

# A BALANCING DOMAIN DECOMPOSITION METHOD BY CONSTRAINTS FOR ADVECTION-DIFFUSION PROBLEMS

XUEMIN TU AND JING LI

**ABSTRACT.** The balancing domain decomposition methods by constraints are extended to solving nonsymmetric, positive definite linear systems resulting from the finite element discretization of advection-diffusion equations. A preconditioned GMRES iteration is used to solve a Schur complement system of equations for the subdomain interface variables. In the preconditioning step of each iteration, a partially sub-assembled finite element problem is solved. A convergence rate estimate for the GMRES iteration is established, under the condition that the diameters of subdomains are small enough. It is independent of the number of subdomains and grows only slowly with the subdomain problem size. Numerical experiments for several two-dimensional advection-diffusion problems illustrate the fast convergence of the proposed algorithm.

## 1. INTRODUCTION

Domain decomposition methods have been widely used and studied for solving large sparse linear systems arising from finite element discretization of partial differential equations. The balancing domain decomposition methods by constraints (BDDC) were introduced by Dohrmann [13] and they represent an interesting redesign of the balancing Neumann-Neumann algorithms; see also Fragakis and Papadrakakis [18] and Cros [12] for related algorithms. Scalable convergence rates for the BDDC methods have been proved by Mandel and Dohrmann [29] for symmetric positive definite problems. Connections and spectral equivalence between the BDDC algorithms and the earlier dual-primal finite element tearing and interconnecting methods (FETI-DP) [16] have been established by Mandel, Dohrmann, and Tezaur [30]; see also Li and Widlund [27], and Brenner and Sung [5]. The BDDC methods have also been extended to solving saddle point problems, e.g., for Stokes equations by Li and Widlund [26], for nearly incompressible elasticity by Dohrmann [14], and for the flow in porous media by Tu [37, 39, 38].

The systems of linear equations arising from the finite element discretization of advection-diffusion equations are nonsymmetric, but usually positive definite. A number of domain decomposition methods have been proposed and analyzed for solving nonsymmetric and indefinite problems. Cai and Widlund [6, 7, 8] studied overlapping Schwarz methods for such problems, using a perturbation approach in

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their analysis, and established that the convergence rates of the two-level overlapping Schwarz methods are independent of the mesh size if the coarse mesh is fine enough. Motivated by the FETI-DPH method proposed by Farhat and Li [17] for solving symmetric indefinite problems, the authors [25] studied a BDDC algorithm for solving Helmholtz equations and estimated its convergence rate using a similar perturbation approach. For some other results using the perturbation approach and for domain decomposition methods, see Xu [42], Vassilevski [40], Gopalakrishnan and Pasciak [20].

For advection-diffusion problems, standard iterative substructuring methods usually do not perform well when advection is strong. Dirichlet and Neumann boundary conditions used for the local subdomain problems in these algorithms are not appropriate because of the loss of positive definiteness of the local bilinear forms. More general boundary conditions need be considered. Therefore, a class of methods have been developed in [9, 10, 36, 19, 31], where additional adaptively chosen subdomain boundary conditions are used to stabilize the local subdomain problems; see also [32, Chapter 6] and the references therein for other similar approaches.

The Robin-Robin algorithm, a modification of the Neumann-Neumann approach for solving advection-diffusion problems, has been developed by Achdou *et al.* [3, 1, 2], where new local bilinear forms corresponding to Robin boundary conditions for the subdomains are used and a coarse level basis function, determined by the solution to an adjoint problem on each subdomain, is added to accelerate the convergence. Equipped with the same type local subdomain bilinear forms with Robin boundary conditions and a similar coarse level basis function, one-level and two-level FETI algorithms were proposed by Toselli [34] for solving advection-diffusion problems. Some additive and multiplicative BDDC algorithms with vertex constraints and edge average constraints have also been studied by Conceição [11]. All these algorithms, based on subdomain Robin boundary conditions, have been shown to be successful for solving advection-diffusion problems, including some advection-dominated cases, but a theoretical analysis is still missing.

In this paper, we develop BDDC algorithms for advection-diffusion problems. As in [2], local subdomain bilinear forms corresponding to Robin boundary conditions are used. The original system of linear equations is reduced to a Schur complement problem for the subdomain interface variables and a preconditioned GMRES iteration is then used. In the preconditioning step of each iteration, a partially sub-assembled finite element problem is solved, for which only the coarse level, primal interface degrees of freedom are shared by neighboring subdomains. The convergence analysis of our BDDC algorithms requires that the coarse level primal variable space contains certain flux average constraints, which depend on the coefficient of the first order term of the problem, across the subdomain interface, in addition to the standard subdomain vertex and edge/face average continuity constraints. A convergence rate estimate for the GMRES iteration is established, under the condition that the diameters of subdomains are small enough. This estimate is independent of the number of subdomains and grows only slowly with the subdomain problem size. A perturbation approach is used in our analysis to handle the non-symmetry of the problem. A key point is to obtain an error bound for the partially sub-assembled finite element problem; we view this problem as a non-conforming finite element approximation. This approach has recently also

been used by the authors [25] in the convergence analysis of a BDDC algorithm for solving interior Helmholtz equations, which are symmetric but indefinite.

The rest of this paper is organized as follows. The advection-diffusion equation and its adjoint form are described in Section 2. In Section 3, the finite element space and a stabilized finite element problem are introduced. The local subdomain bilinear forms and a partially sub-assembled finite element space are introduced in Section 4. In Section 5, an error estimate for the partially sub-assembled finite element problem is proved. The preconditioned interface problem for our BDDC algorithm is presented in Section 6 and its convergence analysis is given in Section 7. To conclude, numerical experiments in Section 8 demonstrate the effectiveness of our algorithm.

## 2. PROBLEM SETTING

We consider the following second order scalar advection-diffusion problem in a bounded polyhedral domain  $\Omega \in \mathbf{R}^d$ ,  $d = 2, 3$ ,

$$(2.1) \quad \begin{cases} Lu := -\nu \Delta u + \mathbf{a} \cdot \nabla u + cu = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

Here the viscosity  $\nu$  is a positive constant. The velocity field  $\mathbf{a}(x) \in (L^\infty(\Omega))^d$  and  $\nabla \cdot \mathbf{a}(x) \in L^\infty(\Omega)$ . The reaction coefficient  $c(x) \in L^\infty(\Omega)$  and  $f(x) \in L^2(\Omega)$ . We define

$$(2.2) \quad \tilde{c}(x) = c(x) - \frac{1}{2} \nabla \cdot \mathbf{a}(x), \quad \tilde{c}_s = \|\tilde{c}(x)\|_\infty, \quad a_s = \|\mathbf{a}(x)\|_\infty, \quad \text{and } c_s = \|c(x)\|_\infty.$$

We also assume that there exists a positive constant  $c_0$  such that

$$(2.3) \quad \tilde{c}(x) \geq c_0 > 0, \quad \forall x \in \Omega.$$

The bilinear form associated with the operator  $L$  is defined, for functions in the space  $H_0^1(\Omega)$ , by

$$(2.4) \quad a_o(u, v) = \int_{\Omega} (\nu \nabla u \cdot \nabla v + \mathbf{a} \cdot \nabla uv + cuv) \, dx,$$

which is positive definite under assumption (2.3). The weak solution  $u \in H_0^1(\Omega)$  of (2.1) satisfies

$$(2.5) \quad a_o(u, v) = \int_{\Omega} f v \, dx, \quad \forall v \in H_0^1(\Omega).$$

Under certain assumption on the shape of  $\Omega$ , e.g.,  $\Omega$  convex, we know that the weak solution  $u$  of the original problem (2.1), as well as the weak solution of the adjoint problem  $L^*u = -\nu \Delta u - \nabla \cdot (\mathbf{a}u) + cu = f$ , satisfies the regularity result,

$$(2.6) \quad \|u\|_{H^2(\Omega)} \leq C \|f\|_{L^2(\Omega)},$$

where  $C$  is a positive constant which depends on the coefficients of the partial differential equation (2.1) and the shape of the domain  $\Omega$ ; cf. [21, Section 9.1].

## 3. FINITE ELEMENT DISCRETIZATION AND STABILIZATION

Let  $\widehat{W} \subset H_0^1(\Omega)$  be the standard continuous, piecewise linear finite element function space on a shape-regular triangulation of  $\Omega$ . We denote an element of the triangulation by  $e$ , and its diameter by  $h_e$ . We set  $h = \max_e h_e$ .

It is well known that the original bilinear form  $a_o(\cdot, \cdot)$  has to be stabilized to remove spurious oscillations in the finite element solution for advection-dominated problems. There are a large number of strategies for this purpose; see [22] and the references therein. Here, we follow [22, 34] and consider the Galerkin/least-squares method (GALS) of [22]. On each element  $e$ , we define the local Peclet number by

$$P_{e_e} = \frac{h_e \|\mathbf{a}\|_{e;\infty}}{2\nu}, \quad \text{where } \|\mathbf{a}\|_{e,\infty} = \sup_{x \in e} |\mathbf{a}(x)|,$$

and we define a positive function  $C(x)$  by

$$(3.1) \quad C(x) = \begin{cases} \frac{\tau h_e}{2\|\mathbf{a}\|_{e;\infty}} & \text{if } P_{e_e} \geq 1, \\ \frac{\tau h_e^2}{4\nu} & \text{if } P_{e_e} < 1, \end{cases} \quad \forall x \in e,$$

where  $\tau$  is a constant. We set  $\tau = 0.7$  in our numerical experiments. Define  $C_s = \|C(x)\|_\infty$ , and we know from the definition of  $C(x)$  that

$$(3.2) \quad C_s = \|C(x)\|_\infty \leq \frac{\tau}{4\nu} h^2.$$

The stabilized finite element problem for solving (2.5) is: find  $u \in \widehat{W}$ , such that

$$(3.3) \quad a(u, v) := a_o(u, v) + \int_\Omega C(x) LuLv \, dx = \int_\Omega f v \, dx + \int_\Omega C(x) f Lv \, dx, \quad \forall v \in \widehat{W}.$$

Here and from now on, the integration over  $\Omega$  in the stabilization terms always represents a sum of integrals over all elements of  $\Omega$ . We note that for all piecewise linear finite element functions  $u$ ,  $Lu = -\nu \Delta u + \mathbf{a} \cdot \nabla u + cu = \mathbf{a} \cdot \nabla u + cu$ , in each element.

The symmetric and skew-symmetric parts of  $a(u, v)$ , respectively, are denoted by

$$(3.4) \quad b(u, v) = \int_\Omega (\nu \nabla u \cdot \nabla v + C(x) LuLv + \tilde{c}uv) \, dx,$$

$$(3.5) \quad z(u, v) = \frac{1}{2} \int_\Omega (\mathbf{a} \cdot \nabla uv - \mathbf{a} \cdot \nabla vu) \, dx.$$

The system of linear equations corresponding to the stabilized finite element problem (3.3) is denoted by

$$(3.6) \quad Au = f,$$

where the coefficient matrix  $A$  is nonsymmetric but positive definite. We denote the symmetric part of  $A$  by  $B$  and its skew-symmetric part by  $Z$ ; they correspond to the bilinear forms  $b(\cdot, \cdot)$  and  $z(\cdot, \cdot)$  in (3.4) and (3.5), respectively. In this paper, we will use the same notation, e.g.,  $u$ , to denote both a finite element function and the vector of its coefficients with respect to the finite element basis; we will also use the same notation to denote the space of finite element functions and the space of their corresponding vectors, e.g.,  $\widehat{W}$ .

4. DOMAIN DECOMPOSITION AND A PARTIALLY SUB-ASSEMBLED FINITE  
ELEMENT SPACE

The original finite element triangulation of  $\Omega$  is decomposed into  $N$  nonoverlapping polyhedral subdomains  $\Omega_i$ ; each subdomain is a union of shape regular elements. The typical diameter of the subdomains is denoted by  $H$ . The nodes on the boundaries of neighboring subdomains match across the subdomain interface  $\Gamma = (\cup \partial\Omega_i) \setminus \partial\Omega$ . The interface  $\Gamma$  is composed of subdomain faces  $\mathcal{F}^l$  and/or edges  $\mathcal{E}^k$ , which are regarded as open subsets of  $\Gamma$ , and of the subdomain vertices, which are end points of edges. In three dimensions, the subdomain faces are shared by two subdomains, and the edges typically by more than two; in two dimensions, each edge is shared by two subdomains. The interface of subdomain  $\Omega_i$  is defined by  $\Gamma_i = \partial\Omega_i \cap \Gamma$ . We denote the space of finite element functions on  $\Omega_i$ , which vanish at the nodes of  $\partial\Omega$ , by  $W^{(i)}$ . The local bilinear and stabilized bilinear forms are defined on  $W^{(i)}$  by

$$(4.1) \quad a_o^{(i)}(u^{(i)}, v^{(i)}) = \int_{\Omega_i} (\nu \nabla u^{(i)} \cdot \nabla v^{(i)} + \mathbf{a} \cdot \nabla u^{(i)} v^{(i)} + cu^{(i)} v^{(i)}) dx,$$

and

$$\begin{aligned} \bar{a}^{(i)}(u^{(i)}, v^{(i)}) &= \int_{\Omega_i} (\nu \nabla u^{(i)} \cdot \nabla v^{(i)} + \mathbf{a} \cdot \nabla u^{(i)} v^{(i)} + cu^{(i)} v^{(i)} + C(x)Lu^{(i)}Lv^{(i)}) dx \\ &= \int_{\Omega_i} (\nu \nabla u^{(i)} \cdot \nabla v^{(i)} + C(x)Lu^{(i)}Lv^{(i)} + \tilde{c}u^{(i)}v^{(i)}) dx \\ &\quad + \frac{1}{2} \int_{\Omega_i} (\mathbf{a} \cdot \nabla u^{(i)} v^{(i)} - \mathbf{a} \cdot \nabla v^{(i)} u^{(i)}) dx + \frac{1}{2} \int_{\Gamma_i} \mathbf{a} \cdot \mathbf{n} u^{(i)} v^{(i)} ds. \end{aligned}$$

We note that, in general, we cannot ensure that the stabilized bilinear form  $\bar{a}^{(i)}(\cdot, \cdot)$  is positive definite on  $W^{(i)}$  since the boundary integral on  $\Gamma_i$  does not vanish and the sign of  $\mathbf{a} \cdot \mathbf{n}$  depends on the orientation of the flow  $\mathbf{a}$  in relation to the external normal direction  $\mathbf{n}$  on  $\Gamma_i$ . We therefore modify  $\bar{a}^{(i)}(\cdot, \cdot)$  as in [2] and introduce

$$(4.2) \quad a^{(i)}(u^{(i)}, v^{(i)}) = \bar{a}^{(i)}(u^{(i)}, v^{(i)}) - \frac{1}{2} \int_{\Gamma_i} \mathbf{a} \cdot \mathbf{n} u^{(i)} v^{(i)} ds,$$

which corresponds to the Robin boundary condition on  $\Gamma_i$ . The assumption (2.3) now ensures that the modified local bilinear forms  $a^{(i)}(\cdot, \cdot)$  are positive definite on  $W^{(i)}$ ,  $i = 1, 2, \dots, N$ . The symmetric and skew-symmetric parts of  $a^{(i)}(u^{(i)}, v^{(i)})$  are represented, respectively, by

$$(4.3) \quad b^{(i)}(u^{(i)}, v^{(i)}) = \int_{\Omega_i} (\nu \nabla u^{(i)} \cdot \nabla v^{(i)} + C(x)Lu^{(i)}Lv^{(i)} + \tilde{c}u^{(i)}v^{(i)}) dx,$$

$$(4.4) \quad z^{(i)}(u^{(i)}, v^{(i)}) = \frac{1}{2} \int_{\Omega_i} (\mathbf{a} \cdot \nabla u^{(i)} v^{(i)} - \mathbf{a} \cdot \nabla v^{(i)} u^{(i)}) dx.$$

We now introduce a partially sub-assembled finite element space, which was introduced by Klawonn, Widlund, and Dryja [24] in their analysis of FETI-DP algorithms for symmetric positive definite problems. The partially sub-assembled finite element space  $\widetilde{W}$  is the direct sum of a coarse level primal subspace  $\widehat{W}_\Pi$ , which is a space of continuous coarse level finite element functions, and a dual subspace  $W_r$ , which is the product of local dual spaces  $W_r^{(i)}$ . The space  $\widehat{W}_\Pi$  corresponds to a few selected subdomain interface degrees of freedom for each subdomain and is

typically spanned by subdomain vertex nodal basis functions, and/or interface edge and/or face basis functions with weights at the nodes of the edge or face. These basis functions will correspond to the primal interface continuity constraints enforced in the BDDC algorithm. To simplify our analysis, we will always assume that the basis has been changed so that we have explicit primal unknowns corresponding to the primal continuity constraints of edges or faces; these coarse level primal degrees of freedom are shared by neighboring subdomains. For more details on the change of basis, see [23, 27]. Each subdomain dual space  $W_r^{(i)}$  corresponds to the subdomain interior and dual interface degrees of freedom and it is spanned by all the basis functions of  $W^{(i)}$  which vanish at the primal degrees of freedom. Thus, functions in the space  $\widetilde{W}$  have a continuous coarse level, primal part and typically a discontinuous dual part across the subdomain interfaces. We have  $\widehat{W} \subset \widetilde{W}$  and we denote the injection operator from  $\widehat{W}$  to  $\widetilde{W}$  by  $\widetilde{R}$ .

We define the bilinear form on the partially sub-assembled finite element space  $\widetilde{W}$  by

$$\widetilde{a}_o(u, v) = \sum_{i=1}^N a_o^{(i)}(u^{(i)}, v^{(i)}), \quad \forall u, v \in \widetilde{W},$$

where  $u^{(i)}$  and  $v^{(i)}$  represent restrictions of  $u$  and  $v$  to subdomain  $\Omega_i$ . Corresponding to the stabilized forms, we define, for all  $u, v \in \widetilde{W}$ ,

$$\widetilde{a}(u, v) = \sum_{i=1}^N a^{(i)}(u^{(i)}, v^{(i)}), \quad \widetilde{b}(u, v) = \sum_{i=1}^N b^{(i)}(u^{(i)}, v^{(i)}), \quad \widetilde{z}(u, v) = \sum_{i=1}^N z^{(i)}(u^{(i)}, v^{(i)}).$$

Denote the partially sub-assembled matrices corresponding to the bilinear forms  $\widetilde{a}(\cdot, \cdot)$ ,  $\widetilde{b}(\cdot, \cdot)$ , and  $\widetilde{z}(\cdot, \cdot)$  by  $\widetilde{A}$ ,  $\widetilde{B}$ , and  $\widetilde{Z}$ , respectively. We have

$$A = \widetilde{R}^T \widetilde{A} \widetilde{R}, \quad B = \widetilde{R}^T \widetilde{B} \widetilde{R}, \quad \text{and} \quad Z = \widetilde{R}^T \widetilde{Z} \widetilde{R}.$$

We note that the use of the modified bilinear form  $a^{(i)}(\cdot, \cdot)$ , defined in (4.2) corresponding to the Robin boundary condition, does not affect the matrix  $A$  of the original problem when it is assembled from  $\widetilde{A}$ , since the additional interface terms in (4.2) cancel.

We define broken norms on the space  $\widetilde{W}$ , by  $\|w\|_{L^2(\Omega)}^2 = \sum_{i=1}^N \|w^{(i)}\|_{L^2(\Omega_i)}^2$  and  $|w|_{H^1(\Omega)}^2 = \sum_{i=1}^N |w^{(i)}|_{H^1(\Omega_i)}^2$ . In this paper,  $\|w\|_{L^2(\Omega)}$  and  $|w|_{H^1(\Omega)}$ , for functions  $w \in \widetilde{W}$ , always represent these broken norms. Since the subdomain bilinear forms  $b^{(i)}(\cdot, \cdot)$ ,  $i = 1, 2, \dots, N$ , are symmetric positive definite on  $W^{(i)}$ , we define  $\|u^{(i)}\|_{B^{(i)}}^2 = b^{(i)}(u^{(i)}, u^{(i)})$ , for any  $u^{(i)} \in W^{(i)}$ . We define  $\|u\|_B^2 = \sum_{i=1}^N \|u^{(i)}\|_{B^{(i)}}^2$ , for any  $u \in \widehat{W}$ , and  $\|w\|_{\widetilde{B}}^2 = \sum_{i=1}^N \|w^{(i)}\|_{B^{(i)}}^2$ , for any  $w \in \widetilde{W}$ . Both  $B$ - and  $\widetilde{B}$ -norms are also well defined for functions in the space  $H^2(\Omega)$ .

Lemmas 4.1 and 4.2 are immediate consequences of the definitions of  $B^{(i)}$ - and  $B$ - norms.

**Lemma 4.1.** *For all  $w^{(i)} \in W^{(i)}$ ,  $i = 1, 2, \dots, N$ ,  $\sqrt{\nu}|w^{(i)}|_{H^1(\Omega_i)} \leq \|w^{(i)}\|_{B^{(i)}}$ , and  $\min\{\sqrt{\nu}, \sqrt{\tilde{c}}\}|w^{(i)}|_{H^1(\Omega_i)} \leq \|w^{(i)}\|_{B^{(i)}}$ .*

**Lemma 4.2.** *There exists a positive constant  $C$ , which is independent of  $H$  and  $h$ , such that for all  $u \in H^2(\Omega)$ ,  $\|u\|_B \leq C\|u\|_{H^2(\Omega)}$ .*

From Lemma 4.1, follows

**Lemma 4.3.** *There exist positive constants  $C_1$  and  $C_2$ , which are independent of  $H$  and  $h$ , such that for all  $u^{(i)}, v^{(i)} \in W^{(i)}$ ,  $i = 1, 2, \dots, N$ ,  $|z^{(i)}(u^{(i)}, v^{(i)})| \leq C_1 \|u^{(i)}\|_{B^{(i)}} \|v^{(i)}\|_{B^{(i)}}$ , and  $|a^{(i)}(u^{(i)}, v^{(i)})| \leq C_2 \|u^{(i)}\|_{B^{(i)}} \|v^{(i)}\|_{B^{(i)}}$ .*

**Lemma 4.4.** *There exists a positive constant  $C$ , which is independent of  $H$  and  $h$ , such that for all  $u, v \in \widehat{W}$ ,  $|z(u, v)| \leq C \|u\|_B \|v\|_{L^2(\Omega)}$ .*

*Proof:* We find, by integration by parts and using Lemma 4.1, that

$$\begin{aligned} |z(u, v)| &\leq \frac{1}{2} \int_{\Omega} |2\mathbf{a} \cdot \nabla uv + \nabla \cdot \mathbf{a} uv| \, dx \\ &\leq C (a_s |u|_{H^1(\Omega)} \|v\|_{L^2(\Omega)} + \|\nabla \cdot \mathbf{a}\|_{\infty} \|u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)}) \leq C \|u\|_B \|v\|_{L^2(\Omega)}. \quad \square \end{aligned}$$

We also have the following approximation property in  $B$ -norm for the finite element space  $\widehat{W}$ .

**Lemma 4.5.** *There exists a positive constant  $C$ , which is independent of  $H$  and  $h$ , such that for all  $u \in H^2(\Omega)$ ,  $\inf_{w \in \widehat{W}} \|u - w\|_B \leq Ch |u|_{H^2(\Omega)}$ .*

*Proof:* We have, for any  $u \in H^2(\Omega)$  and  $w \in \widehat{W}$ , that

$$\begin{aligned} \|u - w\|_B^2 &= b(u - w, u - w) \leq \nu |u - w|_{H^1(\Omega)}^2 + C_s \|L(u - w)\|_{L^2(\Omega)}^2 + \tilde{c}_s \|u - w\|_{L^2(\Omega)}^2 \\ &= \nu |u - w|_{H^1(\Omega)}^2 + C_s \|\nu \Delta u + \mathbf{a} \cdot \nabla(u - w) + c(u - w)\|_{L^2(\Omega)}^2 + \tilde{c}_s \|u - w\|_{L^2(\Omega)}^2 \\ &\leq \nu^2 C_s |u|_{H^2(\Omega)}^2 + (\nu + C_s a_s^2) |u - w|_{H^1(\Omega)}^2 + (C_s c_s^2 + \tilde{c}_s^2) \|u - w\|_{L^2(\Omega)}^2. \end{aligned}$$

We complete the proof by using (3.2) and the following standard finite element approximation results, cf. [35, Lemma B.6],

$$\inf_{w \in \widehat{W}} \left\{ \|u - w\|_{L^2(\Omega)}^2 + h^2 |u - w|_{H^1(\Omega)}^2 \right\} \leq Ch^4 |u|_{H^2(\Omega)}^2. \quad \square$$

For each subdomain interface edge  $\mathcal{E}^k$ , let  $\vartheta_{\mathcal{E}^k}$  be the standard finite element edge cut-off function which vanishes at all interface nodes except those of the edge  $\mathcal{E}^k$  where it takes the value 1. For three-dimensional problems, we denote the finite element face cut-off functions by  $\vartheta_{\mathcal{F}^l}$ , which vanishes at all interface nodes except those of  $\mathcal{F}^l$  where it takes the value 1. Let  $I_h$  be the interpolation operator into the finite element space. In the convergence analysis of our BDDC algorithm for advection-diffusion problems, we require that the coarse level primal subspace  $\widehat{W}_{\Pi}$  satisfies the following assumption.

**Assumption 4.6.** *For two-dimensional problems, the coarse level primal subspace  $\widehat{W}_{\Pi}$  contains all subdomain corner degrees of freedom, and for each edge  $\mathcal{E}^k$ , one edge average degree of freedom and two edge flux average degrees of freedom such that for any  $w \in \widehat{W}$ ,*

$$\int_{\mathcal{E}^k} w^{(i)} \, ds, \quad \int_{\mathcal{E}^k} \mathbf{a} \cdot \mathbf{n} w^{(i)} \, ds, \quad \text{and} \quad \int_{\mathcal{E}^k} \mathbf{a} \cdot \mathbf{n} w^{(i)} s \, ds,$$

respectively, are the same (with a difference of factor  $-1$  corresponding to opposite normal directions) for the two subdomains  $\Omega_i$  that share  $\mathcal{E}^k$ .

For three dimensional problems,  $\widehat{W}_{\Pi}$  contains all subdomain corner degrees of freedom, and for each face  $\mathcal{F}^l$ , one face average degree of freedom and two face

flux average degrees of freedom, and for each edge  $\mathcal{E}^k$ , one edge average degree of freedom, such that for any  $w \in \widetilde{W}$ ,

$$\int_{\mathcal{F}^l} I_h \left( \vartheta_{\mathcal{F}^l} w^{(i)} \right) ds, \quad \int_{\mathcal{F}^l} \mathbf{a} \cdot \mathbf{n} I_h \left( \vartheta_{\mathcal{F}^l} w^{(i)} \right) ds, \quad \text{and} \quad \int_{\mathcal{F}^l} \mathbf{a} \cdot \mathbf{n} I_h \left( \vartheta_{\mathcal{F}^l} w^{(i)} \right) s ds,$$

respectively, are the same (with a difference of factor  $-1$  corresponding to opposite normal directions) for the two subdomains  $\Omega_i$  that share the face  $\mathcal{F}^l$ , and

$$\int_{\mathcal{E}^k} I_h \left( \vartheta_{\mathcal{E}^k} w^{(i)} \right) ds$$

are the same for all subdomains  $\Omega_i$  that share the edge  $\mathcal{E}^k$ .

The following result can be proved under Assumption 4.6.

**Lemma 4.7.** *Let Assumption 4.6 hold. There exist positive constants  $C_1$  and  $C_2$ , which are independent of  $H$  and  $h$ , such that for all  $w^{(i)} \in W^{(i)}$  which vanish at the coarse level primal degrees of freedom,  $\|w^{(i)}\|_{B^{(i)}} \leq C_1 |w^{(i)}|_{H^1(\Omega_i)}$ , and for all  $w \in \widetilde{W}$ ,  $\|w\|_{\widetilde{B}} \leq C_2 |w|_{H^1(\Omega)}$ .*

*Proof:* We recall that, for any  $w^{(i)} \in W^{(i)}$ ,  $Lw^{(i)} = \mathbf{a} \cdot \nabla w^{(i)} + cw^{(i)}$ . We have,

$$\begin{aligned} \|w^{(i)}\|_{B^{(i)}}^2 &= \int_{\Omega_i} \left( \nu |\nabla w^{(i)}|^2 + C(x) (\mathbf{a} \cdot \nabla w^{(i)} + cw^{(i)})^2 + \tilde{c} (w^{(i)})^2 \right) dx \\ &\leq \int_{\Omega_i} \left( \nu |\nabla w^{(i)}|^2 + 2C(x) ((\mathbf{a} \cdot \nabla w^{(i)})^2 + (cw^{(i)})^2) + \tilde{c} (w^{(i)})^2 \right) dx \\ &\leq (\nu + 2C_s a_s^2) |w^{(i)}|_{H^1(\Omega_i)}^2 + (2C_s c_s^2 + \tilde{c}_s) \|w^{(i)}\|_{L^2(\Omega_i)}^2 \leq C_1 |w^{(i)}|_{H^1(\Omega_i)}^2, \end{aligned}$$

where in the last step we use a Poincaré-Friedrichs inequality for  $w^{(i)}$  which has vanishing averages on the subdomain interface.

To prove the second inequality in the lemma, we find that for any  $w \in \widetilde{W}$ ,

$$\|w\|_{\widetilde{B}}^2 = \sum_{i=1}^N \|w^{(i)}\|_{B^{(i)}}^2 \leq (\nu + 2C_s a_s^2) |w|_{H^1(\Omega)}^2 + (2C_s c_s^2 + \tilde{c}_s) \|w\|_{L^2(\Omega)}^2 \leq C_2 |w|_{H^1(\Omega)}^2,$$

where in the last step we use a Poincaré-Friedrichs inequality proved by Brenner in [4, (1.3)] which holds under Assumption 4.6.  $\square$

We will need an error bound for the approximation of partially sub-assembled finite element problems in the analysis of our BDDC algorithm. For this purpose, we make an assumption for our decomposition of the global domain  $\Omega$ .

**Assumption 4.8.** *Each subdomain  $\Omega_i$  is triangular or quadrilateral in two dimensions, and tetrahedral or hexahedral in three dimensions. The subdomains form a shape regular coarse mesh of  $\Omega$ .*

Under Assumption 4.8, we can denote by  $\widehat{W}_H$  the continuous linear, bilinear, or trilinear finite element space on the coarse subdomain mesh, and denote by  $I_H$  the finite element interpolation operator into  $\widehat{W}_H$ . We have the following Bramble-Hilbert lemma; cf. [41, Theorem 2.3].

**Lemma 4.9.** *There exists a constant  $C$ , which is independent of  $H$  and  $h$ , such that for all  $u \in H^2(\Omega)$ ,  $\|u - I_H u\|_{H^t(\Omega_i)} \leq C H^{2-t} |u|_{H^2(\Omega_i)}$ , for all  $t = 0, 1, 2$ , and  $i = 1, 2, \dots, N$ .*

5. ERROR ESTIMATE FOR A PARTIALLY SUB-ASSEMBLED FINITE ELEMENT PROBLEM

In this section, we prove an error bound for the solution of a partially sub-assembled finite element problem.

Given  $g \in L^2(\Omega)$ , we define  $\varphi_g \in H_0^1(\Omega)$  and  $\tilde{\varphi}_g \in \widetilde{W}$  as the solutions to the following problems, respectively,

$$(5.1) \quad a_o(u, \varphi_g) = (u, g), \quad \forall u \in H_0^1(\Omega),$$

$$(5.2) \quad \tilde{a}_o(w, \tilde{\varphi}_g) + \int_{\Omega} C(x)L^*wL^*\tilde{\varphi}_g \, dx = (w, g) + \int_{\Omega} C(x)L^*wg \, dx, \quad \forall w \in \widetilde{W}.$$

We know from (5.1) that  $\varphi_g$  is the weak solution to the adjoint problem  $L^*\varphi_g = g$ , and  $\varphi_g \in H^2(\Omega)$  under the regularity assumption (2.6). We have the following result.

**Lemma 5.1.** *Let Assumption 4.6 hold. For any  $g \in L^2(\Omega)$ , let  $\varphi_g$  be the solution to (5.1) and let  $L_h(q, \varphi_g) = \tilde{a}_o(q, \varphi_g) - (q, g)$ , for  $q \in \widetilde{W}$ . There then exists a constant  $C$ , which is independent of  $H$  and  $h$ , such that for all  $q \in \widetilde{W}$ ,  $|L_h(q, \varphi_g)| \leq CH\mu(H, h)\|\varphi_g\|_{H^2}\|q\|_{\widetilde{B}}$ , where  $\mu(H, h) = 1$ , for two-dimensional problems, and  $\mu(H, h) = 1 + \log(H/h)$ , for three-dimensional problems.*

*Proof:* We give the proof only for the three-dimensional case; the two-dimensional case can be proved in a similar manner. For any  $q \in \widetilde{W}$ , we have

$$\begin{aligned} L_h(q, \varphi_g) &= \tilde{a}_o(q, \varphi_g) - (q, g) \\ &= \sum_{i=1}^N \int_{\Omega_i} \left( \nu \nabla q^{(i)} \nabla \varphi_g + \mathbf{a} \cdot \nabla q^{(i)} \varphi_g + cq^{(i)} \varphi_g - q^{(i)} g \right) dx \\ &= \sum_{i=1}^N \left\{ \int_{\partial\Omega_i} \left( \nu \partial_n \varphi_g q^{(i)} + \mathbf{a} \cdot \mathbf{n} \varphi_g q^{(i)} \right) ds \right. \\ &\quad \left. - \int_{\Omega_i} \left( \nu \Delta \varphi_g q^{(i)} + \nabla \cdot (\mathbf{a} \varphi_g) q^{(i)} - c \varphi_g q^{(i)} + g q^{(i)} \right) dx \right\} \\ &= \sum_{i=1}^N \int_{\partial\Omega_i} \left( \nu \partial_n \varphi_g q^{(i)} + \mathbf{a} \cdot \mathbf{n} \varphi_g q^{(i)} \right) ds \\ &= \sum_{i=1}^N \sum_{\Gamma_{ij} \subset \partial\Omega_i} \int_{\Gamma_{ij}} \left( \nu \partial_n \varphi_g q^{(i)} + \mathbf{a} \cdot \mathbf{n} \varphi_g q^{(i)} \right) ds, \end{aligned}$$

where we use the fact that  $L^*\varphi_g = g$  holds in the weak sense. Here  $\Gamma_{ij}$  represents the boundary faces of  $\Omega_i$ .

Denote the common average of  $q$  on the face  $\mathcal{F}^l$  of  $\Gamma_{ij}$  by  $\bar{q}_{\mathcal{F}^l}$  and its common averages on the edges  $\mathcal{E}^{lk}$  by  $\bar{q}_{\mathcal{E}^{lk}}$ . Since the finite element cut-off functions  $\vartheta_{\mathcal{F}^l}$

and  $\vartheta_{\mathcal{E}^{lk}}$  provide a partition of unity, cf. [35, Section 4.6], we have

$$\begin{aligned} L_h(q, \varphi_g) = & \sum_{i=1}^N \sum_{\Gamma_{ij} \subset \partial\Omega_i} \left\{ \int_{\mathcal{F}^l} \left( \nu \partial_n \varphi_g I_h(\vartheta_{\mathcal{F}^l}(q^{(i)} - \bar{q}_{\mathcal{F}^l})) + \mathbf{a} \cdot \mathbf{n} \varphi_g I_h(\vartheta_{\mathcal{F}^l}(q^{(i)} - \bar{q}_{\mathcal{F}^l})) \right) ds \right. \\ & \left. + \sum_{\mathcal{E}^{lk} \subset \Gamma_{ij}} \int_{\mathcal{F}^l} \left( \nu \partial_n \varphi_g I_h(\vartheta_{\mathcal{E}^{lk}}(q^{(i)} - \bar{q}_{\mathcal{E}^{lk}})) + \mathbf{a} \cdot \mathbf{n} \varphi_g I_h(\vartheta_{\mathcal{E}^{lk}}(q^{(i)} - \bar{q}_{\mathcal{E}^{lk}})) \right) ds \right\}, \end{aligned}$$

where we have also subtracted the constant average values  $\bar{q}_{\mathcal{F}^l}$  and  $\bar{q}_{\mathcal{E}^{lk}}$  from  $q^{(i)}$ , which does not change the sum. Then, from Assumption 4.6, we know that

$$\begin{aligned} L_h(q, \varphi_g) = & \sum_{i=1}^N \sum_{\Gamma_{ij} \subset \partial\Omega_i} \int_{\mathcal{F}^l} \left( \nu \partial_n (\varphi_g - I_H^{(i)} \varphi_g) (I_h(\vartheta_{\mathcal{F}^l}(q^{(i)} - \bar{q}_{\mathcal{F}^l}))) \right) ds \\ & + \sum_{i=1}^N \sum_{\Gamma_{ij} \subset \partial\Omega_i} \int_{\mathcal{F}^l} \left( \mathbf{a} \cdot \mathbf{n} (\varphi_g - I_H^{(i)} \varphi_g) (I_h(\vartheta_{\mathcal{F}^l}(q^{(i)} - \bar{q}_{\mathcal{F}^l}))) \right) ds \\ & + \sum_{i=1}^N \sum_{\Gamma_{ij} \subset \partial\Omega_i} \sum_{\mathcal{E}^{lk} \subset \Gamma_{ij}} \int_{\mathcal{F}^l} \left( \nu \partial_n \varphi_g (I_h(\vartheta_{\mathcal{E}^{lk}}(q^{(i)} - \bar{q}_{\mathcal{E}^{lk}}))) \right) ds \\ & + \sum_{i=1}^N \sum_{\Gamma_{ij} \subset \partial\Omega_i} \sum_{\mathcal{E}^{lk} \subset \Gamma_{ij}} \int_{\mathcal{F}^l} \left( \mathbf{a} \cdot \mathbf{n} \varphi_g (I_h(\vartheta_{\mathcal{E}^{lk}}(q^{(i)} - \bar{q}_{\mathcal{E}^{lk}}))) \right) ds \\ (5.3) \quad & := I_1 + I_2 + I_3 + I_4, \end{aligned}$$

where  $I_H \varphi_g$  represents the interpolation of  $\varphi_g$  into the space  $\widehat{W}_H$  on the coarse subdomain mesh. We will show in the following that each of the four terms in (5.3) can be bounded by  $CH(1 + \log(H/h)) \|\varphi_g\|_{H^2} \|q\|_{\widehat{B}}$ , where  $C$  is a positive constant independent of  $H$  and  $h$ .

For the first term  $I_1$ , from the Cauchy-Schwarz inequality, we have

$$(5.4) \quad |I_1| \leq \nu \sum_{i=1}^N \sum_{\Gamma_{ij} \subset \partial\Omega} \left( \int_{\mathcal{F}^l} |\nabla(\varphi_g - I_H^{(i)} \varphi_g)|^2 ds \int_{\mathcal{F}^l} |I_h(\vartheta_{\mathcal{F}^l}(q^{(i)} - \bar{q}_{\mathcal{F}^l}))|^2 ds \right)^{1/2}.$$

Using a trace theorem and Lemma 4.9, we have for the first factor

$$\begin{aligned} & \int_{\mathcal{F}^l} |\nabla(\varphi_g - I_H^{(i)} \varphi_g)|^2 ds \leq CH \|\nabla(\varphi_g - I_H^{(i)} \varphi_g)\|_{H^1(\Omega_i)}^2 \\ (5.5) \quad & \leq CH \|\varphi_g - I_H^{(i)} \varphi_g\|_{H^2(\Omega_i)}^2 \leq CH |\varphi_g|_{H^2(\Omega_i)}^2. \end{aligned}$$

For the second factor, we have

$$\begin{aligned} & \int_{\mathcal{F}^l} |I_h(\vartheta_{\mathcal{F}^l}(q^{(i)} - \bar{q}_{\mathcal{F}^l}))|^2 ds \leq CH \|I_h(\vartheta_{\mathcal{F}^l}(q^{(i)} - \bar{q}_{\mathcal{F}^l}))\|_{H^1(\Omega_i)}^2 \\ (5.6) \quad & \leq CH(1 + \log \frac{H}{h})^2 \|q^{(i)} - \bar{q}_{\mathcal{F}^l}\|_{H^1(\Omega_i)}^2 \leq CH(1 + \log \frac{H}{h})^2 |q^{(i)}|_{H^1}, \end{aligned}$$

where we have used a trace theorem for the first step, a Poincaré-Friedrichs inequality and [35, Lemma 4.24] in the second, and a Poincaré-Friedrichs inequality in the

last step. Combining (5.4), (5.5), and (5.6), we have the following bound for  $I_1$ ,

$$|I_1| \leq C\nu H(1 + \log \frac{H}{h}) \sum_{i=1}^N |\varphi_g|_{H^2(\Omega_i)} |q|_{H^1(\Omega_i)} \leq C\sqrt{\nu} H(1 + \log \frac{H}{h}) |\varphi_g|_{H^2(\Omega)} \|q\|_{\tilde{B}},$$

where we use the Cauchy-Schwarz inequality and Lemma 4.1 in the last step.

To derive a bound for  $I_2$ , we find from the Cauchy-Schwarz inequality that

$$(5.7) \quad |I_2| \leq \sum_{i=1}^N \sum_{\Gamma_{ij} \subset \partial\Omega} a_s \left( \int_{\mathcal{F}^i} |\varphi_g - I_H^{(i)} \varphi_g|^2 ds \int_{\mathcal{F}^i} |I_h(\vartheta_{\mathcal{F}^i}(q^{(i)} - \bar{q}_{\mathcal{F}^i}))|^2 ds \right)^{1/2}.$$

Using a trace theorem and Lemma 4.9, we have, for the first factor on the right hand side of (5.7),

$$(5.8) \quad \int_{\mathcal{F}^i} |\varphi_g - I_H^{(i)} \varphi_g|^2 ds \leq CH \|\varphi_g - I_H^{(i)} \varphi_g\|_{H^1(\Omega_i)}^2 \leq CH^3 |\varphi_g|_{H^2(\Omega_i)}^2.$$

Combining (5.7), (5.8), and (5.6), and using Lemma 4.1, we have

$$|I_2| \leq Ca_s H^2 (1 + \log \frac{H}{h}) \sum_{i=1}^N |\varphi_g|_{H^2(\Omega_i)} |q|_{H^1(\Omega_i)} \leq Ca_s \frac{H}{\sqrt{\nu}} H(1 + \log \frac{H}{h}) |\varphi_g|_{H^2(\Omega)} \|q\|_{\tilde{B}}.$$

The estimate for  $I_3$  is similar to the estimate for  $I_1$ . Instead of using (5.5) and (5.6), we have, by using a trace theorem,

$$(5.9) \quad \int_{\mathcal{F}^i} |\nabla \varphi_g|^2 ds \leq CH \|\nabla \varphi_g\|_{H^1(\Omega_i)}^2 \leq CH \|\varphi_g\|_{H^2(\Omega_i)}^2,$$

and

$$(5.10) \quad \begin{aligned} & \int_{\mathcal{F}^i} |I_h \vartheta_{\mathcal{E}^{lk}}(q^{(i)} - \bar{q}_{\mathcal{E}^{lk}})|^2 ds \leq Ch \|I_h \vartheta_{\mathcal{E}^{lk}}(q^{(i)} - \bar{q}_{\mathcal{E}^{lk}})\|_{L^2(\mathcal{E}^i)}^2 \\ & \leq Ch(1 + \log \frac{H}{h}) \|I_h \vartheta_{\mathcal{E}^{lk}}(q^{(i)} - \bar{q}_{\mathcal{E}^{lk}})\|_{H^1(\Omega_i)}^2 \leq Ch(1 + \log \frac{H}{h})^2 |q^{(i)}|_{H^1(\Omega_i)}^2. \end{aligned}$$

In the first step of (5.10), we use the fact that  $I_h \vartheta_{\mathcal{E}^{lk}}(q^{(i)} - \bar{q}_{\mathcal{E}^{lk}})$  is different from zero only in the strip of elements next to the edge  $\mathcal{E}^{lk}$ ; in the second and the last steps, we use [35, Lemma 4.16], [35, Corollary 4.20], and a Poincaré-Friedrichs inequality. Combining (5.9) and (5.10), we have

$$|I_3| \leq C\nu\sqrt{Hh}(1 + \log \frac{H}{h}) \sum_{i=1}^N \|\varphi_g\|_{H^2(\Omega_i)} |q|_{H^1(\Omega_i)} \leq C\sqrt{\frac{h\nu}{H}} H(1 + \log \frac{H}{h}) \|\varphi_g\|_{H^2(\Omega)} \|q\|_{\tilde{B}}.$$

Similarly, for  $I_4$ , we have

$$|I_4| \leq Ca_s \sqrt{\frac{h}{\nu H}} H(1 + \log \frac{H}{h}) \|\varphi_g\|_{H^2(\Omega)} \|q\|_{\tilde{B}}. \quad \square$$

**Remark 5.2.** *In the case of two-dimensional problems, only the first two terms in (5.3), corresponding to the edges, appear. The finite element cut-off functions are no longer used in the proof and as a result the factor  $1 + \log(H/h)$  in the bound disappears.*

**Remark 5.3.** *The constant factor in the bound of  $I_2$  in the proof is proportional to  $H/\sqrt{\nu}$ , where  $H$  compensates for the effect of small  $\nu$  in the advection-dominant case. Without using the two face flux average continuity constraints as in Assumption 4.6, this constant factor would become proportional to  $1/\sqrt{\nu}$  instead. For*

two-dimensional problems, the same benefits can be obtained by enforcing the two edge flux average continuity constraints as in Assumption 4.6. Our numerical experiments in Section 8 show the effectiveness of using the two edge flux constraints for two-dimensional examples. The constant factor in the bound of  $I_4$  (only appearing for three-dimensional problems) is proportional to  $\sqrt{h/(\nu H)}$  where  $h/H$  can be used to compensate for the effect of small  $\nu$ ; in fact this factor can be improved to  $\sqrt{hH/\nu}$  by introducing a few extra edge normal flux average constraints, cf. [26, (35)].

**Lemma 5.4.** *There exists a positive constant  $C$ , which is independent of  $H$  and  $h$ , such that for all  $q \in \widetilde{W}$  and  $u \in \widetilde{W} \cup H^2(\Omega)$ ,  $\int_{\Omega} C(x)LqLu \, dx \leq Ch\|q\|_{\widetilde{B}}\|u\|_{\widetilde{B}}$ , and  $\int_{\Omega} C(x)L^*qL^*u \, dx \leq Ch\|q\|_{\widetilde{B}}\|u\|_{\widetilde{B}}$ .*

*Proof:* We only give proof for the first inequality; the second one can be proved in the same way. We have, for any  $q \in \widetilde{W}$  and  $u \in \widetilde{W} \cup H^2(\Omega)$ ,

$$\begin{aligned} \int_{\Omega} C(x)LqLu \, dx &= \int_{\Omega} C(x)(\mathbf{a} \cdot \nabla q + cq)Lu \, dx \\ &\leq \sqrt{\|C(x)\|_{\infty}} \left( \int_{\Omega} (\mathbf{a} \cdot \nabla q + cq)^2 \, dx \right)^{1/2} \left( \int_{\Omega} C(x)(Lu)^2 \, dx \right)^{1/2} \leq Ch\|q\|_{\widetilde{B}}\|u\|_{\widetilde{B}}, \end{aligned}$$

where we use (3.2) in the last step.  $\square$

**Lemma 5.5.** *Let Assumption 4.6 hold.  $\varphi_g$  and  $\widetilde{\varphi}_g$  are solutions of (5.1) and (5.2), respectively, for  $g \in L^2(\Omega)$ . If  $h$  is sufficiently small, then*

$$\|\varphi_g - \widetilde{\varphi}_g\|_{\widetilde{B}} \leq CH\mu(H, h)\|\varphi_g\|_{H^2},$$

where  $C$  is a positive constant, independent of  $H$  and  $h$ , and  $\mu(H, h)$  given in Lemma 5.1.

*Proof:* For any  $\widetilde{\psi} \in \widetilde{W}$ , we have

$$\begin{aligned} \|\widetilde{\varphi}_g - \widetilde{\psi}\|_{\widetilde{B}}^2 &= \widetilde{a}(\widetilde{\varphi}_g - \widetilde{\psi}, \widetilde{\varphi}_g - \widetilde{\psi}) \\ &= \widetilde{a}(\widetilde{\varphi}_g - \widetilde{\psi}, \varphi_g - \widetilde{\psi}) + (\widetilde{a}(\widetilde{\varphi}_g - \widetilde{\psi}, \widetilde{\varphi}_g) - \widetilde{a}(\widetilde{\varphi}_g - \widetilde{\psi}, \varphi_g)) \\ &= \widetilde{a}(\widetilde{\varphi}_g - \widetilde{\psi}, \varphi_g - \widetilde{\psi}) + (\widetilde{a}_o(\widetilde{\varphi}_g - \widetilde{\psi}, \widetilde{\varphi}_g) - \widetilde{a}_o(\widetilde{\varphi}_g - \widetilde{\psi}, \varphi_g)) \\ &\quad + \int_{\Omega} C(x)L(\widetilde{\varphi}_g - \widetilde{\psi})L(\widetilde{\varphi}_g - \varphi_g) \, dx \\ &= \widetilde{a}(\widetilde{\varphi}_g - \widetilde{\psi}, \varphi_g - \widetilde{\psi}) + ((\widetilde{\varphi}_g - \widetilde{\psi}, g) - \widetilde{a}_o(\widetilde{\varphi}_g - \widetilde{\psi}, \varphi_g)) \\ &\quad + \int_{\Omega} C(x)L(\widetilde{\varphi}_g - \widetilde{\psi})L(\widetilde{\varphi}_g - \varphi_g) \, dx - \int_{\Omega} C(x)L^*(\widetilde{\varphi}_g - \widetilde{\psi})L^*(\widetilde{\varphi}_g - \varphi_g) \, dx, \end{aligned}$$

where in the last step we use (5.2) and that  $L^*\varphi_g = g$  holds in the weak sense. Dividing by  $\|\widetilde{\varphi}_g - \widetilde{\psi}\|_{\widetilde{B}}$  on both sides and denoting  $\widetilde{\varphi}_g - \widetilde{\psi}$  by  $q$ , we have, from Lemmas 4.3, 5.4, and 5.1, that

$$\begin{aligned} \|\widetilde{\varphi}_g - \widetilde{\psi}\|_{\widetilde{B}} &\leq C\|\varphi_g - \widetilde{\psi}\|_{\widetilde{B}} + \frac{|(q, g) - \widetilde{a}_o(q, \varphi_g)|}{\|q\|_{\widetilde{B}}} + Ch\|\widetilde{\varphi}_g - \varphi_g\|_{\widetilde{B}} \\ &\leq C\|\varphi_g - \widetilde{\psi}\|_{\widetilde{B}} + CH\mu(H, h)\|\varphi_g\|_{H^2} + Ch\|\widetilde{\varphi}_g - \varphi_g\|_{\widetilde{B}}. \end{aligned}$$

Then, using a triangle inequality, we have

$$\begin{aligned} \|\varphi_g - \tilde{\varphi}_g\|_{\tilde{B}} &\leq \inf_{\tilde{\psi} \in \tilde{W}} \left\{ \|\varphi_g - \tilde{\psi}\|_{\tilde{B}} + \|\tilde{\varphi}_g - \tilde{\psi}\|_{\tilde{B}} \right\} \\ &\leq C \inf_{\tilde{\psi} \in \tilde{W}} \|\varphi_g - \tilde{\psi}\|_{\tilde{B}} + CH\mu(H, h)\|\varphi_g\|_{H^2} + Ch\|\tilde{\varphi}_g - \varphi_g\|_{\tilde{B}} \\ &\leq CH\mu(H, h)\|\varphi_g\|_{H^2} + Ch\|\tilde{\varphi}_g - \varphi_g\|_{\tilde{B}}, \end{aligned}$$

where we use Lemma 4.5 in the last step. If  $h$  is small enough, the second term on the right hand side can be combined with the left hand side and our result is proved.  $\square$

## 6. THE BDDC PRECONDITIONER

The BDDC algorithms and closely related primal versions of the FETI algorithms were proposed by Dohrmann [13], Fragakis and Papadrakakis [18], and Cros [12], for solving symmetric, positive definite problems. The formulation of BDDC preconditioners applies equally well to nonsymmetric problems. In our BDDC algorithm for solving the advection-diffusion problems, the global system of linear equations (3.6) is reduced to a Schur complement problem for the subdomain interface variables and then a preconditioned GMRES iteration is used to solve the interface problem.

We decompose the space  $\widehat{W}$  into  $W_I \oplus \widehat{W}_\Gamma$ , where  $W_I$  is the product of local subdomain spaces  $W_I^{(i)}$ ,  $i = 1, 2, \dots, N$ , corresponding to the subdomain interior variables.  $\widehat{W}_\Gamma$  is the subspace corresponding to the variables on the interface. The original discrete problem (3.6) can be written as: find  $u_I \in W_I$  and  $u_\Gamma \in \widehat{W}_\Gamma$ , such that

$$(6.1) \quad \begin{bmatrix} A_{II} & A_{I\Gamma} \\ A_{\Gamma I} & A_{\Gamma\Gamma} \end{bmatrix} \begin{bmatrix} u_I \\ u_\Gamma \end{bmatrix} = \begin{bmatrix} f_I \\ f_\Gamma \end{bmatrix},$$

where  $A_{II}$  is block diagonal with one block for each subdomain, and  $A_{\Gamma\Gamma}$  corresponds to the subdomain interface variables and is assembled from subdomain matrices across the subdomain interfaces.

Eliminating the subdomain interior variables  $u_I$  from (6.1), we have the Schur complement problem

$$S_\Gamma u_\Gamma = g_\Gamma,$$

where  $S_\Gamma = A_{\Gamma\Gamma} - A_{\Gamma I} A_{II}^{-1} A_{I\Gamma}$ , and  $g_\Gamma = f_\Gamma - A_{\Gamma I} A_{II}^{-1} f_I$ .

Correspondingly, we define a partially sub-assembled Schur complement operator  $\tilde{S}_\Gamma$  as follows. We decompose the space  $\tilde{W}$  into  $W_I \oplus \tilde{W}_\Gamma$ . Here  $\tilde{W}_\Gamma$  contains the coarse level, continuous primal interface degrees of freedom, in the subspace  $\widehat{W}_\Pi$ , which are shared by neighboring subdomains, and the remaining dual subdomain interface degrees of freedom which are in general discontinuous across the subdomain interfaces. Then the partially sub-assembled problem matrix  $\tilde{A}$  can be written in a two by two block form

$$(6.2) \quad \begin{bmatrix} A_{II} & \tilde{A}_{I\Gamma} \\ \tilde{A}_{\Gamma I} & \tilde{A}_{\Gamma\Gamma} \end{bmatrix},$$

where  $\tilde{A}_{\Gamma\Gamma}$  is assembled only with respect to the coarse level primal degrees of freedom across the interface. The partially sub-assembled Schur complement operator  $\tilde{S}_\Gamma$  is defined by  $\tilde{S}_\Gamma = \tilde{A}_{\Gamma\Gamma} - \tilde{A}_{\Gamma I} A_{II}^{-1} \tilde{A}_{I\Gamma}$ . From the definition of  $S_\Gamma$  and  $\tilde{S}_\Gamma$ , we see

that  $S_\Gamma$  can be obtained from  $\widetilde{S}_\Gamma$  by assembling with respect to the dual interface variables, i.e.,

$$S_\Gamma = \widetilde{R}_\Gamma^T \widetilde{S}_\Gamma \widetilde{R}_\Gamma,$$

where  $\widetilde{R}_\Gamma$  is the injection operator from the space  $\widehat{W}_\Gamma$  into  $\widetilde{W}_\Gamma$ . We also define  $\widetilde{R}_{D,\Gamma} = D\widetilde{R}_\Gamma$ , where  $D$  is a diagonal scaling matrix. The diagonal elements of  $D$  equal 1, for the rows of the primal interface variables, and equal  $\delta_i^\dagger(x)$  for the others. Here, for a subdomain interface node  $x$ , the inverse counting function  $\delta_i^\dagger(x)$  is defined by  $\delta_i^\dagger(x) = 1/\text{card}(\mathcal{N}_x)$ , where  $\mathcal{N}_x$  is the set of indices of the subdomains which have  $x$  on their boundaries and  $\text{card}(\mathcal{N}_x)$  is the number of the subdomains in the set  $\mathcal{N}_x$ .

The preconditioned interface problem in our BDDC algorithm is

$$(6.3) \quad \widetilde{R}_{D,\Gamma}^T \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma} S_\Gamma u_\Gamma = \widetilde{R}_{D,\Gamma}^T \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma} g_\Gamma.$$

A GMRES iteration is used to solve (6.3). In each iteration, to multiply  $S_\Gamma$  by a vector, subdomain Dirichlet boundary problems need be solved; to multiply  $\widetilde{S}_\Gamma^{-1}$  by a vector, a partially sub-assembled finite element problem with the coefficient matrix  $\widetilde{A}$  needs be solved, which requires solving subdomain Robin boundary problems and a coarse level problem; cf. [27]. After obtaining the interface solution  $u_\Gamma$ , we find  $u_I$  by solving subdomain Dirichlet problems.

## 7. CONVERGENCE RATE OF THE GMRES ITERATION

In this section, we give a convergence analysis of the GMRES iteration for solving the preconditioned interface problem (6.3) for advection-diffusion problems.

For any  $u_\Gamma \in \widehat{W}_\Gamma$ , we denote its standard discrete harmonic extension to the interior of subdomains by  $u_{\mathcal{H},\Gamma} \in \widetilde{W}$ ; see [35, Section 4.4] for a definition of the discrete harmonic extension. We have the following result on the equivalence of the norms of local discrete harmonic extensions and traces on subdomain boundaries, cf. [35, Lemma 4.10].

**Lemma 7.1.** *There exist positive constants  $c$  and  $C$ , which are independent of  $H$  and  $h$ , such that for all  $u_\Gamma \in \widehat{W}_\Gamma$ , and  $i = 1, 2, \dots, N$ ,*

$$c|u_{\mathcal{H},\Gamma}^{(i)}|_{H^1(\Omega_i)} \leq |u_\Gamma^{(i)}|_{H^{1/2}(\partial\Omega_i)} \leq C|u_{\mathcal{H},\Gamma}^{(i)}|_{H^1(\Omega_i)}.$$

We define another discrete extension of  $u_\Gamma \in \widehat{W}_\Gamma$  to the interior of subdomains by

$$(7.1) \quad u_{\mathcal{A},\Gamma} = \begin{bmatrix} -A_{II}^{-1} \widetilde{A}_{I\Gamma} u_\Gamma \\ u_\Gamma \end{bmatrix} \in \widetilde{W}.$$

The discrete harmonic extension  $u_{\mathcal{H},\Gamma}$  can be obtained from  $u_\Gamma$  by solving subdomain Dirichlet problems corresponding to discrete Laplacian and it minimizes the energy norms of all finite element functions which have the trace  $u_\Gamma$  on the interface.  $u_{\mathcal{A},\Gamma}$  does not have this energy minimization property and it is obtained from  $u_\Gamma$  by solving subdomain advection-diffusion problems with Dirichlet boundary conditions as shown in (7.1). We note that both  $u_{\mathcal{H},\Gamma}$  and  $u_{\mathcal{A},\Gamma}$  are also well defined for  $u_\Gamma \in \widehat{W}_\Gamma$ , and as a result  $u_{\mathcal{H},\Gamma} \in \widetilde{W}$  and  $u_{\mathcal{A},\Gamma} \in \widetilde{W}$ .

We define two bilinear forms for vectors in  $\widehat{W}_\Gamma$  and  $\widetilde{W}_\Gamma$  respectively by

$$(7.2) \quad \langle u_\Gamma, v_\Gamma \rangle_{B_\Gamma} = v_{\mathcal{A},\Gamma}^T B u_{\mathcal{A},\Gamma}, \quad \text{and} \quad \langle u_\Gamma, v_\Gamma \rangle_{Z_\Gamma} = v_{\mathcal{A},\Gamma}^T Z u_{\mathcal{A},\Gamma}, \quad \forall u_\Gamma, v_\Gamma \in \widehat{W}_\Gamma,$$

$$(7.3) \quad \langle u_\Gamma, v_\Gamma \rangle_{\widetilde{B}_\Gamma} = v_{\mathcal{A},\Gamma}^T \widetilde{B} u_{\mathcal{A},\Gamma}, \quad \text{and} \quad \langle u_\Gamma, v_\Gamma \rangle_{\widetilde{Z}_\Gamma} = v_{\mathcal{A},\Gamma}^T \widetilde{Z} u_{\mathcal{A},\Gamma}, \quad \forall u_\Gamma, v_\Gamma \in \widetilde{W}_\Gamma.$$

In general, we use the notation  $\langle p, q \rangle_M$  to represent the product  $q^T M p$ , for any given matrix  $M$  and vectors  $p$  and  $q$ .

From the definitions (7.1), (7.2), and (7.3), follows

**Lemma 7.2.** *For any  $v \in \widetilde{W}$ , denote its restriction to  $\Gamma$  by  $v_\Gamma \in \widetilde{W}_\Gamma$ . Then for any  $u_\Gamma \in \widetilde{W}_\Gamma$  and  $v \in \widetilde{W}$ ,  $\langle u_\Gamma, v_\Gamma \rangle_{\widetilde{S}_\Gamma} = \langle u_{\mathcal{A},\Gamma}, v \rangle_{\widetilde{A}}$  and  $\langle u_\Gamma, v_\Gamma \rangle_{\widetilde{S}_\Gamma} = \langle u_\Gamma, v_\Gamma \rangle_{\widetilde{B}_\Gamma} + \langle u_\Gamma, v_\Gamma \rangle_{\widetilde{Z}_\Gamma}$ . For any  $u_\Gamma \in \widetilde{W}_\Gamma$ ,  $\langle u_\Gamma, u_\Gamma \rangle_{\widetilde{S}_\Gamma} = \langle u_{\mathcal{A},\Gamma}, u_{\mathcal{A},\Gamma} \rangle_{\widetilde{A}} = \langle u_{\mathcal{A},\Gamma}, u_{\mathcal{A},\Gamma} \rangle_{\widetilde{B}} = \langle u_\Gamma, u_\Gamma \rangle_{\widetilde{B}_\Gamma} \geq 0$ , and  $\langle u_\Gamma, u_\Gamma \rangle_{\widetilde{Z}_\Gamma} = 0$ . The same results also hold for functions and the corresponding bilinear forms in the space  $\widehat{W}_\Gamma$ .*

From Lemma 7.2, we define  $B_\Gamma$ - and  $\widetilde{B}_\Gamma$ - norms for elements in the spaces  $\widehat{W}_\Gamma$  and  $\widetilde{W}_\Gamma$  respectively by:  $\|u_\Gamma\|_{B_\Gamma}^2 = \langle u_\Gamma, u_\Gamma \rangle_{B_\Gamma}$ , for any  $u_\Gamma \in \widehat{W}_\Gamma$ , and  $\|w_\Gamma\|_{\widetilde{B}_\Gamma}^2 = \langle w_\Gamma, w_\Gamma \rangle_{\widetilde{B}_\Gamma}$ , for any  $w_\Gamma \in \widetilde{W}_\Gamma$ .

The following two lemmas can be obtained from definitions (7.2) and (7.3), and Lemmas 7.2, 4.3, and 4.4.

**Lemma 7.3.** *There exist positive constants  $C_1$  and  $C_2$ , which are independent of  $H$  and  $h$ , such that for all  $u_\Gamma, v_\Gamma \in \widetilde{W}_\Gamma$ ,  $|\langle u_\Gamma, v_\Gamma \rangle_{\widetilde{Z}_\Gamma}| \leq C_1 \|u_\Gamma\|_{\widetilde{B}_\Gamma} \|v_\Gamma\|_{\widetilde{B}_\Gamma}$ , and  $|\langle u_\Gamma, v_\Gamma \rangle_{\widetilde{S}_\Gamma}| \leq C_2 \|u_\Gamma\|_{\widetilde{B}_\Gamma} \|v_\Gamma\|_{\widetilde{B}_\Gamma}$ . The same results hold for functions and the corresponding bilinear forms in the space  $\widehat{W}_\Gamma$  as well.*

**Lemma 7.4.** *There exists a positive constant  $C$ , which is independent of  $H$  and  $h$ , such that for all  $u_\Gamma, v_\Gamma \in \widehat{W}_\Gamma$ ,  $|\langle u_\Gamma, v_\Gamma \rangle_{Z_\Gamma}| \leq C \|u_\Gamma\|_{B_\Gamma} \|v_{\mathcal{A},\Gamma}\|_{L^2(\Omega)}$ .*

We denote the preconditioned operator  $\widetilde{R}_{D,\Gamma}^T \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma} S_\Gamma$  in (6.3) by  $T$ . The convergence rate of the GMRES iteration can be estimated by using the following result due to Eisenstat, Elman, and Schultz [15].

**Theorem 7.5.** *Let  $c$  and  $C^2$  be two positive parameters such that*

$$(7.4) \quad c \langle u, u \rangle_{B_\Gamma} \leq \langle u, Tu \rangle_{B_\Gamma},$$

$$(7.5) \quad \langle Tu, Tu \rangle_{B_\Gamma} \leq C^2 \langle u, u \rangle_{B_\Gamma}.$$

Then

$$\frac{\|r_m\|_{B_\Gamma}}{\|r_0\|_{B_\Gamma}} \leq \left(1 - \frac{c^2}{C^2}\right)^{m/2},$$

where  $r_m$  is the residual of the GMRES iteration at iteration  $m$ .

**Remark 7.6.** *In our convergence analysis of the GMRES iteration, we use the  $B_\Gamma$ -norm; the analysis in the  $L^2$ -norm is not available yet. In our numerical experiments, we have found that the convergence rates in both the  $B_\Gamma$ - and  $L^2$ - norms are quite similar. For a study of the convergence rates of the GMRES iteration combined with an additive Schwarz method in the Euclidean and energy norms, see Sarkis and Szyld [33].*

We define an interface average operator  $E_{D,\Gamma}$  for functions in the space  $\widetilde{W}_\Gamma$  by  $E_{D,\Gamma}w_\Gamma = \widetilde{R}_\Gamma \widetilde{R}_{D,\Gamma}^T w_\Gamma$ , for any  $w_\Gamma \in \widetilde{W}_\Gamma$ . This operator computes an average of  $w_\Gamma$  across  $\Gamma$ . The following result on the stability of  $E_{D,\Gamma}$  can be found in [24, 23, 28].

**Lemma 7.7.** *Let Assumption 4.6 hold. With  $\Phi(H, h) = C(1 + \log(H/h))$ , where  $C$  is a positive constant which is independent of  $H$  and  $h$ ,  $|(E_{D,\Gamma}w_\Gamma)^{(i)}|_{H^{1/2}(\partial\Omega_i)} \leq \Phi(H, h)|w_\Gamma^{(i)}|_{H^{1/2}(\partial\Omega_i)}$ , for all  $w_\Gamma \in \widetilde{W}_\Gamma$ , and  $i = 1, 2, \dots, N$ .*

**Lemma 7.8.** *Let Assumption 4.6 hold. There then exists a positive constant  $C$ , which is independent of  $H$  and  $h$ , such that for all  $w_\Gamma \in \widetilde{W}_\Gamma$ ,  $\|E_{D,\Gamma}w_\Gamma\|_{\widetilde{B}_\Gamma} \leq C\Phi(H, h)\|w_\Gamma\|_{\widetilde{B}_\Gamma}$ , where  $\Phi(H, h)$  is given in Lemma 7.7.*

*Proof:* It is sufficient to show that  $\|E_{D,\Gamma}w_\Gamma - w_\Gamma\|_{\widetilde{B}_\Gamma} \leq C\Phi(H, h)\|w_\Gamma\|_{\widetilde{B}_\Gamma}$ . Denote  $E_{D,\Gamma}w_\Gamma - w_\Gamma$  by  $v_\Gamma$ . Let  $v_{\mathcal{A},\Gamma}$  and  $v_{\mathcal{H},\Gamma}$  be the extensions defined by (7.1) and the standard discrete harmonic extension of  $v_\Gamma$ , respectively. From the definition of  $v_{\mathcal{A},\Gamma}$ , we know that  $a^{(i)}(v_{\mathcal{A},\Gamma}^{(i)}, q^{(i)}) = 0$ , for any  $q^{(i)} \in W^{(i)}$ , which vanishes at the nodes of  $\partial\Omega_i$ . Take  $q^{(i)} = v_{\mathcal{A},\Gamma}^{(i)} - v_{\mathcal{H},\Gamma}^{(i)}$  and we find

$$a^{(i)}(v_{\mathcal{A},\Gamma}^{(i)}, v_{\mathcal{A},\Gamma}^{(i)} - v_{\mathcal{H},\Gamma}^{(i)}) = 0.$$

Therefore, we have

$$\begin{aligned} \|v_{\mathcal{A},\Gamma}^{(i)} - v_{\mathcal{H},\Gamma}^{(i)}\|_{B^{(i)}}^2 &= |a^{(i)}(v_{\mathcal{A},\Gamma}^{(i)} - v_{\mathcal{H},\Gamma}^{(i)}, v_{\mathcal{A},\Gamma}^{(i)} - v_{\mathcal{H},\Gamma}^{(i)})| = |a^{(i)}(v_{\mathcal{H},\Gamma}^{(i)}, v_{\mathcal{A},\Gamma}^{(i)} - v_{\mathcal{H},\Gamma}^{(i)})| \\ &\leq C\|v_{\mathcal{H},\Gamma}^{(i)}\|_{B^{(i)}}\|v_{\mathcal{A},\Gamma}^{(i)} - v_{\mathcal{H},\Gamma}^{(i)}\|_{B^{(i)}}, \end{aligned}$$

where we use Lemma 4.3 in the last step. Canceling the common factor, we have

$$\|v_{\mathcal{A},\Gamma}^{(i)} - v_{\mathcal{H},\Gamma}^{(i)}\|_{B^{(i)}} \leq C\|v_{\mathcal{H},\Gamma}^{(i)}\|_{B^{(i)}}.$$

Therefore,  $\|v_{\mathcal{A},\Gamma}^{(i)}\|_{B^{(i)}} \leq C\|v_{\mathcal{H},\Gamma}^{(i)}\|_{B^{(i)}}$ . From this and using (7.3) and Lemmas 4.7, 7.1, 7.7, and 4.1, noting that  $v_{\mathcal{H},\Gamma}^{(i)}$  vanishes at the coarse level primal degrees of freedom, we have

$$\begin{aligned} \|E_{D,\Gamma}w_\Gamma - w_\Gamma\|_{\widetilde{B}_\Gamma}^2 &= \|v_{\mathcal{A},\Gamma}\|_{\widetilde{B}}^2 = \sum_{i=1}^N \|v_{\mathcal{A},\Gamma}^{(i)}\|_{B^{(i)}}^2 \leq C \sum_{i=1}^N \|v_{\mathcal{H},\Gamma}^{(i)}\|_{B^{(i)}}^2 \\ &\leq C \sum_{i=1}^N |v_{\mathcal{H},\Gamma}^{(i)}|_{H^1(\Omega_i)}^2 \leq C \sum_{i=1}^N |v_\Gamma^{(i)}|_{H^{1/2}(\partial\Omega_i)}^2 \leq C\Phi^2(H, h) \sum_{i=1}^N |w_\Gamma^{(i)}|_{H^{1/2}(\partial\Omega_i)}^2 \\ &\leq C\Phi^2(H, h) \sum_{i=1}^N |w_{\mathcal{H},\Gamma}^{(i)}|_{H^1(\Omega_i)}^2 \leq C\Phi^2(H, h) \sum_{i=1}^N |w_{\mathcal{A},\Gamma}^{(i)}|_{H^1(\Omega_i)}^2 \\ &\leq C\Phi^2(H, h) \sum_{i=1}^N \|w_{\mathcal{A},\Gamma}^{(i)}\|_{B^{(i)}}^2 = C\Phi^2(H, h)\|w_\Gamma\|_{\widetilde{B}_\Gamma}^2. \quad \square \end{aligned}$$

**Lemma 7.9.** *Let  $w_\Gamma = \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma} S_\Gamma u_\Gamma$ , for  $u_\Gamma \in \widehat{W}_\Gamma$ . Then  $\|w_\Gamma\|_{\widetilde{B}_\Gamma}^2 = \langle u_\Gamma, T u_\Gamma \rangle_{S_\Gamma}$ .*

*Proof:* Since  $\widetilde{R}_{D,\Gamma}^T w_\Gamma = \widetilde{R}_{D,\Gamma}^T \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma} S_\Gamma u_\Gamma = T u_\Gamma$ , we have, using Lemma 7.2,

$$\begin{aligned} \|w_\Gamma\|_{\widetilde{B}_\Gamma}^2 &= \langle w_\Gamma, w_\Gamma \rangle_{\widetilde{S}_\Gamma} = w_\Gamma^T \widetilde{S}_\Gamma w_\Gamma = w_\Gamma^T \widetilde{S}_\Gamma \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma} S_\Gamma u_\Gamma \\ &= w_\Gamma^T \widetilde{R}_{D,\Gamma} S_\Gamma u_\Gamma = \langle u_\Gamma, \widetilde{R}_{D,\Gamma}^T w_\Gamma \rangle_{S_\Gamma} = \langle u_\Gamma, T u_\Gamma \rangle_{S_\Gamma}. \quad \square \end{aligned}$$

**Lemma 7.10.** *Let Assumption 4.6 hold. Let  $w_\Gamma = \tilde{S}_\Gamma^{-1} \tilde{R}_{D,\Gamma} S_\Gamma u_\Gamma$ , for  $u_\Gamma \in \widehat{W}_\Gamma$ . There then exists a positive constant  $C$ , which is independent of  $H$  and  $h$ , such that for all  $u_\Gamma \in \widehat{W}_\Gamma$ ,  $\|w_\Gamma\|_{\tilde{B}_\Gamma}^2 \leq C\Phi^2(H, h)\|u_\Gamma\|_{B_\Gamma}^2$ , where  $\Phi(H, h)$  given in Lemma 7.7.*

*Proof:* We have, from Lemma 7.2,

$$\begin{aligned} \langle Tu_\Gamma, Tu_\Gamma \rangle_{B_\Gamma} &= \langle Tu_\Gamma, Tu_\Gamma \rangle_{S_\Gamma} = \langle \tilde{R}_{D,\Gamma}^T \tilde{S}_\Gamma^{-1} \tilde{R}_{D,\Gamma} S_\Gamma u_\Gamma, \tilde{R}_{D,\Gamma}^T \tilde{S}_\Gamma^{-1} \tilde{R}_{D,\Gamma} S_\Gamma u_\Gamma \rangle_{S_\Gamma} \\ &= \langle \tilde{R}_\Gamma \tilde{R}_{D,\Gamma}^T w_\Gamma, \tilde{R}_\Gamma \tilde{R}_{D,\Gamma}^T w_\Gamma \rangle_{\tilde{S}_\Gamma} = \langle E_D w_\Gamma, E_D w_\Gamma \rangle_{\tilde{S}_\Gamma} = \|E_D w_\Gamma\|_{\tilde{B}_\Gamma}^2. \end{aligned}$$

Then, from Lemmas 7.8, 7.9, and 7.3, we have

$$\begin{aligned} \langle Tu_\Gamma, Tu_\Gamma \rangle_{B_\Gamma} &= \|E_D w_\Gamma\|_{\tilde{B}_\Gamma}^2 \leq C\Phi^2(H, h)\|w_\Gamma\|_{\tilde{B}_\Gamma}^2 = C\Phi^2(H, h)\langle u_\Gamma, Tu_\Gamma \rangle_{S_\Gamma} \\ &\leq C\Phi^2(H, h)\|Tu_\Gamma\|_{B_\Gamma}\|u_\Gamma\|_{B_\Gamma}. \end{aligned}$$

Therefore, we have

$$(7.6) \quad \langle Tu_\Gamma, Tu_\Gamma \rangle_{B_\Gamma} \leq C\Phi^4(H, h)\langle u_\Gamma, u_\Gamma \rangle_{B_\Gamma}.$$

Then, using Lemmas 7.9 and 7.3, and (7.6), we have,

$$\|w_\Gamma\|_{\tilde{B}_\Gamma}^2 = \langle u_\Gamma, Tu_\Gamma \rangle_{S_\Gamma} \leq C\|u_\Gamma\|_{B_\Gamma}\|Tu_\Gamma\|_{B_\Gamma} \leq C\Phi^2(H, h)\|u_\Gamma\|_{B_\Gamma}^2. \quad \square$$

**Lemma 7.11.** *Let  $w_\Gamma = \tilde{S}_\Gamma^{-1} \tilde{R}_{D,\Gamma} S_\Gamma u_\Gamma$ , for  $u_\Gamma \in \widehat{W}_\Gamma$ . Then for all  $v \in \tilde{R}(\widehat{W})$ ,  $\langle w_{\mathcal{A},\Gamma}, v \rangle_{\tilde{A}} = \langle \tilde{R}u_{\mathcal{A},\Gamma}, v \rangle_{\tilde{A}}$ , i.e.,  $\langle w_{\mathcal{A},\Gamma} - \tilde{R}u_{\mathcal{A},\Gamma}, v \rangle_{\tilde{A}} = 0$ .*

*Proof:* For any  $v \in \tilde{R}(\widehat{W})$ , denote its continuous interface part by  $v_\Gamma \in \tilde{R}_\Gamma(\widehat{W}_\Gamma)$ . Given  $u_\Gamma \in \widehat{W}_\Gamma$ , from Lemma 7.2 and the fact that  $\tilde{R}_\Gamma \tilde{R}_{D,\Gamma}^T v_\Gamma = v_\Gamma$ , we have

$$\begin{aligned} \langle w_{\mathcal{A},\Gamma}, v \rangle_{\tilde{A}} &= \langle w_\Gamma, v_\Gamma \rangle_{\tilde{S}_\Gamma} = v_\Gamma^T \tilde{S}_\Gamma w_\Gamma = v_\Gamma^T \tilde{S}_\Gamma \tilde{S}_\Gamma^{-1} \tilde{R}_{D,\Gamma} S_\Gamma u_\Gamma = v_\Gamma^T \tilde{R}_{D,\Gamma} \tilde{R}_\Gamma^T \tilde{S}_\Gamma \tilde{R}_\Gamma u_\Gamma \\ &= \langle \tilde{R}_\Gamma u_\Gamma, \tilde{R}_\Gamma \tilde{R}_{D,\Gamma}^T v_\Gamma \rangle_{\tilde{S}_\Gamma} = \langle \tilde{R}_\Gamma u_\Gamma, v_\Gamma \rangle_{\tilde{S}_\Gamma} = v_\Gamma^T \tilde{S}_\Gamma \tilde{R}_\Gamma u_\Gamma = \langle \tilde{R}u_{\mathcal{A},\Gamma}, v \rangle_{\tilde{A}}. \quad \square \end{aligned}$$

**Lemma 7.12.** *Let Assumption 4.6 hold. Let  $w_\Gamma = \tilde{S}_\Gamma^{-1} \tilde{R}_{D,\Gamma} S_\Gamma u_\Gamma$ , for  $u_\Gamma \in \widehat{W}_\Gamma$ . There then exists a positive constant  $C$ , which is independent of  $H$  and  $h$ , such that for all  $u_\Gamma \in \widehat{W}_\Gamma$ ,*

$$\|w_{\mathcal{A},\Gamma} - u_{\mathcal{A},\Gamma}\|_{L^2(\Omega)} \leq CH\mu(H, h)\Phi(H, h)\|u_\Gamma\|_{B_\Gamma},$$

where  $\mu(H, h)$  and  $\Phi(H, h)$  are given in Lemmas 5.1 and 7.7 respectively.

*Proof:* We have, from (5.1) and (5.2), that

$$\begin{aligned} & (w_{\mathcal{A},\Gamma} - u_{\mathcal{A},\Gamma}, g) \\ &= \tilde{a}_o(w_{\mathcal{A},\Gamma}, \tilde{\varphi}_g) - a_o(u_{\mathcal{A},\Gamma}, \varphi_g) - \int_\Omega C(x) L^* w_{\mathcal{A},\Gamma} L^* (\varphi_g - \tilde{\varphi}_g) dx \\ &= \tilde{a}(w_{\mathcal{A},\Gamma}, \tilde{\varphi}_g) - \tilde{a}(u_{\mathcal{A},\Gamma}, \varphi_g) + \int_\Omega C(x) (L u_{\mathcal{A},\Gamma} L \varphi_g - L w_{\mathcal{A},\Gamma} L \tilde{\varphi}_g \\ &\quad - L^* w_{\mathcal{A},\Gamma} L^* (\varphi_g - \tilde{\varphi}_g)) dx \\ &= \tilde{a}(w_{\mathcal{A},\Gamma} - u_{\mathcal{A},\Gamma}, \tilde{\varphi}_g) - \tilde{a}(u_{\mathcal{A},\Gamma}, \varphi_g - \tilde{\varphi}_g) + \int_\Omega C(x) (L(u_{\mathcal{A},\Gamma} - w_{\mathcal{A},\Gamma}) L \varphi_g \\ &\quad + L w_{\mathcal{A},\Gamma} L (\varphi_g - \tilde{\varphi}_g) - L^* w_{\mathcal{A},\Gamma} L^* (\varphi_g - \tilde{\varphi}_g)) dx \\ &\leq \tilde{a}(w_{\mathcal{A},\Gamma} - u_{\mathcal{A},\Gamma}, \tilde{\varphi}_g) - \tilde{a}(u_{\mathcal{A},\Gamma}, \varphi_g - \tilde{\varphi}_g) + Ch\|w_{\mathcal{A},\Gamma} - u_{\mathcal{A},\Gamma}\|_{\tilde{B}}\|\varphi_g\|_{\tilde{B}} \\ &\quad + Ch\|w_{\mathcal{A},\Gamma}\|_{\tilde{B}}\|\varphi_g - \tilde{\varphi}_g\|_{\tilde{B}}, \end{aligned}$$

where we use Lemma 5.4 in the last step.

Let  $\psi$  be any finite element function in the space  $\widehat{W}$ . Then from Lemma 7.11, we know that  $\tilde{a}(w_{\mathcal{A},\Gamma} - u_{\mathcal{A},\Gamma}, \psi) = 0$ . Therefore,

$$\tilde{a}(w_{\mathcal{A},\Gamma} - u_{\mathcal{A},\Gamma}, \tilde{\varphi}_g) - \tilde{a}(u_{\mathcal{A},\Gamma}, \varphi_g - \tilde{\varphi}_g) = \tilde{a}(w_{\mathcal{A},\Gamma} - u_{\mathcal{A},\Gamma}, \tilde{\varphi}_g - \psi) - \tilde{a}(u_{\mathcal{A},\Gamma}, \varphi_g - \tilde{\varphi}_g).$$

Then, using Lemmas 4.3, 4.2, 4.5, and 5.5, and that  $\|\varphi_g\|_{H^2(\Omega)} \leq C\|g\|_{L^2(\Omega)}$ , we have

$$\begin{aligned} & |(w_{\mathcal{A},\Gamma} - u_{\mathcal{A},\Gamma}, g)| \\ & \leq C(\|w_{\mathcal{A},\Gamma} - u_{\mathcal{A},\Gamma}\|_{\tilde{B}} + \|u_{\mathcal{A},\Gamma}\|_{\tilde{B}})(\|\tilde{\varphi}_g - \psi\|_{\tilde{B}} + \|\varphi_g - \tilde{\varphi}_g\|_{\tilde{B}}) \\ & \quad + Ch(\|w_{\mathcal{A},\Gamma} - u_{\mathcal{A},\Gamma}\|_{\tilde{B}}\|\varphi_g\|_{\tilde{B}} + \|w_{\mathcal{A},\Gamma}\|_{\tilde{B}}\|\varphi_g - \tilde{\varphi}_g\|_{\tilde{B}}) \\ & \leq C(\|w_{\mathcal{A},\Gamma} - u_{\mathcal{A},\Gamma}\|_{\tilde{B}} + \|u_{\mathcal{A},\Gamma}\|_{\tilde{B}})(\|\varphi_g - \psi\|_{\tilde{B}} + 2\|\varphi_g - \tilde{\varphi}_g\|_{\tilde{B}}) \\ & \quad + Ch(\|w_{\mathcal{A},\Gamma} - u_{\mathcal{A},\Gamma}\|_{\tilde{B}}\|\varphi_g\|_{H^2(\Omega)} + \|w_{\mathcal{A},\Gamma}\|_{\tilde{B}}\|\varphi_g - \tilde{\varphi}_g\|_{\tilde{B}}) \\ & \leq CH\mu(H, h)(\|w_{\mathcal{A},\Gamma} - u_{\mathcal{A},\Gamma}\|_{\tilde{B}} + \|u_{\mathcal{A},\Gamma}\|_{\tilde{B}} + \|w_{\mathcal{A},\Gamma}\|_{\tilde{B}})\|g\|_{L^2(\Omega)}. \end{aligned}$$

Therefore, using Lemmas 7.2 and 7.10, we have

$$\begin{aligned} \|w_{\mathcal{A},\Gamma} - u_{\mathcal{A},\Gamma}\|_{L^2(\Omega)} &= \sup_{g \in L^2(\Omega)} \frac{|(w_{\mathcal{A},\Gamma} - u_{\mathcal{A},\Gamma}, g)|}{\|g\|_{L^2(\Omega)}} \\ &\leq CH\mu(H, h)(\|w_{\mathcal{A},\Gamma} - u_{\mathcal{A},\Gamma}\|_{\tilde{B}} + \|u_{\mathcal{A},\Gamma}\|_{\tilde{B}} + \|w_{\mathcal{A},\Gamma}\|_{\tilde{B}}) \\ &= CH\mu(H, h)(\|w_{\Gamma} - u_{\Gamma}\|_{\tilde{B}_{\Gamma}} + \|u_{\Gamma}\|_{\tilde{B}_{\Gamma}} + \|w_{\Gamma}\|_{\tilde{B}_{\Gamma}}) \leq CH\mu(H, h)\Phi(H, h)\|u_{\Gamma}\|_{B_{\Gamma}}. \end{aligned}$$

□

**Lemma 7.13.** *Let Assumption 4.6 hold and let  $v_{\Gamma} = \tilde{R}_{D,\Gamma}^T w_{\Gamma}$ , for  $w_{\Gamma} \in \widetilde{W}_{\Gamma}$ . There then exists a positive constant  $C$ , independent of  $H$  and  $h$ , such that for all  $w_{\Gamma} \in \widetilde{W}_{\Gamma}$ ,*

$$\|v_{\mathcal{A},\Gamma}\|_{L^2(\Omega)}^2 \leq C(H/h)^2\Phi^2(H, h)\|w_{\mathcal{A},\Gamma}\|_{L^2(\Omega)}^2,$$

where  $\Phi(H, h)$  is given in Lemma 7.7.

*Proof:* We only need to show that

$$\|v_{\mathcal{A},\Gamma} - w_{\mathcal{A},\Gamma}\|_{L^2(\Omega)}^2 \leq C(H/h)^2\Phi^2(H, h)\|w_{\mathcal{A},\Gamma}\|_{L^2(\Omega)}^2.$$

From Assumption 4.6, we know for any  $w_{\Gamma} \in \widetilde{W}_{\Gamma}$ ,  $v_{\mathcal{A},\Gamma} - w_{\mathcal{A},\Gamma}$  has zero averages over the subdomain interfaces. Using a Poincaré-Friedrichs inequality, Lemmas 4.1, 7.2, 7.8, and 4.7, and an inverse inequality, we have

$$\begin{aligned} \|v_{\mathcal{A},\Gamma} - w_{\mathcal{A},\Gamma}\|_{L^2(\Omega)}^2 &\leq CH^2\|v_{\mathcal{A},\Gamma} - w_{\mathcal{A},\Gamma}\|_{H^1(\Omega)}^2 \leq CH^2\|v_{\mathcal{A},\Gamma} - w_{\mathcal{A},\Gamma}\|_{\tilde{B}}^2 \\ &= CH^2\|v_{\Gamma} - w_{\Gamma}\|_{\tilde{B}_{\Gamma}}^2 \leq CH^2\Phi^2(H, h)\|w_{\Gamma}\|_{\tilde{B}_{\Gamma}}^2 = CH^2\Phi^2(H, h)\|w_{\mathcal{A},\Gamma}\|_{\tilde{B}}^2 \\ &\leq CH^2\Phi^2(H, h)\|w_{\mathcal{A},\Gamma}\|_{H^1(\Omega)}^2 \leq C(H/h)^2\Phi^2(H, h)\|w_{\mathcal{A},\Gamma}\|_{L^2(\Omega)}^2. \quad \square \end{aligned}$$

**Lemma 7.14.** *Let Assumption 4.6 hold and let  $v_{\Gamma} = Tu_{\Gamma} - u_{\Gamma}$ , for  $u_{\Gamma} \in \widehat{W}_{\Gamma}$ . There then exists a positive constant  $C$  independent of  $H$  and  $h$ , such that for all  $u_{\Gamma} \in \widehat{W}_{\Gamma}$ ,*

$$\|v_{\mathcal{A},\Gamma}\|_{L^2(\Omega)} \leq CH\frac{H}{h}\mu(H, h)\Phi^2(H, h)\|u_{\Gamma}\|_{B_{\Gamma}},$$

where  $\mu(H, h)$  and  $\Phi(H, h)$  are given in Lemmas 5.1 and 7.7, respectively.

*Proof:* Since  $Tu_\Gamma = \tilde{R}_{D,\Gamma}^T w_\Gamma$  and  $\tilde{R}_{D,\Gamma}^T \tilde{R}_\Gamma = I$ , we have  $v_\Gamma = Tu_\Gamma - u_\Gamma = \tilde{R}_{D,\Gamma}^T w_\Gamma - \tilde{R}_{D,\Gamma}^T \tilde{R}_\Gamma u_\Gamma = \tilde{R}_{D,\Gamma}^T (w_\Gamma - \tilde{R}_\Gamma u_\Gamma)$ . Using Lemmas 7.13 and 7.12, we have

$$\|v_{\mathcal{A},\Gamma}\|_{L^2(\Omega)} \leq C \frac{H}{h} \Phi(H, h) \|w_{\mathcal{A},\Gamma} - u_{\mathcal{A},\Gamma}\|_{L^2(\Omega)} \leq CH \frac{H}{h} \mu(H, h) \Phi(H, h)^2 \|u_\Gamma\|_{B_\Gamma}.$$

□

**Theorem 7.15.** *Let Assumption 4.6 hold. There then exists a positive constant  $C$ , which is independent of  $H$  and  $h$ , such that for all  $u_\Gamma \in \tilde{W}_\Gamma$ ,*

$$(7.7) \quad \langle Tu_\Gamma, Tu_\Gamma \rangle_{B_\Gamma} \leq C \Phi^4(H, h) \langle u_\Gamma, u_\Gamma \rangle_{B_\Gamma},$$

and

$$(7.8) \quad c_0 \langle u_\Gamma, u_\Gamma \rangle_{B_\Gamma} \leq \langle u_\Gamma, Tu_\Gamma \rangle_{B_\Gamma}.$$

where

$$(7.9) \quad c_0 = 1 - CH \frac{H}{h} \mu(H, h) \Phi^2(H, h),$$

$\mu(H, h)$  and  $\Phi(H, h)$  are given in Lemmas 5.1 and 7.7 respectively.

*Proof:* The upper bound (7.7) is the same as (7.6).

To prove the lower bound (7.8), we have, from  $\tilde{R}_\Gamma^T \tilde{R}_{D,\Gamma} = I$  and Lemmas 7.2 and 7.3, that

$$\begin{aligned} \langle u_\Gamma, u_\Gamma \rangle_{B_\Gamma} &= \langle u_\Gamma, u_\Gamma \rangle_{S_\Gamma} = u_\Gamma^T \tilde{R}_\Gamma^T \tilde{S}_\Gamma \tilde{S}_\Gamma^{-1} \tilde{R}_{D,\Gamma} S_\Gamma u_\Gamma = \left\langle w_\Gamma, \tilde{R}_\Gamma u_\Gamma \right\rangle_{\tilde{S}_\Gamma} \\ &\leq C \|w_\Gamma\|_{\tilde{B}_\Gamma} \|u_\Gamma\|_{B_\Gamma} = C \langle u_\Gamma, Tu_\Gamma \rangle_{S_\Gamma}^{1/2} \|u_\Gamma\|_{B_\Gamma}. \end{aligned}$$

Here we use Lemmas 7.9 in the last step. Canceling the common factor, we have

$$(7.10) \quad \|u_\Gamma\|_{B_\Gamma}^2 \leq C \langle u_\Gamma, Tu_\Gamma \rangle_{S_\Gamma}.$$

Let  $v_\Gamma = Tu_\Gamma - u_\Gamma$ . We have, from (7.10), Lemmas 7.2, 7.4 and 7.14,

$$\begin{aligned} \langle u_\Gamma, u_\Gamma \rangle_{B_\Gamma} &\leq C \langle u_\Gamma, Tu_\Gamma \rangle_{S_\Gamma} \leq C (\langle u_\Gamma, Tu_\Gamma \rangle_{B_\Gamma} + \langle u_\Gamma, Tu_\Gamma - u_\Gamma \rangle_{Z_\Gamma}) \\ &\leq C \langle u_\Gamma, Tu_\Gamma \rangle_{B_\Gamma} + C \|u_\Gamma\|_{B_\Gamma} \|v_{\mathcal{A},\Gamma}\|_{L^2(\Omega)} \\ &\leq C \langle u_\Gamma, Tu_\Gamma \rangle_{B_\Gamma} + C \left( H \frac{H}{h} \mu(H, h) \Phi^2(H, h) \right) \langle u_\Gamma, u_\Gamma \rangle_{B_\Gamma}, \end{aligned}$$

The second term on the right hand side can be combined with the left hand side and (7.8) is proved. □

**Remark 7.16.** *From the forms of  $\mu(H, h)$  and  $\Phi(H, h)$ , we know that for fixed  $H/h$ , if  $H$  is sufficiently small then  $c_0$  will become positive and be bounded from zero independently of  $H$ . Hence from Theorem 7.5, the convergence rate of the GMRES iteration for (6.3) becomes bounded independently of the number of subdomains. However, the constant  $C$  in the formula of  $c_0$  in (7.9) depends on the coefficients of the partial differential equation (2.1); for very small viscosity value  $\nu$ , it may become impractical for the subdomain diameter  $H$  to satisfy  $c_0 > 0$ .*

## 8. NUMERICAL EXPERIMENTS

We test our BDDC algorithm by solving three advection-diffusion examples in the square domain  $\Omega = [-1, 1]^2$ . These examples were used by Toselli [34] for testing his FETI algorithms.

The domain  $\Omega$  is decomposed into square subdomains and each subdomain into uniform triangles. Piecewise linear finite elements are used in our experiments. The stabilization function  $C(x)$  in (3.3) is defined in (3.1). We also take  $f = 0$  and  $c = 10^{-4}$  in (2.1) in our examples.

A GMRES iteration with the  $L^2$ -norm is used without restart to solve the preconditioned interface problem (6.3). The iteration is stopped when the  $L^2$ -norm of the residual has been reduced by a factor of  $10^{-6}$ ; we have found consistently that the convergence rate using the  $B_\Gamma$ -norm is quite similar to that using the  $L^2$ -norm.

We test two different sets of coarse level primal continuity constraints in our BDDC algorithms. In BDDC-1, only subdomain vertex and edge average continuity constraints are included in the coarse level primal subspace; in BDDC-2, as in Assumption 4.6, two additional edge flux average constraints for each edge are also included in the coarse level variable space. We also compare the performance of our BDDC algorithms with that of the one-level and two-level Robin-Robin algorithms which were developed in [3, 1, 2]. They are denoted by RR-1 and RR-2 in our tables. We do not present numerical results for the one-level and two-level FETI algorithms here; their performances are in fact similar to the Robin-Robin algorithms, cf. [34].

**8.1. Thermal boundary layer (Test Problem I).** We first consider a thermal boundary layer problem. The velocity field  $\mathbf{a}$  in (2.1) is defined by  $\mathbf{a} = (\frac{1+y}{2}, 0)$ . The boundary condition is given by:

$$\begin{aligned} u &= 1, & \begin{cases} x = -1 & -1 < y \leq 1, \\ y = 1, & -1 \leq x \leq 1, \end{cases} \\ u &= 0, & y = -1, \quad -1 \leq x \leq 1, \\ u &= \frac{1+y}{2}, & x = 1, \quad -1 \leq y \leq 1. \end{aligned}$$

In our first set of experiments, reported in Table 1, we fix the subdomain problem size and change the number of subdomains. We see that, for viscosity values  $\nu$  larger than  $10^{-4}$ , the iteration counts of BDDC-2 do not change with an increase of the number of subdomains and that it converges faster than BDDC-1. We believe that for that range of viscosity values, the subdomain diameters in our experiments satisfy that  $c_0$  is positive in Theorem 7.15; cf. Remark 7.16. For smaller viscosity, the improvement in the convergence rate of BDDC-2 over BDDC-1 is no longer clear in this example. In fact  $c_0$  may no longer be positive. We also see from Table 1 that RR-1 and RR-2 converge slower than the BDDC algorithms, and that their iteration counts are more sensitive to an increase of the number of subdomains.

In our second set of experiments, in Table 2, we fix the number of subdomains and change the local subdomain problem size. We see that for all four algorithms, the iteration counts are not sensitive to an increase of the subdomain problem size especially with small viscosity, and that BDDC-2 converges the fastest.

We can also see from Tables 1 and 2 that the iteration counts of all the algorithms are bounded when  $\nu$  goes to zero.

**8.2. Variable flow field (Test Problem II).** We next consider a more complicated flow. The velocity field is  $\mathbf{a} = \frac{1}{2}((1-x^2)(1+y), -(4-(1+y)^2))$ . The

boundary condition is given by:  $u = 1$ , for  $y = -1$  and  $-1 < x < 0$ , with  $u = 0$ , elsewhere on the boundary of  $\Omega$ .

Table 3 gives the iteration counts of the four algorithms, for different number of subdomains with a fixed subdomain problem size. We have similar findings as for the first example in Table 1. We see that BDDC-2 scales well with respect to an increase of the number of subdomains for viscosity values larger than  $10^{-4}$ , and that it converges the fastest among the four algorithms. The improvement in the convergence rate of BDDC-2 over BDDC-1 is obvious, especially when  $\nu > 10^{-5}$ .

In Table 4, we can see that the iteration counts of each algorithm are insensitive to an increase of the subdomain problem size, and they are bounded when  $\nu$  goes to zero.

**8.3. Rotating flow field (Test Problem III).** This example is the most difficult one of the three examples, cf. [34]. Here the velocity field is  $\mathbf{a} = (y, -x)$ . The boundary condition is given by:

$$u = 1, \quad \text{for} \quad \begin{cases} y = -1 & 0 < x \leq 1, \\ y = 1, & 0 < x \leq 1, \\ x = 1, & -1 \leq y \leq 1, \end{cases} \quad \text{with} \quad u = 0, \quad \text{elsewhere on } \partial\Omega.$$

Table 5 gives the iteration counts of the four algorithms for different number of subdomains with a fixed subdomain problem size. We see that BDDC-2 converges much faster than BDDC-1 and the two Robin-Robin algorithms. For the cases where  $\nu > 10^{-5}$ , the iteration counts are almost independent of the number of subdomains. Even when the viscosity  $\nu$  goes to zero, the convergence of BDDC-2 is still very fast, while the convergence rates of BDDC-1 and the two Robin-Robin algorithms are not satisfactory at all.

From Table 6, we see that the iteration counts of all the algorithms increase with an increase of the subdomain problem size; the increase for BDDC-2 is the smallest.

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## REFERENCES

- [1] Yves Achdou, Caroline Japhet, Patric Le Tallec, Frédéric Nataf, François Rogier, and Marina Vidrascu. Domain decomposition methods for non-symmetric problems. In Choi-Hong Lai, Petter E. Bjørstad, Mark Cross, and Olof B. Widlund, editors, *Domain Decomposition Methods in Sciences and Engineering: Eleventh International Conference London, UK*, pages 3–17. DDM.org, 1999. Greenwich, England, July 20–24, 1998.
- [2] Yves Achdou, Patrick Le Tallec, Frédéric Nataf, and Marina Vidrascu. A domain decomposition preconditioner for an advection-diffusion problem. *Comput. Methods Appl. Mech. Engrg.*, 184:145–170, 2000.
- [3] Yves Achdou and Frédéric Nataf. A Robin-Robin preconditioner for an advection-diffusion problem. *C. R. Acad. Sci. Paris*, 325, Série I:1211–1216, 1997.
- [4] Susanne C. Brenner. Poincaré-Friedrichs inequalities for piecewise  $H^1$  functions. *SIAM J. Numer. Anal.*, 41(1):306–324, 2003.
- [5] Susanne C. Brenner and Li-Yeng Sung. BDDC and FETI-DP without matrices or vectors. *Comput. Methods Appl. Mech. Engrg.*, 196(8):1429–1435, 2007.
- [6] Xiao-Chuan Cai. Additive Schwarz algorithms for parabolic convection-diffusion equations. *Numer. Math.*, 60(1):41–61, 1991.
- [7] Xiao-Chuan Cai and Olof Widlund. Domain decomposition algorithms for indefinite elliptic problems. *SIAM J. Sci. Statist. Comput.*, 13(1):243–258, January 1992.

TABLE 1. Iteration counts for changing number of subdomains and  $H/h = 6$  for Test Problem I

| # of Sub. | $4^2$  | $8^2$ | $16^2$ | $32^2$ | $4^2$  | $8^2$ | $16^2$ | $32^2$ |
|-----------|--------|-------|--------|--------|--------|-------|--------|--------|
| $\nu$     | BDDC-1 |       |        |        | BDDC-2 |       |        |        |
| $1e0$     | 3      | 3     | 3      | 3      | 3      | 3     | 3      | 3      |
| $1e-1$    | 5      | 4     | 4      | 4      | 4      | 4     | 4      | 4      |
| $1e-2$    | 6      | 7     | 8      | 7      | 4      | 5     | 5      | 5      |
| $1e-3$    | 5      | 8     | 12     | 18     | 5      | 6     | 7      | 6      |
| $1e-4$    | 5      | 8     | 12     | 20     | 5      | 7     | 11     | 17     |
| $1e-5$    | 5      | 8     | 12     | 21     | 5      | 8     | 12     | 20     |
| $1e-6$    | 5      | 8     | 13     | 21     | 5      | 8     | 12     | 21     |
| $\nu$     | RR-1   |       |        |        | RR-2   |       |        |        |
| $1e0$     | 13     | 45    | 176    | >500   | 6      | 7     | 7      | 6      |
| $1e-1$    | 13     | 36    | 115    | 343    | 9      | 11    | 12     | 12     |
| $1e-2$    | 10     | 18    | 45     | 156    | 8      | 13    | 18     | 23     |
| $1e-3$    | 10     | 14    | 24     | 51     | 9      | 13    | 20     | 30     |
| $1e-4$    | 11     | 16    | 25     | 40     | 10     | 16    | 25     | 41     |
| $1e-5$    | 11     | 17    | 26     | 45     | 11     | 17    | 27     | 46     |
| $1e-6$    | 11     | 17    | 27     | 45     | 11     | 17    | 28     | 46     |

TABLE 2. Iteration counts for  $4 \times 4$  subdomains and changing subdomain problem size for Test Problem I

| $H/h$  | 6      | 12 | 24 | 48 | 6      | 12 | 24 | 48 |
|--------|--------|----|----|----|--------|----|----|----|
| $\nu$  | BDDC-1 |    |    |    | BDDC-2 |    |    |    |
| $1e0$  | 3      | 4  | 5  | 5  | 3      | 4  | 5  | 5  |
| $1e-1$ | 5      | 6  | 6  | 7  | 4      | 5  | 5  | 6  |
| $1e-2$ | 6      | 7  | 8  | 9  | 4      | 5  | 5  | 6  |
| $1e-3$ | 5      | 6  | 7  | 7  | 5      | 5  | 6  | 6  |
| $1e-4$ | 5      | 5  | 6  | 7  | 5      | 5  | 5  | 6  |
| $1e-5$ | 5      | 5  | 5  | 5  | 5      | 4  | 4  | 4  |
| $1e-6$ | 5      | 4  | 4  | 4  | 5      | 4  | 4  | 4  |
| $\nu$  | RR-1   |    |    |    | RR-2   |    |    |    |
| $1e0$  | 13     | 14 | 16 | 16 | 6      | 8  | 10 | 12 |
| $1e-1$ | 13     | 15 | 16 | 18 | 9      | 11 | 13 | 16 |
| $1e-2$ | 10     | 11 | 13 | 15 | 8      | 10 | 11 | 14 |
| $1e-3$ | 10     | 9  | 9  | 10 | 9      | 9  | 9  | 10 |
| $1e-4$ | 11     | 10 | 10 | 10 | 10     | 10 | 10 | 10 |
| $1e-5$ | 11     | 11 | 11 | 11 | 11     | 10 | 11 | 11 |
| $1e-6$ | 11     | 11 | 11 | 11 | 11     | 10 | 11 | 11 |

- [8] Xiao-Chuan Cai and Olof Widlund. Multiplicative Schwarz algorithms for some nonsymmetric and indefinite problems. *SIAM J. Numer. Anal.*, 30(4):936–952, August 1993.
- [9] Claudio Carlenzoli and Alfio Quarteroni. Adaptive domain decomposition methods for advection-diffusion problems. *The IMA Volumes in Mathematics and its Applications*, Springer Verlag, 75:165–186, 1995.

TABLE 3. Iteration counts for changing number of subdomains and  $H/h = 6$  for Test Problem II

| # of Sub. | $4^2$  | $8^2$ | $16^2$ | $32^2$ | $4^2$  | $8^2$ | $16^2$ | $32^2$ |
|-----------|--------|-------|--------|--------|--------|-------|--------|--------|
| $\nu$     | BDDC-1 |       |        |        | BDDC-2 |       |        |        |
| 1e0       | 4      | 4     | 4      | 3      | 2      | 2     | 1      | 1      |
| 1e-1      | 5      | 5     | 5      | 4      | 2      | 2     | 2      | 2      |
| 1e-2      | 6      | 9     | 9      | 8      | 4      | 3     | 3      | 3      |
| 1e-3      | 8      | 12    | 18     | 23     | 6      | 8     | 8      | 7      |
| 1e-4      | 9      | 14    | 25     | 42     | 7      | 11    | 19     | 23     |
| 1e-5      | 9      | 15    | 27     | 50     | 7      | 11    | 22     | 42     |
| 1e-6      | 9      | 15    | 27     | 51     | 7      | 11    | 22     | 45     |
| $\nu$     | RR-1   |       |        |        | RR-2   |       |        |        |
| 1e0       | 13     | 45    | 152    | 390    | 6      | 7     | 7      | 7      |
| 1e-1      | 15     | 32    | 81     | 216    | 9      | 12    | 14     | 14     |
| 1e-2      | 10     | 19    | 41     | 106    | 9      | 14    | 19     | 29     |
| 1e-3      | 13     | 21    | 34     | 64     | 12     | 19    | 29     | 43     |
| 1e-4      | 14     | 27    | 52     | 84     | 14     | 26    | 50     | 76     |
| 1e-5      | 14     | 29    | 63     | 135    | 14     | 28    | 60     | 128    |
| 1e-6      | 14     | 29    | 67     | 156    | 14     | 28    | 64     | 151    |

TABLE 4. Iteration counts for  $4 \times 4$  subdomains and changing subdomain problem size for Test Problem II

| $H/h$ | 6      | 12 | 24 | 48 | 6      | 12 | 24 | 48 |
|-------|--------|----|----|----|--------|----|----|----|
| $\nu$ | BDDC-1 |    |    |    | BDDC-2 |    |    |    |
| 1e0   | 4      | 4  | 5  | 6  | 2      | 1  | 1  | 1  |
| 1e-1  | 5      | 6  | 7  | 8  | 2      | 2  | 2  | 2  |
| 1e-2  | 6      | 7  | 8  | 9  | 4      | 4  | 4  | 4  |
| 1e-3  | 8      | 8  | 8  | 8  | 6      | 6  | 6  | 6  |
| 1e-4  | 9      | 9  | 9  | 9  | 7      | 7  | 7  | 7  |
| 1e-5  | 9      | 9  | 9  | 9  | 7      | 8  | 7  | 7  |
| 1e-6  | 9      | 9  | 9  | 9  | 7      | 8  | 7  | 7  |
| $\nu$ | RR-1   |    |    |    | RR-2   |    |    |    |
| 1e0   | 13     | 15 | 16 | 17 | 6      | 9  | 10 | 11 |
| 1e-1  | 15     | 17 | 18 | 19 | 9      | 11 | 14 | 16 |
| 1e-2  | 10     | 11 | 13 | 14 | 9      | 10 | 11 | 13 |
| 1e-3  | 13     | 12 | 11 | 10 | 12     | 12 | 11 | 10 |
| 1e-4  | 14     | 14 | 13 | 13 | 14     | 14 | 13 | 13 |
| 1e-5  | 14     | 14 | 14 | 14 | 14     | 14 | 14 | 14 |
| 1e-6  | 14     | 14 | 14 | 14 | 14     | 14 | 14 | 14 |

- [10] Marie-Claude Ciccoli. Adaptive domain decomposition algorithms and finite volume/finite element approximation for advection-diffusion equations. *J. Sci. Comput.*, 11(4):299–341, 1996.
- [11] Duilio Conceição. Balancing domain decomposition preconditioners for non-symmetric problems. Technical Report Serie C 46, Instituto de Matemática Pura e Aplicada, Rio de Janeiro, Brazil, May 2006.

TABLE 5. Iteration counts for changing number of subdomains and  $H/h = 6$  for Test Problem III

| # of Sub. | $4^2$  | $8^2$ | $16^2$ | $32^2$ | $4^2$  | $8^2$ | $16^2$ | $32^2$ |
|-----------|--------|-------|--------|--------|--------|-------|--------|--------|
| $\nu$     | BDDC-1 |       |        |        | BDDC-2 |       |        |        |
| 1e0       | 4      | 3     | 3      | 3      | 2      | 2     | 1      | 1      |
| 1e-1      | 5      | 5     | 4      | 4      | 2      | 2     | 2      | 2      |
| 1e-2      | 9      | 9     | 7      | 6      | 4      | 3     | 3      | 3      |
| 1e-3      | 25     | 33    | 30     | 22     | 8      | 7     | 6      | 5      |
| 1e-4      | 38     | 67    | 111    | 112    | 11     | 12    | 14     | 14     |
| 1e-5      | 41     | 84    | 183    | 284    | 12     | 14    | 17     | 24     |
| 1e-6      | 41     | 86    | 199    | 434    | 12     | 14    | 18     | 26     |
| $\nu$     | RR-1   |       |        |        | RR-2   |       |        |        |
| 1e0       | 13     | 45    | 148    | 379    | 6      | 7     | 7      | 7      |
| 1e-1      | 17     | 37    | 92     | 233    | 9      | 11    | 12     | 12     |
| 1e-2      | 28     | 50    | 95     | 218    | 15     | 22    | 31     | 49     |
| 1e-3      | 52     | 94    | 170    | 319    | 34     | 59    | 87     | 96     |
| 1e-4      | 68     | 171   | 360    | >500   | 49     | 114   | 251    | 475    |
| 1e-5      | 71     | 201   | >500   | >500   | 52     | 141   | 359    | >500   |
| 1e-6      | 71     | 205   | >500   | >500   | 53     | 145   | 389    | >500   |

TABLE 6. Iteration counts for  $4 \times 4$  subdomains and changing subdomain problem size for Test Problem III

| $H/h$ | 6      | 12  | 24  | 48  | 6      | 12  | 24  | 48  |
|-------|--------|-----|-----|-----|--------|-----|-----|-----|
| $\nu$ | BDDC-1 |     |     |     | BDDC-2 |     |     |     |
| 1e0   | 4      | 4   | 5   | 6   | 2      | 2   | 1   | 1   |
| 1e-1  | 5      | 6   | 7   | 7   | 2      | 2   | 2   | 2   |
| 1e-2  | 9      | 11  | 12  | 13  | 4      | 4   | 4   | 4   |
| 1e-3  | 25     | 35  | 39  | 41  | 8      | 12  | 14  | 14  |
| 1e-4  | 38     | 72  | 104 | 122 | 11     | 26  | 39  | 45  |
| 1e-5  | 41     | 85  | 156 | 245 | 12     | 33  | 74  | 96  |
| 1e-6  | 41     | 87  | 165 | 290 | 12     | 34  | 88  | 142 |
| $\nu$ | RR-1   |     |     |     | RR-2   |     |     |     |
| 1e0   | 13     | 14  | 15  | 16  | 6      | 8   | 10  | 12  |
| 1e-1  | 17     | 17  | 19  | 21  | 9      | 12  | 14  | 16  |
| 1e-2  | 28     | 30  | 31  | 30  | 15     | 17  | 18  | 20  |
| 1e-3  | 52     | 62  | 68  | 70  | 34     | 43  | 47  | 49  |
| 1e-4  | 68     | 104 | 141 | 166 | 49     | 83  | 110 | 128 |
| 1e-5  | 71     | 119 | 189 | 279 | 52     | 98  | 171 | 250 |
| 1e-6  | 71     | 121 | 197 | 318 | 53     | 100 | 180 | 296 |

- [12] Jean-Michel Cros. A preconditioner for the Schur complement domain decomposition method. In *Domain decomposition methods in science and engineering, Proceedings of the 14th International Conference on Domain Decomposition Methods*, pages 373–380. National Autonomous University of Mexico, 2003.
- [13] Clark R. Dohrmann. A preconditioner for substructuring based on constrained energy minimization. *SIAM J. Sci Comput.*, 25(1):246–258, 2003.

- [14] Clark R. Dohrmann. A substructuring preconditioner for nearly incompressible elasticity problems. Technical Report SAND2004-5393, Sandia National Laboratories, Albuquerque, New Mexico, October 2004.
- [15] Stanley C. Eisenstat, Howard C. Elman, and Martin H. Schultz. Variational iterative methods for nonsymmetric systems of linear equations. *SIAM J. Numer. Anal.*, 20 (2):345–357, 1983.
- [16] Charbel Farhat, Michel Lesoinne, Patrick Le Tallec, Kendall Pierson, and Daniel Rixen. FETI-DP: A dual-primal unified FETI method – part I: A faster alternative to the two-level FETI method. *Internat. J. Numer. Methods Engrg.*, 50:1523–1544, 2001.
- [17] Charbel Farhat and Jing Li. An iterative domain decomposition method for the solution of a class of indefinite problems in computational structural dynamics. *Appl. Numer. Math.*, 54:150–166, 2005.
- [18] Yannis Fragakis and Manolis Papadrakakis. The mosaic of high performance domain decomposition methods for structural mechanics: Formulation, interrelation and numerical efficiency of primal and dual methods. *Comput. Methods Appl. Mech. Engrg.*, 192(35–36):3799–3830, 2003.
- [19] Fabio Gastaldi, Lucia Gastaldi, and Alfio Quarteroni. Adaptive domain decomposition methods for advection dominated equations. *East-West J. Numer. Math.*, 4(3):165–206, 1996.
- [20] Jayadeep Gopalakrishnan and Joseph E. Pasciak. Overlapping Schwarz preconditioners for indefinite time harmonic Maxwell equations. *Math. Comp.*, 72(241):1–15, 2003.
- [21] Wolfgang Hackbusch. *Elliptic Differential Equations. Theory and Numerical Treatment*. Springer-Verlag, Berlin, 1992.
- [22] Thomas J.R. Hughes, Leopoldo P. Franca, and Gregory M. Hulbert. A new finite element formulation for computational fluid dynamics. VIII. The Galerkin/least-squares method for advective-diffusive equations. *Comput. Methods Appl. Mech. Engrg.*, 73(2):173–189, 1989.
- [23] Axel Klawonn and Olof B. Widlund. Dual-primal FETI methods for linear elasticity. *Comm. Pure Appl. Math.*, 59(11):1523–1572, 2006.
- [24] Axel Klawonn, Olof B. Widlund, and Maksymilian Dryja. Dual-primal FETI methods for three-dimensional elliptic problems with heterogeneous coefficients. *SIAM J. Numer. Anal.*, 40(1):159–179, April 2002.
- [25] Jing Li and Xuemin Tu. Convergence analysis of a balancing domain decomposition method for solving interior Helmholtz equations. Technical Report LBNL-62618, Lawrence Berkeley National Laboratory, May 2007.
- [26] Jing Li and Olof B. Widlund. BDDC algorithms for incompressible Stokes equations. *SIAM J. Numer. Anal.*, 44(6):2432–2455, 2006.
- [27] Jing Li and Olof B. Widlund. FETI-DP, BDDC, and Block Cholesky Methods. *Internat. J. Numer. Methods Engrg.*, 66:250–271, 2006.
- [28] Jing Li and Olof B. Widlund. On the use of inexact subdomain solvers for BDDC algorithms. *Comput. Methods Appl. Mech. Engrg.*, 196(8):1415–1428, 2007.
- [29] Jan Mandel and Clark R. Dohrmann. Convergence of a balancing domain decomposition by constraints and energy minimization. *Numer. Linear Algebra Appl.*, 10(7):639–659, 2003.
- [30] Jan Mandel, Clark R. Dohrmann, and Radek Tezaur. An algebraic theory for primal and dual substructuring methods by constraints. *Appl. Numer. Math.*, 54(2):167–193, 2005.
- [31] Frédéric Nataf and François Rogier. Factorization of the convection-diffusion operator and the Schwarz algorithm. *Math. Models Methods Appl. Sci.*, 5(1):67–93, 1995.
- [32] Alfio Quarteroni and Alberto Valli. *Domain Decomposition Methods for Partial Differential Equations*. Oxford Science Publications, 1999.
- [33] Marcus Sarkis and Daniel B. Szyld. Optimal left and right additive Schwarz preconditioning for minimal residual methods with Euclidean and energy norms. *Comput. Methods Appl. Mech. Engrg.*, 196:1507–1514, 2007.
- [34] Andrea Toselli. FETI domain decomposition methods for scalar advection-diffusion problems. *Comput. Methods Appl. Mech. Engrg.*, 190(43-44):5759–5776, 2001.
- [35] Andrea Toselli and Olof B. Widlund. *Domain Decomposition Methods - Algorithms and Theory*, volume 34 of *Springer Series in Computational Mathematics*. Springer Verlag, Berlin-Heidelberg-New York, 2005.
- [36] R. Loredana Trotta. Multidomain finite elements for advection-diffusion equations. *Appl. Numer. Math.*, 21(1):91–118, 1996.
- [37] Xuemin Tu. A BDDC algorithm for a mixed formulation of flows in porous media. *Electron. Trans. Numer. Anal.*, 20:164–179, 2005.

- [38] Xuemin Tu. *BDDC Domain Decomposition Algorithms: Methods with Three Levels and for Flow in Porous Media*. PhD thesis, Courant Institute, New York University, January 2006. TR2005-879, Department of Computer Science, Courant Institute.
- [39] Xuemin Tu. A BDDC algorithm for flow in porous media with a hybrid finite element discretization. *Electron. Trans. Numer. Anal.*, 26:146–160, 2007.
- [40] Panayot S. Vassilevski. Preconditioning nonsymmetric and indefinite finite element matrices. *J. Numer. Linear Algebra Appl.*, 1(1):59–76, 1992.
- [41] Jinchao Xu. *Theory of Multilevel Methods*. PhD thesis, Cornell University, May 1989.
- [42] Jinchao Xu. A new class of iterative methods for nonselfadjoint or indefinite problems. *SIAM J. Numer. Anal.*, 29(2):303–319, 1992.

XUEMIN TU, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, AND LAWRENCE BERKELEY NATIONAL LABORATORY, BERKELEY, CA 94720-3840  
*E-mail address:* [xuemin@math.berkeley.edu](mailto:xuemin@math.berkeley.edu), <http://math.berkeley.edu/~xuemin/>

JING LI, DEPARTMENT OF MATHEMATICAL SCIENCES, KENT STATE UNIVERSITY, KENT, OH 44242  
*E-mail address:* [li@math.kent.edu](mailto:li@math.kent.edu), <http://www.math.kent.edu/~li/>