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ABSTRACT

The connection between Regge poles, bound states and resonances, and asymptotic behavior in momentum transfer is reviewed within the framework of the analytically continued S matrix, and a convergent iteration procedure is given for calculating the position and residue of a Regge pole in terms of a given (generalized) potential. By examining the long-range potential in the $\pi\pi$ system it is inferred that Regge poles should appear in the $I = 0$ and $I = 1$ states, and that the latter pole may be responsible for the ρ meson while the former may well dominate high-energy behavior at low momentum transfer in the crossed channels. The connection of this possibility with forward coherent (diffraction) scattering in general is explored, and a number of experimental predictions are emphasized. Finally it is shown that the short-range forces due to exchange of 4, 6,$\cdots$ pions are likely to be repulsive and must be included in some form if a consistent solution is to be achieved.

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I. INTRODUCTION

In the S-matrix theory of strong interactions, dynamical resonances and bound states have been easily and naturally handled insofar as partial-wave (one-variable) dispersion relations are concerned, but they have been a source of confusion with respect to double-dispersion relations. Froissart[1] showed that partial waves with $J \geqslant 1$ are completely determined by the double-spectral functions; at the same time, as emphasized in the original paper by Mandelstam,[2] resonances or bound states require subtractions in the double-spectral integrals if the usual convergence criteria are applied. The resolution of this dilemma was given by Regge for nonrelativistic potential scattering, where in fact all partial waves are determined by the double-spectral function (even though in the absence of a "crossed" channel, the considerations of Froissart are inapplicable). Regge's explanation is based on the occurrence of poles in the complex angular momentum plane and the association of such poles with resonances and bound states.[3]

The point at issue is essentially the asymptotic behavior of the scattering amplitude as $\cos\theta$ approaches infinity and the energy is kept fixed. This is a highly unphysical region but, as it is here that the double

spectral function fails to vanish, the question is of interest to us. The number of subtractions in $\cos\theta$ which it is necessary to perform depends on the asymptotic behavior. As subtraction terms in $\cos\theta$ are just polynomials in this variable, they correspond to low partial waves, so that the number of partial waves which are undetermined by the double-spectral function depends on the number of subtractions necessary.

In Born approximation, the potential scattering amplitude vanishes asymptotically for large $\cos\theta$, and it is reasonable to suppose that the complete amplitude has this behavior if the potential strength is sufficiently small. It is evident, however, that such a behavior cannot persist as the strength of an attractive force increases since, if there is a bound state of angular momentum ℓ, the scattering amplitude contains a pole term with residue $P_\ell(\cos\theta)$, whose asymptotic behavior is $(\cos\theta)^\ell$. If we assume that the asymptotic behavior does not change suddenly when a bound state appears, we reach the conclusion that the asymptotic behavior becomes progressively more divergent as the strength of attraction increases. Regge's results give one great insight into the nature of this divergence, and show that it does not in fact necessitate undetermined subtraction terms.

Although the existence of Regge poles in the relativistic S matrix has not been established, it appears plausible that they should occur, and we propose here to discuss $\pi\pi$ scattering on such a basis. In particular, we shall show that the $I = 1$, $J = 1$ resonance can plausibly be associated with a Regge pole. It will also be argued that in the $I = 0$ state there may be a Regge pole which, because of Bose statistics and the necessity for an independent subtraction, does not correspond to a resonance or bound state, but which <u>may</u> be connected with high-energy diffraction scattering.

An important practical consequence of an approach in which Regge poles are recognized is that partial-wave calculations for $J \geqslant 1$ are no longer necessary. Computational difficulties associated with nonzero angular momentum in the N/D method thus can be avoided.

We list now those conclusions of Regge that are most important from our point of view.

(a) The elastic scattering amplitude at a fixed energy, if regarded as a function of ℓ, may be analytically continued into the complex ℓ plane for $\mathrm{Re}\,\ell > -1/2$. The only singularities are poles that for positive (physical) kinetic energies are confined to the upper half plane ($\mathrm{Im}\,\ell > 0$); these poles migrate to the real axis for negative kinetic energies.

(b) On the basis of the Sommerfeld-Watson contour representation[4] in the complex ℓ plane, the amplitude may be divided into two parts with different asymptotic behavior. The first part is an integral, along the vertical line $\mathrm{Re}\,\ell = -1/2$, that vanishes as $\cos\theta \to \infty$. The second part consists of pole contributions that generally do not vanish at infinity, these being of the form

$$\sum_i (\beta_i/\sin \pi\alpha_i)\, P_{\alpha_i}(-\cos\theta)\,, \qquad \text{(I.1)}$$

where α_i is the position of the ith pole, in the complex ℓ plane. It may be described as a complex angular momentum for which there exists a bound state at the given energy. Both α_i and β_i depend on the energy. As stated above, each α_i is real for negative kinetic energy but acquires a positive imaginary part for physical energies. (The Sommerfeld-Watson representation is, strictly speaking, valid only for positive kinetic energy, but the conclusions employed here about the connection between Regge poles, bound states and resonances, and asymptotic behavior can be justified by an analytic continuation in E.)

If an individual (physical) partial wave is projected out of (I.1), using the formula[5]

$$\frac{1}{2}\int_{-1}^{+1} P_\ell(z)\, P_\alpha(-z)dz = \frac{1}{\pi}\,\frac{\sin \pi\alpha}{(\alpha-\ell)(\alpha+\ell+1)}\ , \qquad \text{(I.2)}$$

for ℓ integer, $\ell \geqslant 0$, one finds

$$\frac{1}{\pi}\sum_i \frac{\beta_i}{(\alpha_i-\ell)(\alpha_i+\ell+1)}\ , \qquad \text{(I.3)}$$

a result that is immediately interpretable in terms of bound states and resonances. Consider a particular Regge pole and suppose that at some energy $E = E_m$, $\mathrm{Re}\,\alpha$ is equal to an integer $m \geqslant 0$. In the neighborhood of E_m we may write

$$\mathrm{Re}\,\alpha(E) \approx m + (E-E_m)\,(d\,\mathrm{Re}\,\alpha/dE)_{E_m}\ ,$$

$$\mathrm{Im}\,\alpha(E) \approx \mathrm{Im}\,\alpha(E_m)\ , \qquad \text{(I.4)}$$

and

$$\beta(E) \approx \beta(E_m)\ ,$$

so the Regge pole contributes to the ℓth wave a term, for E near E_m ,

$$\frac{1}{\pi}\,\frac{\beta(E_m)/\alpha(E_m)+\ell+1}{m-1+(E-E_m)(\,d\,\mathrm{Re}\,\alpha/dE\,)_{E_m} + i\,\mathrm{Im}\,\alpha(E_m)}\ , \qquad \text{(I.5)}$$

which, for $\ell = m$, has the familiar Breit-Wigner resonance form with a width

$$\Gamma = \mathrm{Im}\,\alpha(E_m)/(d\,\mathrm{Re}\,\alpha/dE)_{E_m}\ . \qquad \text{(I.6)}$$

For negative kinetic energy, $\mathrm{Im}\,\alpha$ vanishes and we have a bound state (i.e., a pole in E on the real E axis).

The above reasoning enables one to extend our previous result that if, at given energy, there existed a bound state of angular momentum ℓ, the scattering amplitude would contain a term behaving asymptotically like $P_\ell(\cos\theta)$. We can now say that if there exists a resonance of angular momentum ℓ (at a given energy), the amplitude will contain a term behaving asymptotically like $P_\alpha(\cos\theta)$, where α is complex and $\mathrm{Re}\,\alpha \sim \ell$. If the resonance is narrow, $\mathrm{Im}\,\alpha$ is small.

Regge was able to prove that $(d\,\mathrm{Re}\,\alpha/dE)_{E_m}$ is positive for a bound-state pole, and gave qualitative arguments to show that the same would be true for sharp resonances--which normally occur at low energies.[3] One may conjecture that when $(d\,\mathrm{Re}\,\alpha/dE)_{E_m}$ is negative one is not dealing with a resonance but with the familiar high-energy return of the phase shift through 90 deg that always occurs in potential scattering. We add the remark that an analysis of the Born series suggests that $\mathrm{Re}\,\alpha \leqslant -1/2$ at sufficiently large $|E|$.

For superpositions of Yukawa potentials all Regge poles are connected with bound states and resonances, and thus may be presumed to have the following general behavior in the complex ℓ plane with E real: For an attractive potential a particular pole passes through $\alpha = -1/2$ at some negative E, and moves to the right along the real axis as E increases. When E reaches zero the pole moves into the upper half plane, perhaps continuing its rightward movement temporarily but eventually swinging back through the vertical line, $\mathrm{Re}\,\ell = -1/2$. For weak potentials the pole will leave the axis before reaching even $\ell = 0$, and there are no bound states.

since it has been discussed in Ref. 8 and will arise again in what follows.

In the following section we review the S-matrix approach to nonrelativistic potential scattering, showing how the Regge poles are to be extracted and how they are related to the partial-wave N/D problem. The final section discusses possible Regge poles in the relativistic $\pi\pi$ amplitude.

II. CALCULATION OF REGGE POLES IN NONRELATIVISTIC POTENTIAL SCATTERING

It has been shown by Blankenbecler et al.[9] that, for nonrelativistic scattering by a superposition of direct Yukawa potentials, the double-spectral function is determined by the following equation, first derived in Ref. 2:

$$\rho(q^2,t) = \frac{1}{2\pi}\,\frac{1}{q}\iint dt'\,dt''\,\frac{\tilde{D}^*(t',q^2)\,\tilde{D}(t'',q^2)}{K^{1/2}(q^2;\,t,t',t'')} \qquad \text{(II.1)}$$

The integration is restricted to that part of the region $t^{1/2} > t'^{1/2} + t''^{1/2}$ for which

$$K(q^2;\,t,t',t'') = t^2 + t'^2 + t''^2 - 2(tt' + tt'' + t't'') - (tt't''/q^2) \qquad \text{(II.2)}$$

is positive. The function $\tilde{D}(t,q^2)$ is the discontinuity in the amplitude in crossing the positive t axis with q^2 fixed: it is related to the potential and to the double-spectral function by

$$\tilde{D}(t,q^2) = v(t) + \frac{1}{\pi}\int dq'^2\,\frac{\rho(q'^2,t)}{q'^2 - q^2}\,, \qquad \text{(II.3)}$$

where $v(t)$ determines the configuration space potential $V(r)$ through the formula

$$V(r) = -\frac{1}{2M^2} \int dt\, v(t) \frac{\exp[-t^{1/2} r]}{r} . \qquad \text{(II.4)}$$

Generally speaking there is some positive threshold t_0, such that $v(t)$ vanishes for $t < t_0$. We shall speak of $t_0^{-1/2}$ as the "range" of the potential.

As has been explained in Ref. 2, the pair of equations (II.1) and (II.3) uniquely determines $\tilde{D}(t,q^2)$ since the nature of the integration region in (II.1) ensures that n iterations give a result exact for $t < (n+1)^2 t_0$. In other words the Born series for $\tilde{D}(t,q^2)$ certainly converges (although not necessarily uniformly in t) regardless of the occurrence of resonances or bound states. It is well known, on the other hand, that the Born series for the scattering amplitude $A(q^2,t)$ does not always converge, a circumstance that at first sight is puzzling if one expects the unsubtracted dispersion relation

$$A(q^2,t) = \frac{1}{\pi} \int dt' \frac{\tilde{D}(t',q^2)}{t' - t} \qquad \text{(II.5)}$$

to be meaningful. In Eq. (1.7) above, however, we have seen that when Regge poles occur with $\mathrm{Re}\,\alpha \geqslant 0$ the integral (II.5) is not defined in the elementary sense but only through analytic continuation; so precisely when the first resonance or bound state appears the possibility of expanding $\tilde{D}$ in a power series no longer implies that A similarly can be expanded.

Nevertheless, a knowledge of $\tilde{D}(t,q^2)$ implies a knowledge of $A(q^2,t)$, as we shall now show, so the iteration of Eqs. (II.1) and (II.3) actually forms the basis for a practical method of calculation--with or without bound states or resonances. The essential point is that, according to Regge,[3]

$$A(q^2,t) = A'(q^2,t) + \sum_i \frac{\beta_i(q^2)}{\sin \pi\, \alpha_i(q^2)} P_{\alpha_i(q^2)}\left(-1 - \frac{t}{2q^2}\right), \tag{II.6}$$

where $A'(q^2,t)$ is the background term that vanishes as $t \to \infty$ (we may also allow A' to contain Regge poles with $\mathrm{Re}\,\alpha_i < 0$). Then by reference to (I.7)

$$\tilde{D}(t,q^2) = \tilde{D}'(t,q^2) - \sum_i \beta_i(q^2)\, P_{\alpha_i(q^2)}\left(1 + \frac{t}{2q^2}\right), \tag{II.7}$$

with the integral

$$A'(q^2,t) = \frac{1}{\pi} \int dt' \frac{\tilde{D}'(t',q^2)}{t' - t}, \tag{II.8}$$

defined in the elementary sense. Thus if it is possible to decompose $\tilde{D}(t,q^2)$ according to (II.7) --so that one has a separate knowledge of $\tilde{D}'(t,q^2)$, $\beta_i(q^2)$, and $\alpha_i(q^2)$ --then one can construct the amplitude $A(q^2,t)$.

An elementary method for determining $\alpha_i(q^2)$ and $\beta_i(q^2)$ may be based on the dominance of the Regge poles over the background term for large t . Suppose that there is only one pole for which $\mathrm{Re}\,\alpha \geqslant 0$; then for sufficiently large t this pole will be dominant in formula (II.7), and one may calculate the position $\alpha_i(q^2)$ and the strength $\beta_i(q^2)$ by equating, at large t , the calculated $\tilde{D}(t, q^2)$ with $-\beta_i P_{\alpha_i}[1 + (t/2q^2)]$. One then subtracts out this pole term at all t to obtain the background term $\tilde{D}'(t,q^2)$. If there is more than one pole, the one for which $\mathrm{Re}\,\alpha_i$ is largest can be determined first and subtracted; the remainder will then be dominated by the pole with the next largest $\mathrm{Re}\,\alpha_i$, and the procedure

can be repeated until all pole parameters are determined. In an actual numerical calculation one may wish to use a more elegant approach, but there seems nothing in principle to prevent the extraction of the necessary information from the iterative solution for $\tilde{D}(t,q^2)$.[10]

Note that when the potential problem is approached in this way there is no need to treat any partial waves separately. In principle an alternative to separating and identifying the Regge poles is to calculate individually by N/D method all waves for $\ell \leqslant (\mathrm{Re}\ \alpha)_{max}$. When these low partial waves are subtracted out of formula (II.5) the remainder of the integral (containing all high waves) converges in the elementary sense. The necessary ingredient for the N/D partial-wave calculation is the discontinuity across the left-hand cut; this is given for the ℓth wave by

$$\mathrm{Im}\ A_\ell(q^2) \underset{q^2 < -t_0/4}{=} \frac{1}{4q^2} \int_{t_0}^{-4q^2} dt\ P_\ell\left(1 + \frac{t}{2q^2}\right) \tilde{D}(t,q^2) \quad , \tag{II.9}$$

and presents no difficulty of principle if, as conjectured in Sec. I, the Regge poles all retreat through the vertical line, $\mathrm{Re}\ \ell = -\frac{1}{2}$, for large $|q^2|$. In this case $\mathrm{Im}\ A_\ell(q^2)$ vanishes sufficiently rapidly as $q^2 \to -\infty$ so that the N/D integral equations are nonsingular. In practice, however, for all $\ell > 0$, delicate cancellations must occur between the right and left cuts to produce the correct threshold behavior. $A_\ell(q^2) \propto (q^2)^\ell$, near $q^2 = 0$. The N/D equations then become awkward from a numerical standpoint, so an approach that does not separate partial waves is preferable.

III. REGGE POLES IN RELATIVISTIC $\pi\pi$ SCATTERING

We now illustrate by a discussion of $\pi\pi$ scattering our conjecture that Regge poles occur quite generally in the relativistic strong-interaction S matrix. Consider the three amplitudes $A^I(s,t)$ which represent pure I scattering $(I = 0, 1, 2)$ in the s channel. Two of the authors have defined a "generalized potential," here to be denoted by $v^I(t,s)$, which is to be used in equations of the type (II.3) and (II.1) in place of the nonrelativistic potential $v(t)$.[8] The "long range" part of the generalized potential, exact for $t < 16\, m_\pi^2$, is associated with 2-pion exchange, and is given by

$$v_{2\pi}^{I}(t,s) = \sum_{I'=0,1,2} \beta_{II'}\, D_{e\ell}^{I'}(t,s) \quad , \qquad \text{(III.1)}$$

where the crossing matrix β has the form,[11]

$$\beta_{II'} = \begin{pmatrix} 1/3 & 1 & 5/3 \\ 1/3 & 1/2 & -5/6 \\ 1/3 & -1/2 & 1/6 \end{pmatrix} , \qquad \text{(III.2)}$$

and $D_{e\ell}^{I}(t,s)$ is the elastic absorptive part for isotopic spin I scattering in the t channel. As discussed below, the imaginary part of $v_{2\pi}^{I}(t,s)$, which develops at large s, produces inelastic scattering in the s channel that is not properly bounded by unitarity. An approximation which replaces v^I by $v_{2\pi}^{I}$ then leads to inconsistencies in the case of actual physical interest. Contributions from $v_{4\pi}^{I}$, $v_{6\pi}^{I}$, etc. must be added to correct this deficiency, but it will be argued below that the low-energy effects of these shorter-range potentials are probably repulsive, so we should be able to discuss qualitative questions on the basis of (III.1).

If Regge poles in fact dominate asymptotic behavior in the relativistic amplitude, as discussed above for the nonrelativistic case, then for the interval in s such that a small number of poles are consistently to the right of all other singularities (and within the region of analyticity in ℓ), it follows that

$$\left.\begin{matrix} A^I(s,t) \\ \\ \tilde{D}^I(t,s) \end{matrix}\right\} \underset{t \to \infty}{\mathcal{X}} t^{\alpha^I(s)} , \qquad \text{(III.3)}$$

if $\alpha^I(s)$ is the position of the pole farthest to the right in the ℓ plane and $\tilde{D}^I(t,s)$ is the discontinuity in $A^I(s,t)$ in crossing the positive t axis. An appropriate general definition of the "strip" region discussed earlier in a qualitative way by two of the authors would be just this interval in s .[12] (The earlier definition of the "strip" was linked to the approximation $v^I \approx v_{2\pi}{}^I$. We now wish to dissolve this link.) If the analogy with nonrelativistic potential scattering holds, we expect $\alpha^I(s)$ to increase with the strength of the generalized potential $v^I(t,s)$, when v^I corresponds to attraction; furthermore, we expect $d\alpha^I/ds$ to be positive for $s < 4$. For $s > 4$, $\alpha^I(s)$ becomes complex, but the real part is continuous and should reach a maximum value at some moderate value of s , eventually falling to a negative value for s sufficiently large.

From the elements of the crossing matrix (III.2) one sees that all contributions to $v_{2\pi}{}^{I=0}$ are attractive and stronger than (or at least as strong as) in the other two I-spin states. Thus if any Regge poles develop, the one standing farthest to the right in the ℓ plane at a given s should be in the $I = 0$ state. If $\alpha^{I=0}(s)$ is still positive for some range of negative s then in the crossed channel (where t corresponds to energy

and s to momentum transfer) the high-energy behavior at fixed (low) momentum transfer evidently will be controlled by the $I = 0$ Regge pole. We now examine the connection between this possibility and constant limits for high-energy total cross sections.

From the optical theorem it follows that

$$D^I(t, s=0) = (q_t \, t^{1/2}/16\pi) \, \sigma_{tot}^I(t) \quad , \qquad \text{(III.4)}$$

where $D^I(t,s)$ is the complete absorptive part in the t channel and σ_{tot}^I is the total cross section, both quantitites for isotopic spin I in the t channel. Then, since

$$\tilde{D}^I(t,s) = \sum_{I'} \beta_{II'} \, D^{I'}(t,s) \quad , \qquad \text{(III.5)}$$

a glance at the elements of $\beta_{II'}$ in (III.2) shows that no cancelation can prevent a behavior

$$\tilde{D}^{I=0}(t,0) \propto t \quad , \quad \text{as} \quad t \to \infty \quad , \qquad \text{(III.6)}$$

if each total cross section σ_{tot}^I approaches a constant. Such asymptotic behavior, pointed out in an earlier paper by two of the authors,[12] implies that

$$\alpha^{I=0}(s=0) = 1 \, . \qquad \text{(III.7)}$$

At first sight this last requirement seems to predict a bound P state of zero total energy, but symmetry requirements eliminate all odd ℓ waves with $I = 0$. Because of the presence of exchange as well as direct forces, the potential determining even physical values of ℓ is different from that determining odd values of ℓ. Nevertheless we must ask the question: Is it reasonable

to expect a direct potential equal to $v^{I=0}$, if it were effective in both odd and even ℓ states, to be sufficiently attractive as to bind a P state? We believe the answer to be affirmative because qualitative arguments have shown that a "bootstrap" mechanism probably can sustain an $I = 1$ P-wave resonance in terms of itself.[13] In other words, a potential

$$v_{2\pi}^{I=1}(t,s) \approx \beta_{11} D_{e\ell}^{I=1}(t,s) \quad , \qquad \text{(III.8)}$$

when $D_{e\ell}^{I=1}$ contains a P-resonance contribution, is attractive and has roughly the required strength to produce the $I = 1$ P resonance in question. Now $\beta_{01} = 2\beta_{11}$, so the corresponding contribution to $v^{I=0}$ is twice as attractive as (III.8) and might well produce a <u>bound</u> P state.

The above argument implies that

$$\alpha^{I=1}(s=0) < 1 \quad , \qquad \text{(III.9)}$$

which is consistent with the experimental requirement that $\mathrm{Re}\,\alpha^{I=1}(s\approx 28) = 1$,[14] and the theoretical expectation that for $s < 28$, $d\,\mathrm{Re}\,\alpha/ds$ is positive. Since $\beta_{21} = -\beta_{11}$, the potential $v^{I=2}$ is probably repulsive and no Regge pole will even appear in the $I = 2$ state. Thus we expect

$$\frac{1}{t}\,\tilde{D}^{I=1,2}(t,0) \xrightarrow[t\to\infty]{} 0 \quad , \qquad \text{(III.10)}$$

and in view of the relation

$$\sigma_{tot}^{I}(t) = (16\,\pi/q_t\,t^{1/2}) \sum_{I'} \beta_{II'}\,\tilde{D}^{I'}(t,0) \quad , \qquad \text{(III.11)}$$

there follows from (III.5) the expectation that

$$\lim_{t \to \infty} \sigma_{tot}^{I=0} = \lim_{t \to \infty} \sigma_{tot}^{I=1}(t) = \lim_{t \to \infty} \sigma_{tot}^{I=2}(t) \quad . \tag{III.12}$$

By such a mechanism, therefore, one expects to achieve both Pomeranchuk conditions.[15]

It may appear strange at first sight that the Pomeranchuk relations should depend on the detailed structure of the crossing matrix. When one realizes, however, that coherent elastic scattering is uniquely associated with states in the crossed channel that have the quantum numbers of the vacuum, then a select role for $I = 0$ in asymptotic considerations is no longer surprising. Pomeranchuk's second condition, after all, is equivalent to the assertion that completely coherent elastic scattering predominates in the forward direction at high energy.

We now remark on two consequences of the assumption that $d\alpha^{I=0}/ds > 0$ for $s < 4$. The first is that in view of (III.7) we expect $\alpha^{I=0}$ to vanish at some negative value of s, a circumstance which would correspond to an unphysical bound S state of imaginary energy. Gell-Mann has pointed out to us that if the residue of such a pole were to vanish there would be no conflict with unitarity.[16] If the residue does not vanish we cannot determine the $I = 0$ S wave from $\tilde{D}^{I=0}$, but must use the N/D method.

The second consequence of the positive derivative of $\alpha^{I=0}$ with respect to s is that the width of the high-energy elastic diffraction peak will shrink indefinitely with increasing energy--albeit only logarithmically. Since the first Pomeranchuk condition ensures that the real part of the amplitude near the forward direction is negligible with respect to the imaginary part, we have

$$\frac{d\sigma^I}{ds} \underset{t \to \infty}{\propto} \left[\frac{1}{t} D^I(t,s) \right]^2 , \qquad s = -2q_t^{\,2}(1 - \cos \theta_t)$$

$$\sim \quad f(s)t^{2(\alpha^{I=0}(s) - 1)} . \tag{III.13}$$

If $\alpha^{I=0}$ is a slowly varying function of s, we may write for small $|s|$

$$\alpha^{I=0}(s) \approx 1 + \epsilon s , \qquad \epsilon > 0 , \tag{III.14}$$

and thus deduce the small momentum-transfer behavior

$$\frac{d\sigma^I}{ds} \xrightarrow[t \to \infty]{} f(s)t^{2\epsilon s}$$

$$= f(s) \exp(s\, 2\, \epsilon\, \ln t) . \tag{III.15}$$

Integration of (III.15) over the elastic diffraction peak yields the related prediction

$$\sigma_{e\ell}^{\;I}/\sigma_{tot}^{\;I} \underset{t \to \infty}{\propto} (\epsilon \ln t)^{-1} . \tag{III.16}$$

Evidently, the rate of shrinkage is small; nevertheless, precision experiments at very high energy should detect such an effect.[17] It is possible to argue, as pointed out by Lovelace,[18] that experiments already are giving support for the form (III.15) through the observed exponential behavior of the tail of diffraction peaks. Such behavior is difficult to understand in any classical model but follows immediately from the Regge pole hypothesis.

As discussed by two of the authors, <u>all</u> forward diffraction peaks (πN, NN, $\pi\pi$, KN, etc.) are controlled by the Regge pole under discussion here, if any are.[19] The universal character of the slope ϵ (and of course higher

derivatives if they can be measured) is another striking feature of our mechanism. One must keep it in mind, of course, that the diffraction peak may well be produced by a more complicated mechanism than envisaged here. Experiments to test the characteristic features of formula (III.15) are therefore of crucial importance. The predictions discussed above are so startlingly nonclassical in nature that their confirmation would provide convincing evidence for the Regge pole hypothesis.

We return finally to discuss the inconsistency in the equations, as they are at present formulated with $v^I = v_{2\pi}{}^I$, in the case where there is a P-wave resonance. The difficulty arises essentially from the equation

$$\rho_{e\ell}{}^I(s,t) = \frac{1}{\pi q_s(q_s{}^2 + 1)^{1/2}} \iint dt'\, dt'' \, \frac{\tilde{D}^{I*}(t',s)\, \tilde{D}^I(t'',s)}{K^{1/2}(q_s{}^2;\, t,t',t'')} , \tag{III.17}$$

which is the relativistic analogue of (II.1). Here $\tilde{D}^I$ is given by

$$\tilde{D}^I(t,s) = v^I(t,s) + \frac{1}{\pi} \int ds' \, \frac{\rho_{e\ell}{}^I(s',t')}{s' - s} , \tag{III.18}$$

the analogue of (II.3), and v^I is in turn given in terms of $D_{e\ell}{}^{I'}(t,s)$ by the crossing equation (III.1). Now, $D_{e\ell}{}^{I=1}(t,s)$ will behave like $\beta_1(t)s^{\alpha_1(t)}$ as s approaches infinity and, if there is a P-wave resonance, the α_1 will be greater than 1 for some values of s. We have indicated that the same may well be true for D^0. From (III.17) one may deduce that if $\tilde{D}^*$ and D behave like $s^{\alpha(t)}$ at large s the contribution to the integral for $\rho_{e\ell}$ from $t' = t'' = t_1$ will behave like $s^{2\alpha(t_1) - 1}$. This value of t' and t'' will contribute if $t \geqslant 4\, t_1$. If $\rho_{e\ell}(s,t)$ behaves like $s^{2\alpha(t_1)-1}$ for $t > 4t_1$, it follows from (III.18), even if subtractions are made, that $D(t,s)$ has the same behavior for such values of t. On

putting this behavior of D into (III.17), we find that $\rho_{e\ell}$ behaves like $s^{4\alpha(t_1)-3}$ when $t > 16\ t_1$. The procedure can be repeated and, if $\mathrm{Re}\ \alpha(t_1) > 1$, it appears that the asymptotic behavior of ρ and D as a function of s becomes worse and worse as t increases.

It is unlikely that the oscillatory behavior of D will decrease the asymptotic behavior of ρ given by (III.17). The simplest way of seeing this is to make a Froissart transformation by which the integral in (III.17) is replaced by another containing a δ function,[10] so that, for any value of t', only one value of t'' contributes. The asymptotic behavior is unchanged by this transformation. Writing this functional relationship as $t'' = t''(t')$, and denoting $\alpha[t''(t')]$ by $\gamma(t')$, we observe that the contribution to the integral on the right-hand side of (III.17) from a particular value of t' behaves like $s^{\alpha^*(t') + \gamma(t')}$. The integral of such a function over t' will ordinarily be dominated by that value of t' for which $\mathrm{Re}\ \alpha + \mathrm{Re}\ \gamma$ is greatest, and cancelations will in general not occur.

If one seeks the physical origin of the inconsistency of our equations, the most likely culprit is the failure of the approximation $v^I \approx v_{2\pi}{}^I$ to put a unitarity bound on inelastic scattering. The trouble develops as soon as the real part of any α^I becomes greater than unity, and Froissart has shown that unitarity requires $\alpha^I(s=0) \leqslant 1$,[1] a constraint that is lacking in our approximation. To remedy the problem one must take some account in the inelastic processes of multipion exchange. An exact treatment is, of course, out of the question, but it may be possible somehow to impose the correct unitarity bound. In terms of our generalized potential, $v^I(t,s)$, the required unitarity damping in the inelastic part comes about through 4π, 6π, etc. contributions; it seems plausible that such contributions appear as repulsive forces, since their effect has to limit the magnitude

of $\alpha(s)$ at low energy. One may speculate, in fact, that there may be a universal repulsive core in all two-body forces due to exchange of multiparticle systems with the quantum numbers of the vacuum. It is for these quantum numbers that the Froissart limit is most closely approached, so the compensating reaction of multiparticle contributions should here be the strongest.

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FOOTNOTES AND REFERENCES

* Work done under the auspices of the U. S. Atomic Energy Commission.
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