2s_{1i} + C_i}{2s_{5k} + D_k} = - \frac{2s_{5i} + D_i}{2s_{1k} + C_k} . \tag{64}$$

We thus have three relations,

$$s_{1i} - \nu_{ik} s_{1k} = - \frac{1}{2} (C_i - \nu_{ik} C_k) = \frac{\nu_{ik}}{2} (C_k - \nu_{ki} C_i) , \tag{65a}$$

$$s_{5i} - \nu_{ik} s_{5k} = - \frac{1}{2} (D_i - \nu_{ik} D_k) = \frac{\nu_{ik}}{2} (D_k - \nu_{ki} D_i) , \tag{65b}$$

$$s_{1i} + \nu_{ik} s_{5k} = - \frac{1}{2} (C_i + \nu_{ik} D_k) = - \frac{\nu_{ik}}{2} (D_k + \nu_{ki} C_i) . \tag{65c}$$

We see that for all three types of integrals the problem is formally the same, namely: Given an integral

$$\int dz'_1 \int dz'_2 \frac{f(z'_1, z'_2)}{(z'_1 - z_1)(z'_2 - z_2)} ,$$

introduce a new variable ν by $z_1 - \nu z_2 = a$ and decompose the above integral into a sum of integrals, each of which depends on only a single one of the variables z_1, z_2. We define $\bar{z}_1 = \nu z'_2 + a$, $\bar{z}_2 = \nu^{-1} (z'_1 - a)$ and then find

$$(z'_1 - z_1)(z'_2 - \bar{z}_2) + (z'_1 - \bar{z}_1)(z'_2 - z_2) = - \nu^{-1} (z'_1 - \nu z'_2 - a)^2$$

$$= (z'_1 - \bar{z}_1)(z'_2 - \bar{z}_2) , \qquad (66)$$

and therefore

$$\frac{1}{(z'_1 - \bar{z}_1)(z'_2 - z_2)} + \frac{1}{(z'_1 - z_1)(z'_2 - \bar{z}_2)} = \frac{1}{(z'_1 - z_1)(z'_2 - z_2)} . \qquad (67)$$

Thus

$$\int dz'_1 \int dz'_2 \frac{f(z'_1, z'_2)}{(z'_1 - z_1)(z'_2 - z_2)}$$

$$= \int dz'_1 \frac{f_1(z'_1, \nu, a)}{z'_1 - z_1} + \int dz'_2 \frac{f_2(z'_2, \nu, a)}{z'_2 - z_2} , \qquad (68)$$

with

$$f_1(z'_1, \nu, a) = \int dz'_2 \frac{f(z'_1, z'_2)}{z'_2 - \nu^{-1} z'_1 + \nu^{-1} a} , \qquad (69a)$$

$$f_2(z'_2, \nu, a) = \int dz'_1 \frac{f(z'_1, z'_2)}{z'_1 - \nu z'_2 - a} . \qquad (69b)$$

This outlines the procedure by which all integrals of Relation (60) can be decomposed into one-dimensional dispersion integrals with fixed ν. We obtain the following dispersion relations for the absorptive parts $A^{(1i)}$, $A^{(5i)}$ that occur in (59):

$$A^{(1i)}(s'_{1i}, \nu_3, s_{23}, s_{34}, s_{42})$$

$$= \frac{1}{\pi} \int_{\mu_c^2}^{\infty} ds'_{1k} \frac{A^{(1i,1k)}(s'_{1i}, s'_{1k}, s_{23}, s_{34}, s_{42})}{s'_{1k} - \nu_{ki}\left(s'_{1i} + \frac{1}{2} C_i - \frac{1}{2} \nu_{ik} C_k\right)}$$

$$+ \frac{1}{\pi} \int_{\mu_c^2}^{\infty} ds'_{1\ell} \frac{A^{(1i,1\ell)}(s'_{1i}, s'_{1\ell}, s_{23}, s_{34}, s_{42})}{s'_{1\ell} - \nu_{\ell i}\left(s'_{1i} + \frac{1}{2} C_i - \frac{1}{2} \nu_{i\ell} C_\ell\right)}$$

$$+ \frac{1}{\pi} \int_{\mu_c^2}^{\infty} ds'_{5k} \frac{A^{(1i,5k)}(s'_{1i}, s'_{5k}, s_{23}, s_{34}, s_{42})}{s'_{5k} + \nu_{ki}\left(s'_{1i} + \frac{1}{2} C_i + \frac{1}{2} \nu_{ik} D_k\right)}$$

$$+ \frac{1}{\pi} \int_{\mu_c^2}^{\infty} ds'_{5\ell} \frac{A^{(1i,5\ell)}(s'_{1i}, s'_{5\ell}, s_{23}, s_{34}, s_{42})}{s'_{5\ell} + \nu_{\ell i}\left(s'_{1i} + \frac{1}{2} C_i + \frac{1}{2} \nu_{i\ell} D_\ell\right)} . \tag{70}$$

$$A^{(5i)}(s'_{5i}, \nu_3, s_{23}, s_{34}, s_{42})$$

$$= \frac{1}{\pi} \int_{\mu_c^2}^{\infty} ds'_{1k} \frac{A^{(1k,5i)}(s'_{1k}, s'_{5i}, s_{23}, s_{34}, s_{42})}{s'_{1k} + \nu_{ki} (s'_{5i} + \frac{1}{2} D_i + \frac{1}{2} \nu_{ik} C_k)}$$

$$+ \frac{1}{\pi} \int_{\mu_c^2}^{\infty} ds'_{1\ell} \frac{A^{(1\ell,5i)}(s'_{1\ell}, s'_{5i}, s_{23}, s_{34}, s_{42})}{s'_{1\ell} + \nu_{\ell i} (s'_{5i} + \frac{1}{2} D_i + \frac{1}{2} \nu_{i\ell} C_\ell)}$$

$$+ \frac{1}{\pi} \int_{\mu_c^2}^{\infty} ds'_{5k} \frac{A^{(5k,5i)}(s'_{5k}, s'_{5i}, s_{23}, s_{34}, s_{42})}{s'_{5k} - \nu_{ki} (s'_{5i} + \frac{1}{2} D_i - \frac{1}{2} \nu_{ik} D_k)}$$

$$+ \frac{1}{\pi} \int_{\mu_c^2}^{\infty} ds'_{5\ell} \frac{A^{(5\ell,5i)}(s'_{5\ell}, s'_{5i}, s_{23}, s_{34}, s_{42})}{s'_{5\ell} - \nu_{\ell i} (s'_{5i} + \frac{1}{2} D_i - \frac{1}{2} \nu_{i\ell} D_\ell)} . \tag{71}$$

These equations as they stand, however, are correct only when we have no contributions from the one-particle intermediate states. In general the dispersion relation (60) will have to be supplemented by additional terms corresponding to the graphs of Fig. 2. These terms arise from the contribution to the absorptive amplitudes as given by (55), (56) from one-particle intermediate states. They are represented by pole terms with residues being the product of a renormalized coupling constant and a scattering amplitude at some unphysical point. Apart from these two factors, Eqs. (55) and (56) predict also a step function $\theta(q_{10} + q_{i0})$ resp. $\theta(-q_{50} - q_{i0})$, which comes from the requirement that the intermediate state be a state of positive energy. In the specific coordinate system that we use the arguments of the step functions can be expressed in terms of s variables,

$$q_{10} + q_{i0}\Big|_{s_{1i} = \mu_\alpha^2} = \frac{1}{2\Delta}\left(\Delta^2 + C_i + 2\mu_\alpha^2\right), \tag{72}$$

$$-\left(q_{50} + q_{i0}\right)\Big|_{s_{5i} = \mu_\beta^2} = \frac{1}{2\Delta}\left(\Delta^2 + D_i + 2\mu_\beta^2\right) . \tag{73}$$

Here we have used (37) through (39) and (61) through (63). The quantities Δ, C_i, D_i depend only on the fixed variables s_{23}, s_{34}, s_{42} and the particle masses.[5] These step functions are not obtained in the framework of a perturbation calculation, and therefore we suppose that they would not have appeared had we carried out the analytical continuation of (55) and (56) somewhat more carefully. In all equations that are to follow we shall ignore the step functions.

The scattering amplitudes, which occur in the residues of the pole terms, themselves satisfy dispersion relations. For example, we obtain by the usual methods for the amplitude $T(q_\beta, q_k; q_i, q_l)$ of the reaction $l + i \rightarrow k + \beta$ the following dispersion relation:

$$T(q_\beta, q_k; q_i, q_l) = -\sum_{\alpha=1}^{m} \left\{ \frac{g_{\beta k\alpha}\, g_{\alpha il}}{\mu_\alpha^2 - s_{li}} + \frac{g_{\beta i\alpha}\, g_{\alpha kl}}{\mu_\alpha^2 - s_{lk}} \right\}$$

$$+ \frac{1}{\pi} \int_{\mu_c^2}^{\infty} ds'_{li} \, \frac{a^{(li,k\beta)}(s'_{li}, s_{ik})}{s'_{li} - s_{li}}$$

$$+ \frac{1}{\pi} \int_{\mu_c^2}^{\infty} ds'_{lk} \, \frac{a^{(lk,\, i\beta)}(s'_{lk}, s_{ik})}{s'_{lk} - s_{lk}} . \tag{74}$$

$T(q_\beta, q_k; q_i, q_l)$ depends, of course, only on the invariants s_{li}, s_{lk}, s_{ik}

subject to the condition $s_{1i} + s_{1k} + s_{ik} = m_1^2 + m_i^2 + m_k^2 + m_\beta^2$. The right-hand side has been furnished as usual with small imaginary parts. The absorptive amplitudes in (74) are in accordance with (22) through (26), defined by

$$a^{(1i,k\beta)}(s_{1i}, s_{ik})$$

$$= -\frac{1}{2} \cdot (2\pi)^3 \cdot \sqrt{2|q_{10}| \cdot 2|q_{\beta 0}|} \int d^4x\, e^{-iq_k x} \langle -q_\beta | j_k(x)\, j_i(0) | q_1 \rangle \tag{75a}$$

$$a^{(1k,i\beta)}(s_{1k}, s_{ik})$$

$$= -\frac{1}{2} \cdot (2\pi)^3 \cdot \sqrt{2|q_{10}| \cdot 2|q_{\beta 0}|} \int d^4x\, e^{-iq_i x} \langle -q_\beta | j_i(x)\, j_k(0) | q_1 \rangle . \tag{75b}$$

We can now write down an explicit expression for the contribution of the pole terms to the dispersion relation (59):

$$\sum_{(i,k,\ell)} \sum_{\alpha,\beta=1}^{m} \frac{g_{1i\alpha}\, g_{\alpha k\beta}\, g_{\beta\ell 5}}{(\mu_\alpha^2 - s_{1i})(\mu_\beta^2 - s_{5\ell})}$$

$$- \sum_{(i,k,\ell)} \sum_{\alpha=1}^{m} \frac{g_{1i\alpha}}{\mu_\alpha^2 - s_{1i}} \cdot \frac{1}{\pi} \int_{\mu_c^2}^{\infty} ds'_{5\ell} \frac{a^{(5\ell,k\alpha)}(s'_{5\ell}, s_{k\ell})}{s'_{5\ell} - \bar{s}_{5\ell}}$$

$$- \sum_{(i,k,\ell)} \sum_{\beta=1}^{m} \frac{g_{5\ell\beta}}{\mu_\beta^2 - s_{5\ell}} \cdot \frac{1}{\pi} \int_{\mu_c^2}^{\infty} ds'_{1i} \frac{a^{(1i,k\beta)}(s'_{1i}, s_{ik})}{s'_{1i} - \bar{s}_{1i}} . \tag{76}$$

Here $\sum_{(i,k,\ell)}$ indicates a summation over all permutations of (2,3,4) and the

quantities $\bar{s}_{5\ell}$ resp. $\bar{s}_{1i}$ are the values of $s_{5\ell}$ resp. s_{1i} at the points, described by $\nu_{i\ell}$, s_{23}, s_{34}, s_{42} and $s_{1i} = \mu_\alpha^2$ resp. $s_{5\ell} = \mu_\beta^2$. They are given (cf. Eq. (65c)) by

$$\bar{s}_{5\ell} = - \frac{\nu_{\ell i}}{2} (C_i + \nu_{i\ell} D_\ell + 2 \mu_\alpha^2) , \tag{77}$$

$$\bar{s}_{1i} = - \frac{\nu_{i\ell}}{2} (D_\ell + \nu_{\ell i} C_i + 2 \mu_\beta^2) . \tag{78}$$

In writing the first term of (76) we have used (67) and combined by pairs the double-pole terms arising from (59) and (74). Therefore the first term of (76) is independent of the ν variables. This however, is not the case for the second and third term of (76). Here the counter terms, which are required in order to form expressions not dependent on ν variables, but analytic in the two independent variables s_{1i}, $s_{5\ell}$, are ordinary dispersion integrals in s_{1i} and $s_{5\ell}$, which might be contained in the third and fourth term of (59). Thus, supplementing the double-dispersion relation (60) by the expression (76), <u>but with $s_{5\ell}$ instead of $\bar{s}_{5\ell}$ and with s_{1i} instead of $\bar{s}_{1i}$</u>, we find that the one-dimensional dispersion relation (59) follows from this by the procedure discussed above. The resulting expressions for the absorptive parts are (70) and (71), but supplemented by

$$- \sum_{\beta=1}^{m} \left\{ \frac{g_{5k\beta}\, a^{(1i,\ell\beta)}(s'_{1i}, s_{i\ell})}{\mu_\beta^2 + \nu_{ki} (s'_{1i} + \frac{1}{2} C_i + \frac{1}{2} \nu_{ik} D_k)} + \frac{g_{5\ell\beta}\, a^{(1i,k\beta)}(s'_{1i}, s_{ik})}{\mu_\beta^2 + \nu_{\ell i} (s'_{1i} + \frac{1}{2} C_i + \frac{1}{2} \nu_{i\ell} D_\ell)} \right\} \tag{70'}$$

$$- \sum_{\alpha=1}^{m} \left\{ \frac{g_{1k\alpha}\, a^{(5i,\ell\alpha)}(s'_{5i},\ s_{i\ell})}{\mu_\alpha^{\,2} + \nu_{ki}\,(s'_{5i} + \frac{1}{2} D_i + \frac{1}{2} \nu_{ik}\, C_k)} \right.$$

$$\left. + \frac{g_{1\ell\alpha}\, a^{(5i,k\alpha)}(s'_{5i},\ s_{ik})}{\mu_\alpha^{\,2} + \nu_{\ell i}\,(s'_{5i} + \frac{1}{2} D_i + \frac{1}{2} \nu_{i\ell}\, C_\ell)} \right\} \qquad (71')$$

It is clear that from the double-dispersion relation (60) supplemented by the one-particle terms only the one-dimensional relation (59) can be derived, but none of the nine others, which are listed symbolically in Table I. This is so because in the other relations different combinations of s variables are kept fixed. Mandelstam's representation (1) of amplitudes for reactions of type $1 + 2 \rightarrow 3 + 4$ might suggest as a possible general rule that the amplitudes for more general reactions are likewise analytic functions--except for certain cuts--of all variables on which they depend, and that they possess representations in terms of multiple-dispersion integrals analogous to those of Mandelstam. From such a general representation it should then be possible to derive all one-dimensional dispersion relations, for example in the case of the reaction $1 + 2 \rightarrow 3 + 4 + 5$ and related channels, all the one-dimensional relations listed on Table I.

In Mandelstam's representation we have a double integral for each pair of independent s variables. This suggests that in the present case (n=5), where we have five independent variables, we have to write a fivefold dispersion integral for each quintuple of independent s variables. Since we have 10 different s variables, we can form from then $\binom{10}{5} = 252$ quintuples, but not all of them consist of independent variables. Using (7c) and (8), we find that there are four distinct classes of combinations of five independent

s variables ($i_1 i_2 i_3 i_4 i_5$ denoting an arbitrary permutation of 1,2,3,4,5),

$$s_{i_1 i_2}\ s_{i_2 i_3}\ s_{i_3 i_4}\ s_{i_4 i_5}\ s_{i_5 i_1} \quad \text{(12 combinations)} \quad (2+2+2+2+2),$$

$$s_{i_1 i_2}\ s_{i_1 i_3}\ s_{i_1 i_4}\ s_{i_2 i_3}\ s_{i_2 i_4} \quad \text{(30 combinations)} \quad (3+3+2+2+0),$$

$$s_{i_1 i_2}\ s_{i_1 i_3}\ s_{i_1 i_4}\ s_{i_2 i_3}\ s_{i_2 i_5} \quad \text{(60 combinations)} \quad (3+3+2+1+1),$$

$$s_{i_1 i_2}\ s_{i_1 i_3}\ s_{i_1 i_4}\ s_{i_2 i_5}\ s_{i_3 i_5} \quad \text{(60 combinations)} \quad (3+2+2+2+1), \tag{79}$$

all together 162 combinations. Thus we would suppose that for n=5 the generalized Mandelstam representation would consist of 162 fivefold dispersion integrals. It would seem conceivable, however, that one or several of the classes (79) do not contribute nonvanishing terms (eventually after we have made certain approximations). For example, when we write down the contributions of lowest order (i.e., fifth-order) perturbation theory in the form used by Tarski (17), we find that only those combinations of s variables occur which belong to the first class in (79). This might suggest that in lowest-order perturbation theory we have a representation, if there exists one at all, in terms of the 12 fivefold dispersion integrals belonging to the first class.

By keeping one s variable fixed we could derive from the five-dimensional dispersion relation 10 different four-dimensional relations. There would be two types of three-dimensional dispersion relations according to whether the pair of fixed s variables is s_{ij}, $s_{k\ell}$ with i, j, k, ℓ all different, or whether it is s_{ik}, $s_{k\ell}$. From each relation of the first

type we could derive two of the one-dimensional dispersion relations, from each one of the second type only a single one-dimensional relation. Likewise from each of the four-dimensional representations we could derive four of the one-dimensional relations.

We will close with some remarks on the evaluation of the unitarity condition for the process $1 + 2 \rightarrow 3 + 4 + 5$. Even when we restrict ourselves to the approximation whereby only two-particle intermediate states are considered, the two-dimensional dispersion relation (60) must be used. This is because we have only three independent energy-type variables, so that the integration over intermediate states involves at least two of the five variables on which the amplitude depends. Unlike the analogous problem in Mandelstam's theory (1), the resulting integral is too complicated to be evaluated explicitly, but some of its properties can be discussed by representing it as an integral over Feynman parameters and using Tarski's methods (17). It would also seem that for an exact evaluation of the three-particle contributions to the unitarity condition for a process of type $1 + 2 \rightarrow 3 + 4$ one has to use the four-dimensional dispersion representation of the amplitudes for $1 + 2 \rightarrow a + b + c$ and $a + b + c \rightarrow 3 + 4$.

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FOOTNOTES

1. We use the metric, in which $x^2 = x_0^2 - x_1^2 - x_2^2 - x_3^2$.

2. Here we have denoted by q_1, q_2 and q'_1, q'_2 the four-momenta of Particles 1 and 2 before and after the scattering, and chosen the sign of q'_1 and q'_2 according to the convention of Section 2, i.e., such that we have $q_1 + q_2 + q'_1 + q'_2 = 0$.

3. Kibble(2) has shown that in "small" parts of the physical region this inequality is not necessarily satisfied.

4. s_{1n} does not appear explicitly in the arguments, because it can be expressed by $s_{23}, \cdots, s_{n-2}, s_{n-1}$ (cf. Eq. (7c)).

5. For $n=4$ the expressions corresponding to (72) and (73) are always positive except for some processes that involve vertices with anomalous thresholds, e.g., certain strange-particle reactions.

FIGURE CAPTIONS

Fig. 1 The diagram of a general reaction.

Fig. 2 Graphs representing the pole terms in the dispersion relation (59). If, for example, we consider pion production in inelastic pion-nucleon scattering and keep the nucleons in the state vectors, then the solid lines represent the nucleon lines and the broken lines meson lines, and $m_\alpha = m_\beta$ = nucleon mass.

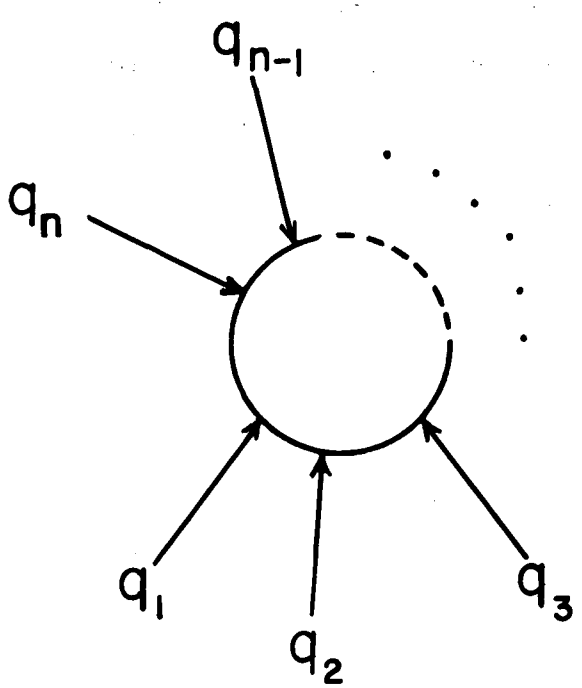

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Fig. 1

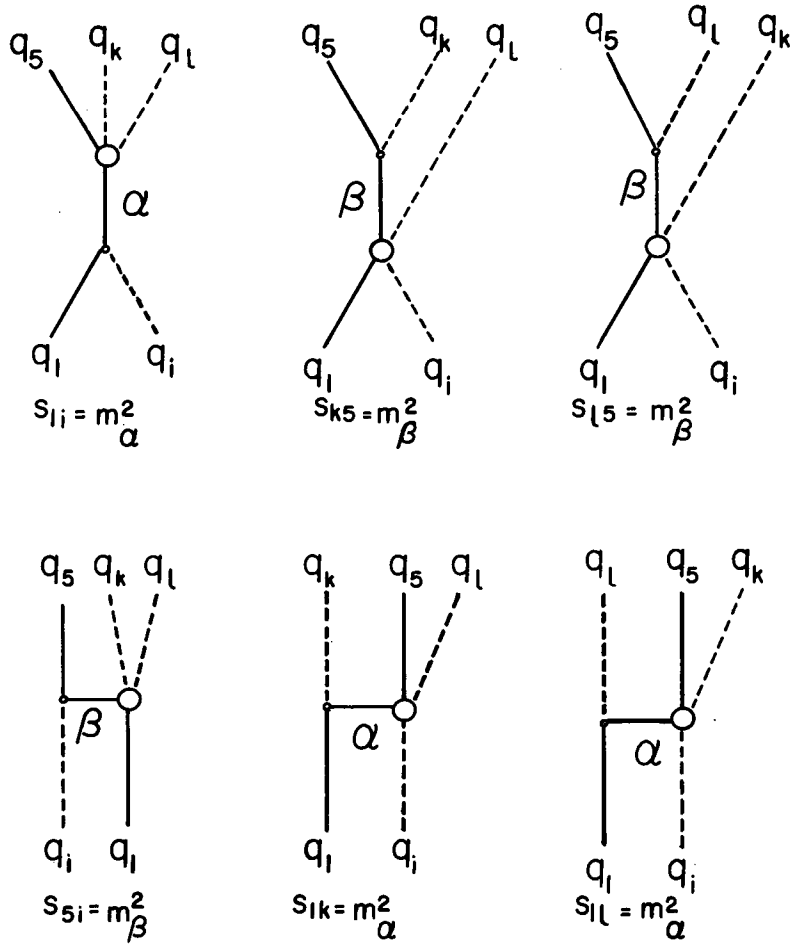

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Fig. 2

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