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CALCULATION OF CYCLOTRON TRIM-COIL CURRENTS FOR FIELD OPTIMIZATION BY LINEAR PROGRAMMING METHODS

Alper A. Garren

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ABSTRACT

Determination of the current settings in the circular pole-face windings of fixed-frequency cyclotrons for optimization of the magnetic field may be solved with the mathematical technique of linear programming. The resulting fields are optimal with regard to particle phase lag as well as radial and vertical stability, and the currents do not exceed preassigned limits in magnitude.

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1. INTRODUCTION

The purpose of the circular pole-face windings of trim coils in fixed-frequency variable-energy spiral-ridge cyclotrons is the adjustment of the radial profile of the magnetic field for each energy and each type of particle to be accelerated. In particular one wishes the magnetic field to approximate the isochronous field as well as possible. This paper describes a method for determining current settings of the trim and main coils that minimize the maximum phase lag or lead the particles will experience during acceleration, while keeping these currents within allowed limits. A generalization is made to keep the radial gradient of the corrected field within acceptable limits, so as to avoid instabilities of the betatron oscillations.

To determine the proper trim-coil settings one needs the following information: the radial profile of the desired isochronous field for the energy and particle required, the uncorrected radial profile at an appropriate main-coil setting, the change in this uncorrected field produced by small changes in the main-coil current, and the contributions to the radial field profile resulting from unit current in each trim coil. This information must be determined at the same main-coil setting as was used in measuring the uncorrected field. Further, one must specify the allowable limits for the trim-coil currents, and the charge, mass, and energy gain per turn of the ions.

The procedure rests upon the assumption that the dependence of the field on the trim-coil currents and on small changes in main-coil current is linear in those currents. If the currents involved are too large, the field can be remeasured with the calculated currents in the coils, and the procudure repeated to correct the first-approximation field.

The method can obviously be extended to other similar problems in magnetic-field adjustment, such as magnet shimming.

A system of IBM 709 codes has been assembled to carry out the calculations necessary for choosing the proper current settings. These codes include an Oak Ridge orbit code;[1] a Berkeley code for calculating isochronous fields;[2] the SCROL linear programming code,[3] which calculates the current settings; and Berkeley codes for calculating the input quantities for SCROL and for predicting the corrected fields, etc. The latter codes and the use of the total complex of codes are described in a separate report.[4] These reports and codes will be available on request. The results of these calculations on the fields measured in the 88-inch cyclotron will be presented in a separate report.[5]

2. ISOCHRONOUS FIELD SHAPE

It is assumed that we are given a measured median-plane uncorrected field with certain radially dependent azimuthal variations. Let the azimuthal average of this field be denoted by $B(R)$. We first ask what average field, $B_s(R,\omega)$, would keep the particles in synchronism with a prescribed time-independent rf frequency on the dees, assuming that we do not alter the azimuthal field variations.

Let the particle's rest mass and charge be m_0 and eZ, respectively. The synchrounous field at $R = 0$, $B_{s0}(\omega) = B_s(0, \omega)$, may be determined by

observing that when $R \to 0$ the azimuthal variations go to zero, and the particle becomes nonrelativistic. Hence the particle rotation frequency in the field $B_{s0}(0)$ is $\omega_p = (eZB_{s0})/(m_0 c)$. Since the field is assumed synchronous, the rf frequency f must be related to the particle rotation frequency $\omega_p/2\pi$ by

$$\begin{aligned} \omega &\equiv 2\pi h f \\ &= \omega_p \equiv \frac{eZB_{s0}(\omega)}{m_0 c}, \end{aligned} \tag{2.1}$$

if the particles are to be accelerated by the hth harmonic of the fundamental rf frequency.

In the following discussions we make use of these 'cyclotron' units:

$$\begin{aligned} &\text{Time unit:} && T = \omega^{-1} \\ &\text{Length unit:} && a = c/\omega \\ &\text{Magnetic field unit:} && B_0 = B_{s0}. \end{aligned} \tag{2.2}$$

In these units we denote time, radius, and magnetic field by τ, r, and b, respectively. These units are related to similar quantities in conventional units by

$$\begin{aligned} \tau &= \omega t, \\ r &= \frac{\omega}{c} R, \\ b(r) &= B(R)/B_{s0}. \end{aligned} \tag{2.3}$$

Also, we use for velocity, momentum, and energy, respectively, the dimensionless quantities

$$\begin{aligned} \beta &= v/c, \\ p &= \frac{P}{m_0 c} = \beta\gamma, \\ \gamma &= \frac{E_{Total}}{m_0 c^2} = \frac{1}{\sqrt{1-\beta^2}} = \sqrt{1+p^2}. \end{aligned} \tag{2.4}$$

If the field had no azimuthal variations the equilibrium orbits would be circles, of radius related to particle momentum by

$$p = r\,b(r). \tag{2.5}$$

In an azimuthally varying field we regard this relation as <u>defining</u> the so called equilibrium radius as a function of momentum. The equilibrium (or closed) orbit for that momentum will oscillate about the circle whose radius is the equilibrium radius defined by (2.5), and in general it will have a different path length from that circle. If we call $\sigma = \frac{\Delta s}{2\pi r}$ the fractional change in path length from that of the equilibrium circle, then the length of the equilibrium orbit (in one turn) is

$$s = 2\pi r\,[1 + \sigma(r)]. \tag{2.6}$$

An expression σ in terms of a Fourier representation of the field has been in a report by Lloyd Smith and the author [reference 6, Eq. (17)]. A good approximation to the general expression given there is

$$\sigma \approx -\frac{1}{(1+G)(N^2-1)}\left(F^2 + \frac{R}{2}\frac{dF^2}{dR}\right), \tag{2.7}$$

where F is the flutter, defined by

$$F^2(R) = \frac{1}{2\pi}\int_0^{2\pi}\frac{[B(R,\theta) - B(R)]^2}{B^2(R)}\,d\theta, \quad B(R) = \frac{1}{2\pi}\int_0^{2\pi} B(R,\theta)\,d\theta, \tag{2.8}$$

and G is the gradient index, $G = \frac{R}{B}\frac{dB}{dR}$.

The time τ to complete one revolution is given, using (2.4) to (2.6), by

$$\tau = \frac{s}{\beta} = \frac{2\pi r\,[1 + \sigma(r)]}{p/\gamma} = 2\pi\,\frac{[1 + \sigma(r)]}{b(r)\sqrt{1-\beta^2(r)}}. \tag{2.9}$$

If the field is isochronous, then $b(r)$ is such that $\tau = 2\pi$.

Hence

$$\beta_s(r) = s/\tau = r\,[1 + \sigma(r)] \tag{2.10}$$

and the isochronous field $b_s(r)$ is given by

$$b_s(r) = \frac{1 + \sigma(r)}{\sqrt{1 - r^2[1+\sigma(r)]^2}}. \tag{2.11}$$

This expression for the synchronous field is exact. However, in practice one does not know σ exactly. We compute b_s to sufficient accuracy with the help of two IBM 704 codes, and magnetic measurements of $B(R,\theta)$ on a polar grid. The first code, called DORO,[2] calculates σ approximately by using Eqs. (2.7) and (2.8). It then substitutes this value of σ in Eq. (2.10) to calculate b_s. If desired, DORO will then modify the original field points with a purely radial correction so that $b(r) \rightarrow b_s(r)$. The modified grid field values are then used as input for an Oak Ridge orbit code, No. 1482,[1] which calculates the frequencies of oscillation about the equilibrium orbit, and the revolution time. We find that for the 88-inch cyclotron this first approximation to b_s using Eq. (2.7) gives b_s good to about two parts in 10^4. Higher accuracy can be obtained by the following iteration procedure. The time values $\frac{\tau(r)}{2\pi}$, calculated by the orbit code are read back into the computor, and the DORO code is run again, but this time it multiplies the $b_s(r)$ calculated from Eqs. (2.7) and (2.11) by these time values. One such iteration gives improvement of about a factor of ten.

3. PHASE ERROR

A. Nonsynchronous Field Error

Having determined what the radial field profile should be for isochronism, we now wish to know the phase errors that will be induced by departures of the field from the isochronous field given by Eq. (2.11).

Let the actual field depart from the synchronous field by

$\Delta b = b - b_s$, so that

$$b(r) = b_s(r)\left[1 + \frac{\Delta b(r)}{b_s(r)}\right]. \tag{3.1}$$

From Eqs. (2.4) and (2.5) we have

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} = \sqrt{1+p^2} = \sqrt{1+r^2b^2(r)}, \tag{3.2}$$

and substituting this in Eq. (2.9) we get

$$\tau = 2\pi\,\frac{1+\sigma(r)}{b(r)}\sqrt{1+r^2b^2(r)}\,. \tag{3.3}$$

Next we substitute expression (3.1) for $b(r)$, and Eq. (2.11) for $b_s(r)$ into Eq. (3.3) to obtain

$$\tau = 2\pi\,\frac{\sqrt{1+r^2(1+\sigma)^2\,\frac{\Delta b}{b_s}\left(2+\frac{\Delta b}{b_s}\right)}}{1+\frac{\Delta b}{b_s}}, \tag{3.4}$$

or to first order in $\Delta b/b_s$:

$$\tau = 2\pi\left\{1 - \frac{\Delta b}{b_s}\left[1 - r^2(1+\sigma)^2\right]\right\}. \tag{3.5}$$

Since an increase of τ from 2π represents a phase lag of the particle relative to the rf, the phase lag per turn, $d\phi/dn$, is given from Eqs. (3.5) and (2.11) by

$$-\frac{d\Phi}{dn} = 2\pi\frac{\Delta b}{b_s}\left[1-r^2(1+\sigma)^2\right] = 2\pi\,\Delta b\,\frac{\left[1-r^2(1+\sigma)^2\right]^{3/2}}{1+\sigma}. \tag{3.6}$$

To obtain the phase lag as a function of radius, we write

$$\frac{d\Phi}{dr} = \frac{d\gamma}{dr}\cdot\frac{d\Phi}{dn}\Big/\frac{d\gamma}{dn}. \tag{3.7}$$

The energy gain per turn, for hth harmonic operation, is

$$\frac{d\gamma}{dn} = \epsilon_0\cos(h\Phi), \quad \epsilon_0 = \frac{\Delta E}{m_0c^2}, \tag{3.8}$$

where ΔE is the maximum energy gain per turn. To obtain $\frac{d\gamma}{dr}$ we use the synchronous value of β, Eq. (2.10), in Eq. (2.4) for γ:

$$\frac{d\gamma}{dr} = \frac{d}{dr}\frac{1}{\sqrt{1-r^2(1+\sigma)^2}} = \frac{r[1+\sigma(r)][1+\sigma(r)+r\sigma'(r)]}{[1-r^2(1+\sigma)^2]^{3/2}} \tag{3.9}$$

By substituting Eqs. (3.6), (3.8), and (3.9) in Eq. (3.7), multiplying both sides by $\cos(h\phi)\,dr$, and integrating from zero to r, one obtains the change in the sine of the phase lag: (Note that as here defined, ϕ is positive if the ions lag behind the rf, negative if they lead it.)

$$S(r) \equiv \sin[h\Phi(r)] - \sin[h\Phi(0)]$$
$$= -\frac{2\pi h}{\epsilon_0}\int_0^r \Delta b(\rho)[1+\sigma(\rho)+\rho\sigma'(\rho)]\rho\, d\rho\,. \tag{3.10}$$

Equation (3.10) is exact to first order in $\frac{\Delta b}{b_s}$. Its meaning is that the fluctuations in $\sin(h\phi)$ are the same for all particles, regardless of starting phase--so that the phase width of the beam in terms of $\sin(h\phi)$ is constant with radius.

B. Frequency Error

Equation (3.10) gives the phase error arising when the actual field is different from the synchronous field appropriate to the rf frequency f. We will find it useful to know the additional phase error that will ensue if the frequency is shifted, leaving the field as it was.

Let ω be the original frequency, ω' the actual frequency, and $\delta\omega=\omega'-\omega$:

$$\omega' = \omega(1+\frac{\delta\omega}{\omega}).$$

Then from (2.1) the new normalizing field is

$$B_{s0}(\omega') = B_{s0}(\omega)(1+\frac{\delta\omega}{\omega}),$$

so that the actual field with the new normalization is

$$b' = \frac{B}{B'_{s0}} = \frac{B_{s0}}{B'_{s0}}\, b = \frac{\omega}{\omega'}\, b.$$

From (2.3) and (2.11) the synchronous field is

$$b_s(R, \omega) = \frac{1 + \sigma(R)}{\sqrt{1 - \{\frac{\omega R}{c}[1 + \sigma(R)]\}^2}},$$

so that the synchronous field for the frequency ω' is

$$b_s' = b_s(R, \omega + \delta\omega) = b_s(R, \omega) + \frac{\partial b_s(R, \omega)}{\partial \omega}\delta\omega + \cdots$$

$$= b_s(R, \omega)[1 + p^2(R)\frac{\delta\omega}{\omega} + \cdots],$$

where we have used $\beta = r[1 + \sigma(r)]$ and $p^2 = \beta^2/(1 - \beta^2)$. Hence, relative to the actual frequency ω', the field error becomes

$$\Delta b' = b' - b_s' = \frac{\omega}{\omega'}[\Delta b - (1 + p^2)b_s\frac{\delta\omega}{\omega} + \cdots].$$

The length unit has also changed, so that from (2.3) we have

$$r' = \frac{\omega'}{\omega} r .$$

Now we can obtain the phase error by substituting primed variables instead of unprimed ones in Eq. (3.10):

$$S'(r') = -\frac{2\pi h}{\epsilon_0}\int_0^{r'} \Delta b'(\rho')k(\rho')\rho' d\rho',$$

where

$$k(\rho) = 1 + \sigma(\rho) + \rho\,\sigma'(\rho). \tag{3.11}$$

Expressing this in terms of the old variables, and keeping only first order terms, we get

$$S(r, \omega + \delta\omega) = S'(r') = \frac{-2\pi h}{\epsilon_0}\int_0^r \frac{\omega}{\omega'}[\Delta b(\rho) - (1+p^2)b_s(\rho)\frac{\delta\omega}{\omega} + \cdots]k(\rho)(\frac{\omega'}{\omega})^2\rho\, d\rho .$$

But

$1 + p^2 = [1 - p^2(1 + \sigma)^2]^{-1}$, so to first order in Δb, $\frac{\delta\omega}{\omega}$:

$$S(r, \omega + \delta\omega) = S(r, \omega) + \frac{\delta\omega}{\omega}\{S(r, \omega) + \frac{2\pi h}{\epsilon_0}\int_0^r \frac{1 + \sigma(\rho)}{[1 - \rho^2(1+\sigma)^2]^{3/2}} k(\rho)\rho\, d\rho\}, \tag{3.12}$$

where $S(r,\omega)$ is given by (3.10).

Expressions (3.10) and (3.12) give us the sine of the phase lag, which concerns the particles, in terms of the field and frequency errors, which we can control with the trim coils.

C. Mass Error

Now suppose we leave the field as it was, but accelerate a particle with a slightly different mass from the original one. (We might also need to know the effect of using a slightly incorrect mass in the calculations.) What will be the additional phase error resulting from the mass change? This depends partly on what we do with the frequency. It is best to keep the field-normalizing constant unchanged: $B_{s0} = B_{s0}'$.

Let the original mass be m, the new one m'. Then from (2.1), (2.3), and (3.8) we have $\omega' = \frac{m}{m'}\,\omega$, $r' = \frac{\omega'}{\omega}\,r = \frac{m}{m'}\,r$, $\epsilon' = \frac{m}{m'}\,\epsilon$, $b' = b$--while from the preceding discussion of frequency error

$$b_s' = b_s\left(1 + p^2\,\frac{\delta\omega}{\omega}\right) = b_s\left(1 - p^2\,\frac{\delta m}{m}\right),$$

so

$$\Delta b' = b' - b_s' = b - b_s\left(1 - p^2\,\frac{\delta m}{m}\right) = \Delta b + p^2\,b_s\,\frac{\delta m}{m}\,.$$

Then from (2.10), (3.10), and $p^2 = \beta^2/(1-\beta^2)$:

$$S'(R) = -\frac{2\pi h}{\left(\frac{m}{m'}\,\epsilon\right)}\left(\frac{m}{m'}\right)^2 \int_0^{\frac{\omega R}{c}} \left(\Delta b + p^2 b_s\,\frac{\delta m}{m}\right) k(\rho)\,\rho\,d\rho$$

$$= \left(1 - \frac{\delta m}{m}\right) S(R) - \frac{\delta m}{m}\,\frac{2\pi h}{\epsilon_0} \int_0^{\frac{\omega R}{c}} \frac{[1+\sigma(\rho)]^3\,k(\rho)}{[1-\rho^2(1+\sigma)^2]^{3/2}}\,\rho^3 d\rho\,. \quad (3.13)$$

For 60-Mev deuterons in the 88-inch cyclotron, the coefficient of $\frac{\delta m}{m}$ in the last term is about 100. Hence if $\frac{\delta m}{m} < 10^{-3}$, the additional phase error is less than 6 deg, provided the frequency is shifted according to $\omega' = \left(1 - \frac{\delta m}{m}\right)\omega$, and the field is unchanged.

4. EXPLICIT EXPRESSION OF FIELD, PHASE SHIFT, AND RADIAL GRADIENT IN TERMS OF COIL CURRENTS

Since we wish to use our control of the trim- and main-coil currents to limit the maximum excursion in phase shift, and to hold the radial field gradient within the bounds needed for stability of the betatron oscillations, we must first express these quantities in terms of the coil currents.

A. Definitions

I = main-coil current setting at which measurements of main- and trim-coil effects are made.

$I + w_1$ = actual main-coil setting.

w_i = current in ith trim coil ($i = 2, \cdots, m$).

ω = frequency for which synchronous field is calculated.

$w_{m+1} = \delta\omega/\omega$ = relative frequency shift from base value.

$B^s(R)$ = synchronous field for frequency ω--calculated from measured $B(R, \theta)$.

$B(R, w_1, w_2, \ldots, w_m)$ = actual azimuthal average of the field with currents w_i in the coils.

$B(R) = B(R, 0, 0, \cdots, 0)$ = the measured "iron" field with current I in main coil, and no current in trim coils.

$B^i(R) = \left[\frac{\partial B(R, w)}{\partial w_i}\right]_{\vec{w} = 0}$ = measured change in field per unit change in the ith current $i = 1, \ldots, m$.

$\vec{w} = \{w_1, w_2, \cdots, w_m\}$ = set of all adjustable currents.

$\Delta B(R, \vec{w}) = B(R, \vec{w}) - B^s(R)$ = residual field error, for frequency ω.

$B^0(R) = B(R) - B^s(R)$ = uncorrected field error.

B. Quantities to be Obtained by Measurement or Prior Calculation

The necessary magnet measurements will give us the quantities $B(R,\theta)$ and the $B^i(R,\theta)$, from which the azimuthal averages $B(R)$ and $B^i(R)$ may be computed. The synchronous field $B^s(R)$ can be calculated from the uncorrected field $B(R,\theta)$ if we ignore the small changes in flutter that the currents w_i will introduce. This is done, for example by the DORO code mentioned previously.[2] One has to choose a value of ω--our method here has been to choose it to correspond to a synchronous field B^s such that $B^s = B$ at a specified radius. Because such a choice is rather arbitrary we shall give the mathematical algorithm a chance to alter it by introducing a frequency shift $\delta\omega$ from this value. Finally, we need the path-length factor $\sigma = \Delta s/s_0$, which is also calculated by the DORO code.

C. Residual Field Error

We expand $B(R,\vec{w})$ in a Taylor series about $\vec{w} = 0$:

$$B(R,\vec{w}) = B(R) + \sum_{i=1}^{m} \left(\frac{\partial B}{\partial w_i}\right)_{\vec{w}=0} \cdot w_i + \cdots \tag{4.1}$$

$$= B(R) + \sum_{i=1}^{m} B^i(R) w_i + \cdots .$$

By using the last two expressions in Subsec. 4-A with (4.1), one obtains the residual field error to first order in terms of the w_i:

$$\Delta B(R,\vec{w}) = B^0(R) + \sum_{i=1}^{m} B^i(R)\, w_i . \tag{4.2}$$

D. Phase Error

The phase error can now be expressed in terms of the w_i by substituting (4.2) in (3.10) or (3.12), using (2.3). The result is

$$S(R, \omega + \delta\omega) = \sin[h\Phi(R)] - \sin[h\Phi(0)]$$
$$= A^0(R) + \sum_{i=1}^{m+1} A^i(R) w_i, \tag{4.3}$$

where

$$A^i(R) = -\frac{2\pi h}{\epsilon_0}\left(\frac{\omega}{c}\right)^2 \frac{1}{B_{s0}} \int_0^R B^i(R')\left[1 + \sigma(R') + R' \frac{d\sigma(R')}{dR'}\right] R'\, dR',$$

$$i = 0, 1, \cdots, m+1 \tag{4.4}$$

and

$$B^{m+1}(R) = B^0(R) - \frac{(1 + \sigma(R))\, B_{s0}}{\left(1 - \left\{\frac{\omega R}{c}[1 + \sigma(R)]\right\}^2\right)^{3/2}}.$$

E. Radial Gradient

The influence of the trim coils on betatron stability arises chiefly through their contribution to the radial gradient of the azimuthal average of the magnetic field, since they do not affect the flutter very much. Specifically, the quantity involved (usually denoted by -n or k) is

$$G = \frac{R}{B}\frac{dB}{dR}.$$

If we substitute (4.2) into this definition, subtract off the synchronous value of G, G^s, then to first order in the w_i:

$$\Delta G(R, \vec{w}) = G(R, \vec{w}) - G^s(R) = D^0(R) + \sum_{i=1}^{m} D^i(R) w_i, \tag{4.5}$$

where

$$D^0(R) = \frac{R}{B^s}\frac{dB^0}{dR} - \frac{RB^0}{(B^s)^2}\frac{dB}{dR},$$

$$D^i(R) = \frac{R}{B^s}\left[\left(1 - \frac{B^0}{B^s}\right)\frac{dB^i}{dR} - \frac{B^i}{B^s}\frac{dB}{dR}\right], \tag{4.6}$$

$$G^s(R) = \frac{R}{B^s}\frac{dB^s}{dR}.$$

Alternatively, one could expand about the uncorrected field B rather than the synchronous field B^s. In that case

$$D^0 = (1 + \frac{B^0}{B^s}) \frac{R}{B} \frac{dB^0}{dR} - D \frac{B^0}{B^s} = D - G^s,$$

$$D^i = \frac{R}{B} \frac{dB^i}{dR} - D \frac{B^i}{B}, \tag{4.7}$$

$$D = \frac{R}{B} \frac{dB}{dR}.$$

The expansion (4.6) is probably best, but we have so far used (4.7). The difference is not too important.

F. Third Radial Derivative

The quantity

$$H = \frac{R^3}{B} \frac{d^3B}{dR^3} \tag{4.8}$$

is also an important one in beam stability. Proceeding as with G, one obtains

$$H(R, \vec{w}) = H^s(R) + T^0(R) + \Sigma\, T^i(R) w_i, \tag{4.9}$$

$$H^s = \frac{R^3}{B^s} \frac{d^3B^s}{dR^3},$$

$$T^0 = \frac{R^3}{B^s} \frac{d^3B^0}{dR^3} - \frac{R^3B^0}{(B^s)^2} \frac{d^3B}{dR^3}, \tag{4.10}$$

$$T^i = \frac{R^3}{B^s} [(1 - \frac{B^0}{B^s}) \frac{d^3B^i}{dR^3} - \frac{B^i}{B^s} \frac{d^3B}{dR^3}].$$

G. Summary

We will, from now on, deal with all quantities at a finite set of radii R_k, and denote these with an index k [i.e., $\psi_k \equiv \psi(R_k)$]. Listed below are the first order expressions for the quantities defined above, in the field obtained with currents $I + w_1, w_2, \cdots, w_m$, and frequency shift $w_{m+1} = \frac{\delta\omega}{\omega}$:

Field: $$B_k(\vec{w}) = B_k + \sum_{i=1}^{m} B_k^{\,i} w_i \qquad (4.11\ 1)$$

Field error: $$\Delta B_k(\vec{w}) = B_k^{\,0} + \sum_{i=1}^{m} B_k^{\,i} w_i \qquad (4.11\ 2)$$

Sine of phase shift: $$S_k(\vec{w}) = A_k^{\,0} + \sum_{i=1}^{m+1} A_k^{\,i} w_i \qquad (4.11\ 3)$$

First radial gradient index: $$G_k(\vec{w}) = G_k^{\,s} + D_k^{\,0} + \sum_{i=1}^{m} D_k^{\,i} w_i \qquad (4.11\ 4)$$

Third radial gradient index: $$H_k(\vec{w}) = H_k^{\,s} + T_k^{\,0} + \sum_{i=1}^{m} T_k^{\,i} w_i. \qquad (4.11\ 5)$$

We now have at hand, in Eqs. (4.11), the explicit dependence on the coil currents of the most important adjustable properties of the magnetic field. The relations are all linear in the independent variables w_i. It is possible to state quite a varied set of criteria for an optimal field, each of which can be expressed as a so-called "linear programming" (L. P.) problem. There exist algorithms and computer codes for the solution of such problems. In the balance of this report a number of possible criteria for optimal cyclotron fields will be translated into the form of L. P. problems. From these formulations it will be clear how to construct the input data for an L. P. code. The computor will then calculate currents w_i that optimize the field according to the criteron chosen.

5. THE LINEAR PROGRAMMING PROBLEM FOR MINIMIZING THE PHASE ERROR

In this section, criteria for optimal fields based on minimizing the phase shift only will be put in the form of L. P. problems. We start with a definition of the standard form of L. P. problem used. For further details a text on linear programming may be consulted, for example, that by Gass.[7]

A. Standard Form for Linear Programming Problem

Primal Problem (A4): Choose values of $x^j, j = 1, \cdots, n$ that minimize z, where

$$z = \sum_{j=1}^{n} a_j^0 x^j, \tag{5.101}$$

subject to

$$\sum_{j=1}^{n} a_j^i x^i = b^i, \quad i = 1, \cdots, m (m < n) \tag{5.102}$$

$$x^j \geqslant 0, \quad j = 1, \cdots, n. \tag{5.103}$$

Dual Problem (A4'): Choose values of w_i, $i = 1, \cdots, m$, that minimize y, where

$$y = \sum_{i=1}^{m} w_i b^i, \tag{5.201}$$

subject to

$$\sum_{i=1}^{m} w_i a_j^i \geqslant -a_j^0, \quad j = 1, \cdots, n. \tag{5.202}$$

---

* We will label L. P. problems with or without primes, according as they are cast in the dual or primal form, respectively.

The primal and dual problems are closely related, in that the solutions of each give the <u>same</u> values of the "objective forms" or "costs," i.e., $z = y$, and if one is solved the solution of the other is obtained as a byproduct. In this report all criteria for optimal fields will be cast in the dual form (A4'). The IBM 704 code used by us, SCROL,[3] solves problems given in the primal form A4, but also provides the solution of the dual problem (A4').*

<u>Standard Tableau.</u> For each problem in this report, cast in the form (A4'), we will also display the b^i and a_j^i in tabular form appropriate to the input of the problem for SCROL in the form A4, as follows:

	i \ j	1				j				n	
	0	a_1^0				a_j^0				a_n^0	First B
w_1	$q+1$	a_1^1	.	.	.	a_j^1	.	.	.	a_n^1	b^1
.	.	.	.	.	.	.	.	.	.	.	.
w_i	$q+i$	a_1^i	.	.	.	a_j^i	.	.	.	a_n^i	b^i
.	.	.	.	.	.	.	.	.	.	.	.
w_m	$q+m$	a_1^m	.	.	.	a_j^m	.	.	.	a_n^m	b^m

Actually, in the SCROL code, the column indices j may be any symbol of up to five characters, but the row indices i must be consecutive. However, the number q must be $\geqslant 1$. The data are input into the code by columns, starting with the b^i, then the $a_1^i, a_2^i, \ldots, a_n^i$.

---

* Our first test of the L. P. method employed an IBM 650 code written by Mr. Roger Symons of IBM for calculating optimal blends of chicken feeds! This code required putting the problem in a somewhat different form, and could not handle more than seven coils because of the small memory of IBM 650.

If a code different from SCROL is used, it may be necessary to put the problem originally given in dual form (A4') into primal form. This can be done by first adding constants to the w_i to make the variables always positive, and then by introducing "slack variables" to convert the inequalities in (A4') to equalities. The resulting problem will then have the structure of the primal problem (A4).

B. The Phase-Minimization Problem*

The simplest problem we can solve is that obtained from the criterion that the largest phase excursion the particle experiences during acceleration be minimized, subject to practical limits on the currents. For the present we fix the frequency ω, so that $w_{m+1} = 0$.

Besides the variables w_i, we introduce into the problem a new variable λ, defined to be an upper bound to the phase excursions $|S_k|$ experienced by the beam. Specifically, we make λ obey

$$-\lambda \leq S_k \leq \lambda, \qquad k = 1, \cdots, K, \tag{5.3}$$

where the S_k are given in terms of the currents w_i by (4.113). Next we force the currents w_i to lie between upper and lower bounds $W_i^{\pm}$:

$$W_i^- \leq w_i \leq W_i^+, \tag{5.4}$$

and finally we specify that λ should be minimized subject to (5.3) and (5.4). Putting these requirements together, and using (4.113) for S_k, gives us an L. P. problem in the form (A4'):

L. P. 1': Choose values of w_i $(i=1, \cdots, m)$ and λ, that minimize y, where

$$y = \lambda, \tag{5.501}$$

---

* The method used is partly inspired by an article by James E. Kelley, Jr., reference 8.

Subject to

$$\lambda + \sum_{i=1}^{m} A_k^{\,i} w_i \geqslant - A_k^{\,0}, \quad k = 1, \cdots, K,$$

$$\lambda - \sum_{i=1}^{m} A_k^{\,i} w_i \geqslant A_k^{\,0}, \tag{5.502}$$

$$w_i \geqslant W_i^{\,-}, \quad i = 1, \cdots, m,$$

$$-w_i \geqslant -W_i^{\,+}. \tag{5.503}$$

The optimal solution to L. P. 1' will give values of the w_i that make λ, the largest phase excursion, a minimum. Since particles are lost whenever $|\sin \Phi| > 1$, this criterion does the most to avoid loss.

The standard tableau for this problem is shown in Table I.

Table I. Standard tableau for phase-minimization problem L. P. 1 (the dual to the original problem L. P. 1'), showing input format required by the SCROL Code.[a]

	BETA →	x^{LS1} · x^{LSk} · x^{LSK}	x^{US1} · x^{USk} · x^{USK}	x^{LW1} · x^{LWi} · x^{LWm}	x^{UW1} · x^{UWi} · x^{UWm}	
PI ↓	i \ j	LS1 · LSk · LSK	US1 · USk · USK	LW1 · LWi · LWm	UW1 · UWi · UWm	FIRST B
z	0	A_1^0 · A_k^0 · A_K^0	$-A_1^0$ · $-A_k^0$ · $-A_K^0$	$-W_1^-$ · $-W_i^-$ · $-W_m^-$	W_1^+ · W_i^+ · W_m^+	
w_1	q + 1	A_1^1 · A_k^1 · A_K^1	$-A_1^1$ · $-A_k^1$ · $-A_K^1$	1 · 0 · 0	-1 · 0 · 0	0
w_2	q + 2	A_1^2 · A_k^2 · A_K^2	$-A_1^2$ · $-A_k^2$ · $-A_K^2$	0 · 0 · 0	0 · 0 · 0	0
·	·	· · · · ·	· · · · ·	· · · · ·	· · · · ·	·
w_i	q + i	A_1^i · A_k^i · A_K^i	$-A_1^i$ · $-A_k^i$ · $-A_K^i$	0 · 1 · 0	0 · -1 · 0	0
·	·	· · · · ·	· · · · ·	· · · · ·	· · · · ·	·
w_m	q + m	A_1^m · A_k^m · A_K^m	$-A_1^m$ · $-A_k^m$ · $-A_K^m$	0 · 0 · 1	0 · 0 · -1	0
$\lambda = w_{m+1}$	q + (m + 1)	1 · 1 · 1	1 · 1 · 1	0 · 0 · 0	0 · 0 · 0	1

[a] The q in the i indices should be set equal to 1. The column indices refer to the constraint equations from which they are derived: LSk stands for lower limit of S_k, USk for upper limit of S_k, LWi for lower limit of w_i, and UWi for upper limit of w_i. The constraint matrix a_j^i goes from $i = q + 1, \cdots, q + (m + 1)$; $j = LS1, \cdots, UWm$. The "cost" coefficients a_j^0 are in the row $i = 0$, the right-hand-side coefficients b^i are in the column headed FIRST B. Output will list the w_i in the PI column, x^j in the BETA column.

C. Control of Extraction Momentum

So far the frequency ω has been held fixed, but the magnetic field at the extraction radius has not, so the extraction energy is not well defined. To remedy this defect we now let the frequency vary, by letting $w_{m+1} = \delta\omega/\omega$ be a free variable, subject to upper and lower limits $W^{\pm}_{m+1}$. That is, the index i in (5.5) should now run from $i = 1, \cdots, m+1$. Now suppose we wish the momentum at the extraction radius R_L to be p_L. From Eq. (2.5), that means we must have

$$p_L = r_L\, b(r_L),$$

so this is equivalent to specifying the field at R_L. Let us refer p_L to its base value p_L^s obtained with the synchronous field:

$$p_L^s = r_L b^s(r_L) = \frac{\beta_L^s}{\sqrt{1-(\beta_L^s)^2}}, \quad \beta_L^s = \frac{\omega R_L}{c}\,[1 + \sigma(R_L)], \tag{5.6}$$

and write $p_L = p_L^s + \delta p_L$. Then, using the above relations and (4.112), we obtain

$$\delta p_L = P_L^0 + \sum_{i=1}^{m} P_L^i\, w_i, \tag{5.7}$$

where

$$P_L^i = \frac{\omega R_L}{c B_{s0}}\, B_L^i = \frac{eZ}{m_0 c^2}\, R_L\, B_L^i. \tag{5.8}$$

To fix δp_L at a specified value, we write (5.7) as two inequalities:

$$\delta p_L \leqslant P_L^0 + \sum_{i=1}^{m} P_L^i\, w_i, \quad \delta p_L \geqslant P_L^0 + \sum_{i=1}^{m} P_L\, w_i,$$

or

$$\pm \sum_{i=1}^{m} P_L^i\, w_i \geqslant \mp \bar{P}_L^0, \tag{5.9}$$

where

$$\bar{P}_L^0 = P_L^0 - \delta p_L \tag{5.901}$$

Then the revised problem, in which the frequency can vary but the extraction momentum is fixed, is L. P. 2', described below.

L. P. 2': Choose values of $w_1, \cdots, w_{m+1}$ and λ, that minimize λ subject to

$$\pm w_i \geqslant \pm W_i^{\mp},$$

$$\lambda \pm \sum_{i=1}^{m+1} A_k^{\,i} w_i \geqslant \mp A_k^{\,0}, \tag{5.10}$$

$$\pm \sum_{i=1}^{m} P_L^{\,i} w_i \geqslant \mp \overline{P}_L^{\,0}.$$

An alternative procedure is to let δp_L be another variable, $\delta p_L = w_{m+2}$, and then set such bounds as desired about it. Then the problem becomes L. P. 3', described below.

L. P. 3': Choose values of $w_1, \cdots, w_m$; $w_{m+1} = \frac{\delta\omega}{\omega}$; $w_{m+2} = \delta p_L$; and λ, that minimize λ, subject to

$$\pm w_i \geqslant \pm W_i^{\mp}, \qquad i = 1, \cdots, m+2,$$

$$\lambda \pm \sum_{i=1}^{m+1} A_k^{\,i} w_i \geqslant \mp A_k^{\,0}, \quad k = 1, \cdots, K, \tag{5.11}$$

$$\mp \delta p_L \pm \sum_{i=1}^{m} P_L^{\,i} w_i \geqslant \mp P_L^{\,0}.$$

D. Arbitrary Starting Phase and Programmed Radial Phase Dependence

If particles are injected from a source offset from the center of the dee gap, they will start with a nonzero starting phase $\Phi(0)$. This might be desirable in order to obtain electric vertical focusing, which occurs for small radii if Φ is positive. Then the phase as a function of radius is $\Phi(R)$, where $\Phi(R)$ is given by Eq. (4.3).

Since this phase should go from $\Phi(0)$ to zero in a smooth fashion, we

specify a programmed radial phase dependence, $\Phi^P(R)$, which we would like the field to produce. Then we should minimize $|\sin[h\Phi(R)] - S^P(R)|$, where $S^P(R) = \sin[h\Phi^P(R)]$. Rewriting (4.3) as

$$\sin(h\Phi_k) = \sin[h\Phi(0)] + S_k,$$

we now set

$$-\lambda \leq \{\sin[h\Phi(0)] + S_k\} - S_k^P \leq \lambda.$$

Proceeding as in Subsec. 5-B, and again using (4.113) to express S_k in terms of the currents w_i, we arrive at two sets of inequalities like the first two in (5.502), with the difference that the right-hand-side factors A_k^0 are replaced by $\bar{A}_k^0$, where

$$\bar{A}_k^0 = A_k^0 - S_k^P + \sin[h\Phi(0)],$$
$$S_k^P = \sin[h\Phi^P(R_k)], \quad \Phi^P = \text{programmed phase.} \tag{5.12}$$

E. Varying the Limiting Phase Shift with Radius

One might not consider a phase error a equally serious at all radii. For example, one might wish to keep the phase especially small near an electrostatic extractor channel so as to maximize turn separation there. We could introduce a factor α_K, defined as the fraction of the nominal phase maximum λ to be tolerated at radius R_k. That is, we require

$$|S_k| \leq \alpha_k \lambda.$$

This means that the λ's in (5.502) should be replaced by $\alpha_k\lambda$. In the tableau, Table I, this means that 1's in the first 2K columns of the last row, belonging to the variable λ, should be replaced by the α_k. If one does not care at all about phase error at some particular R_k, then α_k should be set very large; if one wishes strictly zero phase error at R_k, then α_k should be set equal to zero. Rather than set $\alpha_k = \infty$ in the former case, the same result can be achieved by simply removing from the matrix the two columns involved, LSk and USk.

F. Summary

We summarize this section with a statement of the L. P. problem obtained when all of the features of this section are included:

L. P. 3': Choose values of the variables w_i, $i=1,\cdots,m+1$, and λ that minimize λ, subject to

$$a_k\lambda \pm \sum_{i=1}^{m+1} A_k^{\,i}\, w_i \geqslant \mp \bar{A}_k^{\,0}, \quad k = 1,\cdots,K, \tag{5.131}$$

$$\pm \sum_{i=1}^{m} P_L^{\,i}\, w_i \geqslant \mp \bar{P}_L^{\,0}, \tag{5.132}$$

$$\pm w_i \geqslant \pm W_i^{\mp}, \quad i = 1,\cdots,m+1. \tag{5.133}$$

The solution of L. P. 3' will give: a main-coil current change w_1; trim-coil currents w_i, $i = 2,\cdots, m$; frequency shift $\frac{\delta\omega}{\omega} \equiv w_{m+1}$; and a parameter λ. This parameter will have been minimized, such that: the magnitude of the phase error $|\sin(h\Phi_k) - S_k^{\,P}|$ at each R_k is less than or equal to $a_k\,\lambda$; the momentum at the extraction radius R_L is $p_L = p_L^{\,s} + \delta p_L$, where $p_L^{\,s}$ is the momentum in the synchronous field at R_L, [Eq. (5.6)]; and the currents are bounded, $-W_i^{-} \leqslant w_i \leqslant W_i^{+}$, $i = 1,\cdots,m+1$. Here $S_k^{\,P}$ is the $\sin(h\Phi)$ desired, and the quantities $A_k^{\,i}$, $\bar{A}_k^{\,0}$, $P_L^{\,i}$, $\bar{P}_L^{\,0}$ are defined by Eqs. (4.4), (5.12), (5.8), and (5.901), respectively.

The tableau for the dual of L. P. 3' is given in Table II.

Table II. Standard tableau for the augmented phase-minimization problem L. P. 3 (dual to original problem L. P. 3').

		Momentum at R_L		Bounds on sine of phase shift		Bounds on currents		
				lower	upper	lower	upper	
w_i	i \ j	LPL	UPL	LS1 · LSk · LSK	US1 · USk · USK	LW1 · LWi · LWM	UW1 · UWi · UWM	FIRST B
Cost row →	0	$\bar{P}_L^0$	$-\bar{P}_L^0$	$\bar{A}_1^0 \cdot \bar{A}_k^0 \cdot \bar{A}_K^0$	$-\bar{A}_1^0 \cdot -\bar{A}_k^0 \cdot -\bar{A}_K^0$	$-W_1^- \cdot -W_i^- \cdot -W_M^-$	$W_1^+ \cdot W_i^+ \cdot W_M^+$	
Variables ↓								
Main-coil current change w_1	q+1	P_L^1	$-P_L^1$	$A_1^1 \cdot A_k^1 \cdot A_K^1$	$-A_1^1 \cdot -A_k^1 \cdot -A_K^1$	1 · 0 · 0	-1 · 0 · 0	0
w_2	q+2	P_L^2	$-P_L^2$	$A_1^2 \cdot A_k^2 \cdot A_K^2$	$-A_1^2 \cdot -A_k^2 \cdot -A_K^2$	0 · 0 · 0	0 · 0 · 0	0
Trim-coil currents	·	·	·	· · · · ·	· · · · ·	· · · · ·	· · · · ·	·
	·	·	·	· · · · ·	· · · · ·	· · · · ·	· · · · ·	·
	·	·	·	· · · · ·	· · · · ·	· · · · ·	· · · · ·	·
w_i	q+i	P_L^i	$-P_L^i$	$A_1^i \cdot A_k^i \cdot A_K^i$	$-A_1^i \cdot -A_k^i \cdot -A_K^i$	0 · 1 · 0	0 · -1 · 0	0
	·	·	·	· · · · ·	· · · · ·	· · · · ·	· · · · ·	·
	·	·	·	· · · · ·	· · · · ·	· · · · ·	· · · · ·	·
	·	·	·	· · · · ·	· · · · ·	· · · · ·	· · · · ·	·
w_m	q+m	P_L^m	$-P_L^m$	$A_1^m \cdot A_k^m \cdot A_K^m$	$-A_1^m \cdot -A_k^m \cdot -A_K^m$	0 · 0 · 0	0 · 0 · 0	0
Frequency $\frac{\delta\omega}{\omega} = w_{m+1}$	q+m+1	0	0	$A_1^{m+1} \cdot A_k^{m+1} \cdot A_K^{m+1}$	$-A_1^{m+1} \cdot -A_k^{m+1} \cdot -A_K^{m+1}$	0 · 0 · 1	0 · 0 · -1	0
Max $\lvert \sin h\phi \rvert = \lambda = w_{m+2}$	q+m+2	0	0	$a_1 \cdot a_k \cdot a_K$	$a_1 \cdot a_k \cdot a_K$	0 · 0 · 0	0 · 0 · 0	1

6. LINEAR PROGRAMMING PROBLEMS INCLUDING RADIAL-GRADIENT CONTROL

A. Specification of Gradient Limits

In Sec. 4, the necessity for keeping the radial gradient index G bounded was mentioned. This is because the frequencies of radial and vertical oscillation, ν_r and ν_z, respectively, obey relations*

$$\nu_r^2(R) = 1 + G(R) + \eta_r(R)$$

and (6.1)

$$\nu_z^2(R) = -G(R) = \eta_z(R),$$

where η_r and η_z express the focusing effects obtained from the azimuthal field variations, and G is given by Eq. (4.5). By processing the field measurements with the Berkeley code DORO[2] and the Oak Ridge Code 1482,[1] one can obtain ν_r and ν_z in the measured field, corrected— in its radial dependence only— for isochronism. One can then examine these results, and decide how much fluctuation in ν_r^2 and ν_z^2 is to be tolerated, and this in turn gives the permissible deviations of G from its isochronous value [which is $G_s(R) \approx \beta^2(R)$].

To be more specific, suppose one wished to impose the conditions, roughly appropriate for a low-energy cyclotron:

$$\nu_z \geqslant 0, \quad \nu_r \geqslant 1, \quad \nu_z \leqslant \frac{1}{2}, \quad \nu_r - 2\nu_z \geqslant 0,$$

referring to linear vertical and radial stabiltiy, first-harmonic-induced vertical instability, and lowest-order coupling resonance, respectively. Rewriting in terms of ν_r^2 and ν_z^2, we have

$$\nu_z^2 \geqslant 0, \quad \nu_z^2 \leqslant 1/4, \quad \nu_r^2 \geqslant 1, \quad \nu_r^2 - 4\nu_z^2 \geqslant 0. \tag{6.2}$$

---

* For more complete expressions in terms of the Fourier coefficients representing the field, see reference 6.

The last of these relations is redundant, so we drop it. Referring to (6.1) and letting subscript s refer to values for the isochronous fields, we have

$$0 \leqslant \nu_z^2 = \nu_{zs}^2 - \Delta G - \Delta\eta_z \leqslant \frac{1}{4},$$

$$\nu_r^2 = \nu_{rs}^2 + \Delta G + \Delta\eta_r \geqslant 1. \tag{6.3}$$

The variations $\Delta\eta_r$ and $\Delta\eta_z$ with respect to gradient changes are comparatively small, and for the moment we will drop them. Hence

$$\Delta G \leqslant \nu_{zs}^2, \tag{6.401}$$

$$\Delta G \geqslant -(\frac{1}{4} - \nu_{zs}^2), \tag{6.402}$$

$$\Delta G \geqslant -(\nu_{rs}^2 - 1). \tag{6.403}$$

If we now define $\pm\gamma_k^{\pm}$ to be the upper and lower bounds to be allowed in the fluctuations of G at R_k, we have from (6.4),

$$\gamma_k^+ = \nu_{zs}^2(R_k),$$

$$\gamma_k^- = \min\{[\nu_{rs}^2(R_k) - 1], (\frac{1}{4} - \nu_{zs}^2)\}. \tag{6.5}$$

For a high-energy machine other resonances would enter, but in a similar fashion one could arrive at the appropriate values of the gradient fluctuation limits $\gamma_k^{\pm}$ to replace (6.5).*

By using (6.5) the relations (6.4) can now be written

$$-\gamma_k^- \leqslant \Delta G_k \leqslant \gamma_k^+.$$

---

* In the 88-inch cyclotron the field has a bump near the center, to enhance vertical focusing, hence $\nu_r - 1$ changes, from negative to positive values at about 6 in. By suitable juggling of the $\nu_k^{\pm}$ one can control the radius for which $\nu_r = 1$, and keep $|\nu_r - 1|$ maximized at other radii.

Since we would like to have the gradient fluctuations even smaller, we replace these relations with

$$-\mu\gamma_\mu^- \leq \Delta G_k \leq \mu\gamma_k^+,$$
$$\mu \leq \mu_{max}, \tag{6.6}$$

where μ is a parameter to be minimized, and μ_{max} is set at some value not greater than one. The gradient fluctuation ΔG_k, from Eqs. (4.5), (4.6), and (4.114), is

$$\Delta G \equiv G_k - G_k^s = D_k^0 + \sum_{i=1}^{m} D_k^i w_i. \tag{6.7}$$

Substituting this expression into (6.6), we obtain the following relations, which serve to keep the gradient fluctuations within proper limits:

$$\mu\gamma_k^{\mp} \pm \sum_{i=1}^{m} D_k^i w_i \geq \mp D_k^0, \quad k = 1, \cdots, K,$$
$$-\mu \geq -\mu_{max} \text{ (and } \mu_{max} \leq 1). \tag{6.8}$$

B. "Effective" Gradient

The above formulation ignores the fact that the fluctuations in G do not completely account for those of ν_z^2 and ν_r^2, since there are terms in η_r and η_z, Eq. (6.1), involving G and its higher derivatives. The effect has been explained qualitatively by H. G. Blosser: since the equilibrium orbits oscillate, they do not see the field gradient index G at one radius, but rather some kind of average G over the radial range they sweep out. This has the beneficial effect of making the particles less sensitive to short-range fluctuations in G. Lloyd Smith has investigated a similar effect in conventional cyclotrons.

Quantitatively this effect is rather well described by a single term in

η_r and η_z, namely $(F^2H) / [2(N^2 - 1)^2]$; where F is the flutter, Eq. (2.8); H is the third-derivative index, Eq. (4.8); and N is the number of sectors.* This term is just what would be obtained by averaging G over the equilibrium-orbit oscillations. We can take it into account by replacing G by an "effective" gradient index $\overline{G}$, where

$$\overline{G} \equiv G + \frac{F^2 H}{2(N^2 - 1)^2}, \tag{6.9}$$

and by using $\Delta\overline{G}$ instead of ΔG in (6.4) and (6.6). Thus if one wishes to satisfy relations (6.3), taking into account the noncircularity of the equilibrium orbit, one must replace the $D_k^{\ i}$ with the quantities $\overline{D}_k^{\ i}$, in relations (6.8), where

$$\overline{D}_k^{\ i} = D_k^{\ i} + \frac{F_k^2 \ T_k^{\ i}}{2(N^2 - 1)^2}. \tag{6.10}$$

Here $F_k^{\ 2}$ is the flutter squared at R_k, and the $T_k^{\ i}$ are given by (4.10). Since the additional term tends to dampen the fluctuations of $\nu_z^{\ 2}$, it is conservative to leave it out.

C. Summary: Combined Phase and Gradient Control

The simultaneous control of the phase error and the gradient is achieved by including the contraints (6.8) with constraints (5.13) of L.P. 3'. There are now two parameters to minimize, λ and μ, so we must put an absolute bound on λ:

$$\lambda \leqslant \lambda_{max}.$$

---

* This term is included in the expressions for $\nu_r^{\ 2}$ and $\nu_z^{\ 2}$ in reference 6, Appendices 2 and 3. The notation there is $G = \mu'$, $H = \mu'''$.

Then we minimize some linear combination $b^{\lambda}\lambda + b^{\mu}\mu$, where b^{λ} and b^{μ} are "cost coefficients" to be assigned. The combined problem for phase error and gradient control is as follows:

L. P. 4': Choose values of the variables ($w_1, \cdots, w_{m+1}$, λ, and μ) that minimize

$$y = b^{\lambda}\lambda + b^{\mu}\mu, \tag{6.111}$$

subject to

$$-\lambda \geqslant -\lambda_{max}, \quad -\mu \geqslant -\mu_{max}, \tag{6.112}$$

$$w_i \geqslant W_i^{-}, \quad -w_i \geqslant -W_i^{+}, \tag{6.113}$$

$$a_k \lambda \pm \sum_{i=1}^{m+1} A_k^{\,i} w_i \geqslant \mp \overline{A}_k^{\,0}, \tag{6.114}$$

$$\pm \sum_{i=1}^{m} P_L^{\,i} w_i \geqslant \mp \overline{P}_L^{\,0}, \tag{6.115}$$

$$\gamma_k^{\mp} \mu \pm \sum_{i=1}^{m} D_k^{\,i} w_i \geqslant \mp D_k^{\,0}. \tag{6.116}$$

Here $i = 1, \cdots, m+1$ and $k = 1, \cdots, K$. The $\overline{A}_k^{\,0}$, $P_L^{\,i}$, $\overline{P}_L^{\,0}$ are defined by Eqs. (5.12), (5.8), and (5.901); $A_k^{\,i}$, and $D_k^{\,i}$ by (4.4) and (4.6); and the $\gamma_k^{\pm}$ by (6.5). The D's in (6.116) may be replaced by $\overline{D}$'s, Eq. (6.10) to control the "effective" gradient.

Problem L. P. 4' will accomplish all that L. P. 3' does, except that the objective form (6.111) can involve μ as well as λ, and in addition it bounds ΔG, the departure of the gradient from its isochronous value, according to $-\mu\gamma_k^{-} \leqslant \Delta G_k \leqslant \mu\gamma_k^{+}$. Furthermore, λ and μ have upper bounds λ_{max} and μ_{max}. Of course, if these are taken too small there may be no feasible solutions.

The tableau for L. P. 4, the dual of L. P. 4', is shown in Table III.

Table III. Tableau for combined phase and gradient control, and optimization problem L.P.4 (dual of L.P.4')

($m = M-1$. Replace D_k^i with $\bar{D}_k^i$ to control effective gradient.)

			Max λ	Max μ	Current limits $i = 1, \cdots, M$		Sin $(h\Phi)$ limits $k = 1, \cdots, K$		Momentum at R_L		Gradient limits $k = 1, \cdots, K$		Cost coeff's
					lower	upper	lower	upper			lower	upper	
		$i \backslash j$	LBDMX	MUMAX	LWi	UWi	LSk	USk	LPL	UPL	LGk	UGk	FIRST B
Cost row →		0	λ_{max}	μ_{max}	$-W_i^-$	W_i^+	$\bar{A}^0$	$-\bar{A}_k^0$	$\bar{P}_L^0$	$-\bar{P}_L^0$	D_k^0	$-D_k^0$	0
Variables ↓													
Main-coil current	w_1	$q+1$	0	0	0	0	A_k^1	$-A_k^1$	P_L^1	$-P_L^1$	D_k^1	$-D_k^1$	0
	w_2	$q+2$	0	0	0	0	A_k^2	$-A_k^2$	P_L^2	$-P_L^2$	D_k^2	$-D_k^2$	0
	·	·	·	·	·	·	·	·	·	·	·	·	·
Trim-coil currents	w_i	$q+i$	0	0	1	-1	A_k^i	$-A_k^i$	P_L^i	$-P_L^i$	D_k^i	$-D_k^i$	0
	·	·	·	·	·	·	·	·	·	·	·	·	·
	w_m	$q+m$	0	0	0	0	A_k^m	$-A_k^m$	P_L^m	$-P_L^m$	D_k^m	$-D_k^m$	0
Frequency $\delta\omega/\omega_s$	w_{m+1}	$q+m+1$	0	0	0	0	A_k^{m+1}	$-A_k^{m+1}$	0	0	0	0	0
Max. phase	w_{m+2}	$q+m+2$	-1	0	0	0	a_k	a_k	0	0	0	0	b^λ
Max. gradient μ	w_{m+3}	$q+m+3$	0	-1	0	0	0	0	0	0	γ_k^-	γ_k^+	b^μ

7. CONTROL OF NONLINEAR EFFECTS RESULTING FROM THE THIRD-DERIVATIVE INDEX

It might be necessary to control the third-derivative index H, Eq. (4.8), other ways besides that specified in the preceding section. We will discuss two such possibilities.

A. Cubic Terms in Vertical Equations

The vertical equation of motion for $y = z/R$ contains the nonlinear term $\frac{1}{6} Hy^3$, so that the equation of motion is*

$$\frac{d^2y}{d\theta^2} + (\nu_z^2 + \frac{1}{6} H y^2)y = 0. \tag{7.1}$$

Now, as we have seen, ν_z^2 contains the term $\frac{-HF^2}{2(N^2-1)^2}$. Hence the total contribution to the vertical focusing by H is

$$H[- \frac{F^2}{2(N^2 - 1)^2} + \frac{1}{6} y^2]. \tag{7.2}$$

The first term predominates so long as $y \leqslant y_1$, where

$$y_1 = \frac{\sqrt{3}\ F}{N^2 - 1}. \tag{7.3}$$

For the 88-inch cyclotron this usually gives a $z_1 = Ry_1 \approx 1$ in., at the values of R for which H is significant, which is larger than the dee height. Hence, at Berkeley, we probably will not need to worry about H so far as this nonlinear effect is concerned, but since the situation could be different in another machine, we will set down the formulation necessary for controlling it.

Suppose one wishes to attempt to insure vertical stability of all particles for which $z \leqslant z_{max}$. As we have seen in the preceding section, the fluctuations in ν_z^2 are given by those of the effective gradient

---

* Strictly speaking, this equation is true for another variable q, related to y and $dy/d\theta$ by a θ-dependent linear transformation. However, $q \approx y$. Also, the factor H should really be $H = \mu''' + \mu'' - \mu'$, where $\mu^{(n)} = (R^n/B)(d^nB/dR^n)$, rather than just μ'''.

[Eq. (6.9)], that is, $\nu_z^2 = \nu_{zs}^2 - \Delta\bar{G}$. Thus to keep the force term in Eq. (7.1) positive for $z \leqslant z_{max}$, we must put

$$\nu_{zs}^2 - \left[\Delta G + \frac{F^2 \Delta H}{2(N^2 - 1)^2}\right] + \frac{1}{6} \left(\frac{z_{max}}{R}\right)^2 \Delta H \geqslant 0 \tag{7.4}$$

or

$$\Delta\tilde{G} \leqslant \nu_{zs}^2, \tag{7.5}$$

where

$$\Delta\tilde{G} = \Delta G + \left[\frac{F^2}{2(N^2 - 1)^2} - \frac{1}{6} \left(\frac{z_{max}}{R}\right)^2\right] \Delta H. \tag{7.6}$$

Relation (7.5) plays the same role as (6.401), only it refers to stability of particles with vertical amplitude up to z_{max} rather than zero. One can proceed from it in exactly the same way as before, to reach the following constraints, which insure that particles with amplitude up to z_{max} will retain vertical focusing:

$$\mu\gamma_k^+ - \sum_{i=1}^{m} \tilde{D}_k^i w_i \geqslant \tilde{D}_k^i, \tag{7.7}$$

$$-\mu \geqslant -\mu_{max} \quad (\text{and } \mu_{max} \leqslant 1),$$

where

$$\tilde{D}_k^i = D_k^i + \left[\frac{F_k^2}{2(N^2 - 1)^2} - \frac{1}{6} \left(\frac{z_{max}}{R_k}\right)^2\right] T_k^i. \tag{7.8}$$

To incorporate this vertical focusing feature into the combined problem, L. P. 4' of the preceding section, one simply adds the constraints (7. 7) to constraints (6.112) to (6.116). This will add K new columns to the tableau, Table III, which will look like those labeled UGk, but the D_k^i will be replaced by the $\tilde{D}_k^i$.

B. Effect on Nonlinear Radial Instability

A discussion of the nonlinear 3-3 resonance which is important in a three-sector machine is given in reference 6. The Hamiltonian for the radial motion, Eq. (52) of reference 6, included two terms involving $\mu''' = H$. However, these terms were dropped before arriving at a final expression relating the amplitude to the phase of the radial oscillations, Eq. (61) of reference 6. If these terms are included, one finds that a given particle will have its radial amplitude A and phase χ always constrained to lie on an invariant curve satisfying

$$\sin \chi = \frac{24\,(\nu_r - 1)}{D}\,\frac{1}{A} - \frac{C}{A^3} + \frac{3}{4}\,\frac{H}{D}\,A, \tag{7.9}$$

where

$$D = \left\{\left[a_3'' + 5a'_3 + \left(3 + \frac{H}{8}\right) a_3\right]^2 + \left[b''_3 + 5b'_3 + \left(3 + \frac{H}{8}\right) b_3\right]^2\right\}^{\frac{1}{2}}. \tag{7.10}$$

Here a_3 and b_3 are the coefficients of the $\cos 3\theta$ and $\sin 3\theta$ terms, respectively, in the Fourier expression of the field; primes denote derivatives with respect to $x = (R - R_0)/R_0$, where R_0 is the equilibrium radius; and C is a constant, determined by the initial values of A and χ. Obviously if H is large it has a profound effect on the character of the radial oscillations. This is particularly important for regenerative extraction, a fact which has been noticed by R. H. Bassel and T. A. Welton at Oak Ridge (private communication). The most important effect arises from the last term in (7.9), which stablizes large-amplitude oscillations that would otherwise be unstable, so they represent particles that will no longer escape the cyclotron.

Consequently, one might wish to attempt to keep H itself within acceptable limits. Let us call these limits $\pm\,\eta_k^{\pm}$, refering to radius R_k. Thus we require

$$-\eta_k^{-} \leq H_k \leq \eta_k^{+}.$$

Again we can introduce a new variable, ξ, which we will try to minimize, and rewrite as

$$-\xi\eta_k^- \leq H_k \leq \xi\eta_k^+, \tag{7.11}$$

$$\xi \leq \xi_{max}, \quad \xi_{max} \leq 1.$$

Substituting Eq. (4.115) for H_k into (7.11), we obtain the following set of inequalities, that serve to bound the third-derivative index H:

$$\xi\eta_k^- + \sum_{i=1}^{m} T_k^i w_i \geq -T_k,$$

$$\xi\eta_k^+ - \sum_{i=1}^{m} T_k^i w_i \geq T_k, \tag{7.12}$$

$$\xi \leq \xi_{max}, \quad \xi_{max} \leq 1.$$

These constraints may be added to those of L.P. 4' of Sec. 6. However, we have also added a new variable, ξ, so the objective form should be expanded to

$$y = b^\lambda \lambda + b^\mu \mu + b^\xi \xi .$$

This concludes all of the physical effects we shall consider in this paper.

8. VARIATION OF THE EXTRACTION MOMENTUM

In the L.P. problems we have considered, all except L.P. 1' permit one to specify p_L, the value of the momentum at the extraction radius R_L. In particular, one has to assign a value to $\overline{P}_L^{\,0} = P_L^{\,0} + \delta p_L$ [see Eqs. (5.6), (5.901)], which appears twice in the i = 0 row. To see how the solutions vary with δp_L one may make use of the parametric cost row (PCR) feature of the SCROL code. We have two values of j for which the a_j^0 vary linearly with $\theta = \delta p_L$, namely j = LPL and j = UPL, for which

$$-\begin{pmatrix} a_{LPL}^{\ 0} \\ \\ a_{UPL}^{\ 0} \end{pmatrix} = \pm\,(P_L^{\ 0} + \theta \cdot 1); \qquad \theta = \delta p_L.$$

The input for SCROL then consists of putting $\pm P_L^{\ 0}$ in the j = LPL, UPL columns of the i = 0 row; and the coefficients of θ, ± 1, in the i = 2 row.* One then specifies the upper and lower bands of $\theta = \delta p_L$ that one wants solutions for. The machine will then compute solutions as functions of δp_L between those limits. That is, it prints out a new solution each time the basis changes; the intermediate solutions vary linearly with δp_L.

Actually, since one wants p_L to be both positive and negative, one will have to do PCR twice, first with the i = 1 row containing ± 1, then with the i = 2 row, which should contain ∓ 1. The sum-row index q must be set equal to 3. The constraint matrix rows will start at q + 1 = 4.

It is contemplated that the measurements described in Sec. 4 will be carried out at a number of discreet main-coil current settings I_1. For each of these one should, by examining the uncorrected field, make an educated guess as to the best value of $B_s(R_L)$, and from this calculate associated values of the rf frequency f and the momentum $p_L^{\ s}$. This can be done by solving for $p_L^{\ s}$ and ω as follows:

$$p_L^{\ s} = \frac{eZ}{m_0 c^2}\, R_L\, B^s(R_L),$$

$$\beta_L^{\ s} = \frac{p_L^{\ s}}{\sqrt{1+(p_L^{\ s})^2}},$$

$$\omega = \frac{c\beta_L^{\ s}}{R_L[1+\sigma(R_L)]}.$$

---

*In the physical problem the right-hand side are $-a_j^{\ 0}$. In the dual to this problem the $a_j^{\ 0}$ become the "cost" row.

Such values of ω and p_L^s are calculated by the DORO code if one includes in the input $R_F = R_L$, and one of the following three quantities: $B_F = B^s(R_L)$, E_F = energy in Mev at R_F, or $p_F = p_L^s$. They constitute base values of ω and p_L for the L.P. problems. Thus there will be one $p_L^s(I_1)$ for each of the measurement values of I_1. Parametric programming will then bridge the gap between these discrete values of p_L^s.

9. SUMMARY OF LINEAR PROGRAMMING PROBLEMS FOR TRIM-COIL ADJUSTMENT

We have now described a considerable variety of conditions one might wish to impose on the magnetic field. One might prefer not to impose all of them, or one might wish to impose others not treated here. For convenience we now state a linear programming problem that contains all of the features discussed in this paper. If the user does not wish to impose all of these conditions, he can omit the corresponding constraints, which will mean omitting the corresponding columns of the total matrix shown in Table IV. For those constraints referring to a radius R_k, one may include only those R_k that are of interest for the particular constraint involved. If the remaining constraints do not involve all of the variables λ, μ, or ξ, then those not involved should be left out of the problem and the corresponding rows omitted from Table IV.

In general these linear programming techniques only provide a method for making certain types of optimizations subject to preassigned conditions. If these conditions are made too stringent the trim coils may be incapable of satisfying all of them, and the L.P. codes will state "No feasible solution." The user should use his judgement to try to impose conditions whose mutual satisfaction is within the capacities of the trim coils, and then trust to the

Table IV. Tableau for general trim-coil-adjustment problem L. P. 5 (dual of L. P. 5'), in SCROL format.[a]

Quantity controlled		Current w_i $i=1,\cdots,m+1$		Max λ	Sin $(h\Phi)$ S_k		Momentum at R_L		Max μ	Gradient G_k		Gradient for Z_{max} $\tilde{G}_k$	Max ξ	Third-derivative index, T_k		
Limiting value. Upper or lower limit		W_i^- lower	W_i^+ upper	λ_{max} upper	$-\lambda\alpha_k$ lower	$\lambda\alpha_k$ upper	p_{sL} lower	p_{sL} upper	μ_{max} upper	$-\mu\gamma_k^-$ lower	$\mu\gamma_k^+$ upper	$\mu\gamma_k^+$ upper	ξ_{max} upper	$-\xi\eta_k^-$ lower	$\xi\eta_k^+$ upper	FIRST B
w_i	i \ j	LWi	UWi	LBDMX	LSk	USk	LPL	UPL	MUMAX	LGk	UGk	UGMk	PSIMX	LHk	UHk	0
(Cost row)	0	$-W_i^-$	W_i^+	λ_{max}	$\bar{A}_k^0$	$-\bar{A}_k^0$	$\bar{P}_L^0$	$-\bar{P}_L^0$	μ_{max}	D_k^0	$-D_k^0$	$-\tilde{D}_k^0$	ξ_{max}	T_k	$-T_k$	0
PCR: $+\delta p_L$	1	0	0	0	0	0	+1	−1	0	0	0	0	0	0	0	0
PCR: $-\delta p_L$	2	0	0	0	0	0	−1	+1	0	0	0	0	0	0	0	0
Σ row	$q=3$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
w_1	$q+1$	0	0	0	A_k^1	$-A_k^1$	P_L^1	$-P_L^1$	0	D_k^1	$-D_k^1$	$-\tilde{D}_k^1$	0	T_k^1	$-T_k^1$	0
w_2	$q+2$	0	0	0	A_k^2	$-A_k^2$	P_L^2	$-P_L^2$	0	D_k^2	$-D_k^2$	$-\tilde{D}_k^2$	0	T_k^2	$-T_k^2$	0
.	.	. . .	. . .	.	. . .	. . .	.	.	.	. . .	. . .	. . .	.	. . .	. . .	.
w_i	$q+i$	1	−1	0	A_k^i	$-A_k^i$	P_L^i	$-P_L^i$	0	D_k^i	$-D_k^i$	$-\tilde{D}_k^i$	0	T_k^i	$-T_k^i$	0
.	.	. . .	. . .	.	. . .	. . .	.	.	.	. . .	. . .	. . .	.	. . .	. . .	.
w_m	$q+m$	0	0	0	A_k^m	$-A_k^m$	P_L^m	$-P_L^m$	0	D_k^m	$-D_k^m$	$-\tilde{D}_k^m$	0	T_k^m	$-T_k^m$	0
$w_{m+4}=\frac{\delta\omega}{\omega}$	$q+m+1$	0	0	0	A_k^{m+1}	$-A_k^{m+1}$	0	0	0	0	0	0	0	0	0	0
$w_{m+2}=\lambda$	$q+m+2$	0	0	−1	α_k	α_k	0	0	0	0	0	0	0	0	0	$b^{m+2}=b^\lambda$
$w_{m+3}=\mu$	$q+m+3$	0	0	0	0	0	0	0	−1	γ_k^-	γ_k^+	γ_k^+	0	0	0	$b^{m+3}=b^\mu$
$w_{m+4}=\xi$	$q+m+4$	0	0	0	0	0	0	0	0	0	0	0	−1	η_k^-	η_k^+	$b^{m+4}=b^\xi$
δp_L	$q+m+5$	0	0	0	0	0	−1	1	0	0	0	0	0	0	0	

[a] The solution of the problem with i=0 cost row and no δp_L row will give extraction momentum $p_L=p_{sL}$. To vary p_L perform PCR with rows i=2, 3 for $\pm\,\delta p_L$, respectively. The index k ranges over the radii R_k at which the quantities in question should be controlled. The i=0 row contains the a_j^0. The q+ith row contains the a_j^i. The last row is used only if one wants to control the extraction momentum in the alternative manner of L. P. 3', Subsec. 5-C, in which case one must replace $\bar{P}_L^0$ by P_L^0 in the cost row.

optimization properties of linear programming to make matters as good as possible.

The specification of what constitutes an acceptable solution is made by one's choice of constraints to be imposed, and the choice of parameters defining the size of such constraints, e.g., λ_{max}, μ_{max}, ξ_{max}, α_k, δp_L, $\gamma_k^{\pm}$, z_{max}, $\eta_k^{\pm}$.

The specification of what constitutes a "best" solution is made by one's choice of the "cost coefficients" b^{λ}, b^{μ}, b^{ξ}.

We state the general trim-coil-adjustment problem in L.P.5' below. The tableau for its dual, L.P.5, is shown in Table IV. The meaning of the quantities in the rows $i = 1, 2$ is explained in Sec. 8.

<u>L.P.5'</u>: <u>General trim-coil-adjustment problem.</u>

Find values of the variables $w_1, \cdots, w_{m+1}$; λ; μ; and ξ that minimize

$$y = b^{\lambda}\lambda + b^{\mu}\mu + b^{\xi}\xi, \tag{9.1}$$

subject to the constraints:

Currents (and frequency) w_i:

$$\pm w_i \geqslant \pm W_i^{\mp}, \quad i = 1, \cdots, m+1. \tag{9.2}$$

Sine of phase lag, S_k [cf. Eqs. (4.4) and (5.12) for defining equations]:

$$-\lambda \geqslant -\lambda_{max}, \tag{9.3}$$

$$\alpha_k \lambda \pm \sum_{i=1}^{m} A_k^i w_i \geqslant \mp \bar{A}_k^0. \tag{9.4}$$

Momentum at extraction radius p_L [Eqs. (5.8) and (5.901)]:

$$\pm \sum_{i=1}^{m} P_L^i w_i \geqslant \mp \bar{P}_L^0. \tag{9.5}$$

Gradient error ΔG_k (for control of "effective" gradient error $\Delta \bar{G}_L$, replace D_k^i by $\bar{D}_k^i$ by $\bar{D}_k^i$) [Eqs. (4.6), (6.10), and (6.5)]:

$$-\mu \geqslant -\mu_{max}, \tag{9.6}$$

$$\gamma_k^{\mp} \mu \pm \sum_{i=1}^{m} D_k^i w_i \geqslant \mp D_k^0. \tag{9.7}$$

Effective gradient error for z_{max}, $\Delta\tilde{G}_k$ [Eq. (7.8)]:

$$\gamma_k^{+} \mu - \sum_{i=1}^{m} \tilde{D}_k^{\,i} w_i \geqslant \tilde{D}_k^{\,0}. \tag{9.8}$$

Nonlinear radial-instability control of H_k:

$$- \xi \geqslant -\xi_{max}, \tag{9.9}$$

$$\eta_k \mp \xi \pm \sum_{i=1}^{m} T_k^{\,i} w_i \geqslant \mp T_k, \tag{9.10}$$

The values of k to be used depend on the radii R_k at which the constraint in question is to be imposed.

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APPENDIX: LINEAR PROGRAMMING

A. Statement of L. P. Problem

We start with the statement of the general linear programming (L. P.) problem. In this statement the symbol $\otimes$ may stand for either $\geqslant$, $\leqslant$, or =:

A1: Choose values of the variables x^j, $j = 1, \cdots, n$ that $\begin{pmatrix}\text{minimize}\\ \text{maximize}\end{pmatrix}$ the objective function

$$z = \sum_{j=1}^{n} c_j x^i \tag{A1.1}$$

subject to the constraints

$$\sum_{j=1}^{n} a_j^i x^j \otimes b^i, \; i = 1, \cdots, m \tag{A1.2}$$

and

$$x^j \geqslant 0 \left(\text{for a particular subset of the } x^j\right) \tag{A1.3}$$

Any value of $\vec{x} = x_1, \cdots, x_n$ that satisfies the constraints of Eqs. (A1.2) and (A1.3) is called a feasible solution. Any feasible solution that minimizes or maximizes the objective function z (sometimes called the cost) is called the optimal feasible solution. Any L. P. problem either has no feasible solutions, or if it has feasible solutions it either has an unbounded solution (that is, z can be made as $\begin{pmatrix}\text{small}\\ \text{large}\end{pmatrix}$ as desired), or it has a finite optimal solution at either one or a set of values of the vector $\vec{x} = \{x^1, \cdots, x^n\}$.

B. Geometrical Interpretation

It is helpful to visualize the problem geometrically (see Fig. 1). Suppose the constraints of Eq. (A1.2) are all inequalitites, and we are considering a two dimensional problem, n = 2. Each inequality of Eq. (A1.2) divides the x^1, x^2 plane into an allowed and a forbidden part, separated by a straight line. Together, Eqs. (A1.2) and (A1.3) define a convex polygonal region K on which the feasible solutions must lie. The objective function z is constant on a family of parallel lines. That z line with the smallest z value with at least one point in K, is the optimal one, and the points of

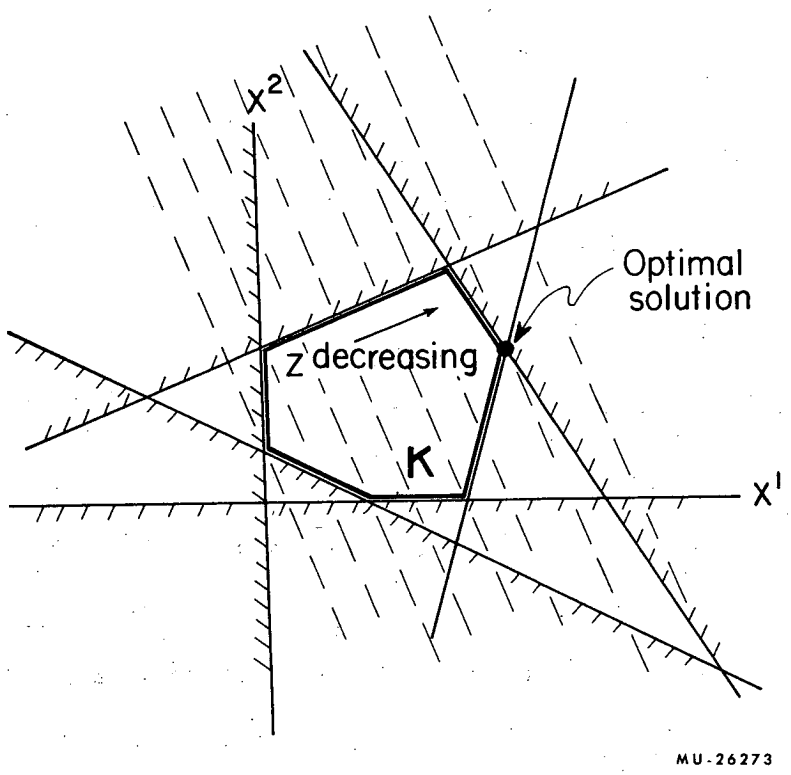

Fig. 1. Geometrical representation of a two-dimensional L.P. problem. The hatched lines represent constraints, which restrict the feasible solutions to the polygonal region K. The dashed lines are lines of constant z. The optimal solution will be a vertex (called an extreme-point solution), or on the line between two vertices.

intersection with K constitute the optimal solutions. Usually the intersection is at a vertex, called an extreme-point solution, but sometimes it is along an edge. In the general case, z assumes its optimal value either at an extreme point or at a set of points which are convex combinations of extreme points where z has the same (optimal) value. (A convex combination is a linear combination with nonnegative coefficients.)

C. Slack Variables

An L.P. problem with inequality constraints may be converted to one with equality constraints by the introduction of new variables, called slacks. Suppose the original problem is

A2: Choose $x^j, j = 1, \cdots, n$, that minimize $z = \sum_{j=1}^{n} c_j x^j$

subject to

$$x^j \geqslant 0, \quad j = 1, \cdots, n, \tag{A2.1}$$

$$\sum_{j=1}^{n} a_j^i x^i \geqslant b^i, \quad i = 1, \cdots, m. \tag{A2.2}$$

We can then subtract from each inequality of (A2.2) a new positive variable x^{n+i}, which will be just the difference between the left and right sides of the inequality, so the problem becomes

A3: Choose variables x^j, $j = 1, \cdots, n + m$, that minimize

$$z = \sum_{j=1}^{n} c_j x^j \equiv \sum_{j=1}^{n+m} \bar{c}_j x^j,$$

subject to

$$x^j \geqslant 0, \quad j = 1, \cdots, n+m, \tag{A3.1}$$

$$\sum_{j=1}^{n} a_j^i x^j - x^{n+i} \equiv \sum_{j=1}^{n+m} \bar{a}_j^i x^j = b^i, \quad i = 1, \cdots, m, \tag{A3.2}$$

and clearly the content of the problem has not been altered. If (A2.2) had $\leq$ relations one would have a term $+x^{n+i}$ in Eq. (A3.2).

The feasible solutions of Eqs. (A3) for which no more than m of the $n + m$ variables x^j is nonzero (or for which at least n of them are zero) are called basic feasible solutions (BFS). These correspond to the extreme-point solutions of (A2), since an extreme point of the polygon defined by (A2.1) and (A2.2) is the point of intersection of n of relations (A2.1) and (A2.2) written as equalities. But each of these relations, written as an equality, is equivalent to one of the x^j, $j = 1, \cdots, n + m$, in Eqs. (A3) being zero. Since there are n such relations, n of the x^j, $j = 1, \cdots, n + m$ are zero, which we have defined as a BFS.

D. Making All Variables Positive

Any variable x^j of Eq. (A1) not constrained to be positive can be replaced by two new positive variables $\bar{x}^j$ and $\bar{\bar{x}}^j$ where $x^j = \bar{x}^j - \bar{\bar{x}}^j$ and $\bar{x}^j \geq 0$, $\bar{\bar{x}}^j \geq 0$. Then the original problem should be restated in terms of $\bar{x}^j$ and $\bar{\bar{x}}^j$, substituting $(\bar{x}^j - \bar{\bar{x}}^j)$ for x^j in Eqs. (A1.1) and (A1.2).

E. Standard Form

By the above methods, any L.P. problem can be cast in the following standard form: (If the original problem calls for maximum z, then change the signs of the c_j.)

A4: Choose values of the variables x^j, $j=1, \cdots, n$, that minimize the objective function.

$$z = \sum_{j=1}^{n} c_j x^j, \tag{A4.1}$$

subject to the constraints

$$\sum_{j=1}^{n} a_j^i x^j = b^i, \quad i = 1, \cdots, m, \quad \text{where } m < n, \tag{A4.2}$$

$$x^j \geq 0, \quad j = 1, \cdots, n. \tag{A4.3}$$

F. Solution of L. P. Problem

Equation (A4) is the form required by the SCROL L.P. program. Of course it is not essential that the methods used by this code to find the optimal solution, if any, be understood by the user. However, the nature of the method can be hinted at here. As suggested above, the only candidates for optimal solutions are the vertices or extremum points of the polygonal region defined by the constraints, which are the same as the BFS defined above. In terms of (A4), each BFS has n - m of the x^j set equal to zero. Thus (A4.3) is reduced to a system of m equations in m unknowns, which has a unique solution. Those columns of $A = \| a_j^i \|$ belonging to the nonzero x^j constitute the basis B (not to be confused with the A's and B's of the text, Sec. 4) corresponding to the BFS, or extreme point. The code proceeds from one extreme point to a neighboring one by removing one column from B, and replacing it by one currently not in B. It has a test to detect when it has arrived at the optimal BFS, and if it has not it chooses the neighboring extreme point that reduces z the most.

One problem that the user must know about is how the code finds an extreme point to start with. Suppose A contains as one square submatrix the unit matrix I. Then it chooses this as its initial basis, for it gives the BFS $\vec{x}_0 = \vec{b}$. If A does not contain all of I it internally augments the problem with so-called artificial variables in such a way as to build I. These artificials are subsequently eliminated. However, to save the machine time, the user should specify if A contains any columns that can be part of I, i.e., columns containing only one element, of value +1. These columns are specified by a special column designation.

G. Dual Programming

Every L. P. problem has a so-called dual problem. Here we shall discuss only the dual problem to a problem in the standard from (A4). Let (A4) be called the primal problem. Then the dual problem may be written as follows*

A4'': Choose values of the variables w_i, $i = 1, \cdots, m$ that maximize the functional

$$y = \sum_{i=1}^{m} b^i w_i,$$

subject to

$$\sum_{i=1}^{m} a_j^i w_i \leq c_j, \quad j = 1, \cdots, n.$$

The following theorem relates the optimal solutions of the primal and dual problems (paraphrasing Gass[7]).

Duality Theorem: If either the primal or dual problem has a finite optimal solution, then so has the other problem, and the extremes of the linear functions are equal, i.e., min z = max y. If either problem has an unbounded solution, then the other problem has no feasible solutions.

The explicit formula for the dual variables in terms of the optimal basis B is

$$w_i = \sum_j c_j (B^{-1})_i^j, \text{ for those } j \text{ in the basis,}$$

whereas the primal variables are given by

$$x^j = \sum_i (B^{-1})_i^j b^i, \quad j \text{ in the basis.}$$

The SCROL program simultaneously computes the solutions to both a primal problem in form (A4) and its dual, (A4''). Hence if one has an original problem in form (A4'') one may solve it by entering problem (A4) into the machine.

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* The dual relationship is reciprocal: if A' is the dual problem to A, then A is the dual problem to A'.

For our purposes it is more convenient to have the dual problem in the following form—obtained by changing the signs of a_j^i and b^i in (A4) and (A4'').

A4': Choose values of the variables w_i, $i = 1, \cdots, m$ that minimize the functional

$$y = \sum_{i=1}^{m} b^i w_i, \tag{A4'.1}$$

subject to

$$\sum_{i=1}^{m} a_j^i w_i \geqslant -c_j, \quad j = 1, \cdots, n. \tag{A4'.2}$$

By comparing the primal problem and its dual we can see the significance of the dual variables ot the primal problem and vice-versa, for by the Duality Theorem, at optimum we have $z = y$. Suppose that in the primal, one varies the c_j slightly. For a small change the original optimal basis will not change, hence the x^j will not change either. Therefore

$$\frac{\partial y}{\partial c_j} = \frac{\partial z}{\partial c_j} = x^j.$$

But in the dual problem the c_j are the right-hand sides of the inequality constraints. Hence the derivative of the objective function (at optimum) with respect to a right-hand-side parameter is equal to the dual variable corresponding to the constraint in question.

H. Parametric Programming

Often one wants to know how the optimal solutions vary as the "cost coefficients" c_j or right-hand-side coefficients b^i vary continuously.

Case 1 (PCR): Suppose in (A4) we let $c_j = d_j + \theta d_j'$, where $\theta \leqslant \theta \leqslant \overline{\theta}$.

Then it can be shown that:

(a) The set of θ for which minimum solutions $\vec{x}$ exist is closed and connected.

(b) Given any finite minimum solution, a set of solutions $\vec{x}$, and associated z values corresponding to all possible values of θ can be found.

(c) Each such solution is minimum over a closed interval of θ (so the basis remains the same in that interval too).

(d) In each such closed interval of θ, the dual solution $\vec{w}$ varies linearly with θ: $w_i(\theta) = s_i + \theta r_i$. The procedure in the SCROL code is to put the d_j in the $i = 0$ row, the d_j^1 in some other row $i < q$, where q is the index of the "sum row," which is a row with zero elements just above the constraint matrix rows.

Case 2 (PLP): On the other hand, if in (A4) the $b^i = e^i + \theta e^{-i}$, the above situation is reversed—the x^j will vary linearly with θ in the various closed intervals, $x^j(\theta) = q^j + \theta p^j$, while in those intervals the dual solutions w_i are fixed.

The SCROL code prints out the solutions $\vec{x}$ and $\vec{w}$, and the value of the functional form $z = y$ at the beginning of each interval of θ in which the basis is fixed. In Case 1 one can trace out $w_i(\theta)$ by joining, with straight lines, these values of w_i given by the code at desired values of θ, while the $x^j(\theta)$ remain constant through each interval. The situation for Case 1 (PCR) is shown in Fig. 2.

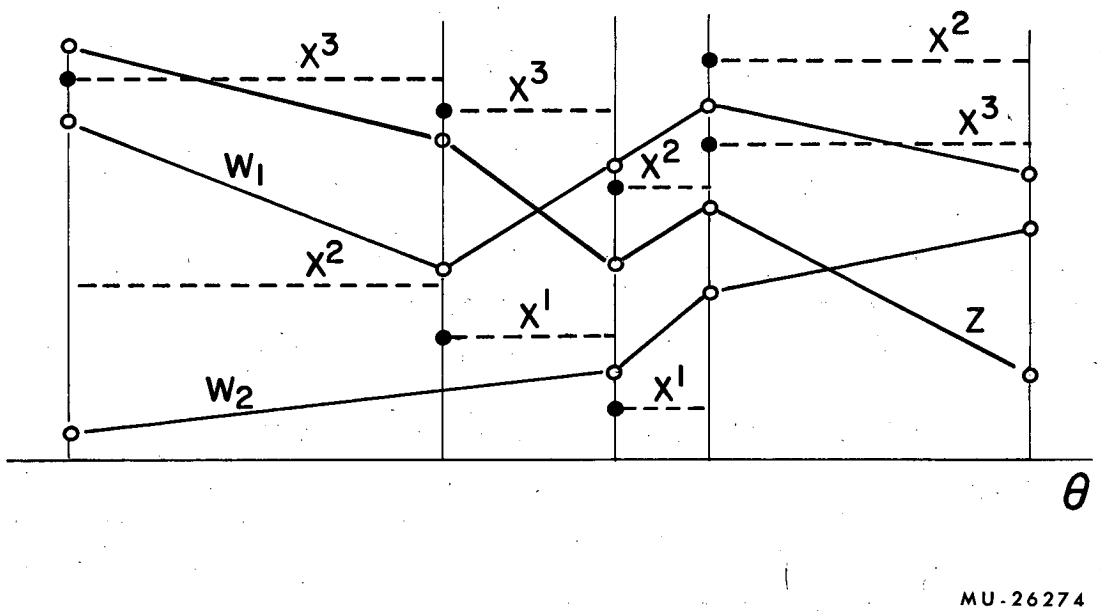

Fig. 2. Parametric programming of the $c_j = d_j + \theta d_j'$.

I. Tableaux for L. P. Problems

The following tableau shows the relation between the primal and dual probems (A4) and (A4''), respectively:

Tableau for standard L. P. problem (A4), showing primal-dual relations.

	x^1 . . . x^j . . . x^n			=
w_1	a_1^1	a_j^1	a_n^1	b^1
w_i	a_1^i	a_j^i	a_n^i	b^i
w_m	a_1^m	a_j^m	a_n^m	b^m
$\leqslant$	c_1	c_j	c_n	max / min

However, all the tableaux in the text refer to the following standard tableau, which in turn is regarded as equivalent to (A4), (A4'), and (A4''):

Standard tableau for standard L. P. problem (A4).

i \ j	1	j	n	
0	c_1	c_j	c_n	
q+1	a_1^1	a_j^1	a_n^1	b^1
q+i	a_1^i	a_j^i	a_n^i	b^i
q+m	a_1^m	a_j^m	a_n^m	b^m

In the IBM 704 codes the cost coefficients c_j are labeled a_j^0, otherwise the notation is about the same as above. However, the first row, here labeled $i = 1$, there has index $i = 2$ or greater. The column indices j are replaced by arbitrary alphanumeric labels. We will sometimes follow that pratice too. The codes print the x^i in a column headed BETA, and the w_i in a column headed PI. The index corresponding to j and the i index are printed in columns headed J and I, respectively.

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