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THE EFFECT OF PAIR INTERACTION ON NUCLEAR MATTER

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THE EFFECT OF PAIR INTERACTION ON NUCLEAR MATTER

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THE EFFECT OF PAIR INTERACTION ON NUCLEAR MATTER

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June 5, 1961

ABSTRACT

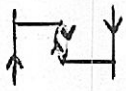
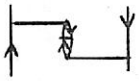
Sawada's Hamiltonian is used once again to study the eigenstates of nuclear matter by adding a momentum dependence to the effective potential. It is shown that the condition for zero sound propagation in nuclear matter is very sensitive to the asymmetry as well as to the average strength of the effective potential between the colliding nucleons at the Fermi surface. And a calculation of the energy of this model Hamiltonian proves that the correction from pair interactions to the first-order Brueckner's expression of nuclear ground-state energy is small.

I. INTRODUCTION

The methods developed by Gell-Mann, Brueckner,[1] Sawada,[2] and Wentzel[3] for a high-density electron gas system can be applied to a variety of problems involving Fermi systems. It is characteristic of these methods that a model Hamiltonian is used which accounts for the interactions between pairs (by a pair one means a particle with momenta $\vec{p} + \vec{q}$ and a hole with momentum $\vec{p}$). Because of its simple quadratic form, this model Hamiltonian can be solved exactly, and the procedure for solving it is quite straightforward.

The first problem to which the method has been applied is that of nuclear matter. Glassgold, Heckrotte, and Watson, in their discussion of the hydrodynamic motions of nuclear matter, have used such a model Hamiltonian.[4] The Coulomb potential between electrons is replaced by the nondiagonal elements of Brueckner's reaction matrix K for nucleons in nuclear matter. The internal degrees of freedom associated with spin and i-spin are also included in their considerations. As a consequence of the internal degrees of freedom, four modes of collective motions are found possible. They are simple compressional waves, spin waves, i-spin waves and coupled spin-i-spin waves. While the i-spin waves can be associated with the Goldhaber-Teller oscillations, the simple compressional waves are found to be unstable, which is of course a drawback of the method.

Brenig has suggested[5] that this instability is due to the impropriety of using the Brueckner's ground-state K-matrix as the effective particle-hole interaction matrix. Indeed, the general effective interaction matrix, written as a perturbation series, contains terms with not only particle-particle potential vertices but also hole-

hole and other complicated potential vertices. In favor of the argument, Sawada has shown the importance of including the "other complicated potential vertices" such as  and  in calculating the effective interaction matrix to obtain stable collective states.[6] However, these treatments have all assumed a momentum-independent, effective interaction potential. Whether the situation will be modified by considering a momentum-dependent, effective interaction matrix we will attempt to determine here.

The effect of the exchange terms, which give considerable contribution for nuclear forces and so should not be neglected, is taken into account by simply adding the minus exchange matrix element to each direct matrix element of the effective interaction operator in the Sawada's Hamiltonian. This procedure is justified at the beginning of Section III.

In Section II, the ground-state energy is expressed as a power series in the effective interaction matrix. With this as a reference state, a model Hamiltonian which consists of a Sawada Hamiltonian and some constant terms, and which reproduces, at the high-density limit, the most important terms of this series, can be easily obtained. An example is given using Brueckner's matrix R_B as the effective reaction matrix. A similar procedure has been used for the pair excited states using Galitskii's matrix, R_G, as the effective reaction matrix. In Section III, the Sawada Hamiltonian is solved by assuming a simple but plausible form (with a few parameters) for the momentum-dependent effective interaction matrix. In the results, one finds that the stability conditions put definite limitations on the values of the parameters introduced in the simple form of the effective interaction

matrix. In Section IV, the Brueckner's R_B-matrix elements as well as the Galitskii's R_G-matrix elements are calculated for a factorable potential with a hard core at small distances. By reducing these matrix elements to the simple phenomenological form, and by comparing the values of those parameters with the stability conditions---since the R_B, R_G-matrix elements are functions of density, the process of reducing and comparing can be carried out at any particular density---one can determine the density intervals in which one or both of the R_B and R_G-matrices yield stable collective states. In these density intervals, one can also calculate the ground-state energy, ϵ_0, of the model Hamiltonian (as obtained in Section II). At densities outside these intervals, one may try other, more appropriate, effective interaction matrices. It is shown that because the equilibrium density of nuclear matter is relatively small, the contribution to the ground-state energy from pair interactions is less than 1 Mev.

II. ESTABLISHMENT OF SAWADA'S HAMILTONIAN FOR THE EIGENENERGIES OF THE NUCLEAR GROUND STATE AND PAIR EXCITED STATES

A. Expansion of the Ground-State Energy In Terms of the Nuclear Reaction Matrix, R.

Consider the Hamiltonian of A nucleons,

$$H = \sum_{i=1}^{A} \frac{k_i^2}{2M} + \frac{1}{2} \sum_{i \neq j}^{A} v_{ij} , \tag{1}$$

and expand the ground-state energy in terms of Goldstone's perturbation series.[7] It is widely accepted that the nucleon-nucleon potential has a high repulsive core at small distances, and the matrix elements of v_{ij} are very large. Therefore it is desirable to expand further in terms of R the nuclear-reaction operator which is defined by

$$R = v + v \frac{Q}{e} R$$

and whose matrix elements are used to describe the "scattering" between the nucleons in the nucleus. The projection operator, Q, and the denominator, e, in the definition of R should be so chosen as to make the expansion converge as quickly as possible. To illustrate the procedure, we write:

$$H(g) = H_0 + H'(g),$$

$$H_0 = \sum_{i=1}^{A} \left(\frac{k_i^2}{2M} + V_i \right),$$

$$H'(g) = \frac{1}{2} \sum_{i \neq j}^{A} v_{ij}(g) - \sum_{i=1}^{A} V_i(g), \tag{2}$$

$$v(g) = gR - g\, v(g) \frac{Q}{e} R,$$

$$V(g) = gV^{I} + g^2 V^{II} + g^3 V^{III} + \cdots, \tag{3}$$

where $V = V(g = 1)$ is the average single-nucleon potential which will also be chosen to make the energy expansion converge as quickly as possible. Here g is introduced as an expansion parameter. At $g = 1$, $H(g = 1)$ is exactly the original Hamiltonian. We shall expand the ground-state energy in terms of $v_{ij}(g)$ according to Goldstone's perturbation expression. Then substituting Eq. (3) into the expansion and writing it in increasing powers of g, we obtain an expansion in terms of R. At the end of the calculation, g will be set equal to unity.

To translate the Hamiltonian into second-quantization language, we use the complete set of eigenfunctions ϕ_ℓ of the single-nucleon Hamiltonian as bases vectors, which satisfies

$$\left(\frac{k^2}{2M} + V \right) \phi_\ell = E_\ell \phi_\ell , \tag{4}$$

where ℓ represents momentum, spin, and i-spin indices. Defining

matrix elements of v and V by

$$\langle t|V(g)|\ell\rangle = \int \phi_t^* (\vec{r})\, V(g)\, \phi_\ell (\vec{r})\, d\vec{r}$$

and

$$\langle rs|v(g)|t\ell\rangle = \int \frac{1}{\sqrt{2}}\left[\phi_r^* (\vec{r}_1)\, \phi_s^* (\vec{r}_2) - \phi_r^* (\vec{r}_2)\, \phi_s^* (\vec{r}_1)\right] v_{12}$$

$$\times \frac{1}{\sqrt{2}}\left[\phi_t(\vec{r}_1)\, \phi_\ell(\vec{r}_2) - \phi_t(\vec{r}_2)\, \phi_r(\vec{r}_1)\right] d\vec{r}_1\, d\vec{r}_2 . \qquad (5)$$

we have

$$H_0 = \sum_\ell E_\ell\, d_\ell^*\, d_\ell$$

and

$$H'(g) = \frac{1}{4} \sum_{rst\ell} \langle rs|v(g)|t\ell\rangle\, d_r^*\, d_s^*\, d_\ell\, d_t$$

$$- \sum_{\ell t} \langle t|V(g)|\ell\rangle\, d_t^*\, d_\ell , \qquad (6)$$

with the usual anticommutation relation for the creation and destruction operators d_ℓ^*, d_ℓ. The vacuum state is defined for all ℓ by

$$d_\ell \mid 0)) = 0. \qquad (7)$$

In carrying out a perturbation expansion for a system of fermions it is convenient to shift the unperturbed ground state so that $|0\rangle\rangle$ is the new vacuum and to introduce explicitly excited particles (a^*, a) and holes (b^*, b):

$$a_\ell = \begin{cases} d_\ell, & \text{for} \quad \ell > \ell_0, \\ 0, & \text{for} \quad \ell < \ell_0, \end{cases}$$

$$b_\ell = \begin{cases} 0, & \text{for} \quad \ell > \ell_0, \\ d_\ell^*, & \text{for} \quad \ell < \ell_0, \end{cases} \tag{8}$$

$$a_\ell | \, 0 \, \rangle\rangle = b_\ell | \, 0 \, \rangle\rangle = 0, \quad \text{for all} \quad \ell.$$

Here ℓ_0 denotes the last single-particle state used to fill up the unperturbed ground state $| \, 0 \, \rangle\rangle$. In these notations, the Hamiltonian takes the form

$$H_0 = \sum_{n<\ell_0} E_n + \sum_{k>\ell_0} E_k \, a_k^* \, a_k - \sum_{n<\ell_0} E_n \, b_n^* \, b_n$$

and

$$H'(g) = \frac{1}{2} \sum_{n_1 n_2 < \ell_0} \langle n_1 n_2 | v(g) | n_1 n_2 \rangle - \sum_{n<\ell_0} \langle n | V(g) | n \rangle$$

$$+ \sum_{\ell t} \left[\sum_{n<\ell_0} \langle tn | v(g) | \ell n \rangle - \langle t | V(g) | \ell \rangle \right] (a_t^* a_\ell - b_\ell^* b_t - a_t^* b_\ell^* - a_\ell b_t)$$

$$+ \frac{1}{4} \sum_{t\ell rs} \langle rs | v(g) | t\ell \rangle \Big[a_r^* a_s^* b_\ell^* b_t^* + b_r b_s a_\ell a_t$$

$$+ a_r^* b_t^* b_s a_\ell + a_s^* b_\ell^* b_r a_t$$

$$- a_r^* b_\ell^* b_s a_t - a_s^* b_t^* b_r a_\ell$$

$$+ a_r^* a_s^* a_\ell a_t + b_\ell^* b_t^* b_r b_s$$

$$+ a_r^* a_s^* b_\ell^* a_t + a_r^* b_s a_\ell a_t$$

$$- a_r^* a_s^* b_t^* a_\ell - a_s^* b_r a_\ell a_t$$

$$+ a_r^* b_\ell^* b_t^* b_s + b_\ell^* b_r b_s a_t$$

$$- a_s^* b_\ell^* b_t^* b_r - b_t^* b_r b_s a_\ell \Big] . \qquad (9)$$

According to Goldstone's theory, the ground-state energy is

$$\epsilon_0(g) = \sum_{n<\ell_0} E_n + \sum_{c=0}^{\infty} \langle\langle 0 | H'(g) \left[\frac{1}{\epsilon_0(0) - H_0} H'(g) \right]^c | 00 \rangle\rangle, \qquad (10)$$

where the sum is over all connected diagrams, $(\epsilon_0(0) = \sum_{n<\ell_0} E_n)$. We have worked out this series in terms of $v(g)$ to the fourth-order terms in Appendix A.

B. Expansion in Terms of Brueckner's K-Matrix Elements

We shall expand the series in Appendix A further in terms of Brueckner's K-matrix elements. In other words, we define the projection operator, Q, the denominator, e, and the average single-particle potential, V, in Eq. (3) such that the R matrix approximately equals Brueckner's K matrix:[8]

$$(rs|v(g)|t\ell) = g(rs|R_B(\delta_i)|t\ell) - g \sum_{k_1k_2>P_F} \frac{(rs|v(g)|k_1k_2)(k_1k_2|R_B(\delta_i)|t\ell)}{E_t + E_\ell - E_{k_i} - E_{k_2} - \delta_i}, \tag{11}$$

$$\begin{aligned} \langle \ell|V^I|k\rangle &= \delta(\ell - k) \sum_{n'<P_F} \langle kn'|R_B(\delta)|kn'\rangle, \quad \text{for} \quad k>P_F, \\ \langle \ell|V^I|n\rangle &= \delta(\ell - n) \sum_{n'<P_F} \langle nn'|R_B(0)|nn'\rangle, \quad \text{for} \quad n<P_F, \\ V^{II} &= V^{III} = \cdots = 0, \end{aligned} \tag{12}$$

where

$$(rs|v(g)|t\ell) = \int \phi_r^*(r_1)\, \phi_s^*(r_2)\, v(g)\, \phi_t(r_1)\, \phi_\ell(r_2)\, dr_1\, dr_2 \quad .$$

Consequently, we have

$$\langle rs|v(g)|t\ell\rangle = (rs|v(g)|t\ell) - (rs|v(g)|\ell t).$$

Single-nucleon wave functions are assumed to be plane waves. This assumption is appropriate for an infinite nucleus (i.e., nuclear matter) and is often used in practical evaluations. In this case, the unperturbed ground state is the degenerate Fermi sea, and P_F is the Fermi momentum.

Let us expand, for example, a few terms in Appendix A as a power series of g according to Eq. (11):

$$(1)\quad 1/2 \sum_{n_1 n_2 < P_F} \langle n_1 n_2 | v(g) | n_1 n_2 \rangle = \frac{g}{2} \sum_{n_1 n_2 < P_F} \langle n_1 n_2 | R_B(\delta_1) | n_1 n_2 \rangle$$

$$- \frac{g^2}{2} \sum_{\substack{n_1 n_2 < P_F \\ k_1 k_2 > P_F}} \frac{(n_1 n_2 | R_B(\delta_2) | k_1 k_2)\langle k_1 k_2 | R_B(\delta_1) | n_1 n_2 \rangle}{E_{n_1} + E_{n_2} - E_{k_1} - E_{k_2} - \delta_1}$$

$$+ \frac{g^3}{2} \sum_{\substack{n_1 n_2 < P_F \\ k_1 \cdots k_4 > P_F}} \frac{(n_1 n_2 | R_B(\delta_3) | k_3 k_4)(k_3 k_4 | R_B(\delta_2) | k_1 k_2)}{(E_{k_1} + E_{k_2} - E_{k_3} - E_{k_4} - \delta_2)}$$

$$\times \frac{\langle k_1 k_2 | R_B(\delta_1) | n_1 n_2 \rangle}{(E_{n_1} + E_{n_2} - E_{k_1} - E_{k_2} - \delta_1)}$$

$$- g^4(\ ) + \cdots ,$$

$$(2)\quad 1/4 \sum_{\substack{n_1 n_2 < P_F \\ k_1 k_2 > P_F}} \frac{\langle n_1 n_2 | v(g) | k_1 k_2 \rangle \langle k_1 k_2 | v(g) | n_1 n_2 \rangle}{E_{n_1} + E_{n_2} - E_{k_1} - E_{k_2}}$$

$$= \frac{g^2}{2} \sum_{\substack{n_1 n_2 < P_F \\ k_1 k_2 > P_F}} \frac{(n_1 n_2 | R_B(\delta_4) | k_1 k_2) \langle k_1 k_2 | R_B(\delta_5) | n_1 n_2 \rangle}{E_{n_1} + E_{n_2} - E_{k_1} - E_{k_2}}$$

$$- \frac{g^3}{2} \sum_{\substack{n_1 n_2 < P_F \\ k_1 \cdot\cdot k_4 > P_F}} \frac{(n_1 n_2 | R_B(\delta_6) | k_3 k_4)(k_3 k_4 | R_B(\delta_4) | k_1 k_2)}{(E_{k_1} + E_{k_2} - E_{k_3} - E_{k_4} - \delta_4)}$$

$$\times \frac{\langle k_1 k_2 | R_B(\delta_5) | n_1 n_2 \rangle}{(E_{n_1} + E_{n_2} - E_{k_1} - E_{k_2})} \; .$$

$$- \frac{g^3}{2} \sum_{\substack{n_1 n_2 < P_F \\ k_1 \cdot\cdot k_4 > P_F}} \frac{(n_1 n_2 | R_B(\delta_4) | k_1 k_2)(k_1 k_2 | R_B(\delta_7) | k_3 k_4)}{(E_{n_1} + E_{n_2} - E_{k_1} - E_{k_2})}$$

$$\times \frac{\langle k_3 k_4 | R_B(\delta_5) | n_1 n_2 \rangle}{(E_{n_1} + E_{n_2} - E_{k_1} - E_{k_2} - \delta_5)}$$

$$+ g^4(\quad) - \cdots ,$$

$$(3)\quad 1/8 \sum_{\substack{n_1 \cdots n_3 < P_F \\ k_1 \cdots k_3 > P_F}} \frac{\langle n_3 n_1 | v(g) | k_3 k_1 \rangle \langle k_3 n_2 | v(g) | n_3 k_2 \rangle}{(E_{n_1} + E_{n_3} - E_{k_1} - E_{k_3})}$$

$$\times \frac{\langle k_2 k_1 | v(g) | n_2 n_1 \rangle}{(E_{n_1} + E_{n_2} - E_{k_1} - E_{k_2})}$$

$$= \frac{g^3}{8} \sum_{\substack{n_1 \cdots n_3 < P_F \\ k_1 \cdots k_3 > P_F}} \frac{\langle n_3 n_1 | R_B(\delta_8) | k_3 k_1 \rangle \langle k_3 n_2 | R_B(\delta_9) | n_2 k_2 \rangle}{(E_{n_1} + E_{n_3} - E_{k_1} - E_{k_3})}$$

$$\times \frac{\langle k_2 k_1 | R_B(\delta_{10}) | n_2 n_1 \rangle}{(E_{n_1} + E_{n_2} - E_{k_1} - E_{k_2})}$$

$$- g^4(\quad) + \cdots ,$$

etc., where δ_i records the effect of previous interactions on a particular interaction vertex. For a matrix element $\langle rs | R_B(\delta_i) | t\ell \rangle$ which appears in a given term, we set $\delta_i = E_{k_a} + E_{k_b} + \cdots - E_{n_c} - E_{n_d} - \cdots$, where $k_a, k_b \cdots$ is greater than P_F ; $n_c, n_d \cdots$ is less than P_F; and the sum is over all $k_a, k_b, \cdots n_c, n_d \cdots$ which appear only once in the matrix elements to the right of $\langle rs | R_B(\delta_i) | t\ell \rangle$. Therefore, in the example above, we have $\delta_1 = \delta_5 = \delta_{10} = 0$, $\delta_2 = \delta_4 = E_{k_1} + E_{k_2} - E_{n_1} - E_{n_2}$, $\delta_3 = \delta_6 = \delta_7 = E_{k_3} + E_{k_4} - E_{n_1} - E_{n_2}$, $\delta_8 = E_{k_1} + E_{k_3} - E_{n_1} - E_{n_3}$, and $\delta_9 = E_{k_1} + E_{k_2} - E_{n_1} - E_{n_2}$. In this way the second term of expansion (1) and the first term of expansion (2) cancel each other, as do the third term of expansion (1)

and the second term of expansion (2), etc. As a result, the expansion of the ground-state energy in terms of powers of g no longer contains terms which correspond to diagrams consisting of the so-called ladder parts. The parameter δ in Eq. (12) can be chosen differently for different purposes. Here we want to equate the first-order term in the g expansion of energy $\epsilon_0(g)$ and the potential energy term in Brueckner's model expression for the nuclear ground-state energy. According to Brueckner and Gammel, the off-the-energy-shell propagation is given by

$$E_k(\Sigma) = \frac{k^2}{2M} + \sum_{n'<P_F} \langle n'k|K(\Sigma)|n'k\rangle$$

$$(n'k|K(\Sigma)|n'k) = (n'k|v|n'k) + \sum_{k_1k_2>P_F} \frac{(n'k|v|k_1k_2)(k_1k_2|K(\Sigma)|n'k)}{\Sigma - \delta' - E_{k_1}(\Sigma) - E_{k_2}(\Sigma)} ,$$

where δ' is some average excitation energy ranging from 0 to $E_{P_F} - E_0$, and Σ is taken to be $E_{n_1} + E_{n_2}$ when $E_k(\Sigma)$ is used in calculating the matrix element $(n_1n_2|K|n_1n_2)$ by using the equation,

$$(n_1n_2|K|n_1n_2) = (n_1n_2|v|n_1n_2) + \sum_{k_1k_2>P_F} \frac{(n_1n_2|v|k_1k_2)(k_1k_2|K|n_1n_2)}{E_{n_1} + E_{n_2} - E_{k_1}(\Sigma) - E_{k_2}(\Sigma)} .$$

In order to make

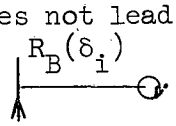
$$\sum_{n_1n_2<P_F} \langle n_1n_2|R_B(0)|n_1n_2\rangle \cong \sum_{n_1n_2<P_F} \langle n_1n_2|K|n_1n_2\rangle ,$$

a reasonable choice for δ should be

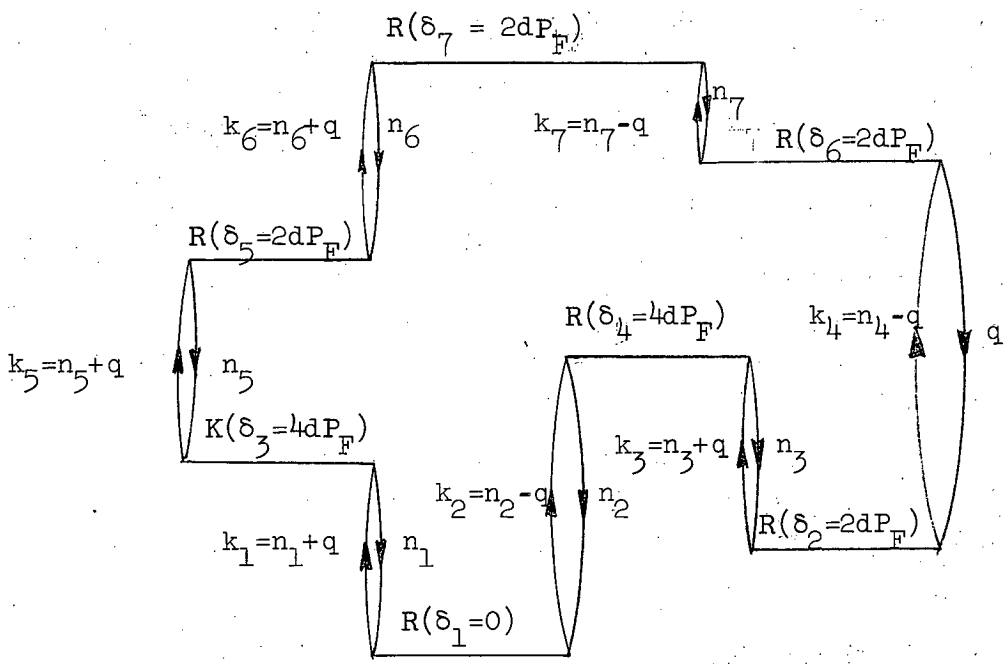
$$\delta = E_k + E_{n'} - \Sigma' + \delta'$$

with

$$\Sigma' = \sum_{n_1 n_2 < P_F} (E_{n_1} + E_{n_2}) \left[\sum_{n_1 n_2 < P_F} \right]^{-1} . \qquad (12')$$

This definition of $\langle k|V|k\rangle$, of course, does not lead to complete cancellation of diagrams with vertices of the form 

It also causes difficulties in our attempt to represent the ring diagrams with $R_B(\delta_i)$ as vertices by a Sawada's Hamiltonian which can have only one R-operator of some kind. To see this, we notice that for ring diagrams, the main contribution comes when the momentum transfer q is small, i.e. when all the particles come from near the Fermi surface. If we write $\lim_{\substack{k \to P_F \\ n \to P_F}} (E_{k_i} - E_{n_i}) = dp_F$, we have, for a typical ring diagram



,

where δ can be any even number of dp_F. From definition (12') with $\delta' = 70$ Mev, $\Sigma' \cong -100$ Mev (Brueckner and Gammel[8]), we expect that dp_F will be $\geqslant 25$ Mev. It seems difficult to establish an average δ among all $\delta_i = 2\ell\, dp_F$, with such a large value for dp_F.

This difficulty will be removed if we have a theory that makes $dp_F = 0$. We can do this by putting $\delta' = 0$ and $\Sigma' = 2E_{n=P_F}$ in definition (12'), or by simply replacing (12') by

$$\delta = E_k - E_{n=P_F} \quad . \tag{12''}$$

Both definitions cause small variation in the first-order energy,[8,9]

$$\sum_{n_1 n_2 < P_F} \langle n_1 n_2 | R_B(0) | n_1 n_2 \rangle = \sum_{n_1 n_2 < P_F} \langle n_1 n_2 | K | n_1 n_2 \rangle - \Delta\epsilon, \tag{13}$$

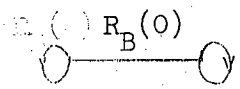
although the matrix element $\langle k | R_B(0) | k, p) = 0$ has a singular point at $k = p_F$ because of the appearance of a zero value in the denominator of Eq. (11). This singularity has a very narrow region, and its effect is negligible. In practice, we may define the matrix element $(k = p_F | R_B(0) | k = p_F, p = 0)$ by the value extrapolated from those computed at, for instance, $k = 0.9\, p_F$ and $k = 1.1\, p_F$ just as Brueckner and his collaborators often did. In this way, the matrix elements of $R_B(0)$ become smooth functions of momenta and can be used as the effective interaction matrix element in studying the effects of pair interaction.

With Eqs. (11), (12), and (12") and the rule to choose δ_i, we obtain then:

$$\epsilon_0(g) = \epsilon_0 + g\epsilon_1 + g^2\epsilon_2 + g^3\epsilon_3 + g^4\epsilon_4 + \cdots ,$$

$$\epsilon_0 = \sum_{n<P_F} \frac{P_n^{\,2}}{2M} ,$$

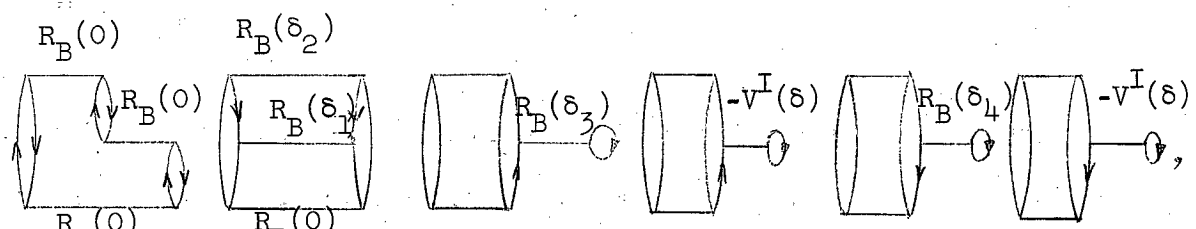
$$\epsilon_1 = \frac{1}{2} \sum_{n_1 n_2 < P_F} \langle n_1 n_2 | R_B(0) | n_1 n_2 \rangle ,$$



,

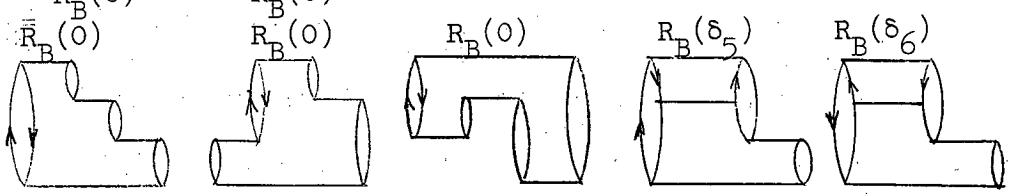
$$\epsilon_2 = 0,$$

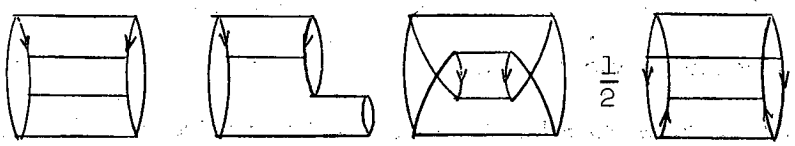
$\epsilon_3 =$



,

$\epsilon_4 =$





$\frac{1}{2}$

$+ \cdots$.

C. Effect of Pair Interaction on the Ground-State Energy

Brueckner's model, i.e.

$$\epsilon_B = \sum_{n<P_F} \frac{P_n^2}{2M} + 1/2 \sum_{n_1 n_2 < P_F} \langle n_2 n_1 | K | n_2 n_1 \rangle, \qquad (15)$$

has been considered successful in describing the nuclear ground state at low densities. Therefore it is expected that the convergence is good in the low-density region. The situation is not so hopeful at high densities. In fact, in the expansion there are terms that contain higher and higher powers of P_F, the parameter that symbolizes the density of the system and that becomes infinity at the high-density limit. Therefore, the convergence of these expansions gets worse with increasing density. Following Hugenholtz's analysis,[10] for any R vertex that is nonsingular in the density region under consideration, the dependence on P_F of a diagram can be estimated analytically at the high-density limit. It can be shown that when $P_F \to \infty$, for each order which is high, the terms that contain the highest power of P_F and so give the main contribution to that order at high density are those which correspond to the ring diagrams with pairs (a particle and a hole) as propagators. If we pick up all the most important terms for each order and then sum them up, we get an expression H_m which gives approximately the ground-state energy of nuclear matter at the high-density limit. In the following, we shall show that the summation $H_m(R_B)$ can be written as a constant term $\epsilon_B - \Delta\epsilon$ and a Sawada's Hamiltonian term $H_s(R_B)$. When the density becomes small, the contribution from $H_s(R_M)$ becomes small, and the summation H_m approaches $\epsilon_B - \Delta\epsilon$, the Brueckner's expression

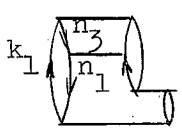
(approximately). Therefore, H_m can be considered to be a good model at both high- and low-density limits. We can also calculate $H_s(K)$ at equilibrium density just to see how large the correction from pair interactions to Brueckner's model will be.

In Eq. (9), among sixteen vertices, only the first six are responsible for ring diagrams. They are:

$$a^*_{\substack{p+q\\ \lambda}}\, a^*_{\substack{p'-q\\ \lambda'}}\, b^*_{\substack{p'\\ \lambda'_0}}\, b^*_{\substack{p\\ \lambda_0}}\,, \quad b_{\substack{p+q\\ \lambda}}\, b_{\substack{p'-q\\ \lambda'}}\, a_{\substack{p'\\ \lambda'_0}}\, a_{\substack{p\\ \lambda_0}}\,,$$

$$a^*_{\substack{p+q\\ \lambda}}\, b^*_{\substack{p\\ \lambda_0}}\, b_{\substack{p'-q\\ \lambda'}}\, a_{\substack{p'\\ \lambda'_0}}\,, \quad a^*_{\substack{p'-q\\ \lambda'}}\, b^*_{\substack{p'\\ \lambda'_0}}\, b_{\substack{p+q\\ \lambda}}\, a_{\substack{p\\ \lambda_0}}\,,$$

$$-a^*_{\substack{p+q\\ \lambda}}\, b^*_{\substack{p'\\ \lambda'_0}}\, b_{\substack{p'-q\\ \lambda'}}\, a_{\substack{p\\ \lambda_0}}\,, \quad -a^*_{\substack{p'-q\\ \lambda'}}\, b^*_{\substack{p\\ \lambda_0}}\, b_{\substack{p+q\\ \lambda}}\, a_{\substack{p'\\ \lambda'_0}}\,, \tag{16}$$

where λ's are the spin and i-spin indices, and conservation of momentum has been represented explicitly. To exclude diagrams of the type 

(which is also from a Hamiltonian with the six terms above; but in which, on the other hand, the pair (k_1, n_1) is broken by the third interaction line), we introduce the following approximate commutation relations, which guarantee that a particle and a hole will be always associated with one another:

$$\left[b_{\substack{p\\ \lambda_0}}\, a_{\substack{p+q\\ \lambda}};\ b_{\substack{p'\\ \lambda'_0}}\, a_{\substack{p'+q'\\ \lambda'}} \right]_- = \left[a^*_{\substack{p+q\\ \lambda}}\, b^*_{\substack{p\\ \lambda_0}};\ a^*_{\substack{p'+q'\\ \lambda'}}\, b^*_{\substack{p'\\ \lambda'_0}} \right]_- = 0,$$

$$\left[b_{\substack{p\\ \lambda_0}}\, a_{\substack{p+q\\ \lambda}};\ a^*_{\substack{p'+q'\\ \lambda'}}\, b^*_{\substack{p'\\ \lambda'_0}} \right]_- = \delta_{pp'}\, \delta_{qq'}\, \delta_{\lambda\lambda'}\, \delta_{\lambda_0\lambda'_0}\,. \tag{17}$$

Let

$$C^*_{\substack{p,q\\ \lambda\lambda_0}} = a^*_{\substack{p+q\\ \lambda}} b^*_{\substack{p\\ \lambda_0}} , \qquad C_{\substack{p,q\\ \lambda\lambda_0}} = b_{\substack{p\\ \lambda_0}} a_{\substack{p+q\\ \lambda}} . \tag{18}$$

Then it follows that

$$C_{\substack{p,q\\ \lambda\lambda_0}} = C^*_{\substack{p,q\\ \lambda\lambda_0}} = 0 \text{ for } |\vec{p}| > P_F \text{ or } |\vec{p}+\vec{q}| < P_F$$

and

$$C_{\substack{p,q\\ \lambda\lambda_0}} | 0 \rangle\rangle = 0 \qquad \text{for all } \vec{p}, \vec{q} .$$

Therefore, we obtain

$$\left[C_{\substack{p,q\\ \lambda\lambda_0}} ; C_{\substack{p',q'\\ \lambda'\lambda'_0}}\right]_- = \left[C^*_{\substack{p,q\\ \lambda\lambda_0}} ; C^*_{\substack{p',q'\\ \lambda'\lambda'_0}}\right]_- = 0,$$

$$\left[C_{\substack{p,q\\ \lambda\lambda_0}} ; C^*_{\substack{p',q'\\ \lambda'\lambda'_0}}\right]_- = \delta_{pp'} \delta_{qq'} \delta_{\lambda\lambda'} \delta_{\lambda_0\lambda'_0} . \tag{19}$$

Form the following interaction from the six terms above:[11]

$$H_{int}(R) = \frac{1}{4} \sum_{\substack{\lambda\lambda_0 \\ \lambda'\lambda'_0}} \sum_{pp'q} \left\langle \begin{matrix} \vec{p}+\vec{q} \\ \lambda \end{matrix} \begin{matrix} \vec{p}'-\vec{q} \\ \lambda' \end{matrix} \middle| R \middle| \begin{matrix} \vec{p} \\ \lambda_0 \end{matrix} \begin{matrix} \vec{p}' \\ \lambda'_0 \end{matrix} \right\rangle$$

$$\times \Bigg\{ C^*_{\substack{p,q \\ \lambda\lambda_0}} C^*_{\substack{p',-q \\ \lambda'\lambda'_0}} + C_{\substack{p+q,-q \\ \lambda_0\lambda}} C_{\substack{p'-q,q \\ \lambda'_0\lambda'}}$$

$$+ C^*_{\substack{p,q \\ \lambda\lambda_0}} C_{\substack{p'-q,q \\ \lambda'_0\lambda'}} + C^*_{\substack{p',-q \\ \lambda'\lambda'_0}} C_{\substack{p+q,-q \\ \lambda_0\lambda}}$$

$$-C^*_{\substack{p,p'-p-q \\ \lambda'\lambda_0}} C_{\substack{p+q,p'-p-q \\ \lambda'_0\lambda}} - C^*_{\substack{p',p-p'+q \\ \lambda\lambda'_0}} C_{\substack{p'-q,p-p'+q \\ \lambda_0\lambda'}}$$

$$- C^*_{\substack{p',p-p'+q \\ \lambda\lambda'_0}} C^*_{\substack{p,p'-p-q \\ \lambda'\lambda_0}} - C_{\substack{p'-q,p-p'+q \\ \lambda_0\lambda'}} C_{\substack{p+q,p'-p-q \\ \lambda'_0\lambda}} \Bigg\} .$$

Let $\vec{q}' = \vec{p}' - \vec{p} - \vec{q}$ be the momentum transfer of the exchange process. Then we can write

$$H_{int}(R) = \frac{1}{4} \sum_{\substack{\lambda\,\lambda_0 \\ \lambda'\lambda'_0}} \sum_{pp'q} \left\langle {}^{\vec{p}+\vec{q}}_{\lambda}\, {}^{\vec{p}'-\vec{q}}_{\lambda'} \middle| R \middle| {}^{\vec{p}}_{\lambda_0}\, {}^{\vec{p}'}_{\lambda'_0} \right\rangle$$

$$\times \Bigg\{ C^*_{\substack{p,q \\ \lambda\,\lambda_0}} C^*_{\substack{p',-q \\ \lambda'\,\lambda'_0}} - C^*_{\substack{p,+q' \\ \lambda'\,\lambda_0}} C^*_{\substack{p' \\ \lambda}}{}_{,-q' \atop \lambda'_0}$$

$$+ C^*_{\substack{p',-q \\ \lambda'\,\lambda'_0}} C_{\substack{p+q,-q \\ \lambda_0\,\lambda}} - C^*_{\substack{p',-q' \\ \lambda\,\lambda'_0}} C_{\substack{p+q',-q' \\ \lambda_0\,\lambda'}}$$

$$+ C^*_{\substack{p,q \\ \lambda\,\lambda_0}} C_{\substack{p'-q,q \\ \lambda'_0\,\lambda'}} - C^*_{\substack{p,q' \\ \lambda'\lambda_0}} C_{\substack{p'-q',q' \\ \lambda'_0\,\lambda}}$$

$$+ C_{\substack{p+q,-q \\ \lambda_0\,\lambda}} C_{\substack{p'-q,q \\ \lambda'_0\,\lambda'}} - C_{\substack{p+q',-q' \\ \lambda_0\,\lambda'}} C_{\substack{p'-q',q' \\ \lambda'_0\,\lambda}} \Bigg\} .$$

In H_{int}, all terms in the second column are identical with those in the first except for $\vec{q} \to \vec{q}'$. If $\vec{q}'$ is replaced by $\vec{q}$ and $\vec{q}$ by $\vec{p}' - \vec{p} - \vec{q}$ in the second column, then we have

$$H_{int}(R) = \frac{1}{2} \sum_{\substack{\lambda\,\lambda_0 \\ \lambda'\lambda'_0}} \sum_{pp'q} \left\langle {}^{\vec{p}+\vec{q}}_{\lambda}\, {}^{\vec{p}'-\vec{q}}_{\lambda'} \middle| R \middle| {}^{\vec{p}}_{\lambda_0}\, {}^{\vec{p}'}_{\lambda'_0} \right\rangle$$

$$\times \Bigg\{ C^*_{\substack{p,q \\ \lambda\,\lambda_0}} C^*_{\substack{p',-q \\ \lambda'\,\lambda'_0}} + C^*_{\substack{p',-q \\ \lambda'\,\lambda'_0}} C_{\substack{p+q,-q \\ \lambda_0\,\lambda}}$$

$$+ C^*_{\substack{p,q \\ \lambda\,\lambda_0}} C_{\substack{p'-q,q \\ \lambda'_0\,\lambda'}} + C_{\substack{p+q,-q \\ \lambda_0\,\lambda}} C_{\substack{p'-q,q \\ \lambda'_0\,\lambda'}} \Bigg\} . \qquad (20)$$

This interaction and the kinetic energy,

$$H_{kin}\left(\sum_{\lambda} E_0\right) = \sum_{\substack{\lambda \\ \lambda_0}} \sum_{q} \sum_{p} \left(E_{\substack{p+q \\ \lambda}} - E_{\substack{p \\ \lambda_0}} \right) C^*_{\substack{p,q \\ \lambda\,\lambda_0}} \; C_{\substack{p,q \\ \lambda\,\lambda_0}} \,, \tag{21}$$

make the sum $H_{kin} + g\, H_{int}$ a Sawada Hamiltonian; we call it $H_s(R)$. If we apply Goldstone's perturbation theory on $H_s(R)$ with commutation relation Eq. (19) between the operator C, C^*, and on

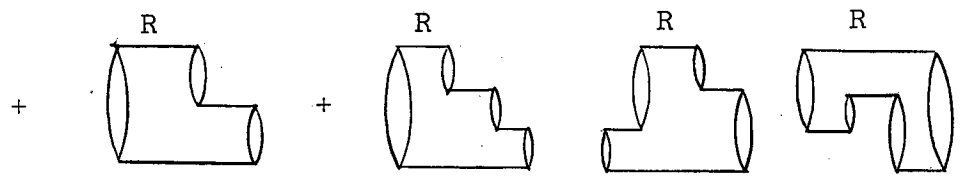
$$H'_s = \sum_{\lambda} \sum_{P>P_F} E_p \, a^+_{\substack{p \\ \lambda}} \, a_{\substack{p \\ \lambda}} - \sum_{\lambda} \sum_{P<P_F} E_p \, a^+_{\substack{p \\ \lambda}} \, b_{\substack{p \\ \lambda}}$$

$$+ \frac{1}{4} \sum_{\substack{\lambda\,\lambda_0 \\ \lambda'\lambda'_0}} \sum_{qpp'} \left\langle \begin{matrix} \vec{p}+\vec{q} \\ \lambda \end{matrix} \; \begin{matrix} \vec{p}'-\vec{q} \\ \lambda' \end{matrix} \middle| R \middle| \begin{matrix} \vec{p} \\ \lambda_0 \end{matrix} \; \begin{matrix} \vec{p}' \\ \lambda'_0 \end{matrix} \right\rangle$$

$$\times \left\{ \begin{array}{l} a^*_{\substack{p+q \\ \lambda}} \, a^*_{\substack{p'-q \\ \lambda'}} \, b^*_{\substack{p' \\ \lambda'_0}} \, b^*_{\substack{p \\ \lambda_0}} + b_{\substack{p+q \\ \lambda}} \, b_{\substack{p'-q \\ \lambda'}} \, a_{\substack{p' \\ \lambda'_0}} \, a_{\substack{p \\ \lambda_0}} \\ \\ a^*_{\substack{p+q \\ \lambda}} \, b^*_{\substack{p \\ \lambda_0}} \, b_{\substack{p'-q \\ \lambda'}} \, a_{\substack{p' \\ \lambda'_0}} + a^*_{\substack{p'-q \\ \lambda'}} \, b^*_{\substack{p' \\ \lambda'_0}} \, b_{\substack{p+q \\ \lambda}} \, a_{\substack{p \\ \lambda_0}} \\ \\ - a^*_{\substack{p+q \\ \lambda}} \, b^*_{\substack{p' \\ \lambda'_0}} \, b_{\substack{p'-q \\ \lambda'}} \, a_{\substack{p \\ \lambda_0}} - a^*_{\substack{p'-q \\ \lambda'}} \, b^*_{\substack{p \\ \lambda_0}} \, b_{\substack{p+q \\ \lambda}} \, a_{\substack{p' \\ \lambda'_0}} \end{array} \right\} \tag{22}$$

with the ordinary anticommutation relation between a's and b's, we get:

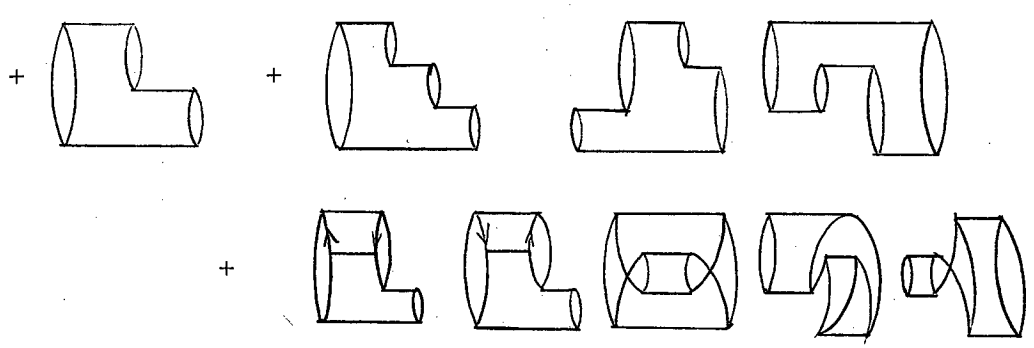
$$\epsilon_{H_s} = \frac{1}{2} \sum_{\substack{pp' < P_F \\ p''p''' > P_F}} \sum_{\lambda's} \frac{|\langle \vec{p}\vec{p}' | R | \vec{p}''\vec{p}''' \rangle|^2}{E_p + E_{p'} - E_{p''} - E_{p'''}}, \quad \vec{p}_1 + \vec{p}' = \vec{p}'' + \vec{p}'''$$



+ fifth-order terms + ···

$$\epsilon_{H'_s} = \frac{1}{4} \sum_{\substack{pp' < P_F \\ p''p''' > P_F}} \sum_{\lambda's} \frac{|\langle \vec{p}\vec{p}' | R | \vec{p}''\vec{p}''' \rangle|^2}{E_p + E_{p'} - E_{p''} - E_{p'''}},$$

$$\vec{p}_1 + \vec{p}' = \vec{p}'' + \vec{p}'''$$



+ fifth-order term + ···.

(23)

Notice that as far as the ring diagrams are concerned, we have

$$\epsilon_{H_s} = (\epsilon_{H'_s})_{\text{ring}} + (\epsilon_{H'_s})_{\text{second-order term}}. \tag{24}$$

Therefore, we can directly use H_s instead of H'_s, with the correction about the second-order term, which is more original but more complicated to evaluate.

Now we are ready to write the model Hamiltonian $H_m(R_B)$. From Eqs. (13), (14), (15), and (23), we have

$$H_m(R_B) = \epsilon_B - \Delta\epsilon + H_s(R_B(0)) - E_2(H_s), \tag{25}$$

where

$$\epsilon_B - \Delta\epsilon = \sum_{\lambda} \sum_{p<P_F} \frac{P^2}{2M} + 1/2 \sum_{\lambda\lambda'} \sum_{pp'<P_F} \left\langle \begin{matrix} p & p' \\ \lambda & \lambda' \end{matrix} \middle| R_B(0) \middle| \begin{matrix} p & p' \\ \lambda & \lambda' \end{matrix} \right\rangle,$$

$$E_2(H_s) = 1/2 \sum_{\substack{pp'<P_F \\ p''p'''>P_F \\ p+p'=p''+p'''}} \frac{\left| \left\langle \begin{matrix} pp' \\ \lambda\lambda' \end{matrix} \middle| R_B(0) \middle| \begin{matrix} p''p''' \\ \lambda''\lambda''' \end{matrix} \right\rangle \right|^2}{E_{\substack{p \\ \lambda}} + E_{\substack{p' \\ \lambda'}} - E_{\substack{p'' \\ \lambda''}} - E_{\substack{p''' \\ \lambda'''}}},$$

and $H_s(R)$ is given by Eqs. (20) and (21).

D. Energy of Pair Excitation in Terms of Galitskii's Reaction Matrix

We can follow a similar procedure for the collective excited states. It is known that a study on the pair excitations can give information about the collective states. Let us assume that $|\phi_{p,g}\rangle\rangle$ and $\epsilon_{p,q}$ are the eigenfunction and eigenvalue of an unperturbed pair state, respectively, and $|\psi(g)\rangle\rangle$ is the eigenfunction of $H(g) = H_0 + H'(g)$ with eigen value $\epsilon(g)$ (spin and i-spin dependences are neglected in the following discussion). Thus we have

$$|\phi_{p,g}\rangle\rangle = a^*_{p+q}\, b^*_p\, |0\rangle\rangle$$

and

$$\epsilon_{p,q} \pm \sum_{n<P_F} \frac{P_n^2}{2M} + E_{p+q} - E_p\,, \tag{26}$$

where the other notations have the meanings defined earlier. If P is the projection operator off the state $|\phi_{p,q}\rangle\rangle$, the Rayleigh-Schrödinger perturbation expansion can be obtained by iterating the equation

$$|\psi(g)\rangle\rangle = |\phi_{p,q}\rangle\rangle + \frac{P}{E_{p,q} - H_0}\left[H'(g) - E(g) + E_{p,q}\right]|\psi(g)\rangle\rangle. \tag{27}$$

Thus, for the energy we have

$$\epsilon(g) = \epsilon_{p,q} + \langle\langle\phi_{p,q}|H'(g)|\psi(g)\rangle\rangle. \tag{28}$$

The solution for $\epsilon(g)$ can be finally written as a power series in g, (i.e., a power series in the nuclear-reaction matrix R) by using

Eq. (3) again with particular choices for Q, e, and V.

Suppose that the projection operator used by Galitskii in his calculation of effective single-particle energy is adopted for Q:[12]

$$(rs|v(g)|m\ell) = g(rs|R_G|m\ell) - g \sum_{k_1k_2>P_F} \frac{(rs|v(g)|k_1k_2)(k_1k_2|R_G|m\ell)}{E'_m + E'_\ell - E'_k - E'_k}$$

$$+ g \sum_{n_1n_2<P_F} \frac{(rs|v(g)|k_1k_2)(k_1k_2|R_G|m\ell)}{E'_m + E'_\ell - E'_n - E'_n}, \tag{29}$$

where Σ' denotes the omission of the terms when the denominator becomes zero, and E'_m is the single-particle energy defined by proper choice of V. Then the second-order term becomes

$$g^2 \sum_{\substack{n<P_F \\ k>P_F}} \frac{\langle p\ p+q|R_G|kn\rangle\langle kn|R_G|p\ p+q\rangle}{E'_{p+q} + E'_p - E'_k - E'_n},$$

which is exactly the result one would get if we apply the expansion process on $H'_s(R_G)$, the Sawada's Hamiltonian with R_G as its vertex operator. This is because terms such as and

are cancelled exactly by using Galitskii's projection operator. For higher terms, the situation becomes more complicated. However, it seems safe to claim that the Galitskii's reaction matrix is the first-order choice for studying pair excitations just as the Brueckner's reaction matrix is for studying ground-state energy.

We shall leave E'_m undefined until Section IV, where we give an approximate solution for both $R_B(0)$ and R_G. Then $R_B(0)$ will be used in calculating $\langle\langle\psi_0|H_s(R_B)|\psi_0\rangle\rangle$, and R_G will be used in estimating the excitation energy of nuclear collective excited states by solving the Sawada's Hamiltonian $H_s(R_G)$ for the excited states.

III. SOLUTION OF H_s AND STABILITY CONDITIONS

A. Eigenvalue Equation for H_s

The commutation relations of Eq. (19) for the pair variables C, C^* lead to the following commutator for $H_s(R)$:

$$\left[H_s, C^*_{\substack{p,q\\ \lambda\\ \lambda_0}}\right] = \left(E_{\substack{p+q\\ \lambda}} - E_{\substack{p\\ \lambda_0}}\right) C^*_{\substack{p,q\\ \lambda\\ \lambda_0}}$$

$$+ \sum_{\substack{\lambda'\\ \lambda'_0}} \sum_{p'} \left\langle \begin{matrix}\vec{p}\\ \lambda_0\end{matrix} \begin{matrix}\vec{p}'+\vec{q}\\ \lambda'\end{matrix} \middle| R \middle| \begin{matrix}\vec{p}+\vec{q}\\ \lambda\end{matrix} \begin{matrix}\vec{p}'\\ \lambda'_0\end{matrix} \right\rangle \left(C^*_{\substack{p',q\\ \lambda'\\ \lambda'_0}} + C_{\substack{p'+q,-q\\ \lambda'_0\\ \lambda'}} \right)$$

$$\left[H_s, C_{\substack{p+q,-q\\ \lambda_0\\ \lambda}}\right] = \left(E_{\substack{p+q\\ \lambda}} - E_{\substack{p\\ \lambda_0}}\right) C_{\substack{p+q,-q\\ \lambda_0\\ \lambda}}$$

$$- \sum_{\substack{\lambda'\\ \lambda'_0}} \sum_{p'} \left\langle \begin{matrix}\vec{p}\\ \lambda_0\end{matrix} \begin{matrix}\vec{p}'+\vec{q}\\ \lambda'\end{matrix} \middle| R \middle| \begin{matrix}\vec{p}+\vec{q}\\ \lambda\end{matrix} \begin{matrix}\vec{p}'\\ \lambda'_0\end{matrix} \right\rangle \left(C^*_{\substack{p',q\\ \lambda'\\ \lambda'_0}} + C_{\substack{p'+q,-q\\ \lambda'_0\\ \lambda}} \right) \tag{27}$$

These commutators are linear in pair variables C, C^*, and therefore we can find a set of new variables:

$$A^*_{qn} = \sum_{\substack{|\vec{p}|<P_F,\lambda \\ |\vec{p}+\vec{q}|>P_F,\lambda_0}} G^{(+)}_{\substack{pqn \\ \lambda \\ \lambda_0}} C^*_{\substack{p,q \\ \lambda \\ \lambda_0}} - \sum_{\substack{|\vec{p}|>P_F,\lambda \\ |\vec{p}+\vec{q}|<P_F,\lambda_0}} G^{(-)}_{\substack{pqn \\ \lambda \\ \lambda_0}} C_{\substack{p+q,-q \\ \lambda \\ \lambda_0}} , \tag{28}$$

such that

$$\left[H_s, A^*_{qn}\right] = \Delta_{qn} A^*_{qn} . \tag{29}$$

In Eq. (28) the numerical coefficients $G^{(+)}$ and $G^{(-)}$ are so defined that $G^{(+)}_{p,q}$ is zero for $|p|>0$, or $|p+q| < 0$, and $G^{(-)}_{p,q}$ is zero for $|p|< 0$ or $|p+q|> 0$. From Eqs. (27), (28), and (29) we may easily show that

$$\left(L_{\substack{p,q \\ \lambda \\ \lambda_0}} = E_{\substack{p+q \\ \lambda}} - E_{\substack{p \\ \lambda_0}} \right)$$

$$\left(\Delta_{qn} - L_{\substack{p,q \\ \lambda \\ \lambda_0}} \right) G^{(\pm)}_{\substack{pqn \\ \lambda \\ \lambda_0}} = \pm \sum_{\substack{\vec{p}'\lambda' \\ \lambda'_0}} \left\langle \begin{matrix} \vec{p}' & \vec{p}+\vec{q} \\ \lambda'_0 & \lambda \end{matrix} \middle| R \middle| \begin{matrix} \vec{p}'+\vec{q} & \vec{p} \\ \lambda' & \lambda_0 \end{matrix} \right\rangle \left(G^{(+)}_{\substack{p'qn \\ \lambda' \\ \lambda'_0}} + G^{(-)}_{\substack{p'qn \\ \lambda' \\ \lambda'_0}} \right) . \tag{30}$$

Equation (29), when applied on the true ground state $|\psi_0\rangle\rangle$ of H_s, can be interpreted as the eigenvalue equation for all the excited states $|\psi_{qn}\rangle$ of H_s if we designate

$$|\psi_{qn}\rangle\rangle = A^*_{qn} |\psi_0\rangle\rangle \tag{31}$$

and Δ_{qn} as the energy of excitation from $|\psi_0\rangle\rangle$ to $|\psi_{qn}\rangle\rangle$. This designation imposes the requirement that the set Eq. (28) is a complete orthonormal set.

Before we proceed to solve Eq. (30), let us calculate the commutators

$$\left[H'_s,\ a^*_{\substack{p+q\\ \lambda}}\ b^*_{\substack{p\\ \lambda_0}}\right] \quad \text{and} \quad \left[H'_s,\ b_{\substack{p+q\\ \lambda}}\ a_{\substack{p\\ \lambda_0}}\right] .$$

Using the ordinary anticommutation relation for the a's and b's we get

$$\left[H'_s,\ a^*_{\substack{p+q\\ \lambda}}\ b^*_{\substack{p\\ \lambda_0}}\right] = \left(E_{\substack{p+q\\ \lambda}} - E_{\substack{p\\ \lambda_0}}\right) a^*_{p+q}\ b^*_p$$

$$+ \sum_{\substack{\lambda'\\ \lambda'_0}} \sum_{\vec{p}'} \left\langle \begin{matrix}\vec{p}\ \vec{p}'+\vec{q}\\ \lambda_0\ \lambda'\end{matrix} \middle| R \middle| \begin{matrix}\vec{p}+\vec{q}\ \vec{p}'\\ \lambda\ \ \lambda'_0\end{matrix} \right\rangle \left(a^*_{\substack{p'+q\\ \lambda'}}\ b^*_{\substack{p'\\ \lambda'_0}} + b_{\substack{p'+q\\ \lambda'}}\ a_{\substack{p'\\ \lambda'_0}}\right)$$

$$+ \text{ terms containing } a^*\ b^*\ b^*\ b, \text{ etc.}$$

and

$$\left[H'_s,\ b_{\substack{p+q\\ \lambda}}\ a_{\substack{p\\ \lambda_0}}\right] = \left(E_{\substack{p+q\\ \lambda}} - E_{\substack{p\\ \lambda_0}}\right) b_{\substack{p+q\\ \lambda}}\ a_{\substack{p\\ \lambda_0}}$$

$$+ \sum_{\substack{\lambda'\\ \lambda'_0}} \sum_{p'} \left\langle \begin{matrix}\vec{p}\ \vec{p}'+\vec{q}\\ \lambda_0\ \lambda'\end{matrix} \middle| R \middle| \begin{matrix}\vec{p}+\vec{q}\ \vec{p}'\\ \lambda\ \ \lambda'_0\end{matrix} \right\rangle \left(a^*_{\substack{p'+q\\ \lambda'}}\ b^*_{\substack{p'\\ \lambda'_0}} + b_{\substack{p'+q\\ \lambda'}}\ a_{\substack{p'\\ \lambda'_0}}\right)$$

$$+ \text{ terms containing } a^*\ a\ b\ a, \text{ etc.} \qquad (32)$$

If we neglect all but the pair terms, these commutators are exactly equal to those in Eq. (27). From this, we can conclude that it is quite safe to simply add the exchange matrix element

$$-\left(\begin{matrix}\vec{p}'+\vec{q} & \vec{p} \\ \lambda' & \lambda_0\end{matrix}\middle| R \middle|\begin{matrix}\vec{p}+\vec{q} & \vec{p}' \\ \lambda & \lambda'_0\end{matrix}\right)$$

to each direct matrix element

$$\left(\begin{matrix}\vec{p} & \vec{p}'+\vec{q} \\ \lambda_0 & \lambda'\end{matrix}\middle| R \middle|\begin{matrix}\vec{p}+\vec{q} & \vec{p}' \\ \lambda & \lambda'_0\end{matrix}\right)$$

in applying Sawada's Hamiltonian to a system for which the effect of exchange terms is not negligible.

B. A Simple Form of R Matrix

Let us assume that the R matrix elements are simple continuous functions of the momenta involved. Then if we use symmetry conditions (rotation and reflection symmetry), we can write the lowest-order term of Taylor's expression of an R matrix element in the following form:

$$(\vec{p}+\vec{q}\,\vec{p}-\vec{q}\,|R|\vec{p}\,\vec{p}') = V_0\left[1 + a_0\vec{p}^{\,2} + b_0\vec{p}^{\,2} + c_0\,\vec{p}\cdot\vec{p}' + d_0\vec{q}^{\,2} + e_0\vec{p}\cdot\vec{q} + f_0\vec{p}'\cdot\vec{q}\right].$$

Further conditions such as

$$(\vec{p}+\vec{q}\,\vec{p}'-\vec{q}\,|R|\,\vec{p}\,\vec{p}') = (\vec{p}'-q\,\vec{p}+\vec{q}\,|R|\,\vec{p}'\,\vec{p})$$

and

$$(\vec{p}+\vec{q}\,\vec{p}'-\vec{q}\,|R|\,\vec{p}\,\vec{p}') = (\vec{p}\,\vec{p}'\,|R|\,\vec{p}+\vec{q}\,\vec{p}'-\vec{q})$$

require that

$$b_0 = a_0,$$

$$-f_0 = e_0 = 2a_0 - c_0 - e_0, \quad \text{so that} \quad -f_0 = e_0 = \frac{1}{2}(2a_0 - c_0),$$

and

$$d_0 = a_0 + b_0 - c_0 + d_0 - e_0 + f_0, \text{ so that } d_0 = d_0 .$$

Then we have

$$(\vec{p} + \vec{q}\ \vec{p}' - \vec{q}\ |R|\ \vec{p}\ \vec{p}')$$

$$= V_0 \left[1 + a_0 \left(\vec{p} + \frac{\vec{q}}{2}\right)^2 + a_0 \left(\vec{p}' - \frac{\vec{q}}{2}\right)^2 + c_0 \left(\vec{p} + \frac{\vec{q}}{2}\right)\cdot\left(\vec{p}' - \frac{\vec{q}}{2}\right)\right.$$

$$\left. + \left(d_0 - \frac{2a_0 - c_0}{4}\right) \vec{q}^{\,2}\right].$$

Accordingly, the exchange element is

$$(\vec{p}' - q\ \vec{p} + \vec{q}\ |R|\ \vec{p}\ \vec{p}') = V_0 \left[1 + \frac{2d_0 + c_0}{2} \left(\vec{p} + \frac{\vec{q}}{2}\right)^2\right.$$

$$\left. + \frac{2d_0 + c_0}{2} \left(\vec{p}' - \frac{\vec{q}}{2}\right)^2 \quad 2d_0 - 2a_0 \left(\vec{p} + \frac{\vec{q}}{2}\right)\cdot\left(\vec{p}' - \frac{\vec{q}}{2}\right) + \frac{2a_0 - c_0}{4} \vec{q}^{\,2}\right],$$

which depends on p, p', and q exactly the same way as the original matrix element does provided we assume $d_0 = a_0 - c_0/2$. The assumption $d_0 = a_0 - c_0/2$ restricts the dependence of $\langle p'_2\ p'_1\ |R|\ p_2\ p_1\rangle$ on momenta to a form $(p'_2\ p'_1\ |R|\ p_2\ p_1) = R\left[f(\vec{p}'_2, \vec{p}'_1), f_2(\vec{p}_2, \vec{p}_1)\right]$. To simplify the resulting analysis, we shall make this assumption, which is not a severe restriction on the form of R, since it is satisfied by all R matrix elements so far used in practical calculations.[13]

Under this assumption, we have

$$(\vec{p} + \vec{q}\,\vec{p}' - q\,|R|\,\vec{p}\,\vec{p}') = (\vec{p}' - q\,\vec{p} + \vec{q}\,|R|\,\vec{p}\,\vec{p}')$$

$$= V_0 \left[1 + a_0 (\vec{p} + \frac{\vec{q}}{2})^2 + a_0 (\vec{p}' + \frac{\vec{q}}{2})^2 + c_0 (\vec{p} + \frac{\vec{q}}{2}) \cdot (\vec{p} - \frac{\vec{q}}{2}) + \frac{2a_0 - c_0}{4} \vec{q}^{\,2} \right]$$

$$= V_0 \left(1 - \frac{c_0 - 2a_0}{4} q^2\right) 1 + \left[\frac{a_0 (\vec{p} + \frac{\vec{q}}{2})^2 + a_0 (\vec{p}' - \frac{\vec{q}}{2})^2 + c_0 (\vec{p} + \frac{\vec{q}}{2}) \cdot (\vec{p}' - \frac{\vec{q}}{2})}{1 - \frac{c_0 - 2a_0}{4} q^2} \right].$$

This expression, of course, is useful for small q only. For large q, we have to add to the above expression terms in higher orders of q, with more parameters. However, the above expression is in such form that a factor that gives the general features of the large q effect of nucleon scatterings can be easily introduced. It is done by replacing the factor

$$V_0 \left[1 - \frac{c_0 - 2a_0}{4} q^2 \right] \text{ by } \quad V_0(q) = \frac{V_0}{1 + \frac{c_0 - 2a_0}{4} q^2} \quad \text{or}$$

$V_0(q) = V_0 \exp\left(- \frac{c_0 - 2a_0}{4} q^2\right)$, or by other similar functions of q. Correspondingly, the factor $\left(1 - \frac{c_0 - 2a_0}{4} q^2\right)$ in the denominator is also to be modified so as to make the over-all matrix element a smooth function of q. For example, we replace

$$\frac{a_0}{1 - \frac{c_0 - 2a_0}{4} q^2}, \quad \frac{c_0}{1 - \frac{c_0 - 2a_0}{4} q^2}$$ by $a_0(q)$, $c_0(q)$, respectively,

where $a_0(q) = a_0$, $c_0(q) = c_0$ or $a_0(q) = \dfrac{a_0}{1 + \frac{c_0 - 2a_0}{4} q^2}$,

$c_0(q) = \dfrac{c_0}{1 + \frac{c_0 - 2a_0}{4} q^2}$, etc. A particular choice of $a_0(q)$, $c_0(q)$

and $V_0(q)$ means a particular way of fixing the parameters of high-order q terms. By means of those arguments, we therefore take

$$\left(\begin{matrix} \vec{p}' & \vec{p}+\vec{q} \\ \lambda'_0 & \lambda \end{matrix}\right| R \left|\begin{matrix} \vec{p}'+\vec{q} & \vec{p} \\ \lambda' & \lambda_0 \end{matrix}\right) = \left[\epsilon_1 + \epsilon_2\, \sigma'\cdot\sigma + \epsilon_3\, \tau'\cdot\tau + \epsilon_4\, \sigma'\cdot\sigma\, \tau'\cdot\tau\right]_{\lambda'_0\lambda', \lambda\lambda_0}$$

$$\times\, (\vec{p}'\, \vec{p} + \vec{q}\, |R|\, \vec{p}' + \vec{q}\, \vec{p})$$

$$\left(\begin{matrix} p+q & p' \\ \lambda & \lambda'_0 \end{matrix}\right| R \left|\begin{matrix} p'+q & p \\ \lambda' & \lambda_0 \end{matrix}\right) = \left[\epsilon_1 + \epsilon_2\, \sigma'\cdot\sigma + \epsilon_3\, \tau'\cdot\tau + \epsilon_4\, \sigma'\cdot\sigma\, \tau'\cdot\tau\right]_{\lambda\lambda', \lambda'_0\lambda_0}$$

$$\times\, (\vec{p}'\, \vec{p} + \vec{q}\, |R|\, p' + \vec{q}\, \vec{p}), \tag{33}$$

where

$$(\vec{p}'\, \vec{p}+\vec{q}\, |R|\, \vec{p}'+q\, \vec{p}) = V_0(q) \left[1 + a_0(q)\, (\vec{p} + \tfrac{q}{2})^2 + a_0(q)\, (\vec{p}' + \tfrac{q}{2})^2\right.$$

$$\left. + c_0(q)\, (p + \tfrac{q}{2})\cdot(\vec{p}' + \tfrac{q}{2})\right]. \tag{34'}$$

This is equally well written as

$$(\vec{p}' \cdot \vec{p} + \vec{q}\ |R|\ \vec{p}' + \vec{q}\ \vec{p}) = V(q)\left[1 + c\ \vec{p} \cdot \vec{p}\right], \tag{34}$$

with

$$V(q) = V_0(q)\ (1 + 2a_0(q)\ P_F^{\ 2})$$

and

$$c = \frac{c_0(q)}{1 + 2a_0(q)\ P_F^{\ 2}}$$

because the matrix element $(\vec{p}'\ \vec{p} + \vec{q}\ |R|\ \vec{p}' + \vec{q}\ \vec{p}\ )$ has its maximum at $q = 0$ and drops to zero as q becomes large just like the Coulomb interaction does. Therefore we expect that the main contribution comes from small q's, and the dependence on p and p' of R for large q's does not alter our final result very much. Consequently we have arrived at Eq. (34) by choosing a simple form which reduces to Eq. (34') for small q.

C. Stability Conditions

In the following, we substitute the above R matrix elements into Eq. (30) and see what values are implied for the parameter Δ_{qn}. We first introduce new notations. Let

$$\rho_\chi = I; \sigma_x, \sigma_y, \sigma_z;\ \tau_x, \tau_y, \tau_z;\ \sigma_x\tau_x,\ \sigma_x\tau_y, \cdots \sigma_z\tau_z$$

and

$$f_\chi = \epsilon_1;\ \epsilon_2, \epsilon_2, \epsilon_2;\ \epsilon_3, \epsilon_3, \epsilon_3;\ \epsilon_4, \epsilon_4, \cdots\ \epsilon_4,$$

corresponding to

$\chi = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16,$

and let

$$G_{\substack{p\\ \lambda\\ \lambda_0}\ qn} = G^{(+)}_{\substack{p\\ \lambda\\ \lambda_0}\ qn} + G^{(-)}_{\substack{p\\ \lambda\\ \lambda_0}\ qn} .$$

Then, from Eqs. (30), (33), and (34) we have

$$G^{(\pm)}_{\substack{p\\ \lambda\\ \lambda_0}\ qn} = \pm \frac{1}{\Delta_{qn} - L_{\substack{p\\ \lambda\\ \lambda_0}\ q+i\eta}} \sum_{\vec{p}'} (\vec{p}', \vec{p} + \vec{q} \,|\, R \,|\, \vec{p}' + \vec{q}\, \vec{p}) \sum_{\chi}^{16}$$

$$f_\chi \left\{ \left[\sum_{\substack{\lambda'\\ \lambda'_0}} (\rho_\chi)_{\lambda'_0\lambda'} G_{\substack{p'\\ \lambda'\\ \lambda'_0}\ qn} \right] (\rho_\chi)_{\lambda\lambda_0} - \sum_{\substack{\lambda'\\ \lambda'_0}} (\rho_\chi)_{\lambda\lambda'} G_{\substack{p'\\ \lambda'\\ \lambda'_0}} (\rho_\chi)_{\lambda'_0\lambda_0} \right\}. \tag{35}$$

The second term inside the braces is from the exchange R matrix element. This term will not cause much complication in decomposing Eq. (35) into that corresponding to different modes, because we have assumed the same momentum dependence for the direct and the exchange R-matrix elements and because we have

$$\sum_{\substack{\lambda \\ \lambda_0}} \left\{ \sum_{\chi} f_\chi \left[\sum_{\substack{\lambda' \\ \lambda'_0}} (\rho_\chi)_{\lambda\lambda'} \, G_{\substack{p'qn \\ \lambda' \\ \lambda'_0}} \, (\rho_\chi)_{\lambda'_0\lambda_0} \right] \right\} (\rho_K)_{\lambda_0\lambda}$$

$$= \sum_{\chi} f_\chi \sum_{\substack{\lambda' \\ \lambda}} G_{\substack{p'qn \\ \lambda' \\ \lambda'_0}} \, (\rho_\chi\rho_K\rho_\chi)_{\lambda'_0\lambda'}$$

$$= \begin{cases} (\epsilon_1 + 3\epsilon_2 + 3\epsilon_3 + 9\epsilon_4) \sum_{\lambda'} G_{\substack{p'qn \\ \lambda' \\ \lambda'}} & \text{for } \rho_K = I, \\[2ex] (\epsilon_1 - \epsilon_2 + 3\epsilon_3 - 3\epsilon_4) \sum_{\substack{\lambda' \\ \lambda'_0}} G_{\substack{p'qn \\ \lambda' \\ \lambda'_0}} \, (\rho_K)_{\lambda'_0\lambda'} & \text{for } \rho_K = \sigma_x, \sigma_y, \sigma_z, \\[2ex] (\epsilon_1 + 3\epsilon_2 - \epsilon_3 - 3\epsilon_4) \sum_{\substack{\lambda' \\ \lambda'_0}} G_{\substack{p'qn \\ \lambda' \\ \lambda'_0}} \, (\rho_K)_{\lambda'_0\lambda'} & \text{for } \rho_K = \tau_x, \tau_y, \tau_z, \\[2ex] (\epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4) \sum_{\substack{\lambda' \\ \lambda'_0}} G_{\substack{p'qn \\ \lambda' \\ \lambda'_0}} \, (\rho_K)_{\lambda'_0\lambda'} & \text{for } \rho_K = \sigma_x\tau_x \cdots \sigma_z\tau_z . \end{cases}$$

Therefore, following the notation of GHW, we write

$$\Gamma^{(\pm)}_{pqn_I} = \sum_{\substack{\lambda \\ \lambda_0}} G^{(\pm)}_{\substack{p\ qn \\ \lambda \\ \lambda_0}} \delta_{\lambda\lambda_0} ,$$

$$\Gamma^{(\pm)}_{pqn_{II}} = \sum_{\substack{\lambda \\ \lambda_0}} G^{(\pm)}_{\substack{p\ qn \\ \lambda \\ \lambda_0}} (\vec{\sigma})_{\lambda_0\lambda} ,$$

$$\Gamma^{(\pm)}_{pqn_{III}} = \sum_{\substack{\lambda \\ \lambda_0}} G^{(\pm)}_{\substack{p\ qn \\ \lambda \\ \lambda_0}} (\vec{\tau})_{\lambda_0\lambda} ,$$

and

$$\Gamma^{(\pm)}_{pqn_{IV}} = \sum_{\substack{\lambda \\ \lambda_0}} G^{(\pm)}_{\substack{p\ qn \\ \lambda \\ \lambda_0}} (\vec{\sigma}\,\vec{\tau})_{\lambda_0\lambda} .$$

We have, from Eq. (35), $\left(L_{\substack{p,q \\ \lambda \\ \lambda_0}} = L_{pq} \text{ for all } \lambda, \lambda_0 \right)$,

$$\Gamma_{pqn_K}^{(\pm)} = \pm \frac{\epsilon'_K}{\Delta_{qn_K} - L_{pq} + i\eta} \sum_{\vec{p}'} (p'\ p + q\ |R|\ p' + q\ p)$$

$$\times \left[\Gamma^{(+)}_{p'qn_K} + \Gamma^{(-)}_{p'qn_K} \right], \tag{36}$$

where K = I, II, III, and IV, and

$$\epsilon'_K = \begin{cases} 4\epsilon_1 - (\epsilon_1 + 3\epsilon_2 + 3\epsilon_3 + 9\epsilon_4) & \text{as } K = \text{I}, \\ 4\epsilon_2 - (\epsilon_1 - \epsilon_2 + 3\epsilon_3 - 3\epsilon_4) & \text{as } K = \text{II}, \\ 4\epsilon_3 - (\epsilon_1 + 3\epsilon_2 - \epsilon_3 - 3\epsilon_4) & \text{as } K = \text{III}, \\ 4\epsilon_4 - (\epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4) & \text{as } K = \text{IV}. \end{cases} \tag{37}$$

Equation (36) is the same as Eq. (6-7) of GHW except for the contribution to ϵ'_K from the exchange term. And, as one would expect, this correction is the same as that obtained by Hatano, who used H'_s with the original anticommutation relations for $a^{*\prime}_s$ and $b^{*\prime}_s$.[14]

To proceed, we write

$$T_{qn_k} = \sum_{\substack{|\vec{p}|<P_F \\ |\vec{p}+\vec{q}|>P_F}} \Gamma^{(+)}_{pqn_k} + \sum_{\substack{|\vec{p}|>P_F \\ |\vec{p}+\vec{q}|<P_F}} \Gamma^{(-)}_{pqn_k} ,$$

$$\hat{q}\, S_{qn_k} = \sum_{\substack{|\vec{p}|<P_F \\ |\vec{p}+\vec{q}|>P_F}} \vec{p}\, \Gamma^{(+)}_{pqn_k} + \sum_{\substack{|\vec{p}|>P_F \\ |\vec{p}+\vec{q}|<P_F}} \vec{p}\, \Gamma^{(-)}_{pqn_k} ,$$

and

$$N_0(\Delta_{qn_K}) = \frac{1}{\Omega}\left(\sum_{\substack{|\vec{p}|<P_F \\ |\vec{p}+\vec{q}|>P_F}} - \sum_{\substack{|\vec{p}|<P_F \\ |\vec{p}+\vec{q}|>P_F}}\right)\frac{1}{\Delta_{qn_K} - L_{p,q} + i\eta},$$

$$\hat{q}\, N_1\,(\Delta_{qn_K}) = \frac{1}{\Omega}\left(\sum_{\substack{|\vec{p}|<P_F \\ |\vec{p}+\vec{q}|>P_F}} - \sum_{\substack{|\vec{p}|<P_F \\ |\vec{p}+\vec{q}|>P_F}}\right)\frac{\vec{p}}{\Delta_{qn_K} - L_{p,q} + i\eta},$$

and

$$\hat{q}\hat{q}\, N_2\,(\Delta_{qn_K}) = \frac{1}{\Omega}\left(\sum_{\substack{|\vec{p}|<P_F \\ |\vec{p}+\vec{q}|>P_F}} - \sum_{\substack{|\vec{p}|>P_F \\ |\vec{p}+\vec{q}|<P_F}}\right)\frac{\vec{p}\,\vec{p}}{\Delta_{qn_K} - L_{p,q} + i\eta}, \tag{38}$$

where Ω is the volume of the system.

Therefore, Eq. (36) becomes

$$T_{qn_K} = \epsilon'_K\, V(q)\, \Omega\, (N_0\, T_{qn_K} + c\, N_1\, S_{qn_K})$$

and

$$S_{qn_K} = \epsilon'_K\, V(q)\, \Omega\, (N_1\, T_{qn_K} + c\, N_2\, S_{qn_K}).$$

Finally, we obtain for Δ_{qn_K}

$$F_K(\Delta_{qn_K}) \equiv 1 - \epsilon'_K V(q)\, \Omega \left[N_0(\Delta_{qn_K}) + c\, \epsilon'_K V(q)\, \Omega \times \frac{\left\{ N_1 (\Delta_{qn_K}) \right\}^2}{1 - c\, \epsilon'_K V(q)\, \Omega\, N_2 (\Delta_{qn_K})} \right] = 0. \tag{39}$$

This equation has roots from $\Delta_{qn_K} = 0$ to $\Delta_{qn_K} = L_{P_F, q}$ for any set of values of ϵ'_K, $V(q)$ and c. This is seen from the fact that $F_K(\Delta_{qn_K})$ changes sign (from $+\infty$ to $-\infty$, or from $-\infty$ to $+\infty$) when Δ_{qn_K} passes through the interval $\left[L_{p_i q} + 0,\ L_{p_{i+1} q} - 0 \right]$. Therefore a zero point of $F_K(\Delta_{qn_K})$ must exist between $L_{p_i, q}$ and $L_{p_{i+1}, q}$. When we take the limit $\Omega \to \infty$, Eq. (39) has a continuous spectrum from 0 to $L_{P_F, q}$ (the maximum value of $L_{p, q}$). These are the scattering solutions. We label them by Δ_{qpK}.

λ

λ_0

Corresponding to each of these solutions, there exists a scattering excited state

$$|\psi_{\substack{qpK\\ \lambda\\ \lambda_0}}\rangle = \left[\sum_{\substack{|\vec{p}'|<P_F\ \lambda'\\ |\vec{p}'+\vec{q}|>P_F\lambda'_0}} G^{(+)}_{\substack{p'\\ \lambda'\\ \lambda'_0}}\ \substack{q\\ \\ \\}\ \substack{pK\\ \lambda\\ \lambda_0}\ c^*_{\substack{p'q\\ \lambda'\\ \lambda'_0}}\right.$$

$$\left. - \sum_{\substack{|\vec{p}|>P_F\\ |\vec{p}+\vec{q}|<P_F}} G^{(-)}_{\substack{p'\\ \lambda'\\ \lambda'_0}}\ \substack{q\\ \\ \\}\ \substack{pK\\ \lambda\\ \lambda_0}\ c_{\substack{p'+q,-q\\ \lambda'_0\\ \lambda'}}\right] |\psi_0\rangle\rangle$$

$$= c^*_{\substack{p,q\\ \lambda\\ \lambda_0}} |\psi_0\rangle\rangle + \left[\sum_{\substack{|\vec{p}'|<P_F,\ \lambda'\\ |\vec{p}'+\vec{q}|>P_F\ \lambda'_0}} G^{(+)}_{\substack{p'\\ \lambda'\\ \lambda'_0}}\ \substack{q\\ \\ \\}\ \substack{pK\\ \lambda\\ \lambda_0}\ c^*_{\substack{p',q\\ \lambda'\\ \lambda'_0}}\right.$$

(principal value)

$$\left. - \sum_{\substack{|\vec{p}'|>P_F,\lambda'\\ |\vec{p}'+\vec{q}|<P_F,\lambda'_0}} G^{(-)}_{\substack{p'\\ \lambda'\\ \lambda'_0}}\ \substack{q\\ \\ \\}\ \substack{pK\\ \lambda\\ \lambda_0}\ c_{\substack{p'+q,-q\\ \lambda'\\ \lambda'_0}}\right] |\psi_0\rangle\rangle .$$

(principal value)

We can also call them pseudo-pair states. If the parameters ϵ', $V(q)$, and c have values such that the equation $F_K\left[\Delta_{qn_K} = \Delta_{qK}\ (\text{a real value} > L_{P_F,q})\right] = 0$ has a solution, then this solution corresponds to a "zero sound" wave:

$$G^{(\pm)}_{\substack{pqK\\ \lambda\\ \lambda_0}} = (\rho_K)_{\lambda\lambda_0}\ \Gamma^{(\pm)}_{pqK} ,$$

$$|\psi_{qK=I}\rangle\rangle = \left[\sum_{\substack{|p|<P_F \\ |p+q|>P_F}} \Gamma^{(+)}_{pqK=I} \sum_{\lambda} c^{*}_{\substack{p,q \\ \lambda \\ \lambda}} - \sum_{\substack{|p|>P_F \\ |p+q|<P_F}} \Gamma^{(-)}_{pqK=I} \sum_{\lambda} c_{\substack{p+q,\ -q \\ \lambda \\ \lambda}}\right] |\psi\rangle\rangle ,$$

$$|\psi_{qK=II}\rangle\rangle = \left[\sum_{\substack{|p|<P_F \\ |p+q|>P_F}} \Gamma^{(+)}_{pqK=II} \sum_{\substack{\lambda \\ \lambda_0}} \bar{\sigma}_{\lambda_0\lambda}\ c^{*}_{\substack{p,q \\ \lambda \\ \lambda_0}} - \sum_{\substack{|p|>P_F \\ |p+q|<P_F}} \Gamma^{(-)}_{pqK=II} \sum_{\substack{\lambda \\ \lambda_0}} \bar{\sigma}_{\lambda_0\lambda}\ c_{\substack{p+q,\ -q \\ \lambda_0 \\ \lambda}}\right] |\psi_0\rangle\rangle,$$

etc. If ϵ', $V(q)$, and c are such that the equation $F_K\left[\Delta_{qn_K} = iI'(\text{an imaginary value})\right] = 0$ has a solution (I' is real), the methods we have used fail. There are cases, of course, in which there are no roots either for $F_K(\Delta_{qn_K} = \Delta_{qK}) = 0$ or for $F_K(\Delta_{qn_K} = iI') = 0$. In these cases, the system is stable although there is no "zero sound" wave.

Let us calculate N_0, N_1, and N_2 for different types of Δ_{qn_K}:

<u>Case (i).</u> For $\Delta_{qn_K} = L_{p,q} = \frac{(p+q)^2}{2M^*} - \frac{p^2}{2M^*} \cong \frac{\vec{p}\cdot\vec{q}}{M^*}$

we calculate

$$N_0\left(\frac{\vec{p}\cdot\vec{q}}{M^*}\right) = -\frac{M^* P_F}{2\pi^2}$$

$$\left[1 + \frac{1}{2}\frac{\vec{p}\cdot\vec{q}}{P_F\, q}\,\ell n\left|\frac{1 - \frac{\vec{p}\cdot\vec{q}}{P_F\, q}}{1 + \frac{\vec{p}\cdot\vec{q}}{P_F\, q}}\right| - i\,\frac{\pi}{2}\,\frac{\vec{p}\cdot\vec{q}}{P_F\, q}\right],$$

$$N_1\left(\frac{\vec{p}\cdot\vec{q}}{M^*}\right) = P_F\,\frac{\vec{p}\cdot\vec{q}}{P_F\, q}\,N_0\,,$$

and

$$N_2\left(\frac{\vec{p}\cdot\vec{q}}{M^*}\right) = -\frac{M^* P_F}{2\pi^2}\,\frac{1}{3}\,P_F^{\,2} + P_F^{\,2}\left(\frac{\vec{p}\cdot\vec{q}}{P_F\, q}\right) N_0\,. \tag{40}$$

<u>Case (ii).</u> For values of Δ_{qn_K} that are real and larger than $L_{P_F q}$, Δ_{qK}, we have

$$N_0(D) = -\frac{M^* P_F}{2\pi^2}\left[1 + \frac{1}{2}\,D\,\ell n\,\frac{D-1}{D+1}\right],$$

where

$$D = \frac{\Delta_{qK}\, M^*}{p_F\, q}\;;$$

$$N_1(D) = P_F\, D\, N_0\,(D);$$

and

$$N_2(D) = -\frac{M^* P_F}{2\pi^2}\,\frac{1}{3}\,P_F^{\,2} + P_F^{\,2}\,D^2\,N_0\,(D). \tag{41}$$

Case (iii). For $\Delta_{qn_K} = iI'$, where I' is real, we have

$$N_0(iI') = -\frac{M^* P_F}{2\pi^2}\left[1 - I \tan^{-1}\frac{1}{I}\right],$$

where

$$I = \frac{I' M^*}{P_F q};$$

$$N_1(iI') = P_F I N_0(iI');$$

and

$$N_2(iI') = -\frac{M^* P_F}{2\pi^2}\frac{1}{3} P_F^2 + P_F^2 I^2 N_0(iI'). \tag{42}$$

For Case (iii), $F_K(iI_K) = 0$ can be reduced to

$$\frac{1}{\lambda_K}\frac{1 + \frac{1}{3} c\, p_F^2 \lambda_K}{1 + \frac{1}{3} c\, p_F^2 \lambda_K + c\, p_F^2 I^2} = I \tan^{-1}\frac{1}{I} - 1 = Q(I), \tag{43}$$

where $\lambda = \frac{M^* P_F}{2\pi^2} \epsilon'_K V(q)\Omega$. The right-hand side is exhibited as a function of I in Fig. 1. We see that $Q(I)$ takes values from 0 to -1 only. If the left-hand side does not fall into this interval, we shall have no roots for Eq. (43), which means no unstable states. Consequently there are many chances for attractive forces to escape from getting unstable states. For example, there are unstable states for the system in which $c = 0$ and λ_K are all small and negative so that $1/\lambda_K$ is less than -1.

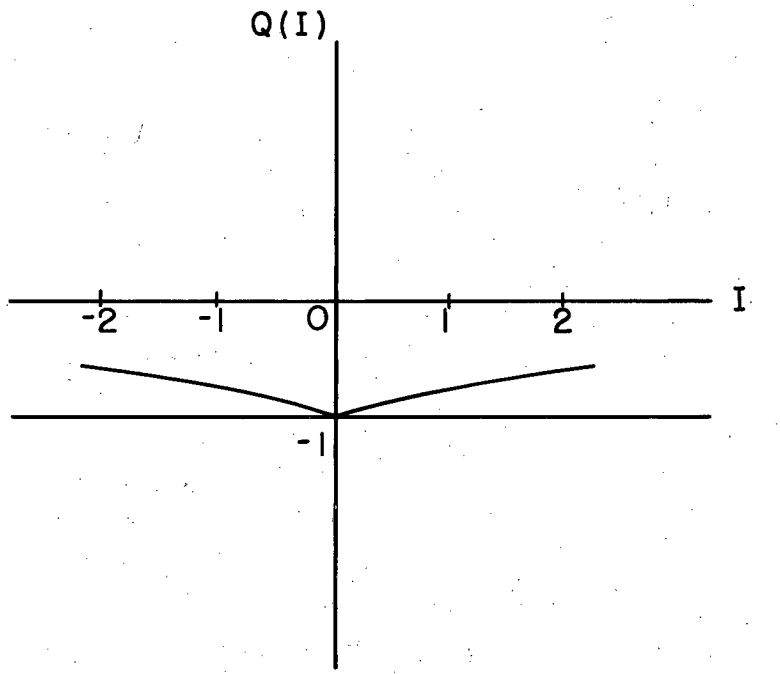


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Fig. 1. Q(I) as a function of I .

For Case (ii), the eigenvalue equation looks like

$$\frac{1}{\lambda_K} \frac{1 + \frac{1}{3} c\, p_F^{\,2} \lambda_K}{1 + \frac{1}{3} c\, p_F^{\,2} \lambda_K + c\, p_F^{\,2} D^2} = \frac{D}{2} \ln \frac{D+1}{D-1} - 1 \equiv Q'(D). \quad (44)$$

If we plot $Q'(D)$ as a function of D in Fig. 2, we can see that $Q'(D)$ takes all positive values for $D > 1$. Therefore, the system can have a "zero sound" wave, if the parameters give a positive value for the left-hand side of Eq. (44).

Because the left-hand side of Eq. (43) and that of Eq. (44) have exactly the same form, we need only discuss one function,

$$f(x) = \frac{1}{\lambda_K} \frac{1 + \frac{1}{3} c\, p_F^{\,2} \lambda_K}{1 + \frac{1}{3} c\, p_F^{\,2} \lambda_K + c\, p_F^{\,2} x^2} .$$

The function $f(x)$ is limited in the different regions in $f(x) \longleftrightarrow x$ plane for different sets of values of λ, $1 + \frac{1}{3} c\, p_F^{\,2} \lambda_K$, and $c\, p_F^{\,2}$ (Fig. 3). For Case 1, $f(x)$ is entirely in the upper half plane. When $f(x)$ intersects with $Q'(x)$ we shall have a $D > 1$ solution. For the second and third cases, $f(x)$ enters into the $(0, -1)$ zone. We expect to have imaginary solution if $f(x)$ intersects with $Q(x)$. For Case 4, there exists an imaginary solution for $\frac{1}{\lambda_K} > -1$. On the other hand, if λ_K is weak enough so that $\frac{1}{\lambda_K} < -1$, we may even get a $D > 1$ solution provided the branch in the first quadrant intersects with $Q'(x)$; otherwise we get no collective solution, but the system is still stable.

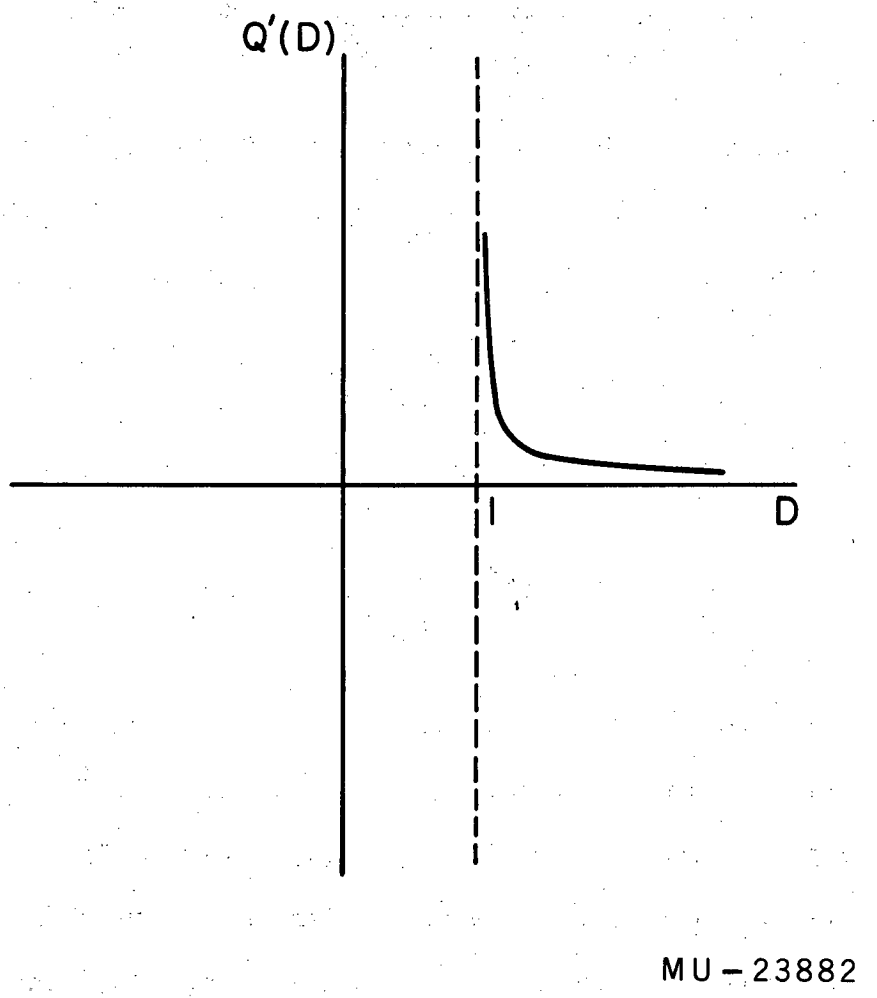

Fig. 2. Q'(D) as a function of D.

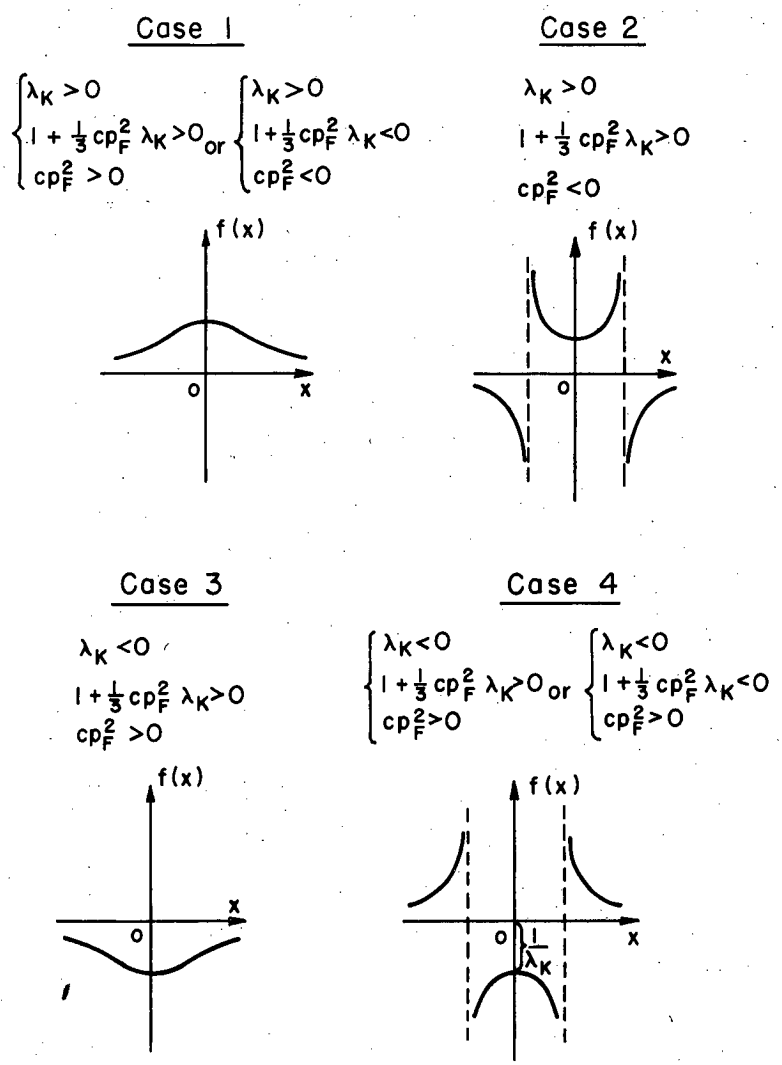

Fig. 3. f(x) as a function of x for different values of λ_K, $1 + \frac{1}{3} cp_F^2 \lambda_K$, and cp_F^2.

The types of solutions for different values of λ_K and $c\ p_F^2$ are listed in detail in Table I. The inequality equations associated with the absence of imaginary solutions can be considered as the stability conditions.

In our example we have assumed the same momentum dependence (i.e. the same parameter c) for all four modes. Therefore the stability conditions (Table I) require that $c\ p_F^2 > \frac{3}{|\lambda_I|}$ or $\frac{3}{2|\lambda_I|} > c\ p_F^2 - \frac{3}{2\lambda_i}$ or $-\frac{3}{\lambda_j} < c\ p_F^2$, where λ_I is the λ value for the simple compressional mode, λ_i is the largest and λ_j is the smallest of the other three positive λ_K's. If we assume different momentum dependences for the four modes, the stability arguments will in general follow the rules listed in Table I.

Table I. Types of solutions for different values of λ_K and $c\,p_F^2$

λ_K	$c\,p_F^2$	Types of solutions
$-3 < \lambda_K < -1$	$c\,p_F^2 < -\dfrac{1}{1-\frac{1}{3}\lvert\lambda_K\rvert}$	imaginary, $D > 1$ solutions
	$-\dfrac{1}{1-\frac{1}{3}\lvert\lambda_K\rvert} < c\,p_F^2 < \dfrac{3}{2\lvert\lambda_K\rvert}$	imaginary solution
	$\dfrac{3}{2\lvert\lambda_K\rvert} < c\,p_F^2 < \dfrac{3}{\lvert\lambda_K\rvert}$	no solution
	$\dfrac{3}{\lvert\lambda_K\rvert} < c\,p_F^2$	imaginary solution
$-1 < \lambda_K < 0$	$c\,p_F^2 < -\dfrac{1}{1-\frac{1}{3}\lvert\lambda_K\rvert}$	$D > 1$ solution
	$-\dfrac{1}{1-\frac{1}{3}\lvert\lambda_K\rvert} < c\,p_F^2 < \dfrac{3}{2\lvert\lambda_K\rvert}$	no solution
	$\dfrac{3}{2\lvert\lambda_K\rvert} < c\,p_F^2 < \dfrac{3}{\lvert\lambda_K\rvert}$	imaginary
	$\dfrac{3}{\lvert\lambda_K\rvert} < c\,p_F^2$	no solution
$0 \leq \lambda_K < 3$	$c\,p_F^2 < -\dfrac{3}{\lambda_K}$	no solution
	$-\dfrac{3}{\lambda_K} < c\,p_F^2 < -\dfrac{3}{2\lambda_K}$	imaginary solution
	$-\dfrac{3}{2\lambda_K} < c\,p_F^2 < -\dfrac{1}{1+\frac{1}{3}\lambda_K}$	no solution
	$-\dfrac{1}{1+\frac{1}{3}\lambda_K} < c\,p_F^2$	$D > 1$ solution

D. Ground-State Energy of H_s

The ground-state energy $\epsilon_0 = \langle\langle \psi_0 | H_s | \psi_0 \rangle\rangle$ can be calculated following the procedure of Sawada et al.[2] Or, if we use the fact that H_s describes a system of linearly coupled oscillators, we can write its expression directly.[3]

Let us introduce

$$\phi_{\substack{p,q\\ \lambda\\ \lambda_0}} = \left(2L_{\substack{p,q\\ \lambda\\ \lambda_0}}\right)^{-1/2} \left(c_{\substack{p,q\\ \lambda\\ \lambda_0}} + c^*_{\substack{p+q,-q\\ \lambda_0\\ \lambda}}\right)$$

and

$$\pi_{\substack{p,q\\ \lambda\\ \lambda_0}} = i\left(L_{\substack{p,q\\ \lambda\\ \lambda_0}}/2\right)^{1/2} \left(c^*_{\substack{p,q\\ \lambda\\ \lambda_0}} - c_{\substack{p+q,-q\\ \lambda_0\\ \lambda}}\right) ,$$

where ϕ, π are canonically conjugated to each other as a consequence of the commutation relations of c and c^*'s. In terms of these variables, H_s becomes:

$$H_{kin} = \frac{1}{2} \sum_q \sum_{\substack{p\\ \lambda\\ \lambda_0}} \left(\pi^*_{\substack{p,q\\ \lambda\\ \lambda_0}} \pi_{\substack{p,q\\ \lambda\\ \lambda_0}} + L^2_{\substack{p,q\\ \lambda\\ \lambda_0}} \phi^*_{\substack{p,q\\ \lambda\\ \lambda_0}} \phi_{\substack{p,q\\ \lambda\\ \lambda_0}} - L_{\substack{p,q\\ \lambda\\ \lambda_0}} \right)$$

and

$$H_{int} = \frac{1}{\Omega} \sum_{q} \sum_{\substack{pp' \\ \lambda\lambda' \\ \lambda_0\lambda'_0}} \left\langle \begin{matrix} p+q & p' \\ \lambda & \lambda' \end{matrix} \middle| R \middle| \begin{matrix} p & p'+q \\ \lambda_0 & \lambda'_0 \end{matrix} \right\rangle$$

$$L^{1/2}_{\substack{p,q \\ \lambda \\ \lambda'_0}} \; \phi^*_{\substack{p,q \\ \lambda \\ \lambda_0}} \; L^{1/2}_{\substack{p',q \\ \lambda' \\ \lambda'_0}} \; \phi_{\substack{p',q \\ \lambda' \\ \lambda'_0}} - \epsilon_c ,$$

where

$$\epsilon_c = + \frac{1}{2\Omega} \sum_{q} \sum_{\substack{|p| < P_F, \lambda \\ |p+q| > P_F, \lambda_0}} \left\langle \begin{matrix} p+q & p \\ \lambda & \lambda_0 \end{matrix} \middle| R \middle| \begin{matrix} p & p+q \\ \lambda_0 & \lambda \end{matrix} \right\rangle .$$

The diagonalization procedure we have described is to transform these linearly coupled oscillators into their normal modes. The ground-state energy is nothing but the zero-point energy of the normal vibrations, namely

$$\epsilon_0(H_s) = \frac{1}{2} \sum_{q} \sum_{n_K} \Delta_{qn_K} - \frac{1}{2} \sum_{q} \sum_{\substack{|p| < P_F \\ |p+q| > P_F}} L_{p,q} - \epsilon_c .$$

The function $F_K(\Delta^2_{qn_K})$ defined in Eq. (39) can be written, in terms of its zeros and its poles $(\xi = \Delta^2)$,

$$F_{Kq}(\xi) = \prod_{n} (\xi - \Delta^2_{qn_K}) \prod_{p} (\xi - L^2_{\substack{p,q \\ \lambda \\ \lambda_0}})^{-1} .$$

Hence we have

$$\epsilon_0(H_s) = \frac{1}{4\pi i} \int \xi^{1/2}\, d\xi \sum_q \sum_K \frac{\partial\, \ell n\, F_{Kq}(\xi)}{\partial\, \xi} - \epsilon_c ,$$

where the closed contour in the complex ξ plane is taken to encircle all zeros (Δ_{qn_K}) and all poles $(L^2_{\substack{p,q\\ \lambda\\ \lambda_0}})$. Integrating by parts, and writing $\xi = \Delta^2$ again, we have

$$\epsilon_0(H_s) = -\frac{1}{4\pi i} \int d\Delta \sum_{qK} \ell n\, F_{Kq}(\Delta^2) - \epsilon_c .$$

Where we go to the limit $\Omega \to 0$, $\ell n\, F_{Kq}(\Delta^2)$ has a cut from 0 to $L_{P_F q}$ along the real Δ axis and possibly a pole at $\Delta_{qK} > L_{P_F q}$ on the real axis. Carrying the contour along both sides of this cut, and changing variables to a new integration variable $\vec{p}$ defined by $\Delta = \frac{\vec{p}\cdot\vec{q}}{M^*}$, we get

$$\epsilon_0(H_s) = \frac{1}{4\pi i} \sum_{qK} \sum_{\substack{\lambda\\ \lambda_0}} \int_{\substack{|p|<P_F\\ |p+q|>P_F}} d\vec{p}\, \frac{1}{M^* \frac{\vec{p}\cdot\vec{q}}{q}}\, \ell n \frac{F_{qK}(L_{\substack{p,q\\ \lambda\\ \lambda_0}} + i\eta)}{F_{qK}(L_{\substack{p,q\\ \lambda\\ \lambda_0}} - i\eta)}$$

$$- \frac{1}{4\pi i} \operatorname{Res} \int d\Delta \sum_{qK} \ell n\, F_{Kq}(\Delta^2) - \epsilon_c .$$

We can substitute Eqs. (39) and (40) into the scattering part of the above expression. Then $\frac{1}{2}\, \ell n \frac{F(L + i\eta)}{F(L - i\eta)}$ becomes an arc tangent. Replacing $\sum_q$ by $4\pi \int q^2\, dq$, and $\int_{\substack{|p|<P_F\\ |p+q|>P_F}} d\vec{p}$ by $2\pi\, P_F^2\, q \int_0^1 x dx$,

where $x = \vec{p} \cdot \vec{q}/P_F q$, we have $\epsilon_0(H_s) = \epsilon_{\text{scatt.}} + \epsilon_{\text{zero sound}} - \epsilon_c$.

Then we have

$$\epsilon_{\text{scatt.}} = \frac{3}{4\pi} \frac{P_F^2}{2M^*} \sum_K \int_0^{q_m} \frac{q^2}{P_F^3} \frac{dq}{P_F} \int_0^1 dx \tan^{-1}$$

$$\times \frac{\frac{\pi}{2} \lambda_K x(1 + \frac{1}{3} c\, p_F^2 \lambda_K + c\, p_F^2 x^2)}{1 + \frac{1}{3} c\, p_F^2 \lambda_K + \lambda_K (1 + \frac{1}{3} c\, p_F^2 \lambda_K + c\, p_F^2 x^2)(1 + \frac{x}{2} \ln \left| \frac{1 - x}{1 + x} \right|)},$$

and

$$\epsilon_{\text{zero sound}} = \frac{1}{2} \sum_K \sum_q (\Delta_{qK} - L_{P_F q})$$

$$= \frac{1}{2} \sum_K \frac{\Omega}{(2\pi)^3} \frac{P_F}{M^*} 4\pi \int q^3 dq \left[D_{qK} - 1\right],$$

where

$$\frac{1}{\lambda_K} \frac{1 + \frac{1}{3} c\, p_F^2 \lambda_K}{1 + \frac{1}{3} c\, p_F^2 \lambda_K + c\, p_F^2 D_{qK}^2} = \frac{1}{2} \ln \frac{D_{qK} + 1}{D_{qK} - 1}$$

and

$$\epsilon_c = -\frac{1}{2} \left(\frac{\Omega}{(2\pi)^3} \right)^2 4\pi \int q^2 dq\, \epsilon'_I V(q)\, 2\pi P_F^2 q \int_0^1 x dx\, (1 + c\, p_F^2).$$

IV. NUMERICAL CALCULATIONS

A. Evaluation of R_B, R_G for a Factorable Potential

In Section III, we have assumed the simple phenomenological form of Eqs. (33) and (34), for the matrix elements of any nucleon reaction operator R. Here, we try to determine the values of the parameters ϵ_K, V_0, a_0, and c, such that Eqs. (33) and (34) represent approximately the matrix elements of a particular reaction operator such as Brueckner's R_B operator and Galitskii's R_G operator.

To calculate the R_B and R_G matrix elements, we assume a nucleon-nucleon potential which is charge-independent, even under inversion in both spin and configuration spaces, and factorable in momentum space:

$$v = 1_{v_{even}} \frac{1 - \sigma'\cdot\sigma}{4} \frac{3 + \tau'\cdot\tau}{4} + 3_{v_{even}} \frac{3 + \sigma'\cdot\sigma}{4} \frac{1 - \tau'\cdot\tau}{4}$$

$$+ 1_{v_{odd}} \frac{1 - \sigma'\cdot\sigma}{4} \frac{1 - \tau'\cdot\tau}{4} + 3_{v_{odd}} \frac{3 + \sigma'\cdot\sigma}{4} \frac{3 + \tau'\cdot\tau}{4} \tag{45}$$

such that

$$(\overline{k', T', M'_T, S', M'_S}|v|\overline{k, T, M_T, S, M_S}) = \delta_{T'T}\,\delta_{S'S}\,\delta_{M'_T M_T}\,\delta_{M'_S M_s}$$

$$\times \begin{cases} 2(\vec{k}'|{}^1v_{even}|\vec{k}) & \text{for} \quad T = 1, \quad S = 0 \\ 2(\vec{k}'|{}^3v_{even}|\vec{k}) & \text{for} \quad T = 0, \quad S = 1 \\ 2(\vec{k}'|{}^1v_{odd}|\vec{k}) & \text{for} \quad T = 0, \quad S = 0 \\ 2(\vec{k}'|{}^3v_{odd}|\vec{k}) & \text{for} \quad T = 1, \quad S = 1 \end{cases}$$

for $I = 1$ and 3, and

$$(\vec{k}'|{}^Iv_{even}|\vec{k}) = \sum_{\ell=even}^{\ell\,max} \sum_{m_\ell=-\ell}^{\ell} Y_\ell^m(\hat{k}')\, Y_\ell^m(\hat{k})(k'|{}^Iv_\ell|k)$$

$$(\vec{k}'|{}^Iv_{odd}|\vec{k}) = \sum_{\ell=odd}^{\ell\,max} \sum_{m_\ell=-\ell}^{\ell} Y_\ell^m(\hat{k}')\, Y_\ell^m(\hat{k})(k'|{}^Iv_\ell|k) \tag{46}$$

$$(\vec{k}'|{}^Iv_\ell|\vec{k}) = \frac{(2\pi)^3}{\Omega}\Big[{}^{I\ell}f_1\ {}^{I\ell}\phi_1(k')\ {}^{I\ell}\phi_1(k) +$$

$$+\ {}^{I\ell}f_2\ {}^{I\ell}\phi_2(k')\ {}^{I\ell}\phi_2(k)\Big]\ , \tag{47}$$

where T, M_T, S, and M_S are the eigenvalues of the total and the z component of two-nucleon i-spin and spin operators, respectively. The f_1's and f_2's are constants and $(\overline{k, T, M_T, S, M_S})$ means an antisymmetrized two-nucleon state. Such a potential allows an exact solution for the R_B and R_G matrix elements. Furthermore, we may treat the "scattering" of each ${}^Iv_\ell$ independently:

$$R = {}^1R_{even}\,\frac{1-\vec{\sigma}'\cdot\vec{\sigma}}{4}\,\frac{3+\vec{\tau}'\cdot\vec{\tau}}{4} + {}^3R_{even}\,\frac{3+\vec{\sigma}'\cdot\vec{\sigma}}{4}\,\frac{1-\vec{\tau}'\cdot\vec{\tau}}{4}$$

$$+ {}^1R_{odd}\,\frac{1-\vec{\sigma}'\cdot\vec{\sigma}}{4}\,\frac{1-\vec{\tau}'\cdot\vec{\tau}}{4} + {}^3R_{odd}\,\frac{3+\vec{\sigma}'\cdot\vec{\sigma}}{4}\,\frac{3+\vec{\tau}'\cdot\vec{\tau}}{4}\,; \tag{48}$$

$$(\vec{k}'|{}^IR_{even}|\vec{k},\vec{P}) = \sum_{\ell=even}^{\ell\,max}\ \sum_{m_\ell=-\ell}^{\ell} Y_\ell^{\,m}(\hat{k}')\,Y_\ell^{*\,m}(\hat{k})\,(k'|{}^IR_\ell|k,P)$$

and

$$(\vec{k}'|{}^IR_{odd}|\vec{k}',\vec{P}) = \sum_{\ell=odd}^{\ell'\,max}\ \sum_{m_\ell=-\ell}^{\ell} Y_\ell^{\,m}(\hat{k}')\,Y_\ell^{*\,m}(\hat{k})\,(k'|{}^IR_\ell|k,P); \tag{49}$$

and

$$(k'|{}^IR_\ell|k,P) = (k'|{}^Iv_\ell|k) + \frac{\Omega}{(2\pi)^3}\int d\vec{k}''\sum_{m_\ell=-\ell}^{\ell}\frac{1}{2\ell+1}\,(k'|{}^Iv_\ell|k'')$$

$$\times\, Y_\ell^{*\,m}(\hat{k}'')\,\frac{Q(\vec{k}'',P)}{e}\,Y_\ell^{\,m}(\hat{k}'')(k''|{}^IR_\ell|k,P)$$

or

$$(k'|{}^IR_\ell|k,P) = (k'|{}^Iv_\ell|k) + \frac{\Omega}{(2\pi)^3}\int d\vec{k}''\,(k'|{}^Iv_\ell|k'')$$

$$\times\,\frac{Q(\vec{k}'',P)}{4\pi e}\,(k''|{}^IR_\ell|k,P). \tag{50}$$

Equation (49) is possible provided we remove the coupling between the directions of $\hat{k}(\hat{k}')$ and $\vec{P}(\vec{P} = \vec{p}_1 + \vec{p}_2$ is the total momentum of the two colliding nucleons) by replacing the real projection operator $Q(\vec{k}'',\vec{P})$ and the denominator $e(\vec{k}'',\vec{P})$ by their angular averages $Q(\vec{k}'',P) = 1/4\pi\int d\vec{p}\,Q(\vec{k}'',\vec{P})$ and $e(\vec{k}'',P) = 1/4\pi\int d\vec{p}\,Q(\vec{k}'',\vec{P})\,e(\vec{k}',\vec{P})$, respectively. This approximation

is based on the argument that the effect of the direction of $\vec{P}$ is small and can be neglected. From Equation (50) and (47), we can easily obtain

$$(k'|{}^{I}R_{\ell}|k,P) = \frac{(2\pi)^3}{\Omega} \left[{}^{I\ell}f_1 \; {}^{I\ell}\phi_1(k') \; {}^{I\ell}X(k,P) + {}^{I\ell}f_2 \; {}^{I\ell}\phi_2(k') \; {}^{I\ell}Y(k,P) \right], \tag{51}$$

where

$$ {}^{I\ell}X(k,P) = \frac{\left[1-{}^{I\ell}f_2 \, {}^{I\ell}I_{12}(k,P)\right] {}^{I\ell}\phi_1(k) + {}^{I\ell}f_2 \, {}^{I\ell}I_{12}(k,P) \, {}^{I\ell}\phi_2(k)}{\left[1-{}^{I\ell}f_1 \, {}^{I\ell}I_{11}(k,P)\right]\left[1-{}^{I\ell}f_2 \, {}^{I\ell}I_{22}(k,P)\right] - {}^{I\ell}f_1 \, {}^{I\ell}f_2 \left[{}^{I\ell}I_{12}(k,P)\right]^2}$$

and

$$ {}^{I\ell}Y(k,p) = \frac{\left[1-{}^{I\ell}f_1 \, {}^{I\ell}I_{11}(k,p)\right] {}^{I\ell}\phi_2(k) + {}^{I\ell}f_1 \, {}^{I\ell}I_{12}(k,P) \, {}^{I\ell}\phi_1(k)}{\left[1-{}^{I\ell}f_1 \, {}^{I\ell}I_{11}(k,P)\right]\left[1-{}^{I\ell}f_2 \, {}^{I\ell}I_{22}(k,P)\right] - {}^{I\ell}f_1 \, {}^{I\ell}f_2 \left[{}^{I\ell}I_{12}(k,P)\right]^2},$$

with

$$ {}^{I\ell}I_{11}(k,P) = \int d\vec{k}'' \; Q(k'',P) \frac{1}{4\pi} \frac{{}^{I\ell}\phi_1^{\,2}(k'')}{e(k'',P)},$$

$$ {}^{I\ell}I_{12}(k,P) = \int d\vec{k}'' \; Q(k'',P) \frac{1}{4\pi} \frac{{}^{I\ell}\phi_1(k'') \; {}^{I\ell}\phi_2(k'')}{e(k'',P)}, \tag{52}$$

and

$$ {}^{I\ell}I_{12}(k,P) = \int d\vec{k}'' \; Q(k'',P) \frac{1}{4\pi} \frac{{}^{I\ell}\phi_2(k'')}{e(k'',P)}.$$

If the constants ${}^{I\ell}f_1$ and ${}^{I\ell}f_2$ and the functions ${}^{I\ell}\phi_1$ and ${}^{I\ell}\phi_2$ are chosen properly, the assumed potential v can reproduce most of the two-nucleon data such as the binding energy of the deuteron and the total and the differential cross sections of n-p and p-p scatterings at energies from zero to 300 Mev (see Appendix B). It is impossible, of course, to explain the polarization experiments by means of a potential of this form. However, the expressions of Eqs. (46) and (47) can be easily modified to include enough parameters to reproduce all the phase shifts that give the best fit found in a phase-shift analysis of complete nucleon-nucleon data. We first expand the plane waves into spherical harmonics:

$$(\vec{k}, S, M_S) = \frac{1}{\sqrt{\Omega}} \; e^{i\vec{k}\cdot\vec{r}} \; \chi_s^{m_s}$$

$$= \frac{1}{\sqrt{\Omega}} \sum_{\ell=0}^{\infty} (2\ell + 1) \, e^{i\pi \ell/2} \, j_\ell(kr) \, P_\ell(\hat{k}\cdot\hat{r}) \, \chi_s^{m_s}$$

$$= \frac{1}{\sqrt{\Omega}} \sum_{\ell=0}^{\infty} \sum_{m_\ell=-\ell}^{\ell} 4\pi \, e^{i\pi \ell/2} \, j_\ell(kr) \, Y_\ell^{m}(\hat{r}) \, Y^{*}{}_\ell^{m}(\hat{k}) \, \chi_s^{m_s}$$

$$= \frac{1}{\sqrt{\Omega}} \sum_{\ell=0}^{\infty} \sum_{m_\ell=-\ell}^{\ell} \sum_{J=|\ell-s|}^{\ell+s} \sum_{M=-J}^{J} e^{i\pi \ell/2} \, j_\ell(kr)$$

$$\times \, Y_\ell^{m}(\hat{r}) \, C(JM;\, \ell m_\ell \, S m_S) \, \mathcal{Y}_{\ell S}^{*JM}(\hat{k}) \, ,$$

where the C's are Clebsch-Gordan coefficients, and the functions $\mathcal{Y}_{\ell S}^{JM}$ are eigenfunctions of J^2, J_z, L^2, and S^2. Now we take

$$(\vec{k}', T', M'_T, S', M'_s | v | \vec{k}, T, M_T, S, M_s) =$$

$$\delta_{TT'}\delta_{ss'}\delta_{M_T M'_T} \begin{cases} \delta_{M_s M'_s} \langle \vec{k}' | {}^{I}v_{even} | \vec{k} \rangle & \text{for } T = 1,\ S = 0 \\ \langle \vec{k}',\ M'_s | {}^{3}v_{even} | \vec{k},\ M_s \rangle & \text{for } T = 0,\ S = 1 \\ \delta_{M_s M'_s} \langle \vec{k}' | {}^{I}v_{odd} | \vec{k} \rangle & \text{for } T = 0,\ S = 0 \\ \langle \vec{k}',\ M'_s | {}^{3}v_{odd} | \vec{k},\ M_s \rangle & \text{for } T = 1,\ S = 1 \end{cases}$$

$$(\vec{k}' M'_s | {}^{3}v_{even} | \vec{k},\ M_s)$$

$$= \frac{(4\pi)^2}{\Omega} \sum_{\ell'=\text{even}}^{\ell\,\max} \sum_{m'_\ell=-\ell'}^{\ell'} \sum_{J'=(\ell'-\ell)}^{\ell'+\ell} \sum_{M'=-J'}^{J'} \sum_{\ell=\text{even}}^{\ell\,\max}$$

$$\times \sum_{M_\ell=-\ell}^{\ell} \sum_{J=(\ell-1)}^{\ell+1} \sum_{M=-J}^{J} e^{-i\,\pi/2\,(\ell'-\ell)}$$

$$C(J'M';\ \ell' M'_\ell,\ 1\ M'_s)\ C(JM;\ \ell M_\ell\ 1\ M_s)$$

$$\times\ \mathcal{Y}^{J'M'}_{\ell' 1}(\hat{k}')\,\mathcal{Y}^{JM}_{\ell 1}(\hat{k}) \int d\vec{r}' \int d\vec{r}\ j_{\ell'}(kr)\ j_\ell(kr)\ Y^{*M'}_{\ell'}(\hat{r}')\ Y^{M}_{\ell}(\hat{r}')$$

$$\times\quad (J'M'\ell'1, \vec{r}'\ |{}^{3}v_{even}|\ J\ M\ \ell\ 1,\ \vec{r})\ .$$

This expression becomes Eqs. (46) and (47) if we assume

$$(J'M'\ell'1,\ \vec{r}|{}^{3}v_{even}|JM\ell 1, \vec{r}) = \delta_{JJ'}\delta_{MM'}\delta_{\ell\ell'}\ {}^{3\ell}f_1\ {}^{3\ell}g_1(\vec{r}')\ {}^{3\ell}g_1(\vec{r})$$

$$+\ {}^{3\ell}f_2\ {}^{3\ell}g_2(\vec{r}')\ {}^{3\ell}g_2(\vec{r})\ ,$$

where g's are arbitrary functions of $\vec{r}$ independent of J. Here, in order to reproduce different phase shifts for different J states (such as ${}^3\delta_\ell$, $J = \ell + 1$, ${}^3\delta_\ell$, $J = \ell - 1$), and those phase shifts with mixed ℓ's, we assume

$$(J'M'\ell'1,\vec{r}'|{}^3v_{even}|JM\ell\,1,\,\vec{r}) = \delta_{JJ'}\delta_{M'M}\Big[{}^{J3\ell\ell'}f_1\;{}^{J3\ell}g_1(r')\;{}^{J3\ell'}g_1(\vec{r})$$

$$+\;{}^{J3\ell\ell'}f_2\;{}^{J3\ell}g_2(r')\;{}^{J3\ell'}g_2(\vec{r})\Big]\;,$$

where the g functions are different for different J's. Suppose that the $g_i(F)$ function does not depend on the azimuthal angle ϕ. Then we can write

$$\int d\hat{r}\;\;Y_\ell^m(\hat{r})\;{}^{J3\ell}g_i\;(\vec{r}) = \delta_{m_0}\frac{4\pi}{\sqrt{4\pi}}\;\;{}^{J3\ell}h'_i(r) = \delta_{m_0}\sqrt{\frac{\pi}{2}}\;{}^{J3\ell}h_i(r)$$

and

$$\int r^2dr\;j_\ell(kr)\;{}^{J3\ell}h_i(r) = {}^{J3\ell}\phi_i(k),$$

where $i = 1$ or 2. Substituting these functions into the long expression of $(\vec{k}',\,M'_s\,|{}^3v_{even}|\vec{k},\,M_s)$, we obtain finally

$$(\vec{k}',M'_s|{}^3v_{even}|\vec{k},\,M_s) = \sum_{\ell'=even}^{\ell\,max}\;\;\sum_{\ell=even}^{\ell\,max}\;\;\sum_{J=(\ell-1)}^{\ell+1}\;(2J+1)\;e^{-i\,\pi/2\,(\ell'-\ell)}$$

$$\times C(JM'_s;\;\ell'01M'_s)\;C(JM_s;\ell01M_s)\,\mathcal{Y}_{\ell'1}^{JM'_s}\;(\hat{k}')\,\mathcal{Y}_{\ell1}^{*JM_s}\;(\hat{k})\;(k'|{}^{J3}v_{\ell'\ell}|k),\tag{46'}$$

where

$$(k'|{}^{J3}v_{\ell'\ell}|k) = \frac{(2\pi)^3}{\Omega} \left[{}^{J3\ell\ell'}f_1 \; {}^{J3\ell}\phi_1(k') \; {}^{J3\ell'}\phi_1(k) \right.$$

$$\left. + \; {}^{J3\ell\ell'}f_2 \; {}^{J3\ell}\phi_2(k') \; {}^{J3\ell'}\phi_2(k) \right] . \qquad (47')$$

We can do the same thing for ${}^{3}v_{odd}$ by introducing ${}^{J3}v_{\ell'\ell}$ with odd ℓ and ℓ'. Now each ${}^{J3}v_{\ell'\ell}$ is connected with a phase shift through the corresponding part of the free scattering reaction matrix, that is

$$(k'|{}^{J3}G_{\ell'\ell}|k) = (k'|{}^{J3}v_{\ell'\ell}|k) + \frac{\Omega M}{(2\pi)^3} \int d\vec{k}'' \, (k'|{}^{J3}v_{\ell'\ell}|k'') \frac{1}{k^2 - k''^2}$$

$$\times (k''|{}^{J3}G_{\ell'\ell}|k) \, (2J+1) \, \frac{1}{3} \sum_{M_s} \sum_{\substack{\ell''=\text{even} \\ \text{or odd}}}^{\ell\,\max} \sum_{\substack{\ell'''=\text{even} \\ \text{or odd}}}^{\ell\,\max}$$

$$e^{i\,\pi/2\,(\ell-\ell')} \; C(JM_s, \ell'' \, 0 \, 1M_s)$$

$$\times C(JM_s, \ell''' \, 0 \, 1M_s) \, \mathcal{Y}^{*JM_s}_{\ell''\ell}(\hat{k}'') \, \mathcal{Y}^{JM_s}_{\ell'''}(\hat{k}'')$$

and

$$(k|{}^{J3}G_{\ell'\ell}|k) = -\frac{8\pi^2}{\Omega \frac{M}{2} k} \tan {}^{J3}\delta_{\ell'\ell} .$$

Therefore, by choosing ${}^{J3}v_{\ell'\ell}$ carefully, we can fit all the phase shifts over a large energy range.

This modified potential gives an analytical solution for the R_B and R_G matrix elements. However, the solution,

$$(\vec{k}', M'_s|^3R_{\substack{\text{even}\\ \text{odd}}}\text{ or }|\vec{k}, M_s, \vec{P})$$

$$= \sum_{\substack{\ell'=\text{even}\\ \text{or odd}}}^{\ell\,\max} \sum_{\substack{\ell'\ \text{even}\\ \text{or odd}}}^{\ell\,\max} \sum_{J=(\ell-1)}^{\ell+1} (4\pi)^2\,(2J+1)\,e^{-i\,\pi/2\,(\ell'-\ell)}$$

$$\times\, C(J\,M'_s;\ \ell'\ 0\ 1\ M'_s)\ C(JM_s;\ \ell o\ 1M_s)$$

$$\mathcal{Y}^{*JM}_{\ell'1}(\hat{k}')\ \ \mathcal{Y}^{*JM_s}_{\ell 1}(\hat{k})\ (k'|^{J3}R_{\ell'\ell}|k, P), \tag{49'}$$

where $(k'|^{J3}R_{\ell'\ell}|k, P)$ can be calculated from $(k'|^{J3}v_{\ell'\ell}|k, P)$ according to an equation similar to Eq. (50), has a spin dependence much more complicated than that in the simple form of Eq. (33). Since our main concern is to see the momentum dependence of the R-matrix elements, we shall avoid this complication and use only the simpler nucleon-nucleon potential of Eqs. (45), (46), and (47).

In Appendix B, we have reached a plausible choice for the constants ${}^{I\ell}f_1$ and ${}^{I\ell}f_2$ and the functions

$$^{I\ell}\phi_1(k) = \frac{1 - {}^{I\ell}\alpha\, k^2}{(k^2 + {}^{I\ell}\beta^2)^{3/2}}\ , \tag{53a}$$

and

$$^{I\ell}\phi_2(k) = \frac{1}{(k^2 + {}^{I\ell}\beta^2)^{1/2}}\ . \tag{53b}$$

The potential parameters are given in Table II.

Table II. Parameters of the nucleon-nucleon potential

Parameters	State				
	I=1(S=0), ℓ=0	I=1, ℓ=2	J=3(S=1), ℓ=0	I=3, ℓ=2	All others
$f_1 \frac{\pi M}{\beta^5} \frac{(1+\alpha\beta^2)^2}{16}$	-0.96	-0.15	-1.23	-0.18	0
$f_2 \frac{\pi M}{\beta} \frac{1}{16}$	$+\infty$	0	$+\infty$	0	0
B in 10^{13}cm.$^{-1}$	1.86	1.0	2.13	1.4	
$\alpha\beta^2$		∞		∞	

The odd- and even-parity states for $\ell > 2$ are neglected in order to simplify the calculations. The positive infinite value for ${}^{10}f_2$ and the negative value for ${}^{10}f_1$ make the S-wave phase shift change sign at a certain energy, a result known as the direct consequence of a hard-core potential. Indeed, if we calculate the expectation value of the partial potential $Y_0^0(k)\, Y_0^0(k')\, {}^1v_0$ for wave packets separated by a distance r, we shall get an energy having a spatial dependence with a large positive core at small r. For D-wave partial potentials, we put ${}^{12}f_2 = 0$. This is because the D-wave states are usually not affected considerably by a hard core at a few hundred Mev.

The angular averages of the projection operators Q_B and Q_G are also different from each other in form. For Brueckner's definition, the integration over the intermediate momentum $\vec{k}''$ is to be carried out so that $|\vec{k}'' \pm \frac{\vec{P}}{2}| \geqq p_F$. Therefore, the corresponding cutoff function $Q_B(\vec{k}'', p)$ should take the form

$$\int d^3\vec{k}''\, Q_B(\vec{k}'', P) \to 4\pi \int_0^{\left(p_F^2 - \frac{p^2}{4}\right)^{1/2}} k''^2\, dk''\, 0$$

$$+ \int_{\left(p_F^2 - \frac{p^2}{4}\right)^{1/2}}^{p_F^2 + \frac{p}{2}} k''^2\, dk''\, 2\pi \int_{\cos^{-1}\frac{k''^2 + \frac{p^2}{4} - p_F^2}{k''p}}^{\pi - \cos^{-1}\frac{k''^2 + \frac{p^2}{4} - p_F^2}{k''p}} \sin\theta''\, d\theta''$$

$$+ 4\pi \int_{p_F + \frac{p}{2}}^{\infty} k''^2\, dk'' . \qquad (54)$$

For Galitskii's definition, the integration over the intermediate momentum $\vec{k}''$ is to be carried out so that for $|\vec{k}'' \pm \frac{\vec{P}}{2}| \geqq p_F$, a

plus sign is inserted in front of the integrand, and for $|\vec{k}'' \pm \frac{\vec{P}}{2}| \leq p_F$, a minus sign is inserted in front of the integrand. Thus we have

$$\int d^3\vec{K}''\, Q_G(\vec{k}'', p) \rightarrow -4\pi \int_0^{p_F - \frac{p}{2}} k''^2\, dk''$$

$$- \int_{p_F - \frac{p}{2}}^{\left(p_F^2 - \frac{p^2}{4}\right)^{1/2}} k''^2\, dk''\, 2\pi \int_{\pi - \cos^{-1} \frac{k''^2 + \frac{p^2}{4} - p_F^2}{k''p}}^{\cos^{-1} \frac{k''^2 + \frac{p^2}{4} - p_F^2}{k''p}} \sin\theta''\, d\theta''$$

$$+ \int_{\left(p_F^2 - \frac{p^2}{4}\right)^{1/2}}^{p_F + \frac{p}{2}} k''^2\, dk''\, 2\pi \int_{\cos^{-1} \frac{k''^2 + \frac{p^2}{4} - p_F^2}{k''p}}^{\pi - \cos^{-1} \frac{k''^2 + \frac{p^2}{4} - p_F^2}{k''p}} \sin\theta''\, d\theta''$$

$$+ 4\pi \int_{p_F + \frac{p}{2}}^{\infty} k''^2\, dk'' . \tag{55}$$

Before we proceed to discuss and make approximations about the denominator $e(k'', p)$ of Eqs. (50) and (52), let us decide first which matrix elements of R_B and R_G are to be calculated. In the interaction

term of Sawada's Hamiltonian, because the operators $C_{p,q}$ and $C^*_{p,q}$ vanish unless we have $|\vec{P}| \leq p_F$ and $|\vec{p} + \vec{q}| \geq p_F$, the summation $\sum_{p_1 p_2 q}$ is actually a sum of four different summations:

$$\sum_{\substack{|\vec{p}_1| < p_F \\ |p_1+q| > p_F \\ |p_2| < p_F \\ |p_2-q| > p_F}} C^*_{p_1,q}\, C^*_{p_2} \qquad \sum_{\substack{|p_1| < p_F \\ |p_1+q| > p_F \\ |p_2-q| < p_F \\ |p_2| > p_F}} C^*_{p_1,q}\, C_{p_2-q;q}$$

$$\sum_{\substack{|p_1+q| < p_F \\ |p_1| > p_F \\ |p_2| < p_F \\ |p_2-q| > p_F}} C^*_{p_2-q}\, C_{p_1+q,-q} \qquad \sum_{\substack{|p_1+q| < p_F \\ |p_1| > p_F \\ |p_2-q| < p_F \\ |p_2| > p_F}} C_{p_1+q,-q}\, C_{p_2-q,q} \,.$$

In the limit $\Omega \to \infty$, these summations become integrations over limited regions in momentum space. For a particular $\vec{q}$ ($\ll p_F$), the four integration areas are shown in Fig. 4. From the shape of the above shaded areas, we see that for most of the contributing matrix elements, $\vec{p}_1$ and $\vec{p}_2$ have either the same or opposite directions. Furthermore, the main contribution to a Sawada's Hamiltonian comes when $|\vec{q}|$ is small. Therefore, $(\vec{p}_1, \vec{p}_1 + \vec{q}|R|\vec{p}_1 + \vec{q}, \vec{p}_1)$ and $(\vec{p}_1, -\vec{p}_1 + \vec{q}|R|\vec{p}_1 + \vec{q}, -\vec{p}_1)$ with $|\vec{q}| \cong 0$, $|\vec{p}_1| \cong p_F$ are two of the characteristic matrix elements and will be calculated in detail. In addition to these two, we shall also calculate $(0, 0|R|0, 0)$, which tells the strength when $\vec{p}_1, \vec{p}_2 = 0$, and corresponds to V_0 of Eq. (34).

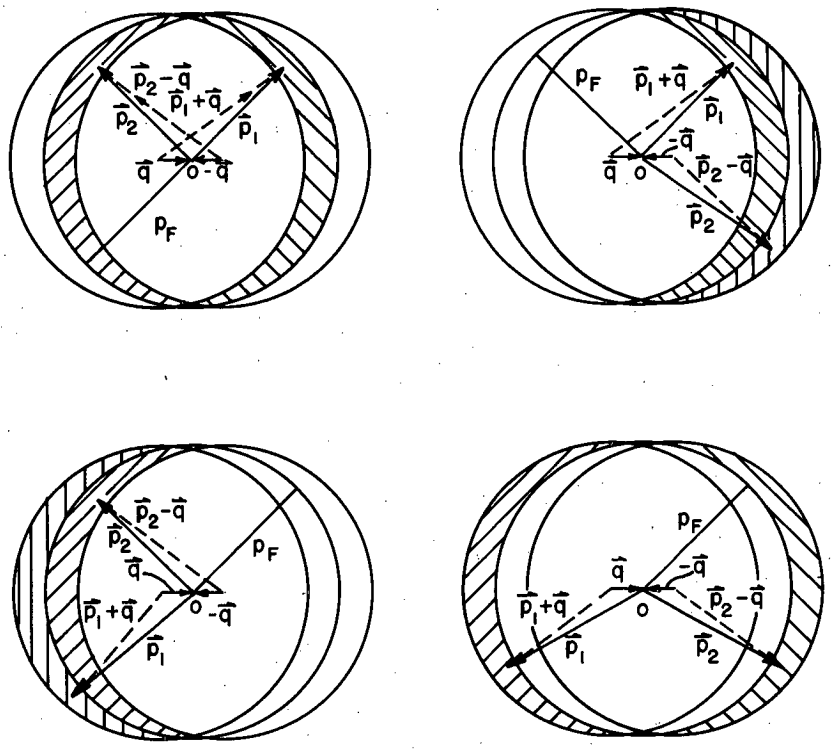

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Fig. 4. The areas of integration for a particular value of $\vec{q}$.

By comparing the calculated results of these three matrix elements with the corresponding forms deduced from Eq. (34), we can determine the values of the parameters V_0, a_0, and c_0.

For $R_B(0)$, we simplify the calculation further by neglecting the self-consistency requirements and assuming the effective mass approximation, $E_p = \frac{p^2}{2M^*} + \text{constant}$. For M^* we adopt the phenomenological form of E_p of Karplus and Watson,[15]

$$E_p = \frac{p^2}{2M} - A\,\frac{60\ \text{Mev}}{\eta^3\ A}\left[1 - 0.21\left(\frac{3}{5\eta^2} + \frac{p^2}{(270)^2}\right]\right.,$$

where η is the conventional radius parameter defined by

$$\Omega = \eta^3\,\Omega_0 = \eta^3 \times \left(\frac{4}{3}\ (1.1\times 10^{-11}\ \text{cm})^3\right). \tag{56}$$

The density of the system is measured (reciprocally) by η, which is related to the Fermi momentum by

$$p_F = \frac{\sqrt[3]{9\pi/8}}{\eta \times 1.1\times 10^{-13}\ \text{cm}} = \frac{1.4}{\eta}\ 10^{13}\ \text{cm}^{-1} = \frac{270}{\eta}\ \frac{\text{Mev}}{c}.$$

Therefore, we have

$$\frac{M}{M^*} = \frac{E_p - \text{constant}}{p^2/2M} = 1 + \frac{0.4}{\eta^3}. \tag{57}$$

The denominator $e(k'', p)$ becomes $(k^2 - k''^2)/M^*$. Now we are ready to calculate ${}^{I\ell}I_{11}$, ${}^{I\ell}I_{12}$, ${}^{I\ell}I_{22}$ in Eq. (52) and then the matrix elements $(0|{}^{I\ell}R_B(0)|0,0)$ $(0|{}^{I\ell}R_B(0)|0, 2p_F)$ and $(p_F|{}^{I\ell}R_B(0)|p_F, 0)$.

The matrix element $(k = p_F|^{I\ell}R_B(0)|k = p_F, 0)$ is actually obtained by extrapolating the computed values at $k = 0.9$ and $k = 1.1\ p_F$. For consistency, the matrix element $(0|^{I\ell}R_B(0)|0, P = 2p_F)$ is also obtained by extrapolating the computed values at $P = 2(0.9\ p_F)$ and $P = 2(1.1\ p_F)$. The results are listed in Table III.

For R_G-matrix elements, we first borrow E_p defined for calculating $R_B(0)$ as the single-particle energy E'_p. In other words, we still use $e_G(k'', P) = \frac{1}{M^*}(k^2 - k''^2)$ with M^* given by Eq. (57). The calculating procedure is the same as before. The results for $\eta = 1$ are given in the first column of Table IV. The matrix element $(0|R_G|0, 2p_F)$ is exactly equal to $(0|R_B(0)|0, 2p_F)$, and the difference between $(p_F|R_G|p_F, 0)$ and $(p_F|R_B(0)|p_F, 0)$ is negligibly small. If, on the other hand, we use a more consistent definition for E'_p, i.e.

$$E'_p = \frac{p^2}{2M} + \sum_{n < P_F} \langle np|R_G|np\rangle, \tag{58}$$

E'_p differs from E_p mainly when p is small, and the effect is to raise E_p to less negative values. In the effective-mass approximation this means a greater effective mass. We take $e_G(k'', p) = 1/M^{*\prime}(k^2 - k''^2)$ and $M/M^{*\prime} = 1 + 0.2/\eta^3$. The result of the calculation at empirical equilibrium density $(\eta = 1)$ is given in the second column of Table IV.

Table III. $R_B(0)$ matrix elements as functions of η in units of $2\pi^2/\Omega M^* p_F$

Matrix elements	States	η				
		0.8	0.9	1.0	1.1	1.2
$(k=0\|R_B(0)\|k=0,P)$	^1D, ^3D	0	0	0	0	0
$(k=0\|R_B(0)\|k=0,P=0)$	^1S	-0.66	-0.87	-1.04	-1.12	-1.16
	^3S	-1.10	-1.34	-1.40	-1.32	-1.10
$(0\|R_B(0)\|0,2p_F)$[a]	^1S	0.88	0.71	0.48	0.32	0.22
	^3S	0.60	0.40	0.22	0.14	0.10
$(k=p_F\|R_B(0)\|k=p_F,0)$[a]	^1S	0.44	0.28	0.05	-0.11	-0.21
	^1D	-0.12	-0.15	-0.16	-0.16	-0.16
	^3S	0.20	-0.00	-0.17	-0.26	-0.31
	^3D	-0.12	-0.13	-0.15	-0.17	-0.18

[a] Extrapolated from computed values at $0.9\ P_F$ and $1.1\ P_F$.

Table IV. R_G Matrix elements in units of $2\pi^2/\Omega M^* p_F$ at $\eta = 1$

Matrix elements	States	$M/M^* = 1.4$	$M/M^* = 1.2$
$(k=0\|R_G\|k=0, P)$	^1D, ^3D	0	0
$(k=0\|R_G\|k=0, P=0)$	^1S	-0.76	-0.88
	^3S	-0.78	-0.88
$(0\|R_G\|0, 2p_F)$ [a]	^1S	0.48	0.40
	^3S	0.22	0.09
$(k=p_F\|R_G(0)\|k=p_F, 0)$ [a]	^1S	0.05	-0.02
	^1D	-0.16	-0.18
	^3S	-0.17	-0.23
	^3D	-0.15	-0.21

[a]Extrapolated from computed values at $0.9\ p_F$ and $1.1\ p_F$.

B. Results

From Eqs. (33), (37), and (48) we can obtain a simple relation between $\epsilon'_K\,(\vec{p}\,\vec{p}+\vec{q}|R|\vec{p}+\vec{q},\ \vec{p})$ and the corresponding matrix elements of ${}^1R_{even}$, ${}^3R_{even}$. We have, for mode I,

$$\epsilon'_I\, R(\vec{p}_1,\ \vec{p}_2,\ \vec{q}) = \frac{1}{2}\left[3(\vec{k}'|{}^1R_{even}|\vec{k},\ \vec{P}) + 3(\vec{k}'|{}^3R_{even}|\vec{k},\ \vec{P})\right];$$

for mode II,

$$\epsilon'_{II}\, R(\vec{p}_1,\ \vec{p}_2,\ \vec{q}) = \frac{1}{2}\left[-3(\vec{k}'|{}^1R_{even}|\vec{k},\ \vec{P}) + (\vec{k}'|{}^3R_{even}|\vec{k},\ \vec{P})\right];$$

for mode III,

$$\epsilon'_{III}\, R(\vec{p}_1,\ \vec{p}_2,\ \vec{q}) = \frac{1}{2}\left[(\vec{k}'|{}^1R_{even}|\vec{k},\ \vec{P}) - 3(\vec{k}'|{}^3R_{even}|\vec{k},\ \vec{P})\right];$$

and for mode IV,

$$\epsilon'_{IV}\, R(\vec{p}_1,\ \vec{p}_2,\ \vec{q}) = \frac{1}{2}\left[-(\vec{k}'|{}^1R_{even}|\vec{k},\ \vec{P}) - (\vec{k}'|{}^3R_{even}|\vec{k},\ \vec{P})\right].$$

In addition we have, for $k = P = 0$, $(k' = k)$,

$$\epsilon'_K\, R(0,0,0) = \epsilon'_K\, V_0(0);$$

for $k = 0$, $P = 2p_F$, $(k' = k)$,

$$\epsilon'_K\, R(p_F,\ p_F,\ 0) = \epsilon'_K\, V_0(0)\left[1 + (2a_0 + c_0)\, p_F^2\right];$$

and for $k = p_F$, $P = 0$, $(k' = k)$,

$$\epsilon'_K\, R(-p_F,\ p_F,\ 0) = \epsilon'_K\, V_0(0)\left[1 + (2a_0 - c_0)p_F^2\right].$$

With these relations, we can calculate $\epsilon'_K\, V_0(0)$, a_0, c_0 for each mode. The values are given in Table V. Accordingly λ_K and cp_F^2 can be easily obtained. They are given in Table VI.

Table V. Values for the parameters ϵ'_K, V_0, a_0, and c_0, where V_0 is in units of $2\pi^2/\Omega M^* P_F$

Mode	Parameters	$R_B(0)$					R_G	
		η=0.8	0.9	1.0	1.1	1.2	1.0	1.0
I	$\epsilon'_I V_0(0)$	-2.64	-3.32	-3.66	-3.66	-3.39	-2.32	-2.64
	$2a_0 p_F^2$	-1.26	-1.0	-0.80	-0.68	-0.57	-0.66	-0.52
	$c_0 p_F^2$	-0.59	-0.50	-0.49	-0.52	-0.60	-0.82	-0.77
II	$\epsilon'_{II} V_0(0)$	0.44	0.64	0.86	1.02	1.19	0.75	0.88
	$2a_0 p_F^2$	-2.07	-1.41	-0.83	-0.79	-0.67	-0.87	-0.87
	$c_0 p_F^2$	-1.25	-0.93	-0.73	-0.62	-0.55	-0.96	-0.71
III	$\epsilon'_{III} V_0(0)$	1.32	1.58	1.58	1.42	1.07	0.79	0.88
	$2a_0 p_F^2$	-1.00	-0.83	-0.71	-0.59	-0.46	-0.49	-0.14
	$c_0 p_F^2$	-0.35	-0.32	-0.36	-0.44	-0.46	-0.67	-0.80
IV	$\epsilon'_{IV} V_0(0)$	0.88	1.11	1.22	1.22	1.13	0.77	0.88
	$2a_0 p_F^2$	-1.26	-1.0	-0.80	-0.68	-0.57	-0.66	-0.52
	$c_0 p_F^2$	-0.59	-0.50	-0.49	-0.52	-0.50	-0.82	-0.77
							$\frac{M}{M^*}$=1.4	$\frac{M}{M^*}$=1.2

Table VI. Values for λ_K and cp_F^2, where

$$\lambda_K = \frac{\Omega M^* p_F}{2\pi^2} \epsilon'_K V_0(0)(1 + 2a_0 p_F^2)$$

and

$$cp_F^2 = c_0 p_F^2 (1 \mp 2a_0 p_F^2)^{-1}$$

Mode	Parameters	$R_B(0)$					R_G	
		η=0.8	0.9	1.0	1.1	1.2	1.0	1.0
I	λ_I	0.70	0	-0.73	-1.17	-1.46	-0.79	-1.26
	cp_F^2	2.27	$+\infty, -\infty$	-2.45	-1.63	-1.40	-2.42	-1.60
II	λ_{II}	-0.47	-0.26	0.15	0.21	0.39	0.10	0.11
	cp_F^2	1.17	2.26	-4.35	-2.95	-1.67	-7.38	-5.35
III	λ_{III}	0	0.27	0.46	0.58	0.58	0.40	0.76
	cp_F^2	$+\infty, -\infty$	-1.88	-1.24	-1.07	-0.85	-1.31	-0.93
IV	λ_{IV}	0.25	0	0.24	0.39	0.49	0.23	0.42
	cp_F^2	2.27	$+\infty, -\infty$	-2.45	-1.63	-1.40	-2.42	-1.57
							$\frac{M}{M^*}$=1.4	$\frac{M}{M^*}$=1.2

We see from Tables V and VI that when the density increases until η reaches a value between 1.1 and 1.0, $1/\lambda_I > -1$, the simple compressional mode (K=1) has an imaginary solution. At $\eta=1$ and with large densities, the simple compressional oscillation appears. In the density region $\eta = 1.0$ to 0.9, the λ and cp_F^2 values of the other three modes fall into Case 2 in Fig. 3. Since $(1 + \lambda\, cp_F^2/3)(cp_F^2)^{-1}$ is smaller than one, the upper branch of function $f(x)$ does not intersect with function $Q'(x)$. There are no zero sound waves for these three modes. On the other hand, if $(1 + \lambda\, cp_F^2/3)$ is larger than $|cp_F^2|$, the lower branch of $f(x)$ does not intersect with $Q(x)$, and the system is stable. We find an unusual situation that instead of the simple compressional mode, the spin, i-spin, and coupled spin-i-spin modes are missing. The stability of the system is, however, established at the nuclear-equilibrium density. The velocity of the simple compressional mode is approximately $p_F/M^* = 0.405$ c.

The Galitskii's projection operator makes little difference for the matrix element at the Fermi surface, and so does not significantly alter the general situation. It yields a slightly larger energy of excitation for the simple compressional mode. The effective mass $(M^*/M = \eta^3/\eta^3 + 0.2)$ still does not provide spin, i-spin, or coupled spin-i-spin oscillations. Furthermore, it causes instability for the simple compressional mode.

The Brueckner's energy and the contribution of ring diagrams are calculated for $\eta = 0.8$, 0.9, and 1.0 (see Table VII). The ring-diagram contribution is of the order of the effect of the rearrangement correction to Brueckner's energy at equilibrium density, which tends to shift toward a higher density.

Table VII. Ground-state energy in Mev.

Ground state energy[a]		η 0.8	0.9	1.0	1.1
$(\epsilon_B + \Delta\epsilon)/A$ [b]		12.52	- 0.28	-12.96	-11.3
$\frac{\epsilon_0(H_s)}{A}$	$-\epsilon_c/A$	- 1.35	- 1.08	- 0.84	
	$\epsilon_{scatt.}/A$	0.66	0.026	- 0.05	
	$\epsilon_{zero\ sound}/A$	0.02	0.009	small	
	$\epsilon_0(H_s)/A$	- 0.67	- 1.04	- 0.89	
$[\epsilon_B + \Delta\epsilon + \epsilon_0(H_s)]/A$ [b]		11.85	- 1.32	-13.85	

[a] Here we have $V_0(q) = V_0(0)/(1 + q^2/\mu^2)$

[b] Here we have

$$\epsilon_B + \Delta\epsilon = \sum_n \frac{p_n^2}{2M} + \frac{1}{2} \sum_{p_1 n_2} (n_1 n_2 | R_B(0) | n_1 n_2)$$

$$= A \left[\frac{3}{5} \frac{p_F^2}{2M} + \frac{1}{2} \frac{\Omega 4\pi}{(2\pi)^3} \frac{p_F^3}{3} \epsilon'_I V_0(0)(1 + \frac{6}{5} a_0 p_F^2) \right] .$$

V. CONCLUDING REMARKS

We have attempted to estimate the effect of pair interaction on the ground-state energy of nuclear matter by using a Sawada's Hamiltonian H_s. The effective interaction has been taken as the Brueckner's reaction matrix, so that a calculation of the lowest energy of H_s gives the net contribution from pair interaction to Brueckner's first-order ground-state energy. The method is shown to be useful, there are no unstable states of H_s found in the neighborhood of equilibrium density, and the calculation is straightforward.

On the other hand, both of the two reaction matrices R_B and R_G calculated in Section IV yield only simple compressional oscillations for a large nucleus, because a strong dependence of the (Fermi) surface matrix elements of R_B and R_G on the total momentum P of the two colliding nucleons appears. The dependences on P and on the relative momentum k of the R_B and R_G matrix elements obtained in Section IV have an order of magnitude $V_0[1 - 0.1\,(2k)^2 - 0.3\,P^2]$. Were the P^2 and the $(2k)^2$ coefficients exchanged, the parameter c_0 would become positive while a_0 would remain the same, and the other three collective modes ($K = 2$, 3, and 4) would appear instead of the simple compressional mode.

It is possible, of course, that we have overemphasized the total momentum dependence by using the factorable nucleon-nucleon potential. Our including the D-wave potential in Eqs. (46) and (47) without making corresponding modifications on the spin dependence of the two-nucleon potential may also be responsible for this situation favoring one single mode. However, since the average strength of the effective interaction at the Fermi surface is insensitive to the type of angular and spin

dependences, we may still conclude from this calculation that the nucleon interactions at the Fermi surface are weak enough to allow collective oscillations.

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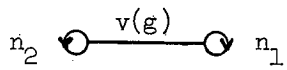
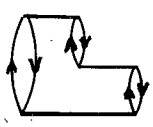
VI. APPENDIXES

A. Expansion of the Ground-State Energy in Terms of $v(g)$

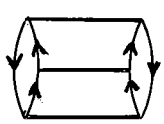
$$\epsilon = \sum_{n<P_F} E_n + \frac{1}{2} \sum_{n_1 n_2 < P_F} \langle n_2 n_1 | v(g) | n_2 n_1 \rangle - \sum_{n<P_F} \langle n | V(g) n \rangle,$$



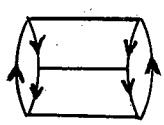
$$+ \frac{1}{4} \sum_{\substack{n_1 n_2 < P_F \\ k_1 k_2 > P_F}} \frac{\langle n_2 n_1 | v(g) | k_2 k_1 \rangle \langle k_2 k_1 | v(g) | n_2 n_1 \rangle}{E_{n_1} + E_{n_2} - E_{k_1} - E_{k_2}}$$



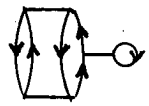
$$+ \left\{ \sum_{\substack{n_1 n_2 n_3 < P_F \\ k_1 k_2 k_3 > P_F}} \frac{\langle n_3 n_1 | v(g) | k_3 k_1 \rangle \langle k_3 n_2 | v(g) | n_3 k_2 \rangle \langle k_2 k_1 | v(g) | n_2 n_1 \rangle}{(E_{n_1} + E_{n_3} - E_{k_1} - E_{k_3})(E_{n_1} + E_{n_2} - E_{k_1} - E_{k_2})} \right.$$



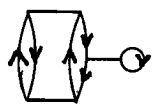
$$+ \frac{1}{8} \sum_{\substack{n_1 n_2 < P_F \\ k_1 \cdots k_4 > P_F}} \frac{\langle n_2 n_1 | v(g) | k_4 k_3 \rangle \langle k_4 k_3 | v(g) | k_2 k_1 \rangle \langle k_2 k_1 | v(g) | n_2 n_1 \rangle}{(E_{n_1} + E_{n_2} - E_{k_3} - E_{k_4})(E_{n_1} + E_{n_2} - E_{k_1} - E_{k_2})}$$



$$+ \frac{1}{8} \sum_{\substack{n_1 \cdots n_4 < P_F \\ k_1 k_2 > P_F}} \frac{\langle n_4 n_3 | v(g) | k_2 k_1 \rangle \langle n_4 n_3 | v(g) | n_2 n_1 \rangle \langle k_2 k_1 | v(g) | n_2 n_1 \rangle}{(E_{n_3} + E_{n_4} - E_{k_1} - E_{k_2})(E_{n_1} + E_{n_2} - E_{k_1} - E_{k_2})}$$

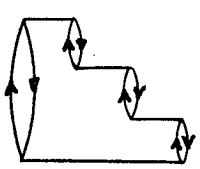


$$+\frac{1}{8}\sum_{\substack{n_1 n_2 < P_F \\ k_1 k_2 k_3 > P_F}} \frac{\langle n_2 n_1 | v(g) | k_3 k_1 \rangle \left[\sum_{k_3 < P_F} \langle k_3 n_3 | v(g) k_2 n_3 \rangle - \langle k_3 | V(g) | k_2 \rangle \right] \langle k_2 k_1 | v(g) | n_2 n_1 \rangle}{(E_{n_1} + E_{n_2} - E_{k_1} - E_{k_2})(E_{n_1} + E_{n_2} - E_{k_1} - E_{k_3})}$$



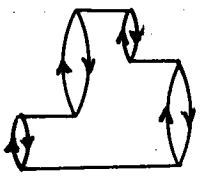
$$+\frac{1}{8}\sum_{\substack{n_1 \cdots n_3 < P_F \\ k_1 k_2 > P_F}} \left(\frac{\langle n_3 n_1 | v(g) | k_2 k_1 \rangle \left[\sum_{n_4 < P_F} \langle n_3 n_4 | v(g) | n_2 n_4 \rangle - \langle n_3 | V(g) | n_2 \rangle \right]}{(E_{n_1} + E_{n_2} - E_{k_1} - E_{k_2})} \right)$$

$$\times \left(\frac{\langle k_2 k_1 | v(g) | n_2 n_1 \rangle}{(E_{n_3} + E_{n_1} - E_{k_1} - E_{k_2})} \right) \Bigg\}$$



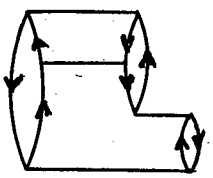
$$+\left\{ \sum_{\substack{n_1 \cdots n_4 < P_F \\ k_1 \cdots k_4 > P_F}} \left(\frac{\langle n_4 n_1 | v(g) | k_4 k_1 \rangle \langle k_4 n_3 | v(g) | n_4 k_3 \rangle}{(E_{n_1} + E_{n_4} - E_{k_1} - E_{k_4})(E_{n_1} + E_{n_3} - E_{k_1} - E_{k_2})} \right) \right.$$

$$\times \left(\frac{\langle k_3 n_2 | v(g) | n_3 k_2 \rangle \langle k_2 k_1 | v(g) | n_2 n_1 \rangle}{(E_{n_1} + E_2 - E_{k_1} - E_{k_2})} \right)$$



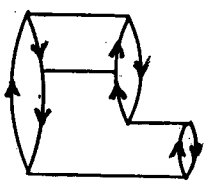
$$+ \sum_{\substack{n_1 \cdots n_4 < P_F \\ k_1 \cdots k_4 > P_F}} \left(\frac{\langle n_4 n_3 | v(g) | k_4 k_3 \rangle \langle k_4 n_1 | v(g) | n_4 k_1 \rangle}{(E_{n_3} + E_{n_4} - E_{k_3} - E_{k_4})(E_{n_1} + E_{n_3} - E_{k_1} - E_{k_3})} \right)$$

$$\times \left(\frac{\langle k_3 n_2 | v(g) | n_3 k_2 \rangle \langle k_2 k_1 | v(g) | n_2 n_1 \rangle}{(E_{n_1} + E_{n_2} - E_{k_1} - E_{k_2})} \right)$$



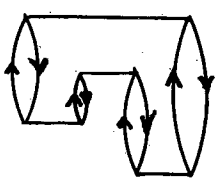
$$+ \sum_{\substack{n_1 \cdots n_4 < P_F \\ k_1 \cdots k_4 > P_F}} \left(\frac{\langle n_4 n_1 | v(g) | k_3 k_4 \rangle \langle k_4 n_3 | v(g) | k_1 n_4 \rangle}{(E_{n_4} + E_{n_1} - E_{k_4} - E_{k_3})(E_{n_3} + E_{n_1} - E_{k_3} - E_{k_1})} \right)$$

$$\times \left(\frac{\langle k_3 n_2 | v(g) | n_3 k_2 \rangle \langle k_2 k_1 | v(g) | n_2 n_1 \rangle}{(E_{n_1} + E_{n_2} - E_{k_1} - E_{k_2})} \right)$$



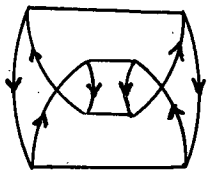
$$+ \sum_{\substack{n_1 \cdots n_4 < P_F \\ k_1 \cdots k_4 > P_F}} \left(\frac{\langle n_4 n_3 | v(g) | k_1 k_4 \rangle \langle n_1 k_4 | v(g) | n_4 k_3 \rangle}{(E_{n_4} + E_{n_3} - E_{k_4} - E_{k_1})(E_{n_3} + E_{n_1} - E_{k_3} - E_{k_1})} \right)$$

$$\times \left(\frac{\langle k_4 n_2 | v(g) | n_3 k_2 \rangle \langle k_2 k_1 | v(g) | n_2 n_1 \rangle}{(E_{n_2} + E_{n_1} - E_{k_2} - E_{k_1})} \right)$$



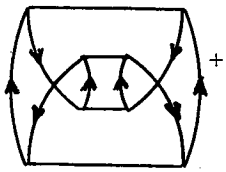
$$+ \sum_{\substack{n_1 \cdots n_4 < P_F \\ k_1 \cdots k_4 > P_F}} \left(\frac{\langle n_3 n_1 | v(g) | k_3 k_1 \rangle \langle n_4 n_2 | v(g) | k_4 k_2 \rangle}{(E_{n_1} + E_{n_3} - E_{k_1} - E_{k_3})(E_{n_1} + E_{n_2} + E_{n_3} + E_{n_4} - E_{k_1} - E_{k_2} - E_{k_3} - E_{k_4})} \right)$$

$$\times \left(\frac{\langle k_4 k_3 | v(g) | n_4 n_3 \rangle \langle k_2 k_1 | v(g) | n_2 n_1 \rangle}{E_{n_1} + E_{n_2} - E_{k_1} - E_{k_2})} \right)$$



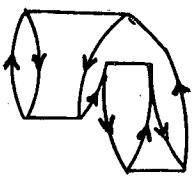
$$+ \frac{1}{16} \sum_{\substack{n_1 \cdots n_4 < P_F \\ k_1 \cdots k_4 > P_F}} \left(\frac{\langle n_2 n_1 | v(g) | k_4 k_3 \rangle \langle n_4 n_3 | v(g) | k_2 k_1 \rangle}{(E_{n_2} + E_{n_1} - E_{k_4} - E_{k_3})(E_{n_1} + E_{n_2} + E_{n_3} + E_{n_4} - E_{k_1} - E_{k_2} - E_{k_3} - E_{k_4})} \right)$$

$$\times \left(\frac{\langle k_4 k_3 | v(g) | n_4 n_3 \rangle \langle k_2 k_1 | v(g) | n_2 n_1 \rangle}{(E_{n_1} + E_{n_2} - E_{k_1} - E_{k_2})} \right)$$



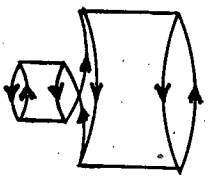
$$+ \frac{1}{16} \sum_{\substack{n_1 \cdots n_4 < P_F \\ k_1 \cdots k_4 > P_F}} \left(\frac{\langle n_4 n_3 | v(g) | k_2 k_1 \rangle \langle k_4 k_3 | v(g) | n_2 n_1 \rangle}{(E_{n_4} + E_{n_3} - E_{k_1} - E_{k_2})(E_{n_1} + E_{n_2} + E_{n_3} + E_{n_4} - E_{k_1} - E_{k_2} - E_{k_3} - E_{k_4})} \right)$$

$$\times \left(\frac{\langle k_4 k_3 | v(g) | n_4 n_3 \rangle \langle k_2 k_1 | v(g) | n_2 n_1 \rangle}{(E_{n_1} + E_{n_2} - E_{k_1} - E_{k_2})} \right)$$



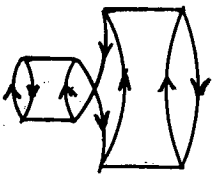
$$+ \frac{1}{4} \sum_{\substack{n_1 \cdots n_4 < P_F \\ k_1 \cdots k_4 > P_F}} \left(\frac{\langle n_4 n_3 | v(g) | k_2 k_4 \rangle \langle k_3 k_1 | v(g) | n_2 n_1 \rangle}{(E_{n_3} + E_{n_4} - E_{k_2} - E_{k_4})(E_{n_1} + E_{n_2} + E_{n_3} + E_{n_4} - E_{k_1} - E_{k_2} - E_{k_3} - E_{k_4})} \right)$$

$$\times \left(\frac{\langle k_4 k_3 | v(g) | n_4 n_3 \rangle \langle k_2 k_1 | v(g) | n_2 n_1 \rangle}{(E_{n_1} + E_{n_2} - E_{k_1} - E_{k_2})} \right)$$



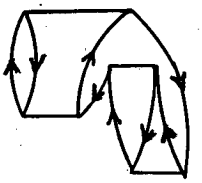
$$+ \frac{1}{4} \sum_{\substack{n_1 \cdots n_4 < P_F \\ k_1 \cdots k_4 > P_F}} \left(\frac{\langle n_3 n_4 | v(g) | k_4 k_2 \rangle \langle n_2 n_1 | v(g) | k_3 k_1 \rangle}{(E_{n_2} + E_{n_1} - E_{k_1} - E_{k_3})(E_{n_1} + E_{n_2} + E_{n_3} + E_{n_4} - E_{k_1} - E_{k_2} - E_{k_3} - E_{k_4})} \right)$$

$$\times \left(\frac{\langle k_4 k_3 | v(g) | n_4 n_3 \rangle \langle k_2 k_1 | v(g) | n_2 n_1 \rangle}{(E_{n_1} + E_{n_2} - E_{k_1} - E_{k_2})} \right)$$



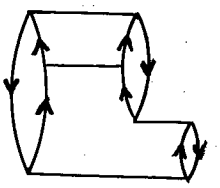
$$+\frac{1}{4}\sum_{\substack{n_1\cdots n_4<P_F\\ k_1\cdots k_4>P_F}}\left(\frac{\langle n_3n_1|v(g)|k_2k_1\rangle\langle n_2n_4|v(g)|k_4k_3\rangle}{(E_{n_3}+E_{n_1}-E_{k_2}-E_{k_1})(E_{n_1}+E_{n_2}+E_{n_3}+E_{n_4}-E_{k_1}-E_{k_2}-E_{k_3}-E_{k_4})}\right)$$

$$\times\left(\frac{\langle k_4k_3|v(g)|n_4n_3\rangle\langle k_2k_1|v(g)|n_2n_1}{(E_{n_1}+E_{n_2}-E_{k_1}-E_{k_2})}\right)$$



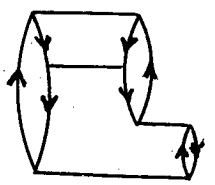
$$+\frac{1}{4}\sum_{\substack{n_1\cdots n_4<P_F\\ k_1\cdots k_4>P_F}}\left(\frac{\langle n_3n_1|v(g)|k_2k_1\rangle\langle n_2n_4|v(g)|k_4k_3\rangle}{(E_{n_2}+E_{n_4}-E_{k_3}-E_{k_4})(E_{n_1}+E_{n_2}+E_{n_3}+E_{n_4}-E_{k_1}-E_{k_2}-E_{k_3}-E_{k_4})}\right)$$

$$\times\left(\frac{\langle k_4k_3|v(g)|n_4n_3\rangle\langle k_2k_1|v(g)|n_2n_1\rangle}{(E_{n_1}+E_{n_2}-E_{k_1}-E_{k_2})}\right)$$



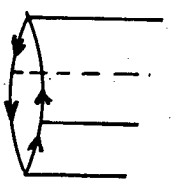
$$+\frac{2}{2}\sum_{\substack{n_1\cdots n_3<P_F\\ k_1\cdots k_5>P_F}}\left(\frac{\langle n_3n_1|v(g)|k_5k_4\rangle\langle k_5k_4|V|k_3k_1\rangle}{(E_{n_3}+E_{n_1}-E_{k_5}-E_{k_4})(E_{n_3}+E_{n_1}-E_{k_3}-E_{k_1})}\right)$$

$$\times\left(\frac{\langle k_3n_2|v(g)n_3k_2\rangle\langle k_2k_1|v(g)|n_2n_1\rangle}{(E_{n_2}+E_{n_1}-E_{k_1}-E_{k_2})}\right)$$



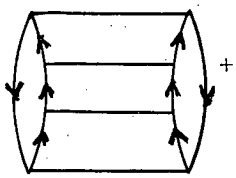
$$\sum_{\substack{n_1\cdots n_5<P_F\\ k_1\cdots k_3>P_F}}\left(\frac{\langle n_5n_4|v(g)|k_3k_1\rangle\langle n_5n_4|v(g)|n_3n_2\rangle}{(E_{n_4}+E_{n_5}-E_{k_1}-E_{k_3})(E_{n_1}+E_{n_3}-E_{k_1}-E_{k_3})}\right)$$

$$\times\left(\frac{\langle k_3n_2|v(g)|n_3k_2\rangle\langle k_2k_1|v(g)|n_2n_1\rangle}{(E_{n_2}+E_{n_1}-E_{k_2}-E_{k_1})}\right)$$



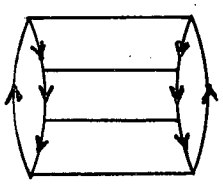
$$+\frac{2}{16}\sum_{\substack{n_1\cdots n_4<P_F\\ k_1\cdots k_4>P_F}}\left(\frac{\langle n_4n_3|v(g)|k_4k_3\rangle\langle n_4n_3|v(g)|n_2n_1\rangle}{(E_{n_4}+E_{n_3}-E_{k_4}-E_3)(E_{n_1}+E_{n_2}-E_{k_4}-E_{k_3})}\right)$$

$$\times\left(\frac{\langle k_4k_3|v(g)|k_2k_1\rangle\langle k_2k_1|v(g)|n_2n_1\rangle}{(E_{n_2}+E_{n_1}-E_{k_2}-E_{k_1})}\right)$$



$$+ \frac{1}{16} \sum_{\substack{n_1 n_2 < P_F \\ k_1 \cdots k_6 > P_F}} \left(\frac{\langle n_2 n_1 | v(g) | k_6 k_5 \rangle \langle k_6 k_5 | v(g) | k_4 k_3 \rangle}{(E_{n_1} + E_{n_2} - E_{k_6} - E_{k_5})(E_{n_1} + E_{n_2} - E_{k_4} - E_{k_3})} \right)$$

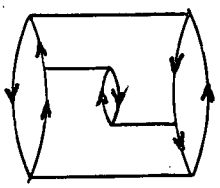
$$\times \left(\frac{\langle k_4 k_3 | v(g) | k_2 k_1 \rangle \langle k_2 k_1 | v(g) | n_2 n_1 \rangle}{(E_{n_1} + E_{n_2} - E_{k_2} - E_{k_1})} \right)$$



$$+ \frac{1}{16} \sum_{\substack{n_1 \cdots n_6 < P_F \\ k_1 k_2 > P_F}} \left(\frac{\langle n_6 n_5 | v(g) | k_2 k_1 \rangle \langle n_6 n_5 | v(g) | n_4 n_3 \rangle}{(E_{n_6} + E_{n_5} - E_{k_2} - E_{k_1})(E_{n_4} + E_{n_3} - E_{k_2} - E_{k_1})} \right)$$

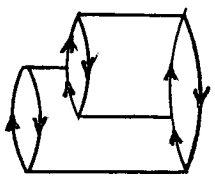
$$\times \left(\frac{\langle n_4 n_3 | v(g) | n_2 n_1 \rangle \langle k_2 k_1 | v(g) | n_2 n_1 \rangle}{(E_{n_2} + E_{n_1} - E_{k_2} - E_{k_1})} \right)$$

+ fourth-order terms containing one or two $\left[\sum_{n < P_F} \langle \ell n | v(g) | \ell n \rangle - \langle \ell | V | \ell \rangle \right]$ factors.



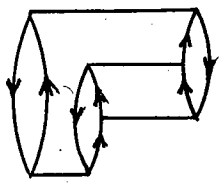
$$+ \frac{1}{2} \sum_{\substack{n_1 \cdots n_3 < P_F \\ k_1 \cdots k_5 > P_F}} \left(\frac{\langle n_2 n_1 | v(g) | k_5 k_3 \rangle \langle n_3 k_5 | v(g) | k_4 k_2 \rangle}{(E_{n_2} + E_{n_1} - E_{k_5} - E_{k_3})(E_{n_1} + E_{n_2} + E_{n_3} - E_{k_2} - E_{k_3} - E_{k_4})} \right)$$

$$\times \left(\frac{\langle k_4 k_3 | v(g) | n_3 k_1 \rangle \langle k_2 k_1 | v(g) | n_2 n_1 \rangle}{(E_{n_1} + E_{n_2} - E_{k_1} - E_{k_2})} \right)$$



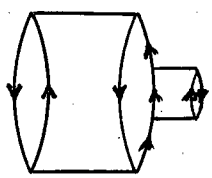
$$+ \sum_{\substack{n_1 \cdots n_3 < P_F \\ k_1 \cdots k_5 > P_F}} \left(\frac{\langle n_3 n_1 | v(g) | k_5 k_3 \rangle \langle k_5 n_2 | v(g) | k_4 k_2 \rangle}{(E_{n_3} + E_{n_1} - E_{k_5} - E_{k_3})(E_{n_1} + E_{n_2} + E_{n_3} - E_{k_2} - E_{k_3} - E_{k_4})} \right)$$

$$\times \left(\frac{\langle k_4 k_3 | v(g) | n_3 k_1 \rangle \langle k_2 k_1 | v(g) | n_2 n_1 \rangle}{(E_{n_1} + E_{n_2} - E_{k_1} - E_{k_2})} \right)$$



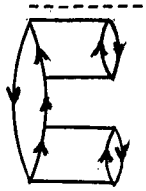
$$+\frac{1}{2}\sum_{\substack{n_1\cdots n_3<P_F\\ k_1\cdots k_5>P_F}}\left(\frac{\langle n_3n_1|v(g)|k_5k_1\rangle\langle k_5n_2|v(g)|k_4k_3\rangle}{(E_{n_1}+E_{n_3}-E_{k_5}-E_{k_1})(E_{n_1}+E_{n_2}+E_{n_3}-E_{k_1}-E_{k_3}-E_{k_4})}\right)$$

$$\times\left(\frac{\langle k_4k_3|v(g)|n_3k_2\rangle\langle k_2k_1|v(g)|n_2n_1\rangle}{(E_{n_1}+E_{n_2}-E_{k_1}-E_{k_2})}\right)$$



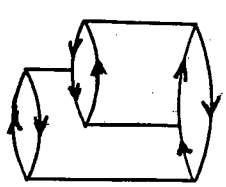
$$+\frac{1}{4}\sum_{\substack{n_1\cdots n_3<P_F\\ k_1\cdots k_5>P_F}}\left(\frac{\langle n_2n_1|v(g)|k_3k_1\rangle\langle k_3n_3|v(g)|k_4k_5\rangle}{(E_{n_2}+E_{n_1}-E_{k_1}-E_{k_3})(E_{n_1}+E_{n_2}+E_{n_3}-E_{k_1}-E_{k_5}-E_{k_4})}\right)$$

$$\times\left(\frac{\langle k_4k_5|v(g)|k_2n_3\rangle\langle k_2k_1|v(g)|n_2n_1\rangle}{(E_{n_1}+E_{n_2}-E_{k_1}-E_{k_2})}\right)$$



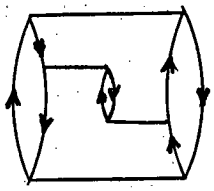
$$+\frac{1}{4}\sum_{\substack{n_1\cdots n_3<P_F\\ k_1\cdots k_5>P_F}}\left(\frac{\langle n_3n_1|v(g)|k_4k_5\rangle\langle k_5k_4|v(g)|k_3n_3\rangle}{(E_{n_1}+E_{n_3}-E_{k_4}-E_{k_5})(E_{n_1}-E_{k_3})}\right)$$

$$\times\left(\frac{\langle k_3n_2|v(g)|k_1k_2\rangle\langle k_2k_1|v(g)|n_2n_1\rangle}{(E_{n_1}+E_{n_2}-E_{k_1}-E_{k_2})}\right)$$



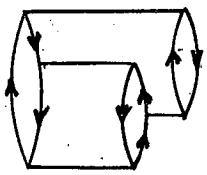
$$+\frac{1}{2}\sum_{\substack{n_1\cdots n_4<P_F\\ k_1\cdots k_4>P_F}}\left(\frac{\langle n_4n_1|v(g)|k_4k_3\rangle\langle n_3n_2|v(g)|n_4k_2\rangle}{(E_{n_4}+E_{n_1}-E_{k_3}-E_{k_4})(E_{n_1}+E_{n_2}+E_{n_3}-E_{k_2}-E_{k_3}-E_{k_4})}\right)$$

$$\times\left(\frac{\langle k_4k_3|v(g)|k_1n_3\rangle\langle k_2k_1|v(g)|n_2n_1\rangle}{(E_{n_1}+E_{n_2}-E_{k_1}-E_{k_2})}\right)$$



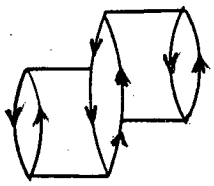
$$+ \sum_{\substack{n_1 \cdots n_4 < P_F \\ k_1 \cdots k_4 > P_F}} \left(\frac{\langle n_4 n_1 | v(g) | k_2 k_4 \rangle \langle n_4 k_3 | v(g) | n_2 n_3 \rangle}{(E_{n_4} + E_{n_1} - E_{k_1} - E_{k_4})(E_{n_1} + E_{n_2} + E_{n_3} - E_{k_2} - E_{k_3} - E_{k_4})} \right)$$

$$\times \left(\frac{\langle k_4 k_3 | v(g) | k_1 n_3 \rangle \langle k_2 k_1 | v(g) | n_2 n_1 \rangle}{(E_{n_1} + E_{n_2} - E_{k_1} - E_{k_2})} \right)$$



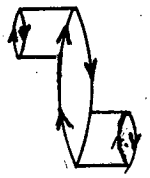
$$+ \frac{1}{2} \sum_{\substack{n_1 \cdots n_4 < P_F \\ k_1 \cdots k_4 > P_F}} \left(\frac{\langle n_4 n_3 | v(g) | k_2 k_3 \rangle \langle n_2 n_1 | v(g) | k_4 n_4 \rangle}{(E_{n_3} + E_{n_4} - E_{k_2} - E_{k_3})(E_{n_1} + E_{n_2} + E_{n_3} - E_{k_2} - E_{k_3} - E_{k_4})} \right)$$

$$\times \left(\frac{\langle k_4 k_3 | v | k_1 n_3 \rangle \langle k_2 k_1 | v(g) | n_2 n_1 \rangle}{(E_{n_1} + E_{n_2} - E_{k_1} - E_{k_2})} \right)$$



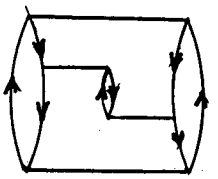
$$+ \frac{1}{4} \sum_{\substack{n_1 \cdots n_4 < P_F \\ k_1 \cdots k_4 > P_F}} \left(\frac{\langle n_4 n_3 | v(g) | k_4 k_3 \rangle \langle n_4 k_2 | v(g) | n_1 n_2 \rangle}{(E_{n_3} + E_{n_4} - E_{k_3} - E_{k_4})(E_{n_1} + E_{n_2} + E_{n_3} - E_{k_2} - E_{k_3} - E_{k_4})} \right)$$

$$\times \left(\frac{\langle k_4 k_3 | v(g) | k_1 n_3 \rangle \langle k_2 k_1 | v(g) | n_2 n_1 \rangle}{(E_{n_1} + E_{n_2} - E_{k_1} - E_{k_2})} \right)$$



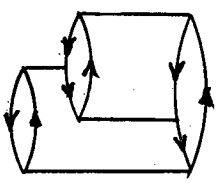
$$+ \frac{1}{4} \sum_{\substack{n_1 \cdots n_4 < P_F \\ k_1 \cdots k_4 > P_F}} \left(\frac{\langle n_4 n_3 | v(g) | k_4 k_3 \rangle \langle k_4 k_3 | v(g) | k_1 n_4 \rangle}{(E_{n_4} + E_{n_3} - E_{k_4} - E_{k_3})(E_{n_3} - E_{k_1})} \right)$$

$$\times \left(\frac{\langle n_3 k_2 | v(g) | n_1 n_2 \rangle \langle k_2 k_1 | v(g) | n_2 n_1 \rangle}{(E_{n_1} + E_{n_2} - E_{k_1} - E_{k_2})} \right)$$



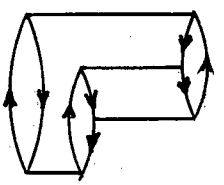
$$+\frac{1}{2}\sum_{\substack{n_1\cdots n_5<P_F\\ k_1\cdots k_3>P_F}}\left(\frac{\langle n_5n_4|v(g)|k_2k_1\rangle\langle n_5k_3|v(g)|n_2n_3\rangle}{(E_{n_5}+E_{n_4}-E_{k_1}-E_{k_2})(E_{n_2}+E_{n_3}+E_{n_4}-E_{k_1}-E_{k_2}-E_{k_3})}\right)$$

$$\times\left(\frac{\langle n_4n_3|v(g)|n_1k_3\rangle\langle k_2k_1|v(g)|n_2n_1\rangle}{(E_{n_1}+E_{n_2}-E_{k_1}-E_{k_2})}\right)$$



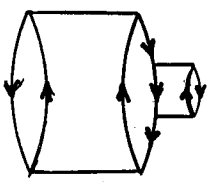
$$+\sum_{\substack{n_1\cdots n_5<P_F\\ k_1\cdots k_3>P_F}}\left(\frac{\langle n_5n_4|v(g)|k_3k_1\rangle\langle n_5k_2|v(g)|n_3n_2\rangle}{(E_{n_5}+E_{n_4}-E_{k_3}-E_{k_1})(E_{n_2}+E_{n_3}+E_{n_4}-E_{k_1}-E_{k_2}-E_{k_3})}\right)$$

$$\times\left(\frac{\langle n_4n_3|v(g)|n_1k_3\rangle\langle k_2k_1|v(g)|n_2n_1\rangle}{(E_{n_1}+E_{n_2}-E_{k_1}-E_{k_2})}\right)$$



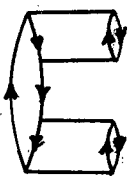
$$+\frac{1}{2}\sum_{\substack{n_1\cdots n_5<P_F\\ k_1\cdots k_3>P_F}}\left(\frac{\langle n_5n_2|v(g)|k_3k_2\rangle\langle n_5k_1|v(g)|n_3n_4\rangle}{(E_{n_5}+E_{n_2}-E_{k_3}-E_{k_2})(E_{n_2}+E_{n_3}+E_{n_4}-E_{k_1}-E_{k_2}-E_{k_3})}\right)$$

$$\times\left(\frac{\langle n_4n_3|v(g)|n_1k_3\rangle\langle k_2k_1|v(g)|n_2n_1\rangle}{(E_{n_1}+E_{n_2}-E_{k_1}-E_{k_2})}\right)$$



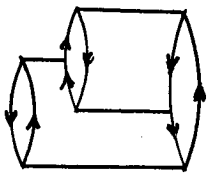
$$+\frac{1}{4}\sum_{\substack{n_1\cdots n_5<P_F\\ k_1\cdots k_3>P_F}}\left(\frac{\langle n_5n_2|v(g)|k_1k_2\rangle\langle n_5k_3|v(g)|n_4n_3\rangle}{(E_{n_5}+E_{n_2}-E_{k_1}-E_{k_2})(E_{n_2}+E_{n_3}+E_{n_4}-E_{k_1}-E_{k_2}-E_{k_3})}\right)$$

$$\times\left(\frac{\langle n_4n_3|v(g)|n_1k_3\rangle\langle k_2k_1|v(g)|n_2n_1\rangle}{(E_{n_1}+E_{n_2}-E_{k_1}-E_{k_2})}\right)$$



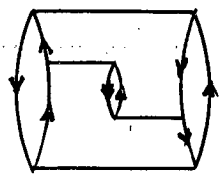
$$+\frac{1}{4}\sum_{\substack{n_1\cdots n_5<P_F\\ k_1\cdots k_3>P_F}}\left(\frac{\langle n_5n_4|v(g)|k_3k_1\rangle\langle n_5n_4|v(g)|k_3n_3\rangle}{(E_{n_5}+E_{n_4}-E_{k_3}-E_{k_1})(E_{n_3}-E_{k_1})}\right)$$

$$\times\left(\frac{\langle n_3k_2|v(g)|n_1n_2\rangle\langle k_2k_1|v(g)|n_2n_1\rangle}{(E_{n_1}+E_{n_2}-E_{k_2}-E_{k_1})}\right)$$



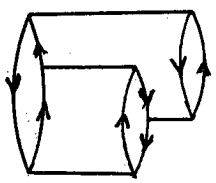
$$+\frac{1}{2}\sum_{\substack{n_1\cdots n_4<P_F\\ k_1\cdots k_4>P_F}}\left(\frac{\langle n_4n_3|v(g)|k_1k_4\rangle\langle k_4n_2|v(g)|k_3k_2\rangle}{(E_{n_4}+E_{n_3}-E_{k_1}-E_{k_4})(E_{n_2}+E_{n_3}+E_{n_4}-E_{k_1}-E_{k_2}-E_{k_3})}\right)$$

$$\times\left(\frac{\langle n_4n_3|v(g)|n_1k_3\rangle\langle k_2k_1|v(g)|n_2n_1\rangle}{(E_{n_1}+E_{n_2}-E_{k_1}-E_{k_2})}\right)$$



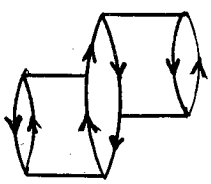
$$+\sum_{\substack{n_1\cdots n_4<P_F\\ k_1\cdots k_4>P_F}}\left(\frac{\langle n_4n_2|v(g)|k_1k_4\rangle\langle k_4n_3|v(g)|k_2k_3\rangle}{(E_{n_4}+E_{n_2}-E_{k_1}-E_{k_4})(E_{n_2}+E_{n_3}+E_{n_4}-E_{k_1}-E_{k_2}-E_{k_3})}\right)$$

$$\times\left(\frac{\langle n_4n_3|v(g)|n_1k_3\rangle\langle k_2k_1|v(g)|n_2n_1\rangle}{(E_{n_1}+E_{n_2}-E_{k_1}-E_{k_2})}\right)$$



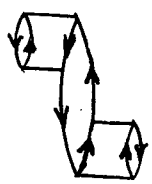
$$+\frac{1}{2}\sum_{\substack{n_1\cdots n_4<P_F\\ k_1\cdots k_4>P_F}}\left(\frac{\langle n_3n_2|v(g)|k_3k_4\rangle\langle k_4n_4|v(g)|k_2k_1\rangle}{(E_{n_3}+E_{n_2}-E_{k_3}-E_{k_4})(E_{n_2}+E_{n_3}+E_{n_4}-E_{k_1}-E_{k_2}-E_{k_3})}\right)$$

$$\times\left(\frac{\langle n_4n_3|v(g)|n_1k_3\rangle\langle k_2k_1|v(g)|n_2n_1\rangle}{(E_{n_1}+E_{n_2}-E_{k_1}-E_{k_2})}\right)$$



$$+\frac{1}{4}\sum_{\substack{n_1\cdots n_4<P_F\\ k_1\cdots k_4>P_F}}\left(\frac{\langle n_4n_3|v(g)|k_4k_3\rangle\langle k_4n_2|v(g)|k_1k_2\rangle}{(E_{n_4}+E_{n_3}-E_{k_4}-E_{k_3})(E_{n_2}+E_{n_3}+E_{n_4}-E_{k_1}-E_{k_2}-E_{k_3})}\right)$$

$$\times\left(\frac{\langle n_4n_3|v(g)|n_1k_3\rangle\langle k_2k_1|v(g)|n_2n_1\rangle}{(E_{n_1}+E_{n_2}-E_{k_1}-E_{k_2})}\right)$$



$$+ \frac{1}{4} \sum_{\substack{n_1 \cdots n_4 < P_F \\ k_1 \cdots k_4 > P_F}} \left(\frac{\langle n_4 n_3 | v(g) | k_4 k_3 \rangle \langle n_3 n_4 | v(g) | n_1 k_4 \rangle}{(E_{n_4} + E_{n_3} - E_{k_4} - E_{k_3})(E_{n_1} - E_{k_3})} \right)$$

$$\times \left(\frac{\langle k_3 n_2 | v(g) | k_1 k_2 \rangle \langle k_2 k_1 | v(g) | n_2 n_1 \rangle}{(E_{n_2} + E_{n_1} - E_{k_2} - E_{k_1})} \right)$$

B. THE FACTORABLE POTENTIAL

We consider all even-parity states for $\ell \leqslant 2$, and neglect all odd-parity states. Let us take:

$$v = {}^1v_{\text{even}} \frac{1-\sigma\cdot\sigma}{4} \frac{3+\tau\cdot\tau}{4} + {}^3v_{\text{odd}} \frac{3+\sigma\cdot\sigma}{4} \frac{1-\tau\cdot\tau}{4} .$$

Then we have

$$\overline{(\vec{k}', T', m'_T, S', m'_S | v | \vec{k}, T, m_T, S, m_S)} = \delta_{TT'}\delta_{SS'}\delta_{m_T m'_T}\delta_{m_S m'_S}$$

$$\times \begin{cases} 2(\vec{k}'|{}^1v_{\text{even}}|\vec{k}) & \text{for } T = 1,\ S = 0 \\ 2(\vec{k}'|{}^3v_{\text{even}}|\vec{k}) & \text{for } T = 0,\ S = 1 \\ 0 & \text{for } T = 0,\ S = 0 \\ 0 & \text{for } T = 1,\ S = 1, \end{cases}$$

$$(\vec{k}'|{}^Iv_{\text{even}}|\vec{k}) = \sum_{\ell=0,2} \sum_{m=-\ell}^{\ell} Y_\ell^m(\hat{k}')\, Y_\ell^m(\hat{k})(k'|{}^Iv_\ell|k),$$

and

$$(k'|{}^Iv_\ell|k) = \frac{(2\pi)^3}{\Omega}\left[{}^{I\ell}f_1\ {}^{I\ell}\phi_1(k')\ {}^{I\ell}\phi_1(k) + {}^{I\ell}f_2\ {}^{I\ell}\phi_2(k')\ {}^{I\ell}\phi_2(k)\right] ,$$

where, for $I = 1$ $(S = 0)$, $\ell = 0$:

$${}^{10}f_1 < 0, \quad {}^{10}f_2 = \infty, \quad {}^{10}\phi_1(k) = \frac{1 - \alpha k^2}{(k^2 + {}^{10}\beta^2)^{3/2}} ,$$

$${}^{10}\phi_2(k) = \frac{1}{(k^2 + {}^{10}\beta^2)^{1/2}} ;$$

for $I = 1$ $(S = 0)$, $\ell = 2$:

$$ {}^{12}f_1 < 0, \quad {}^{12}f_2 = 0, \quad {}^{12}\phi_1(k) = \frac{k^2}{(k^2 + {}^{10}\beta^2)^{3/2}} ; $$

for $I = 3$ $(S = 1)$ $\ell = 0$:

$$ {}^{30}f_1 < 0, \quad {}^{30}f_2 = \infty, \quad {}^{30}\phi_1(k) = \frac{1 - \alpha k^2}{(k^2 + {}^{30}\beta^2)^{3/2}} , $$

$$ {}^{30}\phi_2(k) = \frac{1}{(k^2 + {}^{30}\beta^2)^{1/2}} ; $$

and for $I = 3$ $(S = 1)$ $\ell = 2$:

$$ {}^{32}f_1 < 0, \quad {}^{32}f_2 = 0, \quad {}^{32}\phi_1(k) = \frac{k^2}{(k^2 + {}^{32}\beta^2)^{3/2}} . \qquad \text{(A-1)} $$

1. <u>Existence of a Deuteron State</u>

The Schrödinger equation for a bound state with binding energy $B(= -E)$ of a neutron-proton system can be written as

$$ \left(- \frac{k^2}{M} - B\right) \psi(\vec{k}) = \frac{\Omega}{(2\pi)^3} \int (\vec{k} \,|^3 v_{even} |\vec{k}') \, \psi(\vec{k}') \, d\,\vec{k}' \qquad \text{(A-2)} $$

for $S = 1$. For a factorable potential given above, the deuteron wave function must take the form

$$\psi(\vec{k}) = u(k)\,\mathcal{Y}^{J=1\ M=1}_{\ell=0\ S=1}(\hat{k}) + w(k)\,\mathcal{Y}^{J=1\ M=1}_{\ell=2\ S=1}(\hat{k})$$

$$= u(k)\,Y_0^{\,0}(\hat{k})\,\chi_1^{\,1} + w(k)\left[\sqrt{\frac{6}{10}}\,Y_2^{\,2}(\hat{k})\,\chi_1^{-1} - \sqrt{\frac{3}{10}}\,Y_2^{\,1}(\hat{k})\,\chi_1^{\,0} + \sqrt{\frac{1}{10}}\,Y_2^{\,0}(\hat{k})\,\chi_1^{\,1}\right], \qquad \text{(A-3)}$$

where u and w are the S and D wave functions, respectively. Substituting (A-3) into (A-2), we get

$$(k^2 + \delta^2)\,u(k) = -M\,\frac{\Omega}{(2\pi)^3}\int (k|^3 v_0|k')\,u(k')\,k'^2\,d\,k'$$

and

$$(k^2 + \delta^2)\,w(k) = -M\,\frac{\Omega}{(2\pi)^3}\int (k|^3 v_2|k')\,w(k')\,k'^2\,d\,k'$$

for $BM = \delta^2$. These equations can have the solutions

$$u(k) = N_1\,[k^2 + \delta^2]^{-1}\left[{}^{30}\phi_1(k) - \frac{1 + M\,{}^{30}f_1\,II_{11}}{M\,{}^{30}f_1\,II_{12}}\,{}^{30}\phi_2(k)\right]$$

and

$$w(k) = N_2\,[k^2 + \delta^2]^{-1}\,{}^{32}\phi_1(k) \qquad \text{(A-4)}$$

provided

$$-1 = {}^{30}f_1\,M\left(II_{11} - \frac{II_{12}^{\ 2}}{II_{22}}\right)$$

and

$$-1 = {}^{32}f_1\,M\,(III_{11}), \qquad \text{(A-5)}$$

where

$$II_{11} = \int k^2 dk \frac{{}^{30}\phi_1{}^2(k)}{k^2 + \delta^2} ,$$

$$II_{12} = \int k^2 dk \frac{{}^{30}\phi_1(k)\ {}^{30}\phi_2(k)}{k^2 + \delta^2} ,$$

$$II_{22} = \int k^2 dk \frac{{}^{30}\phi_2{}^2(k)}{k^2 + \delta^2} ,$$

and

$$III_{11} = \int k^2 dk \frac{{}^{32}\phi_1{}^2(k)}{k^2 + \delta^2} .$$

Equation (A-5) can be simplified as

$$1 = - \frac{{}^{30}f_1\ \pi\ M}{{}^{30}\beta^5} \frac{(1 + \alpha\ {}^{30}\beta^2)^2}{16(1 + \frac{\delta}{{}^{30}\beta})^2} \tag{A-6}$$

and

$$1 = - \frac{{}^{32}f_1\ \pi\ M}{2} \left[\frac{\delta^4(\delta - {}^{32}\beta)}{({}^{32}\beta^2 - \delta^2)^3} + \frac{\delta^6}{2\ {}^{32}\beta^2({}^{32}\beta^2 - \delta^2)^2} \right.$$

$$\left. + \frac{3\ {}^{32}\beta}{8({}^{32}\beta^2 - \delta^2)} - \frac{\delta^2}{2\,{}^{32}\beta^3} \right] . \tag{A-7}$$

2. <u>Scattering Properties</u>

Following potential (A-1), the scattering matrix T can be written in a similar form:

$$\overline{(\vec{k}', T', m'_T, S', m'_S | T | \vec{k}, T, m_T, S, m_S)} = \delta_{TT'} \delta_{SS} \delta_{m_T m'_T} \delta_{m_S m'_S}$$

$$\begin{cases} 2(\vec{k}' | {}^1T_{\text{even}} | \vec{k}) & \text{for } T = 1,\ S = 0, \\ 2(\vec{k}' | {}^3T_{\text{even}} | \vec{k}) & \text{for } T = 0,\ S = 1, \\ 0 & \text{otherwise} \end{cases}$$

$$(\vec{k}' | {}^IT_{\text{even}} | \vec{k}) = \sum_{\ell=0,2} \sum_{m=-\ell}^{\ell} Y_\ell^m (\hat{k}')\, Y_\ell^m (\hat{k}) (k' | {}^IT_\ell | k), \quad I = 1,\ 3$$

$$= \sum_{\ell=0,2} \frac{2\ell+1}{4\pi} P_\ell (\hat{k}' \cdot \hat{k}) (k' | {}^IT_\ell | k).$$

By definition, we have

$$(k | {}^IT_\ell | k) \equiv - \frac{8\pi^2}{\Omega\, k(M/2)} \frac{e^{2i\,{}^I\delta_\ell} - 1}{2i} .$$

Therefore the differential cross sections of p-p and n-p scatterings are given by Eqs. (A-8) and (A-9). We are dealing with even-parity states of two identical particles. Thus we have to put a factor 2 in front of each matrix element $(\vec{k} | {}^IT_\ell | \vec{k})$ as a result of antisymmetrization. The p-p state is a $|\vec{k}, T=1, M_T = 1, S, m_S)$ state. The n-p state is a combination of two kinds of states, i.e.

$$\frac{1}{\sqrt{2}}\left[|\vec{k}, T=0, m_T=0, S, m_S) + |\vec{k}, T=1, m_T=0, S, m_S)\right] (\Theta=\hat{k}'\cdot\hat{k}=\text{scattering angle})$$

$$\sigma_{pp}(H) = \frac{\Omega}{\frac{k}{M/2}} \int 2\pi \frac{1}{4} \left|2(\vec{k}'|{}^1T_{even}|\vec{k})\right|^2 \delta\left(\frac{k'^2}{M} - \frac{k^2}{M}\right) \frac{k'^2\, dk'\, \Omega}{(2\pi)^3}$$

$$= \frac{1}{k^2} \left[\sin^2({}^1\delta_0) + 10 \sin {}^1\delta_0 \sin {}^1\delta_2 \cos ({}^1\delta_0 - {}^1\delta_2)\, P_2(\cos\Theta)\right.$$

$$\left. + 25 \sin^2 ({}^1\delta_2)(P_2(\cos\Theta))^2\right], \qquad \text{(A-8)}$$

$$\sigma_{np}(\Theta) = \frac{\Omega}{\frac{k}{M/2}} \int 2\pi \frac{1}{4} \sum_{\substack{S=0,1\\ m_S}} \left|2\left\{\frac{1}{2}\left[(\vec{k}', T=0, m_T=0, S, m_S|T|\vec{k}, T=0, m_T=0, S, m_S)\right.\right.\right.$$

$$\left.\left.\left. \times (\vec{k}', T=1, m_T=0, S, m_S|T|\vec{k}', T=1, m_T=0, S, m_S)\right]\right\}\right|^2 \delta\left(\frac{k'^2}{M} - \frac{k^2}{M}\right) \frac{k'^2 dk'\Omega}{(2\pi)^3}$$

$$= \frac{1}{4k^2}\left[\sin^2({}^1\delta_0) + 10 \sin {}^1\delta_0 \sin {}^1\delta_2 \cos ({}^1\delta_0 - {}^1\delta_2) P_2(\cos\Theta)\right.$$

$$\left. + 25 \sin^2 ({}^1\delta_2)(P_2(\cos\Theta))^2\right]$$

$$+ \frac{3}{4k^2}\left\{\sin^2({}^3\delta_0) + 10 \sin {}^3\delta_0 \sin {}^3\delta_2 \cos ({}^3\delta_0 - {}^3\delta_2) P_2(\cos\Theta)\right.$$

$$\left. + 25 \sin^2 ({}^3\delta_2)\left[P_2(\cos\Theta)\right]^2\right\}. \qquad \text{(A-9)}$$

Consequently, the total cross sections are given by

$$\sigma_{pp} = \frac{4\pi}{k^2} \left[\sin^2 ({}^1\delta_0) + 5 \sin^2 ({}^1\delta_2)\right] \tag{A-10}$$

and

$$\sigma_{np} = \frac{\pi}{k^2} \left[\sin^2 ({}^1\delta_0) + 5 \sin^2 ({}^1\delta_2)\right] + \frac{3\pi}{k^2} \left[\sin^2 ({}^3\delta_0) + 5 \sin^2 ({}^3\delta_2)\right] . \tag{A-11}$$

To express these phase shifts in terms of our potential parameters, we calculate the matrix elements of the free scattering reaction matrix $(k'|{}^I R_\ell|k)$ from

$$(k'|{}^I R_\ell|k) = (k'|{}^I R_\ell|k) + \frac{\Omega M}{(2\pi)^3} \int k''^2 \, dk'' \frac{(k'|{}^I v_\ell|k'')(k''|{}^I R_\ell|k)}{k^2 - k''^2} ,$$

using the potential from Eq. (A-1). In phase-shift representation, we have

$$(k|{}^I R_\ell|k) = - \frac{8\pi^2}{\Omega M/2 \, k} \tan {}^I\delta_\ell .$$

Therefore we can write

$$- \frac{8\pi^2}{\Omega M/2 \, k} \tan {}^1\delta_0 \equiv (k|{}^1 R_0|k) = \frac{(2\pi)^3}{\Omega \frac{\pi}{2} M \, {}^{10}\beta}$$

$$\times \frac{1 + {}^{10}f_1 \frac{\pi M}{{}^{10}\beta^5} \frac{(1+\alpha \, {}^{10}\beta^2)^2}{16} \left(3+2 \frac{k^2}{{}^{10}\beta^2} - \frac{k^4}{{}^{10}\beta^4}\right)\left(1 + \frac{k^2}{{}^{10}\beta^2}\right)^{-3}}{1 + {}^{10}f_1 \frac{\pi M}{{}^{10}\beta^5} \frac{(1 + \alpha \, {}^{10}\beta^2)^2}{16} \left(1 - 2 \frac{k^2}{{}^{10}\beta^2} - 3 \frac{k^4}{{}^{10}\beta^4}\right)\left(1 + \frac{k^2}{{}^{10}\beta^2}\right)^{-3}} \tag{A-12}$$

$$- \frac{8\pi^2}{\Omega\, M/2\ k} \tan {}^1\delta_2 \equiv (k|{}^1R_2|k) = \frac{(2\pi)^3}{\Omega \frac{\pi}{2} M\ {}^{12}\beta}$$

$$\times \frac{{}^{12}f_1 \frac{\pi M}{{}^{12}\beta} \frac{1}{16} \frac{8k^4}{{}^{12}\beta^4} \left(1 + \frac{k^2}{{}^{12}\beta^2}\right)^{-3}}{1 + {}^{12}f_1 \frac{\pi M}{{}^{12}\beta} \frac{1}{16} \left(3 + 10 \frac{k^2}{{}^{12}\beta^2} + 15 \frac{k^4}{{}^{12}\beta^4}\right)\left(1 + \frac{k^2}{{}^{12}\beta^2}\right)^{-3}} \tag{A-13}$$

$$- \frac{8\pi^2}{\Omega\, M/2\ k} \tan {}^3\delta_0 \equiv (k|{}^3R_0|k) = \frac{(2\pi)^3}{\Omega \frac{\pi}{2} M\ {}^{30}\beta}$$

$$\times \frac{1 + {}^{30}f_1 \frac{\pi M}{{}^{30}\beta} \frac{1 + \alpha\, {}^{30}\beta^2)^2}{16} \left(3 + 2 \frac{k^2}{{}^{30}\beta^2} - \frac{k^4}{{}^{30}\beta^4}\right)\left(1 + \frac{k^2}{{}^{30}\beta^2}\right)^{-3}}{1 + {}^{30}f_2 \frac{\pi M}{{}^{30}\beta} \frac{(1 + \alpha\, {}^{30}\beta^2)^2}{16} \left(1 - 2 \frac{k^2}{{}^{30}\beta^2} - 3 \frac{k^4}{{}^{10}\beta^4}\right)\left(1 + \frac{k^2}{{}^{10}\beta^2}\right)^{-3}} \tag{A-14}$$

$$- \frac{8\pi^2}{\Omega\, M/2\ k} \tan {}^3\delta_2 \equiv (k|{}^3R_2|k) = \frac{(2\pi)^3}{\Omega \frac{\pi}{2} M\ {}^{32}\beta}$$

$$\frac{{}^{32}f_1 \frac{\pi M}{{}^{32}\beta} \frac{1}{16} \frac{8k^4}{{}^{32}\beta^4} \left(1 + \frac{k^2}{{}^{32}\beta^2}\right)^{-3}}{1 + {}^{32}f_1 \frac{\pi M}{{}^{32}\beta} \frac{1}{16} \left(3 + 10 \frac{k^2}{{}^{32}\beta^2} + 15 \frac{k^4}{{}^{32}\beta^2}\right)\left(1 + \frac{k^2}{{}^{32}\beta^2}\right)^{-3}} \tag{A-15}$$

At low energy, we have

$$- \frac{1}{k} \tan \delta = \frac{1}{-\frac{1}{a} + \frac{1}{2} r_0 k^2} \cong a + \frac{a^2 r_0}{2} k^2 + \cdots , \tag{A-16}$$

where a is the scattering length, and r_0 is the effective range.

3. Numerical Values for the f and β Parameters

Now we can determine a set of values for f's and β's from the nucleon-nucleon data through the relations we obtained in 1 and 2. We adopt:

singlet scattering length:	${}^1a = -23.7$ fermis;
singlet effective range :	${}^1r_0 = 2.6$ fermis;
triplet scattering length:	${}^3a = 5.4$ fermis;
triplet effective range :	${}^3r_0 = 1.7$ fermis;

from Eqs. (A-12), (A-13), and (A-16) we can determine ${}^{10}f_1$, ${}^{10}\beta$ and ${}^{30}f_1$, ${}^{30}\beta$. Then by using

$$\sigma_{pp}(\Theta = 90\,\text{deg}) = \begin{cases} 5\text{ mb at } E_{lab} = 75\text{ Mev} \\ 3.5\text{ mb at } E_{lab} = 310\text{ Mev} \end{cases}$$

and Eqs. (A-12), (A-8), and (A-13), we can determine ${}^{12}f_1$ and ${}^{12}\beta$. By using

$$\sigma_{np}(\Theta = 90\,\text{deg}) = \begin{cases} 4\text{ mb at } E_{lab} = 90\text{ Mev} \\ 2\text{ mb at } E_{lab} = 310\text{ Mev} \end{cases}$$

and Eqs. (A-12), (A-13), (A-14), (A-9), and (A-15), we can determine ${}^{32}f_1$ and ${}^{32}\beta$. The values are listed in Table II. The conditions (A-6), (A-7), and the total cross section of p-p and p-n scatterings can be checked within reasonable limits by using these results.

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