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ABSTRACT

It is possible to generate a relatively uniform, highly ionized plasma by passing a powerful discharge between electrodes so arranged that the current is forced to flow across an initial strong magnetic field. The magnetic induction due to the discharge causes a bending of the original field. If the discharge is operated with a low-impedance current source, the electric breakdown starts in a limited region near the current-input connections (minimum-inductance path) and propagates as a well-defined front in the manner of a hydromagnetic shock wave. Such a shock is usually compressive, i.e. the sudden increase in temperature and ionization is usually accompanied by an abrupt onset of not only a transverse but also the longitudinal component of plasma flow. Consequently the front must be followed by a rarefaction wave in which the longitudinal flow is brought to rest. The process has certain features in common with gaseous detonations.

In this paper we analyze the phenomenon as a one-dimensional single-fluid hydromagnetic problem, neglecting dissipation behind the wave. We assume zero conductivity in front and thermodynamic equilibrium behind the wave. A full treatment requires numerical methods for solution. However, in the limit of essentially complete ionization behind the front, the problem can be

solved analytically as long as the transverse magnetic field there remains small compared with the longitudinal field. In this case, the front velocity, plasma temperature and density, and the electric field behind the wave, as well as the structure of the rarefaction wave, can be expressed as simple functions of the initial magnetic field, the discharge current, the ionization energy, and the initial gas density. It is interesting to note that, over a certain range of operating conditions, the electric field is relatively independent of the driving current and is primarily determined by the ionization energy per unit mass of the gas. This result is in striking agreement with related observations by Alfvén and coworkers. Finally, conditions are derived for noncompressive waves, hence also for production of uniform plasma.

fields cannot always be neglected. This is particularly true if a magnetic field already exists ahead of the piston. It is thus clearly impossible for a hydromagnetic piston to drive a purely gas dynamic shock into the cold gas strong enough to produce any ionization. The electric field causes currents to flow throughout the ionized region, changing the character of the flow entirely. In effect, the driving field of the piston spreads all the way to the shock front, so that the entire phenomenon always takes on some characteristics of a hydromagnetic shock. We shall use the term hydromagnetic ionizing wave. If the flow behind the wave is steady or if the resistivity is negligible, the electric field must be negligible in the frame of the medium there. Ahead of the wave, however, the electric field in the frame of the unionized gas is, in general, finite.

This fact has interesting consequences. We will demonstrate that the phenomenon has certain features in common with a detonation wave, although the reactions in the gas (dissociation and ionization) are endothermic rather than exothermic. The reason here is that electromagnetic energy from the driving power supply is released in the front, and some of it may be considered as taking the place of the liberated chemical energy. Moreover, just as in combustion fronts, the rate is not uniquely determined by the conservation laws alone since, in contradistinction to the usual hydromagnetic shocks, in the case of our ionizing wave the electric field ahead of the front is not directly linked to the shock velocity. While some conclusions are perfectly general, we restrict our discussion in this paper to situations where a magnetic field exists ahead of the wave. Moreover, we focus our attention on cases where the field is not parallel to the plane of the ionizing front. It is certainly possible to devise experiments in the laboratory in which a hydromagnetic driver is constrained to move in a direction with a component parallel to a magnetic field existing ahead of it,[5] and in some experiments the propagation is exactly along the magnetic field ahead of it.[6] We will show that such an ionizing wave may provide a unique and very useful way of producing a magnetized uniform plasma if certain requirements are fulfilled. In fact, this latter aspect has motivated the present investigation.

THE MODEL

In this paper we restrict ourselves to the analysis of a simplified one-dimensional model. The geometry is best explained with the help of Fig. 1. The gas is considered to be confined between two infinite conducting planes, both parallel to the xz plane. The initial magnetic field is also parallel to the xz plane, the applied electric field is always parallel to the y axis, and everything is assumed to be independent of both the y- and z-coordinates. This means we are looking at plane wave motion and are choosing our x-coordinate along the direction of propagation. It also implies that the viscous drag at the flow boundaries as well as any variation of the electrical conductivity that might appear in the neighborhood of the surfaces are being ignored.

The gas ahead of the wave is, of course, assumed to be at rest, in equilibrium, and nonconducting. Furthermore, we assume that immediately behind the shock the gas is again in thermodynamic equilibrium, so that it obeys an equation of state and so that its relevant physical properties such as composition, electrical conductivity, etc. can be computed from equilibrium considerations. This means we are limiting ourselves to densities high enough to ensure sufficiently rapid equilibration rates. We need not make any assumptions concerning the shock structure in this case other than requiring that the shock thickness is finite and constant. The exact mechanism of ionization is not under discussion here. The requirement of equilibrium behind the front implies that the current there is zero if the flow is steady. This means that the electric field must be zero in the frame of the moving gas behind the front, even if the gas has finite resistivity there. Therefore, the shock relations are always automatically independent of the conductivity.[7]

It is not immediately obvious that a steady wave should propagate in a shock-tube experiment in which, for instance, the current input is kept constant. Since shocks are usually compressive, the front must ordinarily be followed by an expansion wave with its nonsteady flow, unless a suitable additional piston is provided. However, it has been shown that in the limit of negligible dissipation, i.e., isentropic conditions behind the shock front, the flow there can be described as a "centered rarefaction wave".[8] This means that, in this approximation at least, the entire flow pattern spreads at a uniform rate and draws constant total current, so that a steady shock can indeed be driven ahead of it. Accordingly, we shall treat the problem in two steps. First we shall discuss the shock relations under the assumptions of steady flow. Here we shall have to include the effects of dissociation and ionization. Then we shall look at the expansion wave, assuming negligible resistivity, viscosity, and thermal conductivity. Finally we must combine the two regions to describe the entire phenomenon. The model is depicted schematically in Fig. 2. The situation and the analyses here are very similar to those treated by Kemp and Petschek,[9] the only difference being that the latter assume complete dissociation and ionization ahead of the wave, while we require negligible electrical conductivity.

Our model will not be applicable to extremely strong shocks, where the emitted radiation ionizes the gas at large distances from the front.

SHOCK RELATIONS

In accordance with Fig. 2, we distinguish quantities in the regions R_1 and R_2 ahead of and behind the shock by the subsripts 1 and 2, respectively. Since we assume the shock to be steady, it is most convenient to start out by describing the flow in a frame of reference in which the front is stationary (see Fig. 3a). The basic equations are then independent of time and, in our one-dimensional problem may be immediately integrated to give the familiar symmetric jump conditions connecting the quantities in region R_1 and R_2. It is easily shown that these relations do not depend explicitly on any of the irreversible processes occuring in the transition as long as no energy is lost by radiation; i.e., they are true conservation laws. If we denote the velocities in this frame of reference by small letters $\vec{v}_1 = (u_1, o, o)$ and $\vec{v}_2 = (u_2, o, w_2)$, where u_1 and u_2 will be considered negative as indicated in Fig. 3a, the conservation laws are:

for the mass,

$$\rho_1 u_1 = \rho_2 u_2 \tag{1}$$

for the x-momentum,

$$\rho_1 u_1^2 + p_1 + \frac{\mu}{2} H_{z1}^2 = \rho_2 u_2^2 + p_2 + \frac{\mu}{2} H_{z2}^2 \tag{2}$$

for the z-momentum,

$$-\mu H_x H_{z1} = \rho_2 u_2 w_2 - \mu H_x H_{z2} \tag{3}$$

for the energy,

$$\rho_1 u_1 h_1 + E_s H_{z1} = \rho_2 u_2 h_2 + E_s H_{z2} . \tag{4}$$

Here we have expressed the total enthalpy per unit mass as

$$h = e_0 + \frac{\gamma}{\gamma-1}\,\frac{p}{\rho} + \frac{1}{2}\,u^2 + \frac{1}{2}\,w^2. \tag{5}$$

Equation (4) is most easily derived from the complete energy equation as given by Pai.[10] However, we have retained the symbol E_s for the electric field as measured in this frame of reference because the quantities in region R_1 are not directly related to E_s. It should also be noted that only in this frame do we have $E_{s1} = E_{s2} = E_s$; in any other frame moving along the x-direction, there will be a difference between E_1 and E_2 (unless $H_{z1} = H_{z2}$, of course). Furthermore we have expressed the internal energy per unit mass of the gas by two terms: $e = e_0 + p/[(\gamma-1)\rho]$. This means that we are assuming we can describe the plasma as a polytropic ideal gas with an additional "frozen-in" internal energy e_0, as for instance stored in dissociation and ionization. The reason for this idealization will become clear later on. In general, of course, both γ and e_0 will be functions of p and ρ, depending on the composition to be determined from equilibrium considerations.

In addition, we need the field equations for the magnetic and electric quantities. These are

$$H_{x1} = H_{x2} = H_x \tag{6}$$

[Eq. (6) was already used in the derivation of (2), (3), and (4).] and

$$E_s = \mu\,(\mu_2 H_{z2} - w_2 H_x), \tag{7}$$

which follows from the assumed conductivity in region R_2. If region R_1 were also conducting, we would obtain an additional relation, i.e.

$$E_s = \mu u_1 H_{z1} . \tag{8}$$

With $e_0 = 0$ and $\gamma_2 = \gamma_1$ the system (1) to (8) is identical with the one studied previously[1] and derived very elegantly by Lüst.[11]

Since we have to abandon Eq. (8) in our problem, the set is incomplete. In other words, Eq. (1) through (7) are insufficient to determine the quantities in R_2 if those in R_1 are given. We can use these equations, however, to derive a relationship between any two unknown quantities in terms of the given data. We shall then require an additional argument or an additional given datum to close the set and make the problem a determined one. In this sense the situation is very similar to the problem of combustion waves. Actually, in the case of an electrically driven shock tube it is more appropriate to consider the current, i.e. H_{z2}, as independent and u_1, the shock velocity as a dependent variable.

It is instructive and in fact algebraically economical, to express the set (1) to (7) in the laboratory frame of reference before we proceed to reduce these relations to a single equation. As indicated in Fig. 3c, we accomplish this by substituting $u_1 = - U$, $u_2 = -(U-v_2)$, $w_2 = w_2$, $E_1 = E_s + \mu U H_{z1}$, and $E_2 = E_s + \mu U H_{z2}$. The shock relations can then be written in the form

$$\rho_1 U = \rho_2 (U - v_2), \tag{9}$$

$$\rho_1 U v_2 = p_2 - p_1 + \frac{\mu}{2} (H_{z2}^2 - H_{z1}^2), \tag{10}$$

$$-\rho_1 U w_2 = \mu H_x (H_{z2} - H_{z1}), \tag{11}$$

$$\rho_1 U (e_2 - e_1 + \frac{1}{2} v_2^2 + \frac{1}{2} w_2^2) + \frac{\mu U}{2} (H_{z2}^2 - H_{z1}^2) = p_2 v_2 + E_2 H_{z2} - E_1 H_{z1}, \tag{12}$$

and

$$E_2 = E_1 + \mu U (H_{z2} - H_{z1}) = \mu (v_2 H_{z2} - w_2 H_x). \tag{13}$$

Equation (12) is the interesting one. It states that the work done on a unit volume of the undisturbed gas, including the energy change in the magnetic field, has to be provided by both a piston moving with the gas velocity v_2 and the negative divergence of the Poynting vector in the tube. It is the divergence of the Poynting vector which, at least in part, takes the place of the chemical energy released in a combustion wave. The piston, of which either p_2 or v_2 may be specified as the additional datum mentioned before, is necessary to ensure the assumed steady flow. We shall show, however, that here as in the case of detonation waves, the flow is only completely determined by such a piston if its speed exceeds a certain minimum.[12] If no such piston is provided or if the piston is too slow, a region of nonsteady flow in the manner of a rarefaction wave appears between it and the propagating shock front, and the quantity $p_2 v_2$ in Eq. (12) is not determined by the physical piston but by the dynamics of the expansion wave.

The system of Eq. (9) to (13) must still be supplemented by a set of equations which determine

$$e_2 = e_0 + \frac{1}{\gamma_2 - 1} \frac{p_2}{\rho_2} \tag{14}$$

as a function of p_2 and ρ_2. This requires numerical means, and for hydrogen it has essentially been done already.[13] The general solution of the problem, then, also requires numerical means and the discussion of the complete treatment will be the subject of a subsequent paper. In the analysis discussed here we shall simply consider both e_0 and γ_2 as given fixed quantities. The latter is, in fact, a valid approximation if the gas is hot enough to be practically fully dissociated and fully ionized. In this case, we simply have $e_0 = 2e_i + e_d$, the total energy of ionization and dissociation per unit mass, and $\gamma_2 = 5/3$. For hydrogen, the approximation is good if, for instance, p_2 is less than 1 atmos and p_2/ρ_2 is greater than 5×10^8 m^2/sec^2.

SIMPLIFIED SOLUTION

In the following treatment, we shall consider v_2, the x-component of the flow velocity behind the front, as an independent variable. We shall use Eq. (9) to (14) to express U, w_2, p_2 ρ_2, E_2 and hence also E_1 as functions of ρ_1, p_1, γ_1, H_x, H_{z1}, and of H_{z2}, γ_2, e_0 as well as of v_2. Physically, this means that we are specifying the conditions in the undisturbed gas, and the current but not the electric field. If we eliminate in Eq. (12) the quantities w_2, ρ_2, p_2 E_2 and E_1 with the help of Eqs. (9), (10), (11) and (13), we obtain a relation of the fourth degree which is cubic in U and quadratic in v_2. We could solve this for v_2 and study the behavior of $v_2(U)$. However, it turns out to be algebraically much more convenient to introduce a set of new dimensionless variables which simplify the expressions considerably and permit a much more direct inspection of the character of the solutions.

Let us define the following new variables:

$$\left.\begin{aligned}
\Delta H &= H_{z2} - H_{z1} \neq 0 \\
X &= \frac{\rho_1 U v_2}{\mu(\Delta H)^2} \\
Y &= \frac{\rho_1 U^2}{\mu(\Delta H)^2} \\
Z &= \frac{\rho_1 U w_2}{\mu(\Delta H)^2} \\
\Pi &= \frac{p}{\mu(\Delta H)^2}
\end{aligned}\right\} \qquad (15)$$

$$
\left.\begin{aligned}
\epsilon &= \frac{\rho_1 e_0}{\mu(\Delta H)^2} \\
\alpha &= \frac{H_x}{\Delta H} \\
\beta &= \frac{H_{z1} + H_{z2}}{\Delta H} \; .
\end{aligned}\right\} \quad \begin{array}{c}(15)\\ \text{continued}\end{array}
$$

We are not interested in the case $\Delta H = 0$ because this is the ordinary gas dynamic shock. The parameter β can have any value in principle. $\beta = 1$ implies $H_{z1} = 0$, $\beta = -1$ means $H_{z2} = 0$ and $\beta = 0$ refers to $H_{z2} = -H_{z1}$. In analogy to the nomenclature introduced for ordinary hydromagnetic shocks,[8] we shall call these cases magnetic "switch-on", "switch-off", and "transverse" ionizing fronts, respectively. With the above substitutions, the solution takes on the form:

$$
Y = \frac{(\gamma_2+1)X^2 + (\gamma_2 - 1 - \beta + 2\gamma_2\Pi_1)X + (\gamma_2 - 1)\alpha^2}{2X + 2(\gamma_2 - 1)\epsilon + \gamma_2 - 1 - \beta - \frac{2(\gamma_2 - \gamma_1)}{\gamma_1 - 1}\Pi_1} \tag{16}
$$

$$
Z = -\alpha \tag{17}
$$

$$
\rho_2/\rho_1 = \frac{Y}{Y-X} \tag{18}
$$

$$
\Pi_2 = X - \beta/2 + \Pi_1 \tag{19}
$$

$$
\frac{E_2}{\mu U \Delta H} = \frac{E_1}{\mu U \Delta H} + 1 = \frac{\alpha^2 + \frac{1+\beta}{2} X}{Y} \; . \tag{20}
$$

Although this form is still implicit since X contains the dependent variable U, many features of the solutions are easily demonstrated. When E_1, ϵ, and $\gamma_2 - \gamma_1$ are all set equal to zero, these equations are again reduced, of course, to the ones investigated by Bazer and Ericson.[1] In particular, it is readily shown that in such a case X cannot be negative if the entropy is not supposed to diminish across the shock. Also, it is easily seen that under those circumstances X can only be zero if $\beta = 0$, and then we have $Y = \alpha^2$, and $\Pi_2 = \Pi_1$.

None of these inferences can be drawn from Eqs. (16) to (20) if E_1 is allowed to differ from zero. This is the first important conclusion.

We shall now point out some of the general features of Eq. (16), which is plotted for various α in Fig. 4. Of course we are only interested in the region $Y < \alpha^2 + 1/2\,(1+\beta)\,X$ so that E_1 never vanishes.

(a) Equation (16) describes hyperbolas in the X-Y plane. The asymptotes are:

$$X = 1/2\,(1 + \beta - \gamma_2) - (\gamma_2 - 1)\,\epsilon + \frac{\gamma_2 - \gamma_1}{\gamma_1 - 1}\,\Pi_1 \tag{21a}$$

and

$$Y = 1/2\,(\gamma_2+1)\,X + 1/4\,(\gamma_2-1)(1+\beta-\gamma_2) - 1/2\,(\gamma_2^2 - 1)\,\epsilon + (\gamma_2 - 1)\,\frac{\gamma_1+\gamma_2}{\gamma_1 - 1}\,\Pi_1 , \tag{21b}$$

i.e. they do not depend on the parameter α.

(b) When X is very large compared to α^2, ϵ, and γ_2, we have $Y \to \frac{\gamma_2+1}{2}\,X$. This is the ordinary gas dynamic strong shock. We should expect this property because it is clear that the piston in Eq. (12) is doing practically all the work in this case.

(c) The curves Y (X) have minima. The minima have as loci the straight lines

$$Y_m = (\gamma_2+1)\,X - 1/2\,(1+\beta-\gamma_2) + \gamma_2\,\Pi_1 . \tag{22}$$

These are seen to be independent of both α and ϵ. The fact that the $Y(X)$ have minima means that for each set of given conditions ρ_1, p_1, ΔH, etc. the resulting relation $U(v_2)$ has a minimum. Again, this feature is reminiscent of the behavior of detonation waves. However, the analogy should not be stretched too far. One might, for instance, be tempted to identify the minimum with the familiar Chapman-Jouquet point in the theory of gaseous detonations.[12]

The analysis of gaseous combustion waves shows that at the point of minimum propagation speed, the flow velocity of the gas behind the front relative to the front is always exactly sonic, i.e., at that point the rarefaction wave follows the front immediately. Moreover, the entropy behind the front is a minimum when compared to values of entropy on other points along the $U(v_2)$ curve. The analogous conditions are generally not fulfilled for the propagation speeds Y_m of our hydromagnetically driven ionizing fronts. However, in the special case $\beta = -1$, the magnetic switch-off wave, we can show that the analogy is almost complete. This is the second important conclusion.

The proof is elementary. We merely have to express the relative velocity $u_2 = -(U-v_2)$ in terms of our new variables:

$$\frac{\rho_2 u_2^2}{\mu(\Delta H)^2} = Y - X. \tag{23}$$

Substitution from Eqs. (19) and (22) yields for the relative gas speed at the minimum of U

$$(u_2^2)_m = (U-v_2)^2 = \frac{1}{\rho_2}\left[\gamma_2 p_2 + 1/2\,(1+\beta)(\gamma-1)\,\mu\,(\Delta H)^2\right]. \tag{24}$$

The propagation speed, c_2, along the x direction for small disturbances in the plasma in region R_2 is given by the relation[14]

$$c_2^2\left[\frac{\mu}{\rho_2}(H_x^2 + H_{z2}^2) - c_2^2\right] = \frac{\gamma_2 p_2}{\rho_2}\left(\frac{\mu}{\rho_2}H_x^2 - c_2^2\right). \tag{25}$$

Obviously for $H_{z2} = 0$, we have $\beta = -1$, and hence

$$(u_2^2)_m = \frac{\gamma_2 p_2}{\rho_2} = c_2^2 .$$

Likewise, it can be readily shown that the change of entropy per unit mass $ds = 1/T\ [\ de + pd\ 1/\rho]$ taken along the curve $Y(X)$ at the point where $dY = 0$ is given by

$$(T_2 ds_2)_m = \frac{\mu}{2\rho_1}(1+\beta)(H_{z2} - H_{z1})^2\ \frac{dX}{Y}, \tag{26}$$

which, of course, is again zero for $\beta = -1$. We shall therefore call this point in this special case the C-J (Chapman-Jouguet) point and the mode of operation of the ionizing front at this point the C-J ionizing process.

This result is not too surprising since here the magnetic field has no transverse component behind the front so that the gas flow in the x direction is purely acoustic. The energy per unit mass stored in the transverse magnetic field in region R_1, $\mu H_{z1}^2/2\rho_1$, might be expected to be the exact equivalent of the available combustion energy for detonation waves. This is not correct, however. Additional energy must be supplied from the external circuit if a switch-off wave is to propagate. This condition may be connected with the fact that the entropy produced in a switch-off ionizing wave can be shown to be a maximum at the C-J point rather than a minimum.

In the theory of simple gaseous detonation, it is usually argued that the C-J process must occur whenever there is no piston added that moves with a speed $v_2 > (v_2)_m$, the gas flow velocity in the x direction corresponding to the C-J point.[12] The same can be demonstrated here. It is easily verified that, in the case of $\beta = -1$, we have $\gamma_2 p_2 > \rho_2(U-v_2)^2$ for $v_2 > (v_2)_m$. This means that any rarefaction wave existing behind the shock will catch up with and weaken the shock, reducing both U and v_2 either until the flow behind the front is uniform, or until v_2 equals $(v_2)_m$, whichever is reached first. In that case, therefore, the situation $v_2 < (v_2)_m$ is never obtained. Besides, situations with $v_2 < (v_2)_m$ are believed to be unstable, because they involve supersonic flow normal to the front on both sides of the shock.

As a result, we can use Eq. (22) for $\beta = -1$ to express the additional condition for the C-J process. Hence we can eliminate either Y or X from Eq. (16) so that the problem of the switch-off wave is completely determined. However, in order to extend the solution to the general case $-\infty < \beta < +\infty$, we shall postulate here that the relevant physical condition determining the mode of operation according to the arguments in the previous paragraph is

$$U - v_2 = c_2, \tag{27}$$

where c_2 is given by the smallest positive root of Eq. (25). This means region R_2 in Fig. 2 is assumed to be always shrunk to zero length.

Equation (27) can be combined with Eq. (25) and rewritten with the help of our new variables (15) to read

$$(Y-X)\left[\alpha^2 + 1/4(1+\beta)^2 - Y+X\right] = \gamma\Pi_2\,(\alpha^2 - Y+X). \tag{28}$$

Because of Eq. (19) and after some rearrangement, we finally obtain our general subsidiary equation:

$$(1+\beta)^2(Y-X) = 4(\gamma_2 X+X-Y-\gamma_2\beta/2+\gamma_2\pi_1)(\alpha^2+X-Y). \tag{29}$$

The solution of the simultaneous equations (16) and (29) is algebraically rather cumbersome unless $\beta = -1$ or $\alpha = 0$. However, we note that for

$$\alpha^2 >> (1+\beta)^2, \tag{30}$$

we can use as a good approximation

$$Y = (\gamma_2 + 1)\, X - \gamma_2\beta/2 + \gamma_2\, \Pi_1 . \tag{31}$$

A plot of Eq. (31) is also included in the example on Fig. 4. For $\beta = -1$, both Eqs. (29) and (31) are identical with Eq. (22), and then Eq. (31) is valid for all $\alpha > 0$. Certainly for experiments in which $H_x >> H_{z1}$ and $H_x >> H_{z2}$, Eq. (31) is adequate. We may, moreover, always neglect Π_1, because we will certainly need $\Pi_1 << 1$ in ionizing hydromagnetic waves; Π_1 was only carried in our equations for completeness sake. The subscript of γ_2 may then also be dropped. If we now use Eq. (31) to eliminate X from Eq. (16) we obtain the solution for the wave speed

$$Y = (A+B^2)^{1/2} - B, \tag{32}$$

where

$$A = (\gamma^2-1)\,\alpha^2 + \frac{\beta\gamma}{2}\left(\frac{\beta\gamma}{2} + \gamma-1-\beta\right)$$

and

$$B = (\gamma^2-1)\,\epsilon + \frac{\gamma}{2}(\gamma-1-\beta).$$

The terms containing β in this expression are only strictly justified for $(1+\beta)^2 << 1$ because of condition (30).

For $A \gg B^2$, i.e. $\mu H_x \Delta H \gg \rho_1 e_0$, we find

$$U^2 \approx \frac{\mu}{\rho_1} H_x \Delta H \sqrt{\gamma^2 - 1} . \tag{33}$$

For $B^2 \gg A$, on the other hand, we have

$$U \approx \frac{\mu H_x \Delta H}{\rho_1 \sqrt{2 e_0}} \tag{34}$$

In Fig. 5, we show a plot of Y as a function of α for $\beta = -1$, $\gamma = 5/3$ and a variety of values for ϵ according to Eq. (32).

The other quantities of interest—v_2, ρ_2, p_2 and E_2—are most easily expressed in terms of U, the wave speed, by using Eq. (31), (18), (19), and (20). In these, too, we shall ignore p_1 everywhere and drop the subscript of γ_2. From Eq. (31), we obtain immediately

$$v_2 = \frac{U}{\gamma+1} \left(1 + \frac{\beta\gamma}{2Y}\right) \tag{35}$$

and, using Eq. (18),

$$\rho_2 = \rho_1 (1 + 1/2) \left(1 - \frac{\beta}{2Y}\right)^{-1} . \tag{36}$$

According to Eq. (19), p_2 is given by

$$p_2 = \frac{\rho_1 U^2}{\gamma + 1} \left(1 - \frac{\beta}{2Y}\right) . \tag{37}$$

This determines also the temperature behind the front as

$$(RT)_2 = \frac{p_2}{\rho_2} = \frac{\gamma U^2}{(\gamma+1)^2} \left(1 - \frac{\beta}{2Y}\right)^2 . \tag{38}$$

Finally, the electric field in the region R_2 is determined from Eq. (20) to be

$$E_2 = \frac{\mu \Delta H}{U} \left[\frac{\mu}{\rho_1} H_x^2 + \frac{(1+\beta)U^2}{2(\gamma+1)} \left(1 + \frac{\beta\gamma}{2Y}\right) \right] . \tag{39}$$

SPECIFIC CONCLUSIONS

From the set of relations (32) to (39) a number of conclusions concerning these hydromagnetic ionizing fronts may be drawn immediately. First of all, it is easily demonstrated with the help of Eq. (16) that $\alpha^2 \gg Y \gg 1$ if both $\alpha^2 \gg (1+\beta)^2$ and $\alpha^2 \gg 1$ are fulfilled. Equations (32) to (39) therefore show that under these circumstances v_2, ρ_2, p_2, and E_2 do not depend strongly on β. Also, it is seen that in this case the difference between conditions (22) and (31) is negligible. In other words, if the longitudinal magnetic field H_x is much stronger than both H_{z1} and H_{z2}, Eqs. (32 through (39) can be expected to describe the phenomenon rather well, even if the postulate (27) is not the correct one. This is the third important conclusion.

Furthermore, certain interesting features pertaining to the extreme case mentioned above are worth pointing out. Equation (36) in this limit states that ρ_2/ρ_1 is remarkably insensitive to changes in the independent variables, the value being surprisingly low. For examples, for $\gamma = 5/3$, we have $\rho_2/\rho_1 \approx 1.6$.

Substitution for U from Eq. (32) in Eq. (39) shows that E_2 varies only slowly with ΔH. In fact, for $\mu H_x \Delta H << \rho_1 e_0$ Eq. (34) applies, and we have

$$E_2 \approx \mu H_x \sqrt{2e_0}\ , \tag{40}$$

which is independent of the current and gas density. It resembles the findings by Alfvén[15] and Fahleson,[16] although the experiments described by them did not appear to involve distinct fronts producing full ionization, as assumed in our model. Equation (34) when combined with Eq. (11) can also be written

$$w_2^2 = 2e_0. \tag{41}$$

Actually, when Eq. (34) applies, the temperature T_2 is often too low to justify the original assumption of complete ionization.

In Fig. 6, Eq. (39) for the case of $\beta = +1$ is plotted in a nondimensional form, i.e., expressing the quantity $E_2/\mu H_x\sqrt{2e_0}$ as a function of $\Delta H\sqrt{\mu/(\rho_1 e_0)}$ for various values of $H_x\sqrt{\mu/(\rho_1 e_0)}$. The solid curves are fair approximations also for $\beta \neq 1$ provided that $(1+\beta)^2 << \alpha^2$. The predictions of Eqs. (32) through (39) may be compared with the experimental findings of Wilcox et al. in which $\beta = +1$.[4] Although their geometry is not one-dimensional but cylindrical, their observations agree fairly well with some of the major conclusions arrived at here (uniform propagation speed of a distinct front, voltage regulations, etc.,).[17] More extensive comparison between theory and experiment is planned for the near future.

While the magnetic "switch-on" wave is of particular interest to the experimentalist because of the simplicity in instrumentation, the "switch-off" wave is more attractive from the analytical point of view. In addition to the close correspondence to gaseous detonation waves, in the "switch-off" case, we note that both Eqs. (16) and (20) become simplified. In particular, it is interesting to see that, for $\beta = -1$, E_2 has a maximum at the C-J point. This is in agreement with the fact that the entropy produced is a maximum for the C-J ionizing process. Moreover, we recall that for $\beta = -1$, Eqs. (32) through (39) are exact, the only restriction being $\alpha > 0$.

Finally we shall investigate under what conditions v_2 can be zero, i.e. $\rho_2 = \rho_1$. As pointed out before, Eqs. (16) through (20) do not restrict X to values greater than zero if β is permitted to take on values less than zero. In our model of a closed input end of the tube, v_2 can never be negative. If conditions in the front call for $v_2 < 0$, a precompression shock is set up, violating the assumption of gas at rest in region R_1. If the precompression shock is strong enough to ionize the gas, the front will change its character such that v_2 is greater than zero. In a very similar manner, deflagrations are changed into detonations in the case of closed gas-combustion tubes. Therefore, we may set $X = 0$ in both Eqs. (16) and (29) and obtain two simultaneous equations in Y, β, and α:

$$Y_0 = \frac{(\gamma-1)\alpha^2}{2(\gamma-1)\epsilon + \gamma-1-\beta} \tag{42}$$

$$-(1+\beta)^2 \; Y_0 \geqslant 2\,(2Y_0 + \beta\gamma)\,(\alpha^2 - Y_0). \tag{43}$$

We use the symbol $\geqslant$ to allow values $c_2 \geqslant U$ in Eq. (27). If we eliminate Y_0 between Eqs. (42) and (43), we find the minimum condition for $-\beta$ as a function of α and ϵ that makes $v_2 = 0$ possible. We shall not do this here, because it is lengthy and not particularly instructive. However, we may also ask what can be the maximum α for which a switch-off wave, $\beta = -1$, does not yet bring about a compression. This means that, after imposing $\beta + 1 = 0$ in Eqs. (42) and (43), we solve for α. The result is

$$\alpha^2 \leqslant \gamma \left[\epsilon + \frac{\gamma}{2(\gamma-1)}\right]. \tag{44}$$

We may, of course, express this relation as a condition for the minimum admissible value of H_{z1} if H_x, e_0, ρ, and γ are all given:

$$H_{z1}^2 \geqslant \frac{2}{\gamma^2}(\gamma-1)\,(H_x^2 - \frac{\gamma}{\mu}\;\rho e_0). \tag{45}$$

The propagation speed of the front is then given directly by Eq. (42).

The transverse velocity becomes independent of H_x:

$$w_2^2 = 2e_0 + \frac{\gamma\mu}{(\gamma-1)\rho} H_{z1}^2 . \tag{46}$$

The expression for the pressure is simply

$$p_2 = 1/2\ \mu H_{z1}^2 , \tag{47}$$

which imposes a required minimum on H_{z1} to ensure adequate ionization. The electric fields are

$$E_2 = -\mu w_2 H_x \tag{48}$$

and

$$E_1 = E_2 \left(1 - \frac{\rho U^2}{\mu H_x^2}\right).$$

The situation is particularly simple for $\mu H_x^2 \gg \gamma\rho e_0$. In that case, Eq. (45) reduces to

$$H_{z1}/H_x \geqslant \frac{1}{\gamma}\sqrt{2(\gamma-1)} \approx 0.7 \tag{49}$$

for $\gamma = 5/3$. Moreover, both U and the impedance $-E_2/H_{z1}$ become independent of current (the minus sign refers to the fact that, for $\beta < 0$, E is negative if H_{z1} is positive):

$$U^2 \approx \frac{(\gamma-1)\mu}{\gamma\rho} H_x^2 \tag{50}$$

$$E_2 \approx \gamma E_1 \approx -\sqrt{\frac{\gamma\mu}{(\gamma-1)\rho}}\ \mu H_x H_{z1} \tag{51}$$

while

$$w_2^2 \approx \frac{\gamma\mu}{(\gamma-1)\rho} H_{z1}^2 = \frac{2}{\gamma-1}\ \frac{\gamma p_2}{\rho} . \tag{52}$$

It is felt that such a switch-off ionizing wave would be a very suitable means of generating a uniform magnetized plasma. After the plasma is formed, the resulting transverse motion is easily arrested by shorting out E_2 through a suitable resistor so that a simple Alfvén-wave relaxation will take place without disturbing the state of the gas. It would be interesting to try to realize this situation experimentally and to test the various conclusions arrived at in this analysis.

For $v_2 > 0$, however, the front must be followed by a rarefaction wave. A brief discussion of this phenomenon is presented in the next section.

THE RAREFACTION WAVE

As pointed out before, in the analysis of the nonsteady flow behind the front, we shall have to assume isentropic motion. Otherwise the analysis would become very complicated. This problem has already been treated by several authors,[8, 9, 14] and, in the main, we shall merely summarize the results. If we assume plane motion, we can eliminate the time and space differentials in the basic equations of magnetohydrodynamics by the formal operator substitution[14]

$$d = \frac{\partial}{\partial t} + (v+c)\frac{\partial}{\partial x}. \tag{53}$$

As a result, we obtain the so-called "characteristic equations" for the motion, which for our geometry take the following form corresponding to the conservation laws:

Mass

$$c d\rho = \rho dv \tag{54}$$

x-momentum

$$c\rho dv = a_s^2\, d\rho + \mu H_z dH_z \tag{55}$$

z-momentum

$$c\rho dw = -\mu H_x dH_z \tag{56}$$

Energy

$$p\ \rho^{-\gamma} = \text{constant.} \tag{57}$$

Here we have written a_s for the speed of ordinary sound:

$$\frac{dp}{d\rho} = \frac{\gamma p}{\rho} = a_s^2 \tag{58}$$

The field equations are:

$$H_x = \text{constant} \tag{59}$$

$$c dH_z = H_z dv - H_x dw \tag{60}$$

$$E = \mu(vH_z - wH_x). \tag{61}$$

Some authors have used the term "simple magnetosonic waves" for this case.[18] The fact that the substitution (53) indeed eliminates both independent variables from the equations implies that the dependent variables are all constant for given "phases" $x_0 = x - (c+v)t$. In our particular case of the rarefaction wave, all phases coincide at, say, $x = 0$ for $t = 0$, so that we may set $x_0 = 0$ for all variables. Such a phenomenon is called a centered wave. It means that the coordinate of a constant condition, a "phase", is given by $x = (c+v)t$. Inspection of the character of hydromagnetic waves shows that the quantity c here in the case of a rarefaction wave is given by the smallest positive root of Eq. (25). In line with our earlier treatment, we shall describe the wave in the laboratory frame of reference.

The simultaneous solution of Eq. (54) to (60) is complicated only because of the complex nature of the condition (25). The set is easily reduced to two simultaneous equations. In order to obtain explicit answers, however, numerical means have to be used eventually. This has already been done rather completely by Kemp and Petschek,[9] and therefore shall not be repeated here. We shall only demonstrate the almost obvious fact that, for large ratios H_x/H_z, the flow can be approximated by the familiar acoustic solution, in which case an analytic treatment is possible. These solutions will be exact for the switch-off case, where $H_{z2} = 0$.

Let us suppose that, in an actual experiment where such a wave is propagated, the input current is given and constant in time. According to our model, this determines H_{z4}. Equations (54) to (60) then indicate that at any point x moving with constant velocity x/t, H_z is constant. Particularly at a point moving immediately behind the front, $x = Ut$, the transverse field is given by H_{z2} and also is constant in time. Since we already know the relationship between U and H_{z2} from our shock analysis, it is easier to pretend that H_{z2} is given, so that we may compute U, v_2, w_2, p_2, ρ_2, etc. in order to apply them as boundary conditions for the solution of Eq. (54) to (60). The only other condition we know is that at $x = 0$, either $v = v_4 = 0$ or $\rho = \rho_4 = 0$. (In our acoustic approximation, of course, we will never find $\rho = 0$). Integration of our equations then will determine H_{z4}, w_4, p_4, ρ_4, etc. This approach is a standard technique for treating rarefaction waves.

Using Eq. (58) and dropping the subscript 2 which only refers to region R_2, we can write Eq. (25) in the form

$$\frac{a_s^2}{c^2} = 1 + \frac{H_z^2}{H_x^2}\left(1 - \frac{\rho c^2}{\mu H_x^2}\right). \tag{62}$$

For the slow-wave root where we limit ourselves to cases $\rho c^2 << \mu H_x^2$, we may therefore also approximate

$$c^2 \approx a_s^2 = \frac{\gamma p}{\rho} \tag{63}$$

and

$$a_s^2 - c^2 \approx c^2 \frac{H_z^2}{H_x^2}, \tag{64}$$

as long as we have $H_z << H_x$ $(\alpha \gg 1)$.

For Eq. (54), we obtain in that case the well-known acoustic solution using Eq. (57) to eliminate p:

$$c = c_2 + \frac{1}{2}(\gamma-1)(v-v_2). \tag{65}$$

If the expansion wave is attached to the shock as postulated in Eq. (27), we therefore find

$$c = U - \frac{1}{2}(\gamma+1) v_2 + \frac{1}{2}(\gamma-1) v. \tag{66}$$

For c_4, where $v = v_4 = 0$ with Eq. (35), we have

$$c_4 = \frac{1}{2} U (1-\frac{\beta\gamma}{2Y}). \tag{67}$$

In other words the tail of the expansion wave moves at roughly half the speed of the front.

The density ρ_4 is obtained from Eqs. (27), (35), (57), and (63) using Eq. (67):

$$\rho_4 \approx \rho_2 (\frac{\gamma+1}{2\gamma})^{2/(\gamma-1)} \approx 2\rho_1 (\frac{\gamma+1}{2\gamma})^{(\gamma+1)/(\gamma-1)}, \tag{68}$$

where the value of ρ_1 was substituted from Eq. (36)

For $\gamma = 5/3$, this yields $\rho_4 \approx 0.8 \rho_1$.

Therefore it appears that the expansion produced by a hydromagnetic ionizing wave is very mild if H_z is much less than H_x and about half the length of the generated plasma is uniform and without longitudinal motion.

Pressure and temperature in region R_4 may also be immediately computed from Eqs. (57) and (68). The results are

$$p_4 \approx p_2 \left(\frac{\gamma+1}{2\gamma}\right)^{2\gamma/(2-1)} \approx \frac{\rho_1 U^2}{2\gamma} \left(\frac{\gamma+1}{2\gamma}\right)^{(\gamma+1)/(\gamma-1)} \tag{69}$$

and

$$(RT)_4 \approx (RT)_2 \left(\frac{\gamma+1}{2\gamma}\right)^2 \approx \frac{U^2}{4\gamma} \tag{70}$$

where the values of p_2 and $(RT)_2$ are substituted from Eqs. (37) and (38).

Finally we wish to calculate H_{z4} and E_4 (or w_4) in this approximation. Using Eqs. (54), (55), (63), and (64), we find

$$\mu H_x^2 \, dH_z \approx -H_z dp$$

so that we have

$$H_{z4} \approx H_{z2} \exp\left[\frac{\mu}{H_x^2}(p_2 - p_4)\right] \tag{71}$$

$$\approx H_{z2} \left[1 + \frac{\mu}{H_x^2}(p_2 - p_4)\right] .$$

Similarly, we deduce from Eqs. (56) and (60) the approximate solution

$$w_4 \approx w_2 - \frac{H_{z2}}{H_x} v_2$$

so that we have

$$E_4 = -\mu w_4 H_x \approx E_2 . \tag{72}$$

For large H_x/H_{z4}, the net impedance of the shock tube, which we may express as $E_4 (H_{z4} - H_{z1})^{-1}$, is then essentially computed from Eq. (39), where U must be evaluated from Eq. (32). That is, the expansion wave does not contribute appreciably to the electrical behavior. This is fortunate in retrospect, since large current densities at finite conductivity in region R_3 would certainly conflict violently with the assumption of isentropic flow there. We conclude that the major deviation from this idealized model will be caused by the finite viscosity of the plasma, which must definitely cause considerable dissipation. It is therefore essential that the channel in which such a plasma is generated is not too narrow in the direction of the electric field.

This discussion may suffice to outline the principal features of hydromagnetic ionizing waves and of the plasma which can be generated by them. It is felt that a more precise analysis is not warranted at this point because of the drastic simplifying assumptions that had to be made at the outset. The main problems that still need to be investigated most urgently center on the ionizing mechanism itself, which is active in the propagating front and which controls the shock structure and governs the approach to the equilibrium assumed in this paper.

BIBLIOGRAPHY

[1] BAZER, J., et al., Astrophys. J. 129 (1959) 758.

[2] KOLB, A. C., in Proceedings of the Fourth International Conference on Ionization Phenomena in Gases, 2, North-Holland Publishing Co., Amsterdam, Holland (1959) 1021.

[3] DATTNER, A., ibid., p. 1151.

[4] MARSHALL, J., Phys. Fluids 3 (1960) 134.

[5] PATRICK, R. M., Phys. Fluids 2 (1959) 589.

[6] WILCOX, J. M., et al., A Device for Generating a Highly Ionized Hydrogen Plasma, UCRL-9528 (1961), submitted to Rev. Sci. Instr.

[7] LIEPMANN, H. W., et al., in Symposium of Plasma Dynamics, (edited by Clauser, F.) Addison-Wesley Publishing Co., Inc. (1960) 199.

[8] FRIEDRICHS, K. O., et al., Notes on Magnetohydrodynamics VIII, Nonlinear Wave Motion, NYO-6486 (1958).

[9] KEMP, N. H., et al., Phys. Fluids 2 (1959) 599.

[10] PAI, S.-I., Phys. Rev. 105, (1957) 1424.

[11] LÜST, R., Z. Naturforsch. 8a (1953) 277.

[12] COURANT, R., et al., Supersonic Flow and Shock Waves, Interscience Publishers, Inc., New York (1948) 204-232.

[13] GROSS, R. A., et al., Some Properties of a Hydrogen Plasma, Proceedings of the Third Biennial Gas Dynamics Symposium, Northwestern University Press, Chicago, (1959), p. 15.

[14] GRAD, H., in the Magnetodynamics of Conducting Fluids (edited by Bershader, D.) Stanford University Press, Stanford (1959).

[15] ALFVÉN, H., Revs. Modern Phys. 32 (1960) 710.

[16] FAHLESON, U. V., Phys. Fluids 3 (1960) 123.

[17] WILCOX, J. M., Lawrence Radiation Laboratory, University of California, private communication.

[18] AKHIEZER, A. I., et al., Proceedings of the Second International Conference on the Peaceful Uses of Atomic Energy, Geneva, 1958, United Nations, New York (1958), 31, 225.

FIGURES

Fig. 1. Idealized experiment with plane hydromagnetic ionizing waves.

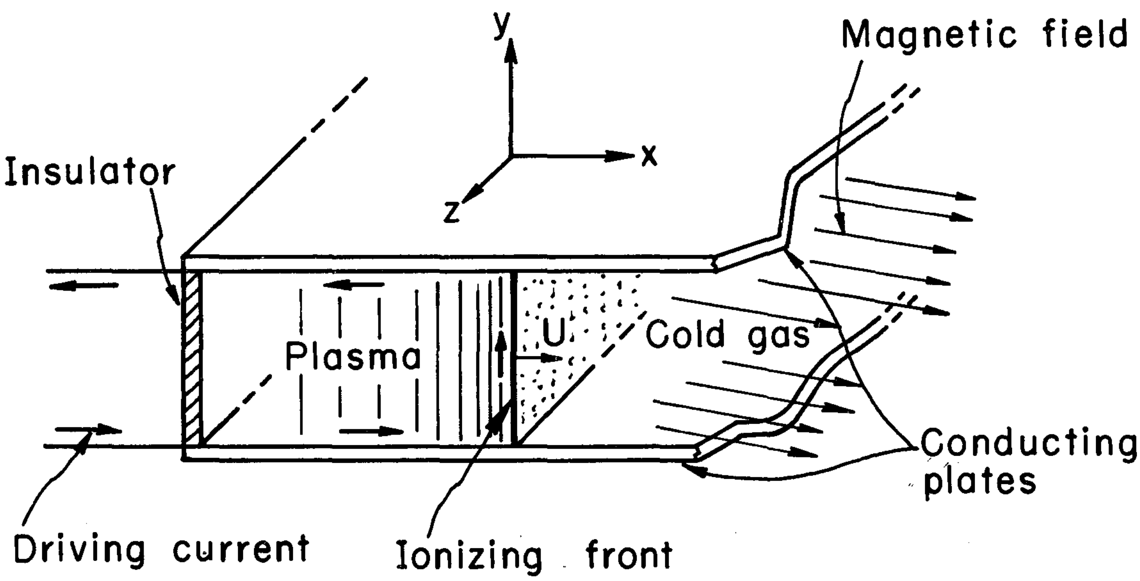
Fig. 2. Model for analysis of hydromagnetic ionizing waves.

Fig. 3. Schematic for shock conditions. Note that in this example the current is in the $+y$ direction so that the velocity w_2 is negative ($-z$ direction).

Fig. 4. Plot of $Y(X)$, Eq. (16), for various values of α^2. This includes plots of Eqs. (21) and (31).

Fig. 5. Plot of $Y(\alpha)$, Eq. (32), for various values of ϵ.

Fig. 6. Plot of $E_2(\Delta H)$, Eq. (39), for various values of $\mu H_x^2/\rho_1$ (made nondimensional).



MU-23727

Fig. 1

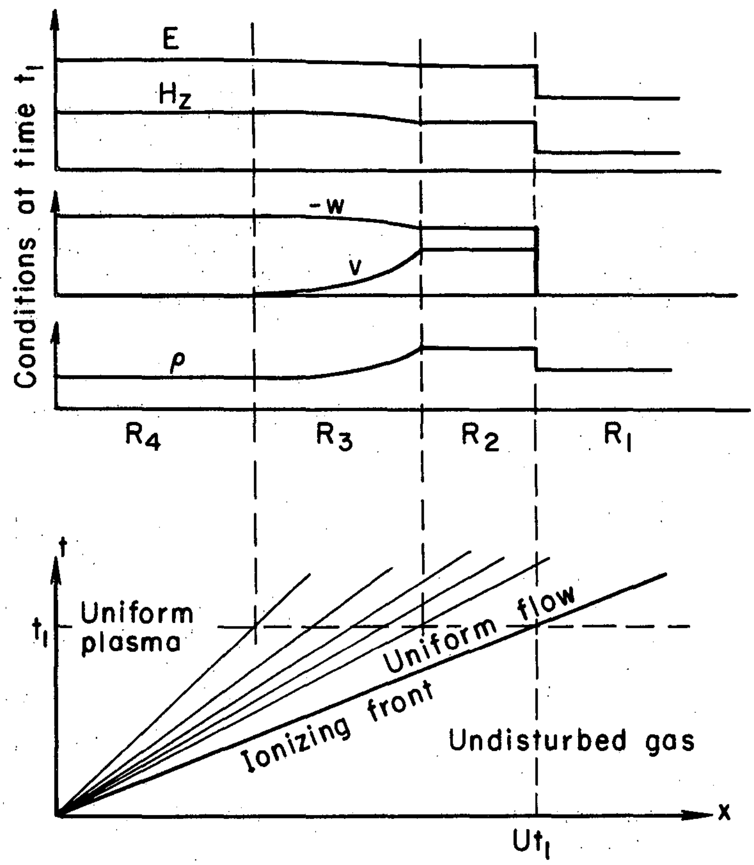


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Fig. 2

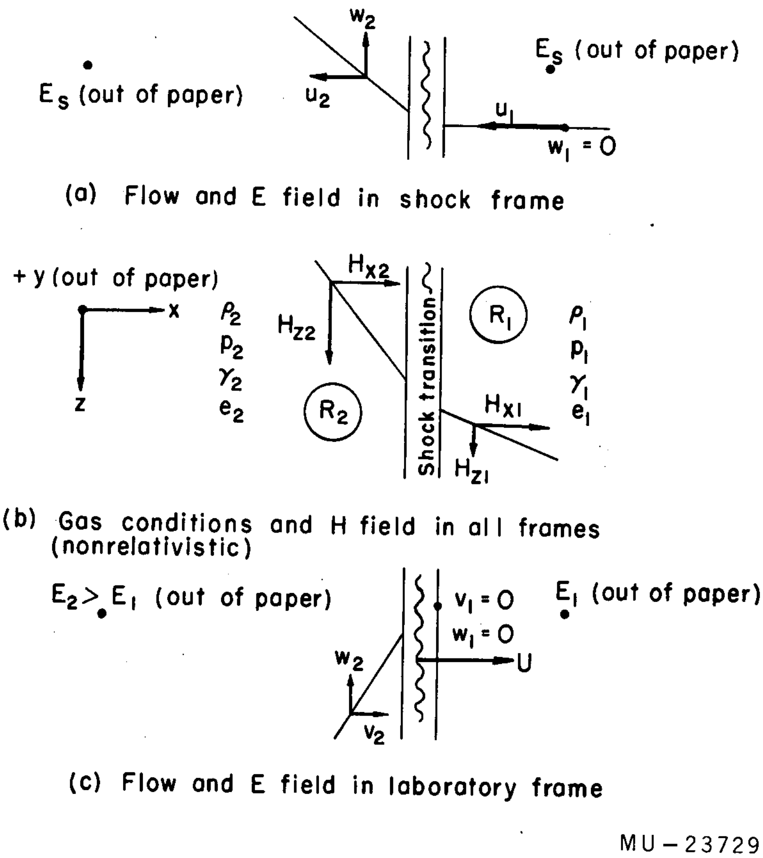
E_s (out of paper)
w_2
u_2
E_s (out of paper)
u_1
$w_1 = 0$
(a) Flow and E field in shock frame
+ y (out of paper)
x
z
ρ_2
p_2
γ_2
e_2
H_{x2}
H_{z2}
R_2
Shock transition
R_1
ρ_1
p_1
γ_1
e_1
H_{x1}
H_{z1}
(b) Gas conditions and H field in all frames (nonrelativistic)
$E_2 > E_1$ (out of paper)
$v_1 = 0$
$w_1 = 0$
U
E_1 (out of paper)
w_2
v_2
(c) Flow and E field in laboratory frame
MU-23729

Fig. 3

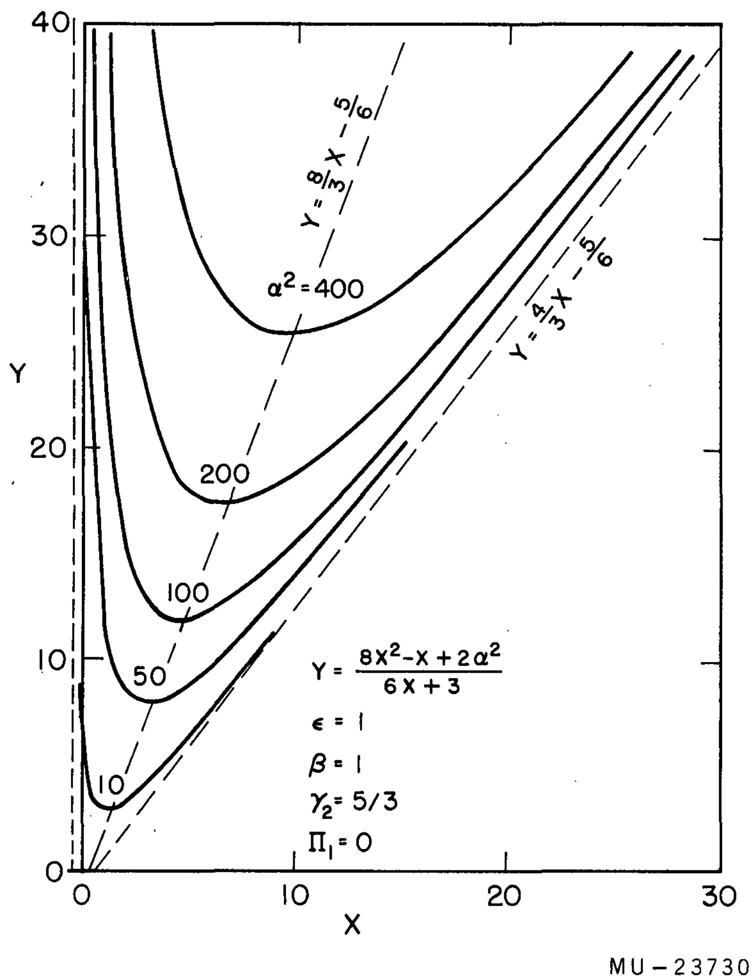
Y = 8/3 X − 5/6
Y = 4/3 X − 5/6
α² = 400
200
100
50
10
Y = (8X² − X + 2α²)/(6X + 3)
ε = 1
β = 1
γ₂ = 5/3
Π₁ = 0
Y
X
0
10
20
30
40
MU-23730

Fig. 4

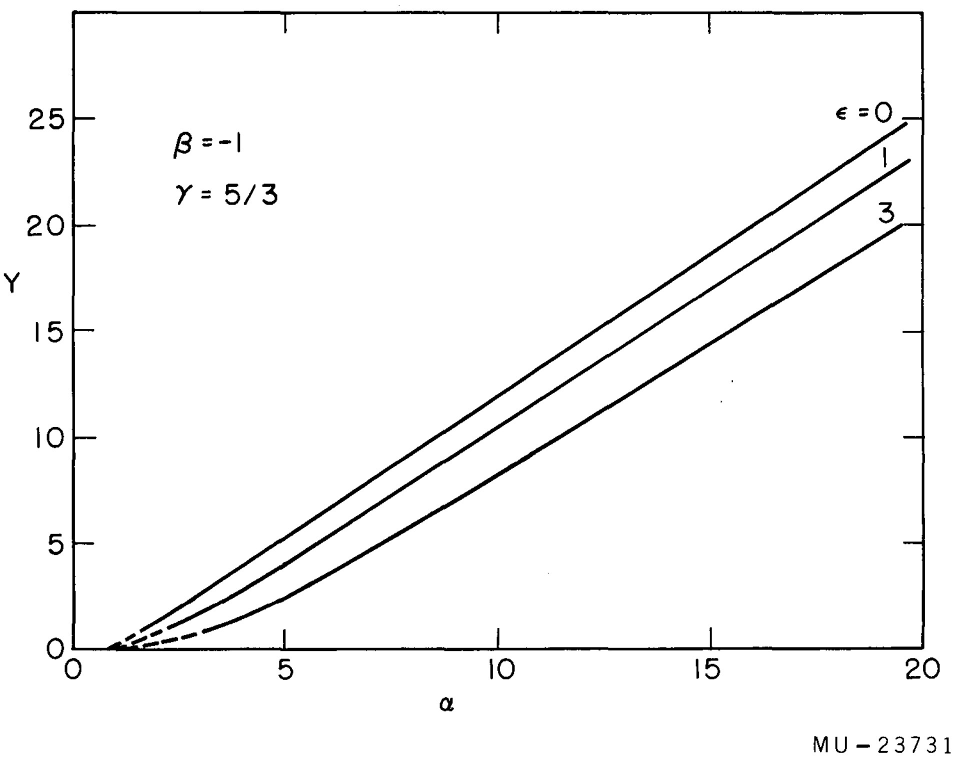

Fig. 5

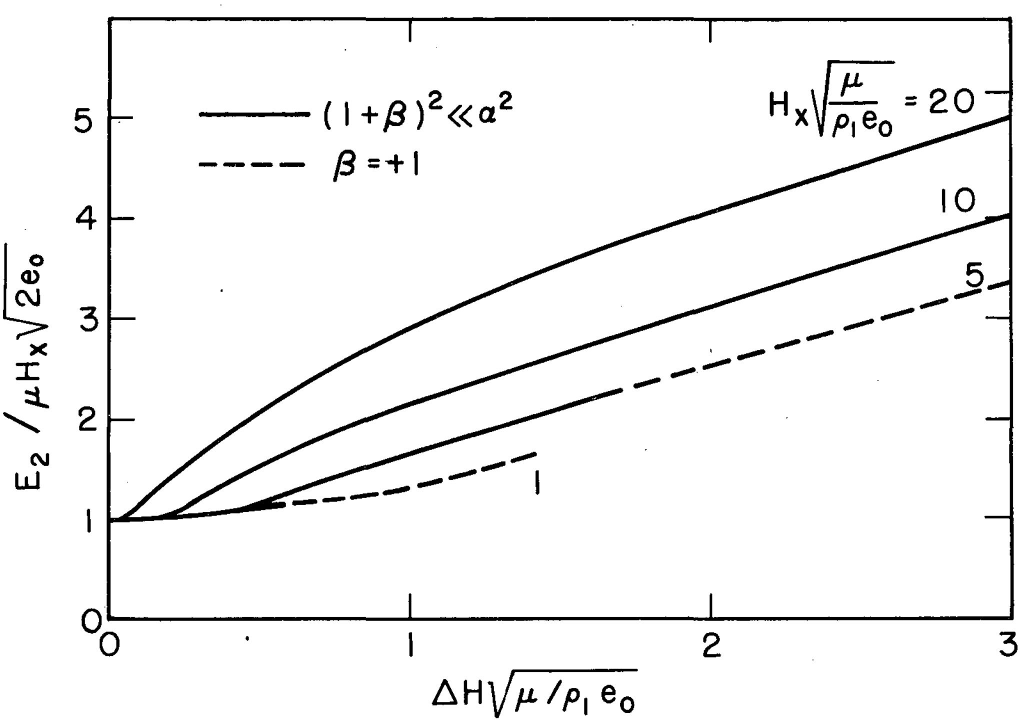

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Fig. 6