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Monopole Condensation and Confining Phase of N = 1 Gauge Theories Via M Theory Fivebrane

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Abstract

The fivebrane of M theory is used in order to study the moduli space of vacua of confining phase $N = 1$ supersymmetric gauge theories in four dimensions. The supersymmetric vacua correspond to the condensation of massless monopoles and confinement of photons. The monopole and meson vacuum expectation values are computed using the fivebrane configuration. The comparison of the fivebrane computation and the field theory analysis shows that at vacua with a classically enhanced gauge group $SU(r)$ the effective superpotential obtained by the "integrating in" method is exact for $r = 2$ but is not exact for $r > 2$. The fivebrane configuration corresponding to $N = 1$ gauge theories with Landau-Ginzburg type superpotentials is studied. $N = 1$ non-trivial fixed points are analyzed using the brane geometry.

1 Introduction

In the last couple of years we have learnt how string theory, M theory and F theory can be used in order to study the non perturbative dynamics of low energy supersymmetric gauge theories in various dimensions. The two main two techniques applied in these studies are geometric engineering [1–3] and brane dynamics. In the first approach one typically compactifies string theory, M theory or F theory on a singular Calabi-Yau d-fold, turns off gravity and studies the gauge theory in the uncompactified dimensions. In the second approach the gauge theory is realized on the world volume of a brane. In this paper we will use the latter framework.

A configuration consisting of an M theory fivebrane wrapping a Riemann surface Σ can be used to study the low energy properties of supersymmetric gauge theories in four dimensions. In [4] the structure of the Coulomb branch of $N = 2$ gauge theories has been determined using the M theory fivebrane[1]. The study of the moduli space of vacua of $N = 1$ SQCD using a configuration with an M theory fivebrane wrapping a Riemann surface was first done in [6–8].

The four dimensional supersymmetric gauge theory obtained from the M theory fivebrane is a compactification of a $(0, 2)$ theory in six dimensions. As discussed in [7], the brane theory has two scales, the radius of the eleventh dimension R and a typical scale of the brane configuration L_{brane}. Correspondingly, there are Kaluza-Klein modes with mass $1/R$, $1/L_{brane}$. The supersymmetric gauge theories that we would like to study have only one scale Λ. In order to correctly obtain these four dimensional field theories we have to find the the values of the parameters R, L_{brane} in which the brane theory and the field theory agree. This, in particular, requires a decoupling of the Kaluza-Klein modes of the brane theory. This is rather difficult and in view of these complications it is not yet precisely clear which quantities in field theory can be reliably computed using the brane theory. It seems, however, that the brane theory can at least be used for the study and computation of holomorphic objects.

In this paper we will use the fivebrane of M theory to study the moduli space of vacua of confining phase $N = 1$ supersymmetric gauge theories in four dimensions. The supersymmetric vacua correspond to the condensation of massless monopoles (dyons) and confinement of photons. At points where there are mutually non-local massless dyons the dyons vevs and the mass gap due to dyon condensation vanish. These points are

[1]The configuration consisting of a type IIA NS 5-brane wrapping a Riemman surface Σ describing the Seiberg-Witten curve was studied in [3]. Four-dimensional abelian gauge theory obtained from a fivebrane wrapping Σ was studied in [5].

candidates for non trivial $N = 1$ fixed points, and we will study them as well.

The paper is organized as follows:

Section 2 containes a brief review of the $N = 2$ moduli space of vacua. Section 3 is devoted to a field theory analysis of the $N = 1$ gauge theory obtained by adding to the $N = 2$ superpotential an $N = 1$ perturbation of the form $\Delta W = \sum_{k=2}^{N_c} \mu_k \mathrm{Tr}(\Phi^k)^2$ where Φ is the scalar chiral multiplet in the adjoint representation of the gauge group. This perturbation lifts the non singular locus of the $N = 2$ Coulomb branch. At the singular locus there are massless monopoles that can condense due to the perturbation, confine the gauge fields and generate a mass gap. We compute the monopole vacuum expectation values. When the number of flavours N_f is greater than zero there is a meson field $\tilde{Q}Q$ that acquires a vaccum expectation value, where $Q, \tilde{Q}$ constitute a quark hypermultiplet. We compute the vev assuming a minimal form of the effective superpotential obtained by the "integrating in" method. The form of the effective superpotential is not completely fixed by symmetries and holomorphy. Possible additional terms are denoted by W_{Delta}, and the assumption that the minimal superpotential is exact has been called $W_\Delta = 0$ in [9].

In section 4 we construct the fivebrane configuration that describes the theory with the superpotential perturbation ΔW. The corresponding type IIA brane configuration in the limit when $\mu_{N_c} \to \infty$ was studied in [10, 11]. In section 5 we use the fivebrane configuration to compute the monopole and meson vacuum expectation values as well as the vev of the baryon operator for $N_f \geq N_c$. The comparison of the fivebrane computation and the field theory analysis shows a complete agreement between the dyons vevs computed from field theory and the dyons vevs computed using the fivebrane. Furthermore, the fivebrane provides a geometrical description of the dyons vevs. The comparison of the meson vevs shows that that at vacua with enhanced gauge group $SU(r)$ the effective superpotential obtained by the "integrating in" method with $W_\Delta = 0$ is exact for $r = 2$ but is not exact for $r > 2$. Also, the fivebrane relation between the vev of the baryon operator and the meson vevs suggests that the baryonic branch of the $N = 2$ theory is split by the perturbation, a phenomenon that does not exist if $N = 2$ is broken to $N = 1$ by a mass term for the adjoint chiral multiplet. In section 6 we study a particular subset of the $N = 2$ Coulomb branch singular locus where a maximal number of mutually local massless monoples become massless. We use the fivebrane configuration to study the Higgs branches that emanate from this locus and relate the meson vev to those obtained when there is only a mass term for the adjoint chiral multiplet. In section 7 we study the

[2]The parameter μ_k is often denoted by g_k/k in the literature.

fivebrane configuration corresponding to $N = 1$ gauge theories with a Landau-Ginzburg type superpotential $\sum_l \mathrm{Tr}(h_l \tilde{Q} \Phi^l Q)$. In section 8 we use the brane geometry to analyze the candidates for $N = 1$ non-trivial fixed points where mutually non-local dyons are massless. As an illustration, the example of $SU(3)$ with $N_f = 2$ is worked out in some more detail in section 9, and section 10 is devoted to a discussion.

The method of using brane dynamics to study supersymmetric field theories in various dimensions has recently been used in many other works [13–37].

2 Preliminaries: $N = 2$ Moduli Space of Vacua

We consider $N = 2$ supersymmetric gauge theory with gauge group $SU(N_c)$ and N_f quark hypermultiplets in the fundamental representation. In terms of $N = 1$ superfields the vector multiplet consists of a field strength chiral multiplet W_α and a scalar chiral multiplet Φ both in the adjoint representation of the gauge group. A quark hypermultiplet consists of a chiral multiplet Q in the N_c and $\tilde{Q}$ in the $\bar{N}_c$ representation of the gauge group. The $N = 2$ superpotential takes the form

$$W = \sqrt{2} \tilde{Q}_i^a \Phi_a^b Q_b^i + \sqrt{2} m_j^i \tilde{Q}_i^a Q_a^j, \tag{2.1}$$

where $a, b = 1, ..., N_c; i, j = 1, ..., N_f$ and the quark mass matrix $m = \mathrm{diag}[m_1, ..., m_{N_f}]$.

The classical R-symmetry group is $SU(2)_R \times U(1)_R$. The bosons in the vector multiplet are singlets under $SU(2)_R$ while the fermions in the vector multiplet form a doublet. The fermions in the hypermultiplet are singlets under $SU(2)_R$ while the scalars in the hypermultiplet form a doublet. The theory is asymptotically free for $N_f < 2N_c$. The instanton factor is proportional to $\Lambda^{2N_c - N_f}$ where Λ is the dynamically generated scale. The $U(1)_R$ symmetry is anomalous and is broken to $Z_{2N_c - N_f}$.

The moduli space of vacua includes the Coulomb and Higgs branches. The Coulomb branch is $N_c - 1$ complex dimensional and is parametrized by the gauge invariant order parameters

$$u_k = \langle \mathrm{Tr}(\phi^k) \rangle, \qquad k = 2, ..., N_c, \tag{2.2}$$

where ϕ is the scalar field in the vector multiplet. Generically along the Coulomb branch the gauge group is broken to $U(1)^{N_c - 1}$. The Coulomb branch structure is corrected by one loop and by instantons. The quantum Coulomb branch parametrizes a family of genus $N_c - 1$ hyperelliptic curves [38–42]

$$y^2 = C_{N_c}^2(v) - \Lambda_{N=2}^{2N_c - N_f} \prod_{i=1}^{N_f} (v + m_i), \tag{2.3}$$

whose period matrix τ_{ij} is the low energy gauge coupling and where $C_{N_c}(v)$ is a degree N_c polynomial in v with coefficients that depend on the gauge invariant order parameters u_k, and m_i $(i = 1, ..., N_f)$ are the quark masses. The polynomial $C_{N_c}(v)$ is (for $N_f < N_c$) given by

$$C_{N_c}(v) = \sum_{i=0}^{N_c} s_i v^{N_c - i} \ , \tag{2.4}$$

where s_k and u_k are related by the Newton formula

$$k s_k + \sum_{i=1}^{k} s_{k-i} u_i = 0, \qquad k = 1, 2, \ldots, N_c \ , \tag{2.5}$$

with $s_0 = 1, s_1 = u_1 = 0$. From this one can derive the following relation (for $j \geq k$)

$$\frac{\partial s_j}{\partial u_k} = -\frac{1}{k} s_{j-k} \ . \tag{2.6}$$

There are two types of Higgs branches [43]: The non-baryonic and the baryonic branches. We will discuss the massless case. The non-baryonic branches are classified by an integer r such that $1 \leq r \leq min\{[N_f/2], N_c - 2\}$. The r-th non-baryonic branch has complex dimension $2r(N_f - r)$. The non-baryonic branches emanate from submanifolds in the Coulomb branch (of dimension $N_c - r - 1$ for the r-th non-baryonic branch) and constitute mixed branches. The effective theory along the root of the baryonic branch is $SU(r) \times U(1) \times U(1)^{N_c - r - 1}$ with N_f massless quarks charged under the first $U(1)$. Generically there are no massless hypermultiplets charged under the last $N_c - r - 1$ $U(1)$'s. There are special points along the root where such massless matter exists. The curve at the r-th non-baryonic branch root takes the form

$$y^2 = v^{2r} \left(C_{N_c - r}(v)^2 - \Lambda_{N=2}^{2N_c - N_f} v^{N_f - 2r} \right) \ . \tag{2.7}$$

There is a single baryonic branch for $N_f \geq N_c$, where generically the gauge group is completely broken. Its complex dimension is $2N_f N_c - 2(N_c^2 - 1)$. The baryonic branch emanates from the origin of the Coulomb branch The effective theory at the root of the baryonic branch is $SU(N_f - N_c) \times U(1)^{2N_c - N_f}$ with N_f massless quarks charged under the $U(1)$'s and $2N_f - N_c$ massless singlets charged under the $2N_c - N_f$ $U(1)'s$. This can be seen by looking at the curve at the baryonic root

$$y^2 = v^{2N_f - 2N_c} \left(v^{2N_c - N_f} + \Lambda_{N=2}^{2N_c - N_f} \right)^2 \ . \tag{2.8}$$

The complete square part of (2.8) is the required degeneration in order to have $2N_f - N_c$ massless hypermultiplets.

The Higgs branches are determined classically, however the points where they intersect each other and the Coulomb branch are modified quantum mechanically.

3 Breaking $N = 2$ to $N = 1$

The $N = 2$ supersymmetry can be broken to $N = 1$ by adding a tree level superpotential perturbation ΔW to the $N = 2$ superpotential (2.1)

$$W = \sqrt{2}\mathrm{Tr}(\tilde{Q}\Phi Q) + \sqrt{2}\mathrm{Tr}(m\tilde{Q}Q) + \Delta W \ , \tag{3.1}$$

where

$$\Delta W = \sum_{k=2}^{N_c} \mu_k \mathrm{Tr}(\Phi^k) \ . \tag{3.2}$$

In this section we will analyze the $N = 1$ field theory that is obtained from the tree level superpotential (3.1).

3.1 Pure Yang-Mills Theory

Dyon Condensation

We consider $N = 2$ pure $SU(N_c)$ Yang-Mills theory perturbed by the superpotential ΔW (3.2). Near points where $N_c - 1$ or less mutually local dyons are massless, each charged with respect to a different $U(1)$, the superpotential describing the low energy theory is

$$W = \sqrt{2} \sum_{i=1}^{N_c-1} \tilde{M}_i A_i M_i + \sum_{k=2}^{N_c} \mu_k U_k \ , \tag{3.3}$$

where A_i are the chiral superfield parts of the $N_c - 1$ $N = 2$ $U(1)$ gauge multiplets, $\tilde{M}_i, M_i$ are the dyon hypermultiplets and U_k are the chiral superfields representing the operators $\mathrm{Tr}(\Phi^k)$ in the low energy theory. The vevs of the lowest components of A_i, M_i, U_k are denoted by a_i, m_i, u_k respectively. Note that without matter it is not possible to have a point in the moduli space where two or more mutually local massless field are charged with respect to the same $U(1)$, since otherwise the $SQED$ effective theory at that point will have flat directions parametrizing a Higgs branch emanating from that point. Therefore the superpotential (3.3) describes correctly the pure Yang-Mills case.

The superpotential (3.3) is in fact exact. To show that we will make use of the holomorphy and global symmetries argument [44]. In order to apply it we need the charges of the fields and parameters of the theory under $U(1)_J \subset SU(2)_R$. The assignment of charges to the parameters and fields which restore $U(1)_R$ will be useful later. The list of charges is given by

$$
\begin{array}{ccc}
 & U(1)_R & U(1)_J \\
A_D & 2 & 0 \\
\tilde{M}_i M_i & 0 & 2 \\
\mu_k & 2-2k & 2 \\
U_k & 2k & 0 \\
\Lambda_{N=2} & 2 & 0
\end{array}
\tag{3.4}
$$

The superpotential W has charge two under $U(1)_J$. This together with the requirement for regularity at $\mu_k = \tilde{M}_i M_i = 0$ and the holomorphy constraint restrict its form to $W = \sum_{i=1}^{N_c-1} \tilde{M}_i M_i F_i(A_i) + \sum_{k=2}^{N_c} \mu_k G_k(A)$. The limits of small μ_k lead to the form (2.1).

The low energy vacua are obtained by imposing the vanishing D-term constraints

$$
|m_i| = |\tilde{m}_i| \ , \tag{3.5}
$$

and the $dW = 0$ constraints

$$
-\frac{\mu_k}{\sqrt{2}} = \sum_{i=1}^{N_c-1} \frac{\partial a_i}{\partial u_k} m_i \tilde{m}_i, \qquad k = 2, ..., N_c \ , \tag{3.6}
$$

and

$$
a_i m_i = a_i \tilde{m}_i = 0, \qquad i = 1, ..., N_c - 1 \ . \tag{3.7}
$$

At a point in the moduli space where no dyons are massless we have $a_i \neq 0, i = 1, ..., N_c - 1$. Therefore (3.7) implies that $m_i = \tilde{m}_i = 0$ and thus $\mu_k = 0$ by (3.6). This is the moduli space of vacua of the $N = 2$ theory and we see that a generic point (non-singular point) in the moduli space is lifted by the perturbation.

We now discuss the singular points in the moduli space. Consider a point in the moduli space where l mutually local dyons are massless. This means that some of the one-cycles shrink to zero and that the genus $N_c - 1$ curve (2.3) degenerates to a genus $N_c - l - 1$ curve. The right hand side of (2.3) takes the form

$$
C_{N_c}^2(v) - \Lambda_{N=2}^{2N_c} = \prod_{i=1}^{l} (v - p_i)^2 \prod_{j=1}^{2N_c-2l} (v - q_j) \tag{3.8}
$$

with p_i and q_j distinct. Equations (3.7) imply that

$$
m_i = \tilde{m}_i = 0, \qquad i = l+1, ..., N_c - 1 \ , \tag{3.9}
$$

while $m_i, \tilde{m}_i$ for $i = 1 \ldots l$ are unconstrained. Assuming the matrix $\partial a_i / \partial u_k$ is non-degenerate there will be a complex $N_c - l - 1$ dimensional moduli space of $N = 1$ vacua which remains after the perturbation.

The matrix $\partial a_i/\partial u_k$ can be explicitly evaluated using the relation

$$\frac{\partial a_i}{\partial s_k} = \oint_{\alpha_i} \frac{v^{N_c-k} dv}{y} \ , \tag{3.10}$$

where the RHS is the integral of a holomorphic one form on the curve (2.3). At a point where the l dyons become massless we have that the cycles $\alpha_i \to 0, i = 1 \dots l$ and (3.10) reduces to a contour integrals around $v = p_i, i = 1 \dots l$ with the result

$$\frac{\partial a_i}{\partial s_k} = \frac{p_i^{N_c-k}}{\prod_{j\neq i}(p_i - p_j)\prod_j (p_i - q_j)^{1/2}}. \tag{3.11}$$

This matrix has indeed maximal rank. Combining (2.6),(3.6) and (3.11) we find the following relation between the parameters μ_k and the dyon vevs $m_i\tilde{m}_i$

$$\frac{-\mu_k}{\sqrt{2}} = \sum_{i=1}^{l}\sum_{j=2}^{N_c} \frac{-1}{k} s_{j-k} p_i^{N_c-j} \frac{m_i\tilde{m}_i}{\prod_{l\neq i}(p_i - p_l)\prod_m (p_i - q_m)^{1/2}}. \tag{3.12}$$

For comparison with results obtained from the branes it will be useful to rewrite this relation in a different form. First, we define

$$\omega_i = \frac{\sqrt{2} m_i\tilde{m}_i}{\prod_{l\neq i}(p_i - p_l)\prod_m (p_i - q_m)^{1/2}}. \tag{3.13}$$

In terms of ω_i the generating function $\sum_{k=2}^{N_c} k\mu_k v^{k-1}$ for the μ_k is given by

$$\begin{aligned}
\sum_{k=2}^{N_c} k\mu_k v^{k-1} &= \sum_{k=2}^{N_c}\sum_{i=1}^{l}\sum_{j=2}^{N_c} v^{k-1} s_{j-k} p_i^{n_c-j}\omega_i \\
&= \sum_{k=-\infty}^{N_c}\sum_{i=1}^{l}\sum_{j=2}^{N_c} v^{k-1} s_{j-k} p_i^{n_c-j}\omega_i + \mathcal{O}(v^0) \\
&= \sum_{i=1}^{l}\sum_{j=2}^{N_c} C_{N_c}(v) v^{j-N_c-1} p_i^{N_c-j}\omega_i + \mathcal{O}(v^0) \\
&= \sum_{i=1}^{l}\sum_{j=-\infty}^{N_c} C_{N_c}(v) v^{j-N_c-1} p_i^{N_c-j}\omega_i + \mathcal{O}(v^0) \\
&= \sum_{i=1}^{l} \frac{C_{N_c}(v)}{(v-p_i)}\omega_i + \mathcal{O}(v^0).
\end{aligned} \tag{3.14}$$

Given a set of perturbation parameters μ_k and a point in the $N = 2$ moduli space of vacua specified by the set p_i, q_j of (3.8), equations (3.14) determine whether this point remains as an $N = 1$ vacuum after the perturbation and determine the vevs of the dyon

fields $m_i\tilde{m}_i$. It is convenient for comparison with the brane picture to define a polynomial $H(v)$ of degree $l-1$ through

$$\sum_{i=1}^{l} \frac{\omega_i}{(v-p_i)} = \frac{2H(v)}{\prod_{i=1}^{l}(v-p_i)}. \tag{3.15}$$

At a given point p_i, q_j, $H(v)$ determines uniquely the dyons vevs

$$m_i\tilde{m}_i = \sqrt{2}H(p_i)\prod_m (p_i - q_m)^{1/2} \ . \tag{3.16}$$

We will see that the brane picture provides a geometrical interpretation of the dyon vevs (3.16).

Maximal Number of Mutually Local Massless Dyons

There are N_c points in the moduli space related to each other by the action of the discrete $\mathbf{Z}_{2N_c}$ R-symmetry group (3.4), where $N_c - 1$ mutually local dyons are massless. At these points $a_i = 0, i = 1, ..., N_c - 1$ and the curve (2.3) degenerates to a genus zero curve. These points correspond to the N_c massive vacua of $N = 1$ pure Yang-Mills theory where the discrete $\mathbf{Z}_{2N_c}$ R-symmetry is spontaneously broken to $\mathbf{Z}_2$. Equations (3.6) can be solved for generic μ_k and these $N = 1$ vacua are generically not lifted. However, it is difficult to see from equations (3.6) whether there exist values of the parameters μ_k for which these vacua are lifted. We will see in section 6 that the brane construction predicts that these points are not lifted for arbitrary values of the parameters μ_k.

3.2 $SU(N_c)$ with N_f Flavours

Consider now the addition of N_f flavours to the $SU(N_c)$ Yang-Mills theory. In the following we will compute the dyon and meson vevs and discuss the structure of the Coulomb and Higgs branches of the $N = 2$ theory after the perturbation.

3.2.1 Dyon Condensation

As in the pure Yang-Mills case, the perurbation (3.2) lifts the non singular locus of the $N = 2$ Coulomb branch. The computation of the dyon vevs along the singular locus is similar to that of the previous subsection. In the following we will compute the dyon vevs at the roots of the non-baryonic branches.

Non-Baryonic Branch

The effective theory along the root of the non-baryonic branch is $SU(r) \times U(1) \times U(1)^{N_c-r-1}$ with N_f massless quarks $\tilde{Q}, Q$ charged under the first $U(1)$ with charge 1. Generically there are no massless hypermultiplets charged under the last $N_c - r - 1$ $U(1)$'s. There are special points along the root where such massless matter exists. Along the r-th non-barynoic branch root there are $2N_c - N_f$ special points related to each other by a $\mathbf{Z}_{2N_c-N_f}$ action and for each $U(1)_i$ factor of the effective theory at the root there is a charged massless hypermultiplet $\tilde{M}_i, M_i$. The superpotential describing $\tilde{M}_i, M_i$ at these points is

$$W = \sqrt{2} \sum_{i=1}^{N_c-r-1} \tilde{M}_i A_i M_i + \sum_{k=2}^{N_c} \mu_k U_k \ . \tag{3.17}$$

These points are similar to the above N_c points in the moduli space of pure Yang-Mills theory where there is a maximal number of mutually local massless dyons and they are not lifted for generic μ_k. In fact, the analysis of the non-baryonic branch roots upon perturbation is analogous to that of the pure Yang-Mills case. If we vary the superpotential (3.17) with respect to $\tilde{M}_i, M_i, U_k$ we get

$$-\frac{\mu_k}{\sqrt{2}} - \sum_{j=N_c-r+1}^{N_c} \frac{\mu_j}{\sqrt{2}} \frac{\partial u_j}{\partial u_k} = \sum_{i=1}^{N_c-r-1} \frac{\partial a_i}{\partial u_k} m_i \tilde{m}_i, \qquad k = 2, ..., N_c - r \ , \tag{3.18}$$

and

$$a_i m_i = a_i \tilde{m}_i = 0, \qquad i = 1, ..., N_c - r - 1 \ . \tag{3.19}$$

The vanishing D-term constraints are $|m_i| = |\tilde{m}_i|$. The new term on the left hand side of (3.18) appears because on the non-baryonic root the U_k with $k > N_c - r$ are no longer independent coordinates, but depend on U_k with $k \leq N_c - r$.

At a point in the non-baryonic branch root where none of the hypermultiplets $\tilde{M}_i, M_i$ are massless we have $a_i \neq 0, i = 1, ..., N_c - r - 1$. Therefore (3.19) implies that $m_i = \tilde{m}_i = 0$ and we are left with $N_c - r - 1$ equations for the $N_c - 1$ parameters μ_k. Thus, there is an r-dimensional space of $N = 1$ vacua after the perturbation. When we discuss the non-baryonic roots in the sequel, we will most of the time put μ_k with $k > N_c - r$ equal to zero, so that the U_k that appear in the superpotential are all independent. With this additional assumption, most of the non-baryonic root is lifted, unless some of the hypermultiplets $\tilde{M}_i, M_i$ are massless.

If some $\tilde{M}_i, M_i$ are massless, the computation of the dyon vevs at the non-baryonic branch root is analogous to the pure Yang-Mills case where the hypermultiplets $\tilde{M}_i, M_i$ correspond to the dyons in that discussion. In order to see that note that if we define $\tilde{y} = y/v^r$ in (2.7) we get at the r-th non-baryonic branch root

$$\tilde{y}^2 = C_{N_c-r}(v)^2 - \Lambda_{N=2}^{2(N_c-r)-(N_f-2r)} v^{N_f-2r} \ . \tag{3.20}$$

A point in the r-th non-baryonic branch root where l mutually local dyons are massless means that the genus $N_c - r - 1$ curve (3.20) degenerates to a genus $N_c - r - l - 1$ curve and it takes the form

$$C_{N_c-r}(v)^2 - \Lambda_{N=2}^{2(N_c-r)-(N_f-2r)} v^{N_f-2r} = \prod_{i=1}^{l}(v-p_i)^2 \prod_{j=1}^{2(N_c-r-l)}(v-q_j) \tag{3.21}$$

with p_i and q_j distinct. Equations (3.19) imply that

$$m_i = \tilde{m}_i = 0, \qquad i = l+1, ..., N_c - r - 1 \ , \tag{3.22}$$

while $m_i, \tilde{m}_i$ for $i = 1 \dots l$ are unconstrained. Repeating the analysis of the pure Yang-Mills case with the relation

$$\frac{\partial a_i}{\partial s_k} = \oint_{\alpha_i} \frac{v^{N_c-k} dv}{\tilde{y}} = \oint_{\alpha_i} \frac{v^{N_c-r-k} dv}{y} \ , \tag{3.23}$$

analogous to (3.14) we get

$$2H(v)\frac{C_{N_c-r}(v)}{\prod_{i=1}^{l}(v-p_i)} + \mathcal{O}(v^0) = \sum_{k=2}^{N_c} k\mu_k v^{k-1} \ . \tag{3.24}$$

The function $H(v)$, the dyon vevs $m_i\tilde{m}_i$ and ω_i are related by (3.13) and (3.15).

Note that the $r = 0$ case is a special case of the above analysis. It corresponds to the general points in the moduli space of vacua which are not at the baryonic or non-baryonic roots.

Baryonic Branch

The effective theory at the root of the baryonic branch is $SU(N_f - N_c) \times U(1)^{2N_c-N_f}$ with N_f massless quarks $\tilde{Q}, Q$ charged under the $U(1)$'s with charge $1/(N_f - N_c)$ and $2N_c - N_f$ massless singlets $\tilde{M}_i, M_i$ charged under only the i-th $U(1)$ with charge -1. The superpotential describing $\tilde{M}_i, M_i$ at the root is

$$W = -\sqrt{2} \sum_{i=1}^{2N_c-N_f} \tilde{M}_i M_i A_i + \sum_{k=2}^{N_c} \mu_k U_k \ . \tag{3.25}$$

For each $U(1)$ factor of the effective theory at the root there is a charged massless hypermultiplet $\tilde{M}_i, M_i$. Thus, the baryonic branch root is similar to the N_c points in the moduli space of pure Yang-Mills theory where there are massless mutually local dyons charged with respect to the different $N_c - 1$ $U(1)'s$. Therefore the baryonic branch is not lifted for generic μ_k. We will see in section 6 that the brane construction predicts that the baryonic branch root is not lifted for arbitrary values of the parameters μ_k.

3.2.2 The Meson Vevs

In the previous subsection we studied the Coulomb branch and the roots of the Higgs branches in the presence of the perturbation ΔW (3.2). However, even if the root of a Higgs branch is not lifted by a perturbation, the structure of the Higgs branch itself may be significantly modified. In the $N = 2$ theory the Higgs branch is determined classically and it does not receive quantum corrections. Geometrically it is a hyperkähler manifold. After adding the $N = 1$ perturbation to the superpotential the structure of the Higgs branch is modified by quantum corrections and geometrically it is a kähler manifold.

In the case when the $N = 1$ perturbation is only a mass term for the adjoint chiral multiplet, $\Delta W = \mu_2 \mathrm{Tr}(\Phi^2)$, the structure of the Higgs branches is the as follows [43, 6]: The baryonic and non-baryonic branches remain, however the non-baryonic branches emanate only from the points in the Coulomb branch which are not lifted. In addition, the structure of the baryonic and non-baryonic branches depends on the parameter μ_2 and they are no longer hyperkähler manifolds. In the presence of the more general perturbation ΔW (3.2) the structure of the Higgs branches is modified more significantly.

In the following we will compute the vev of the meson field $\tilde{Q}Q$ along the singular locus of the Coulomb branch. This vev is generated by the non perturbative dynamics of the $N = 1$ theory, and was clearly zero in the $N = 2$ theory before the perturbation (3.2). The comparison of the vev of the meson calculated by field theory methods with the predictions from the fivebrane of M theory, which will be done in section 5, will provide a method to test the exactness of the field theory description.

One massless dyon

We consider the $N = 1$ superpotential

$$W = \sqrt{2}\mathrm{Tr}(\tilde{Q}\Phi Q) + \sqrt{2}\mathrm{Tr}(m\tilde{Q}Q) + \sum_{k=2}^{N_c} \mu_k \mathrm{Tr}(\Phi^k). \tag{3.26}$$

In order to determine the meson vev we follow the strategy outlined in [45–49]. We first determine a low-energy effective superpotential for a phase with one unbroken confining $SU(2)$. From that, we determine the meson vev. The low-energy effective superpotential is not completely fixed by symmetries and holomorphy alone, there can be corrections to it (denoted by W_Δ in [9]). We will assume in the following that $W_\Delta = 0$. The agreement of the meson vev with the result obtained from the M theory fivebrane in section 5 will justify this assumption.

Classically, the locus with one massless dyon corresponds to the cases where Φ after diagonalization takes the form

$$\Phi_{\rm cl} = {\rm diag}(a_1, a_1, a_3, a_4, \ldots, a_{N_c}) \ , \tag{3.27}$$

with all a_j different. For these values of Φ there is an unbroken $SU(2)$. This form of Φ can be derived by differentiating the superpotential with respect to Φ and putting $Q = \tilde{Q} = 0$ which yields

$$\sum_{k=2}^{N_c} k\mu_k \Phi^{k-1} - \frac{1}{N_c}\sum_{k=2}^{N_c} k\mu_k {\rm Tr}(\Phi^{k-1}) = 0 \ , \tag{3.28}$$

where the second term appears because Φ is traceless. If these equations have a solution then we see that each diagonal component of Φ solves an $N_c - 1^{\rm th}$ order polynomial equation. As there are N_c diagonal components in Φ this implies Φ must have two identical components. Also, since we assumed all a_j were different, these must be in one-to-one correspondence with the solutions of this $N_c - 1^{\rm th}$ order polynomial equation. The sum of the solutions of a polynomial equation $az^k + bz^{k-1} + \ldots = 0$ is $-b/a$, and therefore we find in this case that

$$a_1 + a_3 + a_4 + \ldots + a_{N_c} = -\frac{(N_c - 1)\mu_{N_c - 1}}{N_c \mu_{N_c}}. \tag{3.29}$$

On the other hand, we also know that ${\rm Tr}(\Phi) = 0$, and combining these two equations we see that

$$a_1 = \frac{(N_c - 1)\mu_{N_c - 1}}{N_c \mu_{N_c}}. \tag{3.30}$$

One can also express the other a_j in terms of μ_k but these expressions will not be needed in the present discussion.

Next, we substitute $\Phi = \Phi_{\rm cl} + \delta\Phi$ in the $N = 1$ superpotential and integrate out $\delta\Phi$. This yields an $SU(2)$ $N = 1$ theory with N_f flavors and superpotential

$$W = \sum_{k=2}^{N_c} \mu_k {\rm Tr}(\Phi_{\rm cl}^k) + \sqrt{2}{\rm Tr}(\tilde{Q}(a_1 + m)Q) \ , \tag{3.31}$$

where a_j should be understood as functions of μ_k. The scale matching relation between the scale of this $N = 1$ $SU(2)$ theory and the original broken $SU(N_c)$ theory reads [50]

$$\Lambda_{SU(2),N_f}^{6-N_f} = (N_c \mu_{N_c})^2 \Lambda_{N=2}^{2N_c - N_f}. \tag{3.32}$$

Next, we integrate out the quarks to end up with a pure $SU(2)$ theory, with superpotential

$$W = \sum_{k=2}^{N_c} \mu_k {\rm Tr}(\Phi_{\rm cl}^k) \ , \tag{3.33}$$

and scale

$$\Lambda^6_{SU(2)} = \Lambda^{6-N_f}_{SU(2),N_f} \det(a_1 + m). \tag{3.34}$$

Pure $N = 1$ $SU(2)$ gauge theory has gaugino condensation, and the final proposal for the full exact low-energy effective superpotential is then (3.33) plus a term due to gaugino condensation,

$$W = \sum_{k=2}^{N_c} \mu_k \mathrm{Tr}(\Phi^k_{\mathrm{cl}}) \pm 2\Lambda^3_{SU(2)} \ , \tag{3.35}$$

where the $\pm$ sign reflects the vacuum degeneracy of the pure $N = 1$ $SU(2)$ gauge theory. Again, the assumption that (3.35) is exact is the assumption $W_\Delta = 0$. In terms of the original $N = 2$ scale W reads

$$W(\mu_k, m) = \sum_{k=2}^{N_c} \mu_k \mathrm{Tr}(\Phi^k_{\mathrm{cl}}) \pm 2N_c \mu_{N_c} \det(a_1 + m)^{1/2} \Lambda^{N_c - N_f/2}_{N=2} \ , \tag{3.36}$$

Taking the mass matrix m diagonal, we find that the vacuum expectation value of the meson $M_i = \mathrm{Tr}(\tilde{Q}_i Q_i)$

$$M_i = \frac{\partial W}{\partial m_i} = \frac{\pm}{\sqrt{2}} N_c \mu_{N_c} \Lambda^{N_c - N_f/2}_{N=2} \frac{\det(a_1 + m)^{1/2}}{(a_1 + m_i)} \tag{3.37}$$

with a_1 given by (3.30). The $\pm$ sign corresponds to two possible values of the gauge invariant order parameters u_k (2.2) parametrizing the singularities of the same $N = 2$ curve.

More than one massless dyon

In the following we will generalize the previous discussion to the case with more than one massless dyon. In field theory, such a situation can be realized by taking

$$\Phi_{\mathrm{cl}} = (a_1^{r_1}, \ldots, a_k^{r_k}) \ , \tag{3.38}$$

by which we mean that Φ_{cl} has r_1 times the eigenvalue a_1, r_2 times the eigenvalue a_2, etc. The unbroken gauge group will be $SU(r_1) \times \ldots \times SU(r_k) \times U(1)^{k-1}$. Again, the logic will be to first integrate out the adjoint superfield Φ, and after that to integrate out the quarks.

Integrating out the adjoint superfield proceeds in two steps. First we integrate all components of Φ that satisfy $[\Phi_{\mathrm{cl}}, \Phi] \neq 0$. The remaining components of Φ will consist of fields transforming under the adjoint of each $SU(r_j)$, and some neutral components that we ignore. In the second step we integrate out the adjoints of each of the $SU(r_j)$.

After the first step level matching gives the following relation between the original $N=2$ scale $\Lambda_{N=2}$ and the scale $\Lambda_{1,i}$ of the $SU(r_i)$ gauge theory with adjoint and matter

$$\Lambda_{N=2}^{2N_c-N_f} = \Lambda_{1,i}^{2r_i-N_f} \prod_{j\neq i}(a_j - a_i)^{2r_j}. \tag{3.39}$$

In the second step, we integrate out the adjoint in each $SU(r_j)$. The level matching for this step involves the mass of the adjoint, which follows by expanding the superpotential to second order around $\Phi_{\rm cl}$

$$m_{{\rm adj},i} = \sum_{l=2}^{N_c} l(l-1)\mu_l a_i^{l-2}. \tag{3.40}$$

In order to evaluate this quantity, we use the fact that all a_i are solutions of the polynomial equation (see (3.28))

$$\sum_{l=2}^{N_c} l\mu_l x^{l-1} - \frac{1}{N_c}\sum_{k=2}^{N_c} k\mu_k {\rm Tr}(\Phi^{k-1}) = 0. \tag{3.41}$$

Therefore, we can write

$$\sum_{l=2}^{N_c} l\mu_l x^{l-1} - \frac{1}{N_c}\sum_{k=2}^{N_c} k\mu_k {\rm Tr}(\Phi^{k-1}) = \phi(x)\prod_{j=1}^{k}(x-a_j)\ , \tag{3.42}$$

where $\phi(x)$ is some polynomial of order $N_c - 1 - k$. Differentiating this identity with respect to x and taking $x = a_i$ we find that

$$m_{{\rm adj},i} = \phi(a_i)\prod_{j\neq i}(a_i - a_j). \tag{3.43}$$

The level matching relation between the scale $\Lambda_{i,1}$ of the $SU(r_i)$ theory with the adjoint and the scale $\Lambda_{i,2}$ of the $SU(r_i)$ theory with the adjoint integrated out is

$$\Lambda_{i,2}^{3r_i-N_f} = \Lambda_{i,1}^{2r_i-N_f} m_{{\rm adj},i}^{r_i}\ , \tag{3.44}$$

and using (3.39) and (3.43) we find

$$\Lambda_{i,2}^{3r_i-N_f} = \phi(a_i)^{r_i}\prod_{j\neq i}(a_j - a_i)^{r_i-2r_j}\Lambda_{N=2}^{2N_c-N_f}. \tag{3.45}$$

At this moment we have pure $N=1$ gauge theory with matter. As in the case with only one massless dyon, the last step is to integrate out the quarks. The scale Λ_i of the resulting pure $N=1$ $SU(r_i)$ gauge theory is now

$$\Lambda_i^{3r_i} = \det(m+a_i)\Lambda_{i,2}^{3r_i-N_f} = \det(m+a_i)\phi(a_i)^{r_i}\prod_{j\neq i}(a_j-a_i)^{r_i-2r_j}\Lambda_{N=2}^{2N_c-N_f}. \tag{3.46}$$

The final low-energy effective superpotential is now again the classical result with an additional term due to gaugino condensation,

$$W = \sum_{k=2}^{N_c} \mu_k \mathrm{Tr}(\Phi_{\mathrm{cl}}^k) + \sum_i r_i \omega_i \Lambda_i^3 \ , \tag{3.47}$$

where ω_i is an r_i-th root of unity. As in the one massless dyon case, the assumption that (3.47) is exact means that we are assuming that $W_\Delta = 0$ in [9]. For the meson vev we get

$$\sqrt{2} M_j = \frac{\partial W}{\partial m_j} = \sum_i \frac{\omega_i \phi(a_i)}{m_j + a_i} \det(m + a_i)^{1/r_i} \prod_{j \neq i} (a_j - a_i)^{1 - 2r_j/r_i} \Lambda_{N=2}^{\frac{2N_c - N_f}{r_i}} . \tag{3.48}$$

Note that the sum over i in (3.48) extends only over those i for which $r_i > 1$. In the case of one massless dyon we had $r_1 = 2$ and $r_j = 1$ for $j > 1$ and (3.48) reduces to (3.37).

Non-Baryonic Branch

The non-baryonic root is a special case of the previous analysis for more than one massless dyon. At the r-th non-baryonic root Φ_{cl} takes the form

$$\Phi_{\mathrm{cl}} = (a_1^r, a_2, \ldots, a_{N_c - r}) \ , \tag{3.49}$$

and the classical unbroken gauge group is $SU(r) \times U(1)^{N_c - r - 1}$. The meson vev takes the form

$$\sqrt{2} M_j = \frac{\omega_i \phi(a_1)}{m_j + a_1} \det(m + a_1)^{1/r} \prod_{j \neq 1} (a_j - a_1)^{1 - 2/r} \Lambda_{N=2}^{\frac{2N_c - N_f}{r}} . \tag{3.50}$$

summary: The perturbation ΔW (3.2) does not lift the codimension one singular locus of the $N = 2$ Coulomb branch where dyons become massless. The condensation of dyons along the singular locus confines photons. Equations (3.13),(3.14), (3.15) and (3.24) determine the vevs of the condensed dyons. For $N_f > 0$ the meson gets a vev. Assuming $W_\Delta = 0$ the equations (3.37), (3.48) and (3.50) give the meson vevs along the singular locus of the Coulomb branch. These results will be compared with the M theory fivebrane in section 5. We will see that when the unbroken gauge group is $SU(2)$ there is an agreement between the result for the meson vev in this section and the one derived via the M theory fivebrane . When the unbroken gauge group is $SU(r), r > 2$ we will find disagreement with the computation using the M theory fivebrane. This indicates that $W_\Delta \neq 0$ in this case.

4 Brane Configuration

4.1 Type IIA Picture

Consider type IIA string theory in flat space-time where x^0 denotes the time coordinate and $x^1, \ldots, x^9$ denote the space coordinates. Consider first the type IIA picture of the $N = 2$ gauge theory. We consider the brane configuration of [4] that preserves eight supercharges. The brane configuration depicted consists of two NS 5-branes with worldvolume coordinates $x^0, x^1, x^2, x^3, x^4, x^5$, N_c D4 branes suspended between them with worldvolume coordinates x^0, x^1, x^2, x^3, x^6 and N_f D6 branes with worldvolume coordinates $x^0, x^1, x^2, x^3, x^7, x^8, x^9$.

The D4 brane is finite in the x^6 direction and we consider the four dimensional $N = 2$ supersymmetric gauge theory on its worldvolume coordinates x^0, x^1, x^2, x^3. The theory has an $SU(N_c)$ gauge group and N_f hypermultiplets in the fundamental representation of the gauge group [4]. The Higgs branch is described by D4 branes suspended between D6 branes. The motion of a D4 brane between two D6 branes is parametrized by two complex parameters, the x^7, x^8, x^9 coordinates together with the gauge field component A_6 in the x^6 coordinate. The brane configuration is invariant under the rotations in the x^4, x^5 and x^7, x^8, x^9 directions — $U(1)_{4,5}$ and $SU(2)_{7,8,9}$. These are interpreted as the classical $U(1)$ and $SU(2)$ R-symmetry groups of the four-dimensional theory on the brane worldvolume.

In the following we will consider a perturbations of the form ΔW (3.2). The brane configuration that realizes an $N = 1$ theory with a superpotential (3.1) has been constructed and studied in [10, 11] [1] . It consists of NS 5-brane with worldvolume coordinates $x^0, x^1, x^2, x^3, x^4, x^5$, $N_c - 1$ NS' 5-branes with worldvolume coordinates $x^0, x^1, x^2, x^3, x^8, x^9$ N_c D4 branes suspended between them with worldvolume coordinates x^0, x^1, x^2, x^3, x^6 and N_f D6 branes with worldvolume coordinates $x^0, x^1, x^2, x^3, x^7, x^8, x^9$.

Let us introduce the complex coordinates $v = x^4 + ix^5, w = x^8 + ix^9$. We will take the N_c NS' 5-branes to stretch in the (v, w) coordinates. The minima of the superpotential (3.1) label the separation of the NS' branes in the v direction in the construction of [10, 11], a fact that we can reproduce from our brane configuration in the limit where we send the coefficients in ΔW to infinity.

[1] The type IIA brane configuration of [10, 11] corresponds to large coefficients in ΔW (3.2). The M theory fivebrane configuration that we will use does not have this restriction.

4.2 Fivebrane Configuration

It has been shown in [4] that the $N = 2$ brane configuration is described in M theory as a (generically) smooth fivebrane with worldvolume $R^4 \times \Sigma$ where Σ is the genus $N_c - 1$ curve that determines the structure of the $N = 2$ Coulomb branch. Denote $s = (x^6 + ix^{10})/R, \quad t = exp(-s)$ where x^{10} is the eleventh coordinate of M theory which is compactified on a circle of radius R. The curve Σ is given by an algebraic equation in (v, t) space, which for $N = 2$ $SU(N_c)$ SQCD with N_f flavors is given by

$$t^2 - 2C_{N_c}(v)t + \Lambda_{N=2}^{2N_c - N_f} \prod_{i=1}^{N_f} (v + m_i) = 0, \tag{4.1}$$

where t is related to y in (2.3) by $t = y + C_{N_c}(v, u_k)$. In the M theory configuration, $SU(2)_{7,8,9}$ is preserved but $U(1)_{4,5}$ is broken.

Let us now construct the configuration of M theory fivebrane that corresponds to the type IIA brane realization of the superpotential perturbation ΔW (3.2). The left NS 5-brane corresponds to the asymptotic region $v \to \infty, t = y \sim v^{N_c}$, the right $N_c - 1$ NS' 5-branes correspond to the asymptotic region $v \to \infty, t \sim \Lambda_{N=2}^{2N_c - N_f} v^{N_f - N_c}$. The boundary conditions that we will impose are

$$\begin{array}{lll} w & \to & \sum_{k=2}^{N_c} k\mu_k v^{k-1} \qquad \text{as } v \to \infty,\ t \sim \Lambda_{N=2}^{2N_c - N_f} v^{N_f - N_c} \\ w & \to & 0 \qquad\qquad \text{as } v \to \infty,\ t \sim v^{N_c} \end{array} . \tag{4.2}$$

Alternatively, if the $N_c - 1$ NS' 5-branes were located at the left and the NS 5-brane at the right the boundary conditions would read

$$\begin{array}{lll} w & \to & \sum_{k=2}^{N_c} k\mu_k v^{k-1} \qquad \text{as } v \to \infty,\ t \sim v^{N_c} \\ w & \to & 0 \qquad\qquad \text{as } v \to \infty,\ t \sim \Lambda_{N=2}^{2N_c - N_f} v^{N_f - N_c} \end{array} \tag{4.3}$$

An alternative argument for the boundary condition (4.2) is the following. In the $N = 2$ case the left and right NS 5-branes are parallel, both having $w = 0$, and the motion of the D4-brane between the two NS 5-branes is the degree of freedom corresponding to the adjoint scalar field Φ. When we deform one of the two NS 5-branes, such motion is no longer possible, and $N = 2$ supersymmetry is broken to $N = 1$. If we nevertheless try to move the D4-brane without changing its direction, we have to take it off one of the two NS 5-branes. This will give rise to a potential energy that gives a mass to the adjoint scalar. The mass is proportional to the distance in the $(8, 9)$ direction between the D4-brane and the NS 5-brane from which it is being disconnected. If we keep one NS

5-brane at $w = 0$, and deform the other from $w = 0$ to $w \equiv w(v)$, then the mass of the adjoint is $w(< \Phi >)$, because we can identify $< \Phi >$ with v. The mass for the adjoint one gets from the superpotential is $W'(\Phi)$, and by matching these two we see that the boundary conditions that we have to impose on the NS 5-brane in order to describe ΔW is $w(v) \sim \Delta W'(v) = \sum_{k=2}^{N_c} k\mu_k v^{k-1}$ as $v \to \infty$.

In the fivebrane configuration $SU(2)_{7,8,9}$ is broken to $U(1)_{8,9}$ if the parameter μ_k is assigned the $U(1)_{4,5} \times U(1)_{8,9}$ charge $(2-2k, 2)$. We list below the charges of the coordinates and parameters.

$$\begin{array}{ccc} & U(1)_{4,5} & U(1)_{8,9} \\ v & 2 & 0 \\ w & 0 & 2 \\ t & 2N_c & 0 \\ x & 2N_c & 0 \\ \mu_k & 2-2k & 2 \\ \Lambda_{N=2} & 2 & 0 \end{array} \tag{4.4}$$

Comparing (3.4) and (4.4) we see that $U(1)_{45} = U(1)_R$ and $U(1)_{89} = U(1)_J$.

We do not expect to be able to construct the brane configuration for arbitrary values of u_k's. Consider a perturbation of the form $\sum_{k=2}^{N_c-l+1} \mu_k \mathrm{Tr}(\Phi^k)$. We have seen that from the field theory point of view that such a perturbation lifts most of the Coulomb branch of the $N = 2$ theory. The moduli space of vacua that remains is the singular part of the $N = 2$ Coulomb branch where l or more mutually local dyons become massless. In the M-theory picture, it is possible to construct the corresponding brane only when the (v, t) curve degenerates to a genus $g \leq N_c - l - 1$ curve. In order to see that, note that w is a function on the (v, t) curve. The boundary conditions (4.3) or (4.2) mean that that w is a meromorphic function of the (v, t) which has a pole of order $N_c - l$ at one point. Such a function exists only when the (v, t) curve is equivalent to a genus $g \leq N_c - l - 1$ curve.

We will now analyze in detail the possible fivebrane configurations that satisfy the right boundary conditions. For this purpose we make two assumptions. First, we assume that the equation defining the $N = 2$ curve remains unchanged, i.e. is still given by (4.1). The second assumption is that w will be a rational function of t and v. With these assumptions, one can classify the set of allowed functions w that satisfy the appropriate boundary conditions, as we will now demonstrate. For simplicity, we will denote $C_{N_c}(v, u_k)$ by C and $\Lambda_{N=2}^{2N_c-N_f} \prod_{i=1}^{N_f}(v + m_i)$ by G.

First, notice that any rational function of t and v can, using (4.1), be rewritten in the

form

$$w(t,v) = \frac{a(v)t + b(v)}{c(v)t + d(v)}. \tag{4.5}$$

Let us denote the two solutions of (4.1) by $t_\pm(v)$. It is straightforward to determine that

$$w(t_+(v),v) + w(t_-(v),v) = \frac{2acG + 2adC + 2bcC + 2bd}{c^2G + 2cdC + d^2} \ , \tag{4.6}$$

and

$$w(t_+(v),v) - w(t_-(v),v) = \frac{2(ad-bc)S\sqrt{T}}{c^2G + 2cdC + d^2} \ , \tag{4.7}$$

where

$$C^2(v) - G(v) \equiv S^2(v)T(v) \ , \tag{4.8}$$

has been decomposed so that $T(v) = \prod_j (v - b_j)$ with all b_j distinct. Except for the boundary conditions on w as $v \to \infty$, we should also require that w has no poles for a finite value of v since there are no other infinite NS 5-branes in the type IIA picture. This is equivalent to the requirement that $w(t_+(v),v) \pm w(t_-(v),v)$ has no poles. This then implies that there should exist polynomials $H(v), N(v)$ such that

$$\frac{2acG + 2adC + 2bcC + 2bd}{c^2G + 2cdC + d^2} = 2N \ , \tag{4.9}$$

$$\frac{2(ad-bc)S}{c^2G + 2cdC + d^2} = H \ . \tag{4.10}$$

By shifting $a \to a + Nc$, $b \to b + Nd$, (4.5) and (4.9) become

$$\begin{aligned} w &= N + \frac{a(v)t + b(v)}{c(v)t + d(v)} \ , \\ 0 &= 2acG + 2adC + 2bcC + 2bd \ , \\ H &= \frac{2(ad-bc)S}{c^2G + 2cdG + d^2} \ . \end{aligned} \tag{4.11}$$

The second of these equations can be decomposed as

$$a(cG + dC) + b(d + cC) = 0 \ , \tag{4.12}$$

which implies that

$$cG + dC = -be, \qquad d + cC = ae \ , \tag{4.13}$$

for some (possibly) rational function e. Solving for c, d and substituting this back in the third equation in (4.11) we find that it reduces to $e = 2S/H$. If we now assume that H, a and c are given, we can solve for b and d using (4.13) with $e = 2S/H$. This then provides

us with the most general rational function w which does not contain any poles for finite v. The result is

$$w = N + \frac{at + cHST - aC}{ct - cC + aS/H} \ , \tag{4.14}$$

where N, H, a, c are arbitrary polynomials. Quite interestingly, w is independent of a and c. One can verify that the difference between w's with different choices of polynomials a and c is in fact proportional to $t^2 - 2Ct + G$ and therefore zero. The simplest cases are to take either $a = 0$ or $c = 0$. In particular, if $c = 0$,

$$w = N + \frac{H}{S}(t - C) \ . \tag{4.15}$$

Having determined the most general solution for w that has no poles, our next task is to impose the boundary conditions. These can be read of from

$$w(t_\pm(v), v) = N \pm H\sqrt{T}. \tag{4.16}$$

As $v \to \infty$ and $t = t_-(v) \sim v^{N_c}$, we want that $w \to 0$. This completely fixes N:

$$N(v) = [H(v)\sqrt{T(v)}]_+ \ , \tag{4.17}$$

where $[f(v)]_+$ denotes the part of $f(v)$ with non-negative powers of v, in a power series expansion around $v = \infty$.

With the choice (4.17) for N, it is guaranteed that $w(t_-(v), v) = \mathcal{O}(v^{-1})$.

In the other asymptotic region, $v \to \infty$ and $t = t_+(v) \sim v^{N_f - N_c}$, w behaves as

$$w = [2H(v)\sqrt{T(v)}]_+ + \mathcal{O}(v^{-1}). \tag{4.18}$$

Thus, the minimal choice $H = 1$ implies that $w \sim v^{k-1}$, if the order of T is $2k - 2$. For all other choices of H, w grows faster than this. This clearly shows the relation between the genus of the degenerate Riemann surface, and the minimal power needed in the superpotential.

Note that imposing the other boundary condition (4.3) simply corresponds to the choice $N(v) = -[H(v)\sqrt{T(v)}]_+$.

Finally, we note that w satisfies the following important equation:

$$w^2 - 2Nw + N^2 - TH^2 = 0. \tag{4.19}$$

If TH^2 is of order $2k - 2$, then $N^2 - TH^2$ is at most of order $k - 2$. In particular, if the genus of the degenerate Riemann surface is zero and we choose $H = 1$, then (4.19) becomes

$$w^2 + (av + b)w + c = 0 \ , \tag{4.20}$$

for some constants a, b, c. Solving for v yields the expression for v as a function of w obtained in [6].

In the limit where we make the coefficients in ΔW large, both N and H go to infinity. In (4.19), $N^2 - TH^2$ vanishes as $\Lambda_{N=2} \to 0$, and the correct limit is one where we send N and H to infinity, and $\Lambda_{N=2}$ to zero in such a way that the ration $(N^2 - TH^2)/N$ has a finite limit. In this limit, (4.19) becomes $w = (N^2 - TH^2)/(2N)$, and $w \to \infty$ whenever $N \to 0$. Thus the locations of the NS' branes in the v-directions are given by the solutions of $N(v) = 0$. Except for the constant term $N(v)$ is just the derivative of the superpotential, showing that the locations of the NS' in the v-direction are given by the solutions of $W'(v) = \text{const}$. Except for the constant, this reproduces the picture of [10, 11].

summary: The brane configuration can be constructed only at the singular locus of the $N = 2$ Coulomb branch. At a point in the Coulomb branch where the (v, t) curve degenerates and takes the form (3.8) the fivebrane configuration is described by

$$w = [H(v)\sqrt{T(v)}]_+ \pm H(v)\sqrt{T(v)} \ , \tag{4.21}$$

or equivalently

$$w^2 - 2Nw + N^2 - TH^2 = 0, \tag{4.22}$$

where we decomposed $C_{N_c}(v)^2 - \Lambda_{N=2}^{2N_c - N_f} \prod_{i=1}^{N_f}(v + m_i) = S(v)^2 T(v)$, $S(v) = \prod_{i=1}^{l}(v - p_i)$, $T(v) = \prod_{i=1}^{2N_c - 2l}(v - q_i)$ with all q_i distinct, $H(v)$ is a polynomial in v of degree $l - 1$, and $N(v) = [H(v)\sqrt{T}(v)]_+$. $[f(v)]_+$ denotes the part of $f(v)$ with non-negative powers of v and $\pm$ refer to the two asymptotic limits $t \to 0, \infty$.

In following sections we will determine the function $H(v)$ and derive from the fivebrane configuration the vevs for the dyons, meson and baryon fields.

5 Comparison to Field Theory

We will now study the brane description of the superpotential perturbation ΔW (3.2) of the $N = 2$ theory, and compare the results to the field theory analysis in section 3.

5.1 Pure Yang-Mills Theory

Consider a point in the $N = 2$ moduli space of vacua where the (v, t) curve degenerates to a genus $N_c - l - 1$ curve. i.e. the curve takes the form (3.8). As shown in the previous

section, the most general deformation of the brane is (see (4.15))

$$w = N(v) + H(v)\frac{t - C_{N_c}(v)}{\prod_{i=1}^{l}(v - p_i)} \ , \tag{5.1}$$

where $H(v)$ and $N(v)$ are arbitrary polynomials of v. Consider the deformation of the left NS 5-brane. We have to impose the boundary conditions (4.3). As shown in the previous section, the boundary condition $w \to 0$ as $v \to \infty$ and $t \sim \Lambda_{N=2}^{2N_c} v^{-N_c}$ implies that N has to be given by (4.17), explicitly

$$N = \left[H \prod_{j=1}^{2N_c-2} (v - q_j)^{1/2} \right]_+ \ , \tag{5.2}$$

where $[f(v)]_+$ denotes the part of $f(v)$ with non-negative powers of v. The second boundary condition in (4.3) shows that as $v \to \infty$, $t \sim v^{N_c}$, w should behave as $w \to \sum_{k=2}^{N_c} k\mu_k v^{k-1}$. Thus, the relation between $H(v)$ and the values of μ_k can be determined by expanding w as given in (4.18) in powers of v. Using that $T = (t - C)/S$ and $t = 2C_{N_c}(v) + \mathcal{O}(v^{-N_c})$ we find

$$w = 2H(v)\frac{C_{N_c}(v)}{\prod_{i=1}^{l}(v - p_i)} + \mathcal{O}(v^0) = \sum_{k=2}^{N_c} k\mu_k v^{k-1} \ , \tag{5.3}$$

which determines $H(v)$ in terms of μ_k.

Equations (5.3) are precisely the field theory equations (3.14) and (3.15) that determine the $N = 1$ moduli space of vacua after perturbation and the dyon vevs. We see that the M theory fivebrane describes correctly the fact that only the singular part of the $N = 2$ Coulomb branch is not lifted and reproduces the equations that determine the vevs of the massless dyons along the singular locus. The geometrical interpretation of the dyon vevs (3.16) will be given at the end of section 5.2.1.

5.2 $SU(N_c)$ with N_f Flavours

The computation of the dyons vevs along the singular locus is similar to that of the previous subsection. Since $t = 2C_{N_c}(v) + \mathcal{O}(v^{N_f - N_c})$, (5.3) is only valid if $N_f \leq N_c$. However, this does not contradict (3.14), because that result assumes the form of the curve (2.3) which is no longer correct for $N_f \geq N_c$. We have not checked the case $N_f > N_c$ in detail, but believe the correspondence between the field theory and five-brane will still be valid, in particular the dyon vevs are still given by (3.16) where $H(v)$ is the polynomial entering the description of the five brane geometry.

In the following we will compute the dyons vevs at the roots of the non-baryonic branches, for the cases where $N_f - 2r \leq N_c - r$.

5.2.1 Dyon Condensation

Non-Baryonic Branch

Consider a point at the r-th non-baryonic branch root where the $(v, \tilde{t} = \tilde{y} + C_{N_c-r}(v))$ curve (3.20) degenerates to a genus $N_c - r - l - 1$ curve and takes the form (3.21). Using the results in section 4 in the presence of matter, the most general deformation of the brane in our case is (see (4.15))

$$w = N(v) + H(v)\frac{\tilde{t} - C_{N_c-r}(v)}{\prod_{i=1}^{l}(v - p_i)} \ , \tag{5.4}$$

where $H(v)$ and $N(v)$ are arbitrary polynomials of v. We restrict $H(v)$ to be at most of order $l-1$, as discussed in section 3.2.1. Consider the deformation of the left NS 5-brane. The boundary condition $w \to 0$ as $v \to \infty$ and $\tilde{t} \sim \Lambda_{N=2}^{2(N_c-r)} v^{-N_c+r}$ implies again (4.17), i.e.

$$N = \left[H \prod_{j=1}^{2(N_c-r-1)} (v - q_j)^{1/2} \right]_+ \ . \tag{5.5}$$

The second boundary condition, which says that as $v \to \infty$, $\tilde{t} \sim v^{N_c-r}$, w should behave as $w \to \sum_{k=2}^{N_c} k\mu_k v^{k-1}$, yields again the relation between $H(v)$ and μ_k by expanding (4.18) in powers of v. Using $\tilde{t} = 2C_{N_c-r}(v) + \mathcal{O}(v^{N_f-r-N_c})$ we obtain

$$w = 2H(v)\frac{C_{N_c-r}(v)}{\prod_{i=1}^{l}(v - p_i)} + \mathcal{O}(v^0) = \sum_{k=2}^{N_c} k\mu_k v^{k-1} \ , \tag{5.6}$$

which determines $H(v)$.

Equations (5.6) are precisely the field theory equations (3.24) that determine the structure of the non-baryonic branch after perturbation and the dyon vevs. We see that the fivebrane in M theory describes correctly the moduli space of vacua of the $N = 1$ theory corresponding to the non-baryonic, as well as the $r = 0$ case which corresponds to the general points in the moduli space of vacua which are not at the baryonic or non-baryonic roots.

Baryonic Branch

At the baryonic branch root the (v, t) curve degenerates at $2N_c - N_f$ points and factorizes into two rational curves:

$$\begin{aligned} C_L : \qquad t &= v^{N_c}, \qquad w = 0, \\ C_R : \qquad t &= \Lambda_{N=2}^{2N_c-N_f} v^{N_f-N_c}, \qquad w = 0 \ . \end{aligned} \tag{5.7}$$

This makes the construction of the fivebrane configuration easy in this case. In order to satisfy the boundary conditions that describe the $N_c - 1$ left NS' 5-branes we simply replace $t = v^{N_c}, w = 0$ by $t = v^{N_c}, w = \sum_{k=2}^{N_c} k\mu_k v^{k-1}$ in order to satisfy the boundary conditions. The fivebrane curve factorizes into two rational curves:

$$\begin{aligned} \tilde{C}_L : \qquad t &= v^{N_c}, \qquad w = \sum_{k=2}^{N_c} k\mu_k v^{k-1} \\ C_R : \qquad t &= \Lambda_{N=2}^{2N_c - N_f} v^{N_f - N_c}, \qquad w = 0 \ . \end{aligned} \tag{5.8}$$

The field theory analysis suggested that the baryonic branch root is not lifted for generic μ_k. We see from the brane picture that the baryonic branch root is not lifted for arbitrary values of the parameters μ_k.

Geometrical Interpretation of the Dyon Vevs

The dyon vev (3.16) $m_i \tilde{m}_i$ is equal to $\sqrt{2}(H\sqrt{T})(p_i)$. This is equal (up to a factor of $\sqrt{2}$) to the difference between the two finite values of w (4.21) as we take $v = p_i$. The singular $N = 2$ curve (3.8), (4.1) has a double point at $v = p_i, t = C_{N_c}(p_i)$, caused by the existence of a massless dyon. After the perturbation ΔW (3.2) this double point splits into two separate points in (v, t, w) space, and the distance between the points in the w direction is exactly the dyon vev of the dyon that became massless at this point in the $N = 2$ theory. Thus, giving a vev to the field that was responsible for the singularity resolves the singularity. This is analogous to the resolution of the conifold singularity [51], where black holes rather than dyons cause and resolve the singularity. This provides a simple geometrical interpretation of the dyon vevs in the brane picture. In the case of Argyres-Douglas points (see section 8), the singularity is worse than a double point, the dyon vev vanishes at the singularity and the singularity is not completely resolved.

5.2.2 The Meson Vevs

In [6] the eigenvalues of the meson matrix were identified with the values of w at $t = 0, v = -m_i$. One way to see this identification is to note that in the Type IIA set-up in which all the D6-branes are sent to the infinity $x^6 = +\infty$, there are N_f semi-infinite D4-branes ending on the right NS 5-brane from the right. The values of w at $t = v = -m_i$ are the asymptotic positions in the $w = x^8 + ix^9$ direction of these semi-infinite D4-branes. Moreover, the order of the zero of t at these values of w is the the number of D4-branes at these values in the limit $x^6 \to +\infty$.

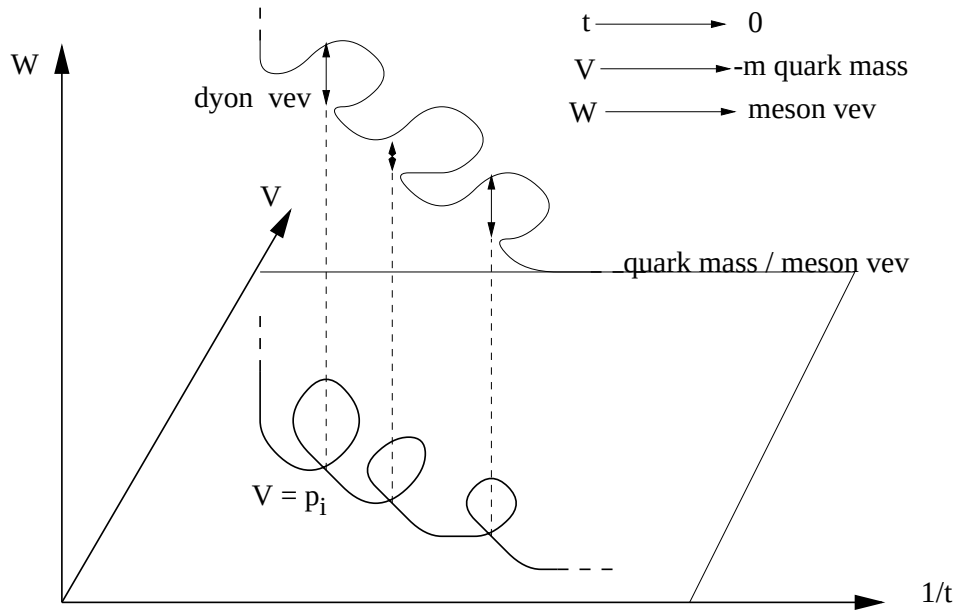

Figure 1: The Dyon and Meson Vevs in the Brane Geometry

The $U(N_f)$ symmetry associated with these N_f semi-infinite D4-branes is a global symmetry of the four-dimensional field theory on the D4-branes which are finite in the x^6 direction. When the D4-branes are separated from each other in the w direction the global symmetry is broken. The only quantity with $U(1)_{8,9}$ charge 2 that can break the $U(N_f)$ flavor symmetry is the meson vev $M^i_j = \widetilde{Q}^i Q_j$.

One massless dyon

Let us now compute the finite values of w at $t = 0, v = -m_i$ and compare to the meson vevs.

At the locus where there is one massless dyon we have that

$$C^2_{N_c}(v) - \Lambda^{2N_c - N_f}_{N=2} \prod_{i=1}^{N_f}(v + m_i) = (v - p)^2 T(v). \tag{5.9}$$

and the function w is given by

$$w = [H\sqrt{T(v)}]_+ \pm H\sqrt{T(v)}\ , \tag{5.10}$$

where H is a constant. We assume $N_f < N_c$. Then we have the following important simplification

$$\sqrt{T(v)} = \frac{C_{N_c}(v)}{v - p} + \mathcal{O}(v^{-1}). \tag{5.11}$$

We can always decompose

$$C_{N_c}(v) = C_{N_c}(p) + (v-p)\tilde{C}_{N_c}(v) \ , \tag{5.12}$$

for some $N_c - 1$ degree polynomial $\tilde{C}_{N_c}$, and we see that

$$\sqrt{T(v)} = \tilde{C}_{N_c}(v) + \mathcal{O}(v^{-1}) \quad \longrightarrow \quad [\sqrt{T(v)}] = \tilde{C}_{N_c}(v). \tag{5.13}$$

We are interested in the finite values of w as $v \to -m_i$ and $y \to 0$. Using (5.13) we find

$$w_i = w(v \to -m_i) = H\tilde{C}_{N_c}(-m_i) \pm H\sqrt{T(-m_i)}. \tag{5.14}$$

Using (5.9) it is clear that $\sqrt{T(-m_i)} = C_{N_c}(-m_i)/(-p-m_i)$ and using the decomposition (5.12) we find that

$$\sqrt{T(-m_i)} = \frac{C_{N_c}(p)}{-p-m_i} + \tilde{C}_{N_c}(-m_i). \tag{5.15}$$

Inserting this back in (5.14), and taking the minus rather than the plus sign (which is the sign that corresponds to $t \to 0$) we get

$$w_i = H\frac{C_{N_c}(p)}{p+m_i} \ , \tag{5.16}$$

To work out $C_{N_c}(p)$ we insert $v = p$ in (5.9) and from that obtain

$$w_i = H\Lambda_{N=2}^{N_c-N_f/2}\frac{\det(p+m)^{1/2}}{(p+m_i)}. \tag{5.17}$$

The last thing we have to do is to compute p and H. The asymptotic behavior of w for large v is

$$w \sim 2H\frac{C_{N_c}(v)}{v-p} \sim 2Hv^{N_c-1} + 2Hpv^{N_c-2} + \dots \ , \tag{5.18}$$

and this should be equal to $N_c\mu_{N_c}v^{N_c-1} + (N_c-1)\mu_{N_c-1}v^{N_c-2} + \dots$ from which we derive that

$$2H = N_c\mu_{N_c}, \qquad p = \frac{(N_c-1)\mu_{N_c-1}}{N_c\mu_{N_c}} \ , \tag{5.19}$$

which then finally gives us for the finite value of w

$$w_i = \frac{1}{2}N_c\mu_{N_c}\Lambda_{N=2}^{N_c-N_f/2}\frac{\det(a_1+m)^{1/2}}{(a_1+m_i)} \ . \tag{5.20}$$

Comparing (5.20) and (3.37) we see that up to a factor of $\sqrt{2}$, the values of w at $t = 0, v = -m_i$ are exactly the meson vevs derived from field theory assuming the low energy effective superpotential (3.35), which was derived using the "integrating in" method with $W_\Delta = 0$.

More than one massless dyon

Let us now compare the result (3.48) for the case with more than one massless dyon to the finite values of w. If there are l massless dyons the $N=2$ curve can be factorized as

$$C_{N_c}(v)^2 - \Lambda_{N=2}^{2N_c-N_f}\prod_{i=1}^{N_f}(v+m_i) = \prod_{i=1}^{l}(v-p_i)^2 T(v) \tag{5.21}$$

In this case, w is given by

$$w = [H(v)\sqrt{T(v)}] \pm H(v)\sqrt{T(v)}, \tag{5.22}$$

where $H(v)$ is a polynomial of degree $l-1$. As in the case with one massless dyon, for $N_f \leq N_c$ we can write

$$H(v)\sqrt{T(v)} = \frac{H(v)C(v)}{\prod_{i=1}^{l}(v-p_i)} + \mathcal{O}(v^{-1}). \tag{5.23}$$

We can always decompose

$$H(v)C(v) = G_0(v) + G_1(v)\prod_{i=1}^{l}(v-p_i) \tag{5.24}$$

with $G_0(v)$ a polynomial of order $l-1$. The same derivation that led to (5.16) shows that in this case the j-th finite value of w is equal to

$$w_j = \frac{G_0(-m_j)}{\prod_{i=1}^{l}(m_j+p_i)}. \tag{5.25}$$

It remains to determine $G_0(-m_j)$. Taking $v=p_i$ in (5.24) and (5.21) we find

$$G_0(p_i) = H(p_i)\sqrt{\prod_{j=1}^{N_f}(m_j+p_i)}\Lambda_{N=2}^{N_c-N_f/2} \tag{5.26}$$

These are l linear equations for the l coefficients of G_0 and can be used to completely determine G_0 in terms of p_i and m_j, and also to determine $G_0(-m_j)$. We omit the details of this calculation, but just give the final result for w_j

$$w_j = \sum_{i=1}^{l}\frac{H(p_i)\det(m+p_i)^{1/2}}{\prod_{t\neq i}(p_t-p_i)(m_j+p_i)}\Lambda_{N=2}^{N_c-N_f/2}. \tag{5.27}$$

Comparing (5.27) and (3.48) we see that if $r_1 = \ldots = r_l = 2$ and $r_j = 1$ for $j > l$, then (3.48) agrees with (5.27), if we identify ϕ with H and a_i with p_i. Therefore it seems that l massless dyons can be described by an $SU(2)^l$ confining $N=1$ theory, and the "integrating in" method works with $W_\Delta = 0$.

It is interesting to note that the above algebraic results continue to agree even when $2l > N_c$, in which case the field theoretical derivation is senseless. This may indicate that the low-energy effective superpotential is still fine, although its derivation is not.

When we compare (5.27) and (3.48) with at least one of the r_i greater than two we see a disagreement. In this case there is a corresponding classical enhancement of the gauge group to $SU(r_i)$. We interpret the disagreement as an indication that the effective low energy superpotential (3.47) does not provide an exact description and $W_\Delta \neq 0$ in the "integrating in" method. Notice that (5.27) has a well-defined limit if we send $p_i \to p_j$, as long as $p_i + m_j \neq 0$. The case when $p_i + m_j = 0$ is briefly commented upon in section 9.

Non-Baryonic Branch

At a generic point in the r-th non-baryonic root, with l additional massless dyons, the fivebrane result for the meson vevs can be read from (5.27)

$$w_j = \sum_{i=1}^{l} \frac{H(p_i)\det(m+p_i)^{1/2}}{\prod_{t\neq i}(p_t - p_i)(m_j + p_i)} \Lambda_{N=2}^{N_c - N_f/2}. \tag{5.28}$$

This result is the same as one would obtain for a confining $SU(2)^l$ subgroup in $SU(N_c - r)$, but does not at all resemble the result (3.50) for a confining $SU(r)$ subgroup in $SU(N_c)$, nor does it look like any of the results obtained with a confining $SU(r) \times SU(2)^l$ in $SU(N_c)$. As we discussed previously this is as an indication that the effective low energy superpotential (3.47) does not provide an exact description at the r-th non-baryonic branch root and $W_\Delta \neq 0$ in the "integrating in" method.

5.2.3 The Baryon Vev

The vev of the baryon operators vanishes on the non-baryonic branches and is non zero on the baryonic branch. The curve at the baryonic root consists of two branches (5.8). In [6] the vev for the baryon operator $\tilde{B}B$ when the vev of the meson matrix vanishes was identified (up to chiral rotation) with the distance between these two branches at $w = 0$. This identification reproduced upon varying w, namely giving a vev to the meson, the correct field theory equation. In our case we do not have the field theory result. However, we can still get the brane prediction for the corresponding equation. We get

$$\Delta t = v^{N_c} - \Lambda_{N=2}^{2N_c - N_f} v^{N_f - N_c} \ , \tag{5.29}$$

where $\sum_{k=2}^{N_c} k\mu_k v^{k-1} = m$, $\Delta t = \tilde{B}B$ and m is the meson vev.

We see that for a given vev m for the meson, the baryon vev can take several values. This indicates that the baryonic branch splits in several parts, which is a novel phenomena that we do not have when we only add a mass for the adjoint chiral multiplet. Another possible interpretation of this phenomenon is that we are not using the right degrees of freedom, namely that we should use several meson and baryon operators to describe the branch.

6 Maximal Number of Mutually Local Massless Dyons

In [6] the fivebrane description of the $N = 2$ theory perturbed by adding a mass term for the adjoint chiral multiplet was studied. With such a perturbation, most of the Coulomb branch is lifted besides a discrete set of points. These points lie on orbits of $\mathbf{Z}_{2N_c-N_f}$ and correspond to points in the moduli space of vacua with the maximal number (N_c-1) of mutually local massless dyons. The root of the baryonic branch as well as the baryonic branch itself remain. The non baryonic branches remain but instead of being mixed branches they emanate from points. The r-th non-baryonic branch that emanated from a submanifold of dimension N_c-r-1 in the Coulomb branch is now emanating from $2N_c-N_f$ points related to each other by $\mathbf{Z}_{2N_c-N_f}$ [1]. At these points there are N_c-r-1 additional mutually local massless dyons.

The above points are characterized geometrically by the fact that the genus N_c-1 hyperelliptic curve degenerates to a genus zero curve. In this section we construct the fivebrane configuration, corresponding to the $N = 2$ theory perturbed by the superpotential ΔW (3.2), at these points. We have $T(v) = v^2 + av + b$ for certain a, b. Following the same arguments as in [6] there exists a coordinate q such that

$$\begin{aligned} v &= \frac{(q-q_+)(q-q_-)}{q} \\ t &= q^{N_c-N_f}(q-q_+)^r(q-q_-)^{N_f-r}. \end{aligned} \tag{6.1}$$

the values of the parameters $q_\pm$ are determined by the requirement that v, t in (6.1) satisfy (4.1) and take the form [6]

$$q_+ = -\frac{N_c-N_f+r}{N_c-r}q_-$$

[1] The $r = N_f/2$ case (N_f even) is exceptional, the $\mathbf{Z}_2$ subgroup is unbroken and the $\mathbf{Z}_{2N_c-N_f}$ orbit consists of $N_c - N_f/2$ points.

$$q_- = \left(\frac{(-1)^{r+N_c}}{4} \left(\frac{N_c - r}{N_c - N_f + r} \right)^{N_c - N_f + r} \right)^{\frac{1}{2N_c - N_f}} \Lambda_{N=2} \ . \tag{6.2}$$

Consider now the superpotential deformations ΔW (3.2). w will be a meromorphic function on the Riemann surface, and is therefore a function of q. This function can be explicitly determined and reads

$$w = \sum k\mu_k \left[\left(\frac{(q - q_+)(q - q_-)}{q} \right)^{k-1} \right]_> \ , \tag{6.3}$$

where now $[a(q)]_>$ denotes the terms with a positive power of q in the power series expansion of $a(q)$ (i.e. we also drop the constant piece). If there is only a perturbation with $\mu_2 \neq 0$, we find $w = 2\mu_2 q$ as expected. Using the explicit expression for w as a function of q we can now determine the two finite values of q as $v \to 0$, i.e. if $q \to q_\pm$. We get that $w_\pm$ is the constant term in the power series in x of

$$w_\pm = \sum_k k\mu_k (-1)^k (x + q_\pm) \frac{(x + q_+)^{k-2}(x + q_-)^{k-2}}{x^{k-2}} \Bigg|_{x^0} \ . \tag{6.4}$$

When only $\mu_2 \neq 0$ we see that $w_\pm = 2\mu_2 q_\pm$ in agreement with [6].

6.1 Pure Yang-Mills

In section 3 we noted that there are N_c points in the moduli space related to each other by the action of the discrete $\mathbf{Z}_{2N_c}$ R-symmetry group (3.4), where $N_c - 1$ mutually local dyons are massless. It was not clear from the field theory analysis under what conditions these vacua are not lifted. The brane picture suggests that these points are not lifted for arbitrary values of the perturbation parameters μ_k. The fivebrane configuration at one of this points takes the form

$$\begin{aligned} v &= t^{1/N_c} - \Lambda_{N=2}^2 (4t)^{-1/N_c} \\ w &= \sum_{k=2}^{N_c} k\mu_k [(t^{1/N_c} - \Lambda_{N=2}^2 (4t)^{-1/N_c})^{k-1}]_> \ , \end{aligned} \tag{6.5}$$

while the configurations at the other $N_c - 1$ points are constructed by applying the Z_{2N_c} action. Again, $[f(t)]_>$ means keeping only the positive powers of t.

6.2 $SU(N_c)$ with N_f Flavours

Massless Quarks

For each $r < N_f/2$ there are $(2N_c - N_f)$ solutions (6.1), (6.2) and (6.3) related by the $\mathbf{Z}_{2N_c-N_f}$ action, while for $r = N_f/2$ (N_f even) there are $(N_c - N_f/2)$ solutions.

The values $w_\pm$ of w at $t = 0$ (6.4) are interpreted as the $(r, N_f - r)$ eigenvalues of the meson matrix $M^i_j = \widetilde{Q}^i Q_j$. The r-th Higgs branch emanating from the $2N_c - N_f$ points where the meson matrix takes the diagonal form with these eigenvalues consists of the orbits of the complexified flavor group $GL(N_f, \mathbf{C})$ acting on this diagonal meson matrix. The diagonal meson matrix is invariant under the subgroup $GL(r, \mathbf{C}) \times GL(N_f - r, \mathbf{C})$ which implies that the Higgs branch is the homogeneous space $GL(N_f, \mathbf{C})/(GL(r, \mathbf{C}) \times GL(N_f - r, \mathbf{C}))$. It has a complex dimension $2r(N_f - r)$ which is the dimension of the r-th non-baryonic branch. It is interesting to note that quations (6.4) express the eigenvalues of the meson matrix for a generic perturbation ΔW in terms of the meson vevs in the presence of only a mass term for the adjoint chiral multiplet $\mu_2 \mathrm{Tr}(\Phi^2)$.

Massive Quarks

A similar construction of the fivebrane configuration can be done when in the presence of a quark mass term $m \sum_i \widetilde{Q}^i Q_i$. In this case, there is no baryonic branch and the curve C does not factorize. The difference between the massive and massless fivebrane configuration is that v is replaced by $v + m$ in (6.1)

$$v + m = \frac{(q - q_+)(q - q_-)}{q} \ , \tag{6.6}$$

and the relation between $q_\pm$ in (6.2) is modified to

$$q_+ = -\frac{1}{N_c - r} \left((N_c - N_f + r) q_- + m \right) \ . \tag{6.7}$$

In addition, we need to modify (6.3), by replacing $(q - q_+)(q - q_-)/q$ by $(q - q_+)(q - q_-)/q - m$, and (6.4) has to be modified accordingly. For every $r = 0, 1, \ldots, [N_f/2]$, there are $2N_c - N_f$ solutions, which however are not related by the discrete R-symmetry group $Z_{2N_c-N_f}$ which is broken by the quark mass term. As in the massless case, the r-th Higgs branch emanating from these points has complex dimension $2r(N_f - r)$.

More complicated mass terms for the quarks can also be treated, but the equation for t as a function of q and the equations for the meson vevs will become much more complicated.

7 Landau-Ginzburg Type Deformations

$N = 1$ gauge theories with a LG type superpotential

$$W = \sum_l \mathrm{Tr}(h_l \tilde{Q} \Phi^l Q) \ , \tag{7.1}$$

have been studied in [52, 53]. These theories have a Coulomb branch parametrized by the gauge invariant order parameters constructed from the vev of Φ. The Coulomb branch parametrizes a family of hyperelliptic curves

$$t^2 - 2C_{N_c}(v)t + \Lambda_{N=2}^{2N_c - N_f} \det(m + \sum h_l v^l) = 0 \ . \tag{7.2}$$

The low energy gauge coupling of the $N = 1$ theory is the period matrix of the hyperelliptic curve (7.2). However, the theory has only $N = 1$ supersymmetry and therefore the special geometry structure of $N = 2$ theories no longer exists and the Kähler potential is not encoded in the curve.

In the following we will discuss the M theory fivebrane configurations that correspond to the $N = 1$ theories with the superpotential (7.1). The simplest example that gives rise to an $N = 1$ theory with a superpotential of the form (7.1) is the one obtained by rotating simultaneously the two NS 5-branes in the IIA picture of the $N = 2$ theory from (x^4, x^5) to (x^8, x^9) by a fixed angle. Let us construct the rotated brane configuration in M theory. The $N = 2$ brane configuration can be written in the form [4]

$$\begin{aligned} tz &= \Lambda_{N=2}^{2N_c - N_f} \prod_{i=1}^{N_f} (v + m_i), \\ t + z &= 2C_{N_c}(v), \\ w &= 0 \ . \end{aligned} \tag{7.3}$$

The first of these equations describes N_f D6-branes located at $t = z = 0$, $v = -m_i$. Recall that in M theory, the D6-branes are Kaluza-Klein Monopoles described by a Taub-NUT space [54]. As a complex surface, the Taub-NUT space is the same as the ALE space of the A_{n-1}-type described by the first equation in (7.3). The last two equations in (7.3) describe the geometry of the Riemann surface. When we rotate the two NS-branes in the type IIA picture by an angle ϕ in the (x^8, x^9) direction, we keep the D6 branes fixed and therefore in the M theory description we do not change the ALE space defining equation.

After the rotation, the last two equations in (7.3) are modified to

$$t + z = 2C_{N_c}(v(\phi)), \tag{7.4}$$

$$w(\phi) = 0 \ , \tag{7.5}$$

where $v(\phi)$, $w(\phi)$ are the rotated coordinates

$$w(\phi) = w\cos\phi - v\sin\phi, \qquad v(\phi) = v\cos\phi + w\sin\phi. \tag{7.6}$$

If we eliminate w from (7.4), and introduce the new variable $\tilde{v} = v(\phi)$, the rotated fivebrane configuration is

$$\begin{aligned} tz &= \Lambda_{N=2}^{2N_c-N_f}\prod_{i=1}^{N_f}(\cos\phi\tilde{v} + m_i), && (7.7)\\ t+z &= 2C_{N_c}(\tilde{v}), && (7.8)\\ w &= \tilde{v}\sin\phi. && (7.9) \end{aligned}$$

The first and second equation describe the Coulomb branch of $N = 1$ SYM with a $W = \sqrt{2}\cos\phi\mathrm{Tr}(\tilde{Q}\Phi Q)$ superpotential (7.2) [1]. If we work to first order in ϕ, $\cos\phi \sim 1$ and the only nontrivial modification is that $w \sim \tilde{v}\phi$. This is an example of one of the functions w found in (4.15), namely the trivial case where $H(v) = 0$ and $N(v) = \phi v$.

This correspondence is very suggestive and suggests that in general the case with $H(v) = 0$ and $N(v)$ some polynomial should correspond to $N = 1$ deformations with superpotentials $\mathrm{Tr}(\tilde{Q}\Phi^l Q)$. However, it is also clear that in order to go beyond infinitesimal deformations we need to also deform the relation between t, z and v. Therefore, the complete deformation is presumably described in terms of two first-order differential equations that control the change of w and the t, z, v equation as functions of some deformation parameter.

As a preliminary check of this idea, suppose the deformed brane is simply given by replacing v and w by functions $\tilde{v}(v, w)$ and $\tilde{w}(v, w)$ in the second and third equation in (7.3). The change of variables should be holomorphic and preserve the complex structure

$$d\tilde{v} \wedge d\tilde{w} = dv \wedge dw. \tag{7.10}$$

We can use the third equation $\tilde{w}(v, w) = 0$ to solve for w as a function of v, $w = w(v)$. Substituting this in the second equation and denoting $h(v) = \tilde{v}(v, w(v))$ we find for the deformed geometry

$$\begin{aligned} yz &= \Lambda_{N=2}^{2N_c-N_f}\prod_{i=1}^{N_f}(v + m_i),\\ y+z &= 2C_{N_c}(h(v)),\\ \tilde{w} &= 0\ . && (7.11) \end{aligned}$$

[1] The theory with $\cos\phi = 0$ has a superpotential $W = 0$ when the masses m_i are zero. It was argued that it corresponds to a non trivial fixed point [50]. In this case the curve that describes the fivebrane factorizes to $t = 0, z = 2C_{N_c}$ and $z = 0, t = 2C_{N_c}$. It would be interesting to study the physics of this fivebrane configuration.

If h has a left inverse $p(h(v)) = v$, we can introduce a new variable $v' = h(v)$, and extract from (7.11) the curve

$$t^2 - 2C_{N_c}(v')t + \Lambda_{N=2}^{2N_c-N_f} \det(m + p(v')) = 0 \ , \tag{7.12}$$

which is of the form (7.2). It would be interesting to derive the form of $\tilde{v}, \tilde{w}$ that gives rise to a specific polynomial $p(v')$ from first principles.

Taking the curve in (7.11) as a starting point, functions $\tilde{w}$ on it are again classified by (4.15), and one expects that suitable functions $\tilde{w}$ correspond to adding superpotential terms $\mu_k \mathrm{Tr}(\Phi^k)$ as in the pure $N = 2$ case. In field theory, adding superpotentials of the form (7.2) and $\mu_k \mathrm{Tr}(\Phi^k)$ leads to meson vevs which are given by (3.48) but with $a_i + m$ replaced by $p(a_i) + m = m + \sum h_l a_i^l$. The same result can be derived from the brane if the function $H(v)$ entering $\tilde{w}$ is given by

$$H(v) = \prod_j \left(\frac{p(v) - p_j}{v - h(p_j)} \right) h'(v)\phi(h(v)) \tag{7.13}$$

and $N(v)$ is defined by (cf. (5.24)

$$C(v)H(v) = G_0(v) + N(v) \prod_{i=1}^{l} (v - p_i) \tag{7.14}$$

where $G_0(v)$ is a polynomial of degree $l-1$. We leave a further study of these deformations to a future work.

8 Superconformal Field Theory

As is well known, $N = 2$ field theories have $N = 2$ critical points where mutually non-local degrees of freedom become massless. The simplest case is pure $SU(3)$ gauge theory which has two points in its moduli space where mutually non-local degrees of freedom become massless [55]. More general cases have been studied in [56–58]. At all critical points certain dyons become massless, and these theories therefore can be broken to $N = 1$ by suitable superpotentials $\sum \mu_k \mathrm{Tr}(\Phi^k)$. For $SU(3)$, it was argued in [55] that the corresponding $N = 1$ theories might correspond to non-trivial $N = 1$ fixed points, for the points of highest criticality in $SU(N_c)$ see [46]. Here we will rederive these arguments from the brane picture, and study the brane configurations that describe $N = 1$ fixed points.

To show why the perturbed $N = 2$ theories are candidates for $N = 1$ fixed points, we need to analyze the dyon vevs (3.16). In the neighborhood of an $N = 2$ critical point,

there is a dyon condensate, and the $U(1)$ under which the dyon is charged has a gap. For a non-trivial $N = 1$ fixed point, a necessary but not sufficient condition is that this gap has to vanish as we approach the critical point.

As n of the q_m approach the p_i, say $q_m = p_i + \epsilon$ for $m = 1, \ldots, n$ while keeping H fixed, we find that the dyon vev $m_i \tilde{m}_i$ (3.16) behaves as $\epsilon^{n/4}$, and this indeed vanishes as $\epsilon \to 0$. Geometrically, this is not a double point, and it is not completely resolved.

Let us now study the brane configuration at $N = 1$ critical points in more detail. Near a point where a dyon becomes massless the brane geometry is described by

$$(w - N(v))^2 = T(v)H(v)^2 \tag{8.1}$$

$$(t - C_{N_c}(v))H(v) = (w - N(v))S(v). \tag{8.2}$$

We assume here that $H(v)$ is nonzero at the point, say $v = p_i$, where the dyon becomes massless. If $H(v)$ has a first order zero at this point, we can divide the second relation by $(v - p_i)$ and still have a good description. If $H(v)$ has a second order or higher order zero the two equations do not describe the brane geometry near $v = p_i$. Notice that the first equation in (8.1) cannot be replaced by the equation for the $N = 2$ curve, because then the second equation at $v = p_i$ would then reduce to $0 = 0$ and not constrain w, and this is incorrect. Although the $N = 2$ curve was singular at $v = p_i$, the two equations in (8.1) describe a smooth brane geometry, indicating that there is no longer any massless matter at the point $v = p_i$, and indeed there is a dyon condensate in the $N = 1$ theory.

Now consider the case where

$$C_{N_c}(v)^2 - \Lambda_{N=2}^{2N_c - N_f} \det(v + m) = (v - p)^{2k} \prod_{i=1}^{l} (v - p_i)^2 T(v) \ , \tag{8.3}$$

where we distinguish two cases, either $T(v)$ has a single zero at $v = p$ or it is nonzero at $v = p$. The deformed brane configuration is still given by (8.1), where $H(v)$ is of order $k + l - 1$. The matrix of derivatives of (8.1) with respect to (w, v, t) at reads

$$J = \begin{pmatrix} 2(w - N(v)) & 2(w - N)N'(v) - (TH^2)'(v) & 0 \\ -S(v) & tH'(v) - (C_{N_c}H)'(v) - wS'(v) + (NS)'(v) & H(v) \end{pmatrix}. \tag{8.4}$$

One readily verifies that this matrix has rank two at $v = p$, $t = C_{N_c}(p)$ and $w = N(p) \pm H(p)\sqrt{T(p)}$, unless $H(p) = 0$. It does not matter whether $T(v)$ is zero at $v = p$ or not. This implies for example that the Argyres-Douglas points in $SU(3)$ pure gauge theory, perturbed by a $\mathrm{Tr}(\Phi^3)$ perturbation which corresponds to a case where $k = 1$, $T(p) = 0$ and $H = \text{const}$, are described by a smooth brane configuration [1]. It is quite intriguing that

[1] Of course, the singular point $t = 0$ is infinitely far away, but even if we include the point $t = 0$ the brane configuration is still smooth.

smooth brane configuration can give rise to an $N = 1$ fixed point. One could consider this as an indication that the Argyres-Douglas points perturbed by a $\mathrm{Tr}(\Phi^3)$ superpotential do in fact correspond to conventional field theories. However, as we will discuss shortly, one cannot necessarily draw conclusions based on local properties of the five-brane alone.

We briefly indicate the nature of the singularity in the brane in the singular case where $H(p) = 0$. We write that $H(v) = (v-p)^r \tilde{H}(v)$, and assume that $r \leq k$. We will also decompose $T(v) = (v-p)^\nu \tilde{T}(v)$ with $\nu = \pm 1$, and $S(v) = (v-p)^k \tilde{S}(v)$. In terms of the new variables $\tilde{v} = v - p$, $\tilde{w} = w - N(v)$, and $y = t - C_{N_c}(v)$, the geometry reads

$$\tilde{w}^2 = \tilde{v}^{2r+\nu} \tilde{T}(\tilde{v}) \tilde{H}(\tilde{v})^2 \tag{8.5}$$

$$y \tilde{H}(\tilde{v}) = \tilde{v}^{k-r} \tilde{S}(\tilde{v}) \tilde{w} \tag{8.6}$$

and near $v = p$ we effectively have

$$\tilde{w}^2 = \tilde{v}^{2r+\nu}, \qquad y = \tilde{v}^{k-r} \tilde{w}. \tag{8.7}$$

This clearly describes a singular brane configuration. In order to determine whether this brane configuration corresponds to a non-trivial $N = 1$ fixed point, we would need to determine the dimensions of some of the operators at the fixed point. In the next section we will discuss how this can in principle be done using the fivebrane configuration. A detailed calculation will be left for future work.

8.1 Scaling Dimensions, the Kähler Potential and the Superpotential

At the $N = 1$ fixed point, there is an unbroken $U(1)_R$ symmetry. Unfortunately, it is an accidental symmetry, making it hard to determine the dimensions of the chiral fields that give rise to relevant perturbations. In $N = 2$ theories, the dimensions can be determined using the fact that contour integrals of the Seiberg-Witten differential needs to have dimension one [55–58]. In the $N = 1$ theories discussed in [59], the holomorphic three-form of Calabi-Yau manifolds could be used to fix the dimensions of operators. What distinguishes these two cases from the present one is that in both the dimensions could be determined by a local calculation near the singularity in the Riemann surface and the Calabi-Yau manifold respectively. In the present case, the complete global structure of the fivebrane seems relevant in order to determine the normalization of fields, and this explains why smooth brane configurations can still correspond to non-trivial $N = 1$ theories.

If there is a physical quantity with a fixed dimension which depends only on a neighborhood of the point $\tilde{v} = \tilde{w} = \tilde{t} = 0$, then we can read of the relative dimensions of $\tilde{v}, \tilde{w}$

and $\tilde{t}$ from (8.7) using the obvious $U(1)_R$ symmetry it possesses (and using the fact that $\Lambda_{N=2}$ has $U(1)_R$ weight zero). The overall normalization of operators would then follow from the scaling behavior of this physical quantity[1].

However, if there is no physical quantity which depends only a neighborhood of the point $\tilde{v} = \tilde{w} = \tilde{t} = 0$, then in order to determine the dimensions of operators, we need to explicitly compute another object whose dimension is known. Examples are the Kähler potential, which has dimension two, and the superpotential, which has dimension three. The Kähler potential is a non-holomorphic quantity and its computation using the brane geometry is still an open problem. The complication in performing this computation is the need for a consistent decoupling of the Kaluza-Klein modes.

The computation of the superpotential seems an easier task. In [7] the following expression for the superpotential was proposed. Let $\Sigma \subset \mathbf{R}^5 \times S^1$ be the Riemann surface part of the fivebrane, and $\Sigma_0 \subset \mathbf{R}^5 \times S^1$ be another Riemann surface which has the same asymptotic behavior at infinity. If $B \subset \mathbf{R}^5 \times S^1$ is a three-manifold with boundary $\Sigma - \Sigma_0$, then

$$W(\Sigma) - W(\Sigma_0) = \int_B \Omega \tag{8.8}$$

where Ω is the holomorphic three-form

$$\Omega = R\frac{dt}{t} \wedge dv \wedge dw. \tag{8.9}$$

This superpotential has been computed in some examples in [7, 35]. One of the problems with this definition of the superpotential is the dependence on the the surface Σ_0. We are only interested in the behavior of the superpotential as a function of some parameter, say g. If we change a parameter it is possible that we change the asymptotic behavior of the fivebrane, in which case we would have to choose a new surface Σ_0 as well. The dependence of Σ_0 on g is not fixed by anything, making the outcome of this calculation highly ambiguous. Keeping Σ_0 fixed anyway would lead to the result

$$\frac{\partial W}{\partial g} = \int_\Sigma \Omega_2 \tag{8.10}$$

[1] At the point of highest criticality in pure $SU(N_c)$ gauge theory with the most singular choice of H in (8.7), we have $2r + \nu = N_c - 2$ and $k - r = 1$. Then (8.7) would yield the following relations between dimensions (indicated by square brackets): $2[\tilde{w}] = 2[\tilde{H}] + (N_c - 2)[\tilde{v}]$ and $[y] + [\tilde{H}] = [\tilde{w}] + [\tilde{v}]$. In addition the superpotential contains $\tilde{H}\mathrm{Tr}(\Phi^{N_c})$ leading to an additional relation $[\tilde{H}] + N_c[\tilde{v}] = 3$. From this one can derive that $[y] = \frac{N_c(3-[\tilde{w}])}{N_c+2}$ and $[\tilde{v}] = \frac{2(3-[\tilde{w}])}{N_c+2}$. At the $N = 2$ point the dimensions of $\tilde{v}, y$ [57] are recovered if we take $[\tilde{w}] = 2$, the standard dimension of a meson in the UV. If for some reason the dimension of $\tilde{w}$ would remain two even in the perturbed theory, the dimensions of operators in the $N = 1$ points would be the same as those at the $N = 2$ points.

with

$$\Omega_2 = \frac{R}{t}\left(\frac{\partial t}{\partial g}dv \wedge dw + \frac{\partial v}{\partial g}dw \wedge dt + \frac{\partial w}{\partial g}dt \wedge dv\right). \tag{8.11}$$

On the other hand, in all examples considered in [7, 35] the superpotential turns out to be, roughly speaking, a weighted sum of contour integrals

$$W \sim \sum a_i R \oint_{C_i} vw\frac{dt}{t} \tag{8.12}$$

where the C_i are contour integrals around the various infinite 'tubes' that stick out of the Riemann surface Σ. A reduction of the expression in (8.11) to one of the type (8.12) will be useful in order to arrive at an expression for the superpotential which does correctly reproduce the field theory superpotentials without having to choose an additional surface Σ_0. With such an expression it would be quite easy to determine for example the dimensions of operators at an $N = 1$ critical point. Notice that (8.12) does depend on the asymptotic behavior of the brane, and does not seem to care whether the brane has a singular point or not. We will not carry out the computation here but comment how we can recognize several pieces of the field theory superpotential in (8.12). As $v \to -m_i$ and $w \to w_i$, there is an infinite tube stretching out with $t \to 0$. The contour integral around this tube contributes in (8.12) a term proportional to $m_i w_i$. Here we recognize the term in the superpotential which is simply the mass term for the meson M_i. This provides another explanation why the finite values of w as $v \to -m_i$ should be related to the meson vevs.

Another term can be seen in the case where we have the maximal number of mutually massless dyons, in the parametrization (6.1) and (6.3). Then there is an infinite tube with $q \to 0$, and the contour integral around that tube yields a term proportional to $q_+ q_- w_1$, where w_1 is the coefficient of the term in w that is linear in q in (6.3). It is a highly non-trivial result that this is proportional to $\sum_k \mu_k u_k$, where u_k is the vev of $\mathrm{Tr}(\Phi^k)$ at the point with the maximal number of massless dyons. The explicit form of u_k is given in (10.1). Thus it seems that a suitable version of (8.12) does reproduce precisely the field theory superpotential.

9 Example: $SU(3)$ with $N_f = 2$

To illustrate several of the things discussed in the paper, we will now discuss the example of $SU(3)$ gauge theory with two flavors and superpotential $W = \mu_2 \mathrm{Tr}(\Phi^2) + \mu_3 \mathrm{Tr}(\Phi^3)$ in some detail. The $N = 2$ theory is described by the curve

$$t^2 - 2t(v^3 - \frac{u_2}{2}v - \frac{u_3}{3}) + \Lambda^4_{N=2}(v + m_1)(v + m_2) = 0 \tag{9.1}$$

The first thing is to find the locus in moduli space where a dyon becomes massless. At this locus, we have

$$(v^3 - \frac{u_2}{2}v - \frac{u_3}{3})^2 - \Lambda_{N=2}^4(v+m_1)(v+m_2) = (v-a)^2 T(v) \tag{9.2}$$

for some a. From this we find that u_2, u_3 should be given by

$$\begin{aligned} u_2 &= 6a^2 - b^{-1}(2a+m_1+m_2)\Lambda_{N=2}^2 \\ u_3 &= -6a^3 + \frac{3}{2}ab^{-1}(2a+m_1+m_2)\Lambda_{N=2}^2 - 3b\Lambda_{N=2}^2 \end{aligned} \tag{9.3}$$

where

$$b = \pm\sqrt{(a+m_1)(a+m_2)}. \tag{9.4}$$

We will take the plus sign in b from now on, the discussion with a minus sign in b proceeds in a similar way. Now recall that the deformed brane configuration is described by (see (4.15) and (4.19), H must in this case be equal to a constant)

$$w^2 - 2N(v)w + N(v)^2 - H^2T(v) = 0 \tag{9.5}$$

$$(w - N(v))(v-a) = (t - C_3(v))H \tag{9.6}$$

and by explicit computation we find

$$N(v) = H(v^2 + av + (a^2 - \frac{u_2}{2})) \tag{9.7}$$

$$N^2 - H^2T = H^2(-2a^3 + au_2 + \frac{2}{3}u_3)v + H^2(-4a^4 + \Lambda_{N=2}^4 + 2a^2u_2 + \frac{4}{3}au_3) \tag{9.8}$$

For the dyon vev we find

$$(m\tilde{m})^2 = H^2\sqrt{T(a)} = H^2(6a\Lambda_{N=2}^2 b + \frac{(m_1+m_2)^2}{4b^2}\Lambda_{N=2}^4). \tag{9.9}$$

The asymptotic behavior of w tells us which values of μ_2, μ_3 the deformed brane corresponds to. From (9.7) we see that $w \sim 2Hv^2 + 2Hav$ and therefore

$$3\mu_3 = 2H, \qquad 2\mu_2 = 2Ha \tag{9.10}$$

consistent with what we found in general in (5.18).

Using (9.3) and (9.7) we can solve explicitly for w as $v = -m_1$ or $v = -m_2$. We find that in both cases one of the two solutions is equal to

$$w_i = H\Lambda_{N=2}^2 \frac{b}{a+m_i} \tag{9.11}$$

which confirms (5.20).

Let us now consider what happens when we tune the parameters further, so that we get more massless particles. On the one hand, we can tune parameters so that mutually non-local dyons become massless. This happens whenever $T(v)$ will be divisible by additional factors of $(v-a)$. The point of highest criticality is reached when

$$u_2 = 15a^2, \quad u_3 = -60a^3, \quad m_1 + m_2 = -\frac{16}{5}a, \quad m_1 m_2 = 4a^2 \tag{9.12}$$

and

$$a^4 = \frac{4}{405}\Lambda^4. \tag{9.13}$$

For these values of the parameters the equation for the $N=2$ curve reads

$$y^2 = (v-a)^5(v+5a) \tag{9.14}$$

where $y = t - C_3(v)$. One way to approach this point of highest criticality is to start with a point with two mutually local massless dyons where $y^2 = (v-a)^2(v-b)^2 T(v)$, and then to tune the parameters in such a way that $b \to a$ and $T(v)$ becomes divisible by $(v-a)$. When there are two mutually local massless dyons, supersymmetry is unbroken for each value of μ_2 and μ_3 in the superpotential, and this shows that the same is true for the point of highest criticality. The equations describing the brane configuration at this point read (where $\tilde{w} = w - N(v)$ and $\tilde{v} = v - a$)

$$\tilde{w}^2 = H(\tilde{v})^2 \tilde{v}(\tilde{v} + 6a), \qquad yH(\tilde{v}) = \tilde{v}^2 \tilde{w}. \tag{9.15}$$

This brane configuration is smooth unless $H(\tilde{v}) \sim \tilde{v}$, in which case it reduces to (near $\tilde{v} = 0$) $\tilde{w}^2 = 6a\tilde{v}^3$, $y = \tilde{v}\tilde{w}$. To reach this singular brane configuration we need to take the parameters in the superpotential such that $2\mu_2 = 3a\mu_3$. We expect therefore a qualitative difference between the different $N=1$ theories obtained by perturbing the highest $N=2$ critical point, depending on whether $2\mu_2 = 3a\mu_3$ or not. One could speculatethat only in the latter case we obtain an $N=1$ superconformal field theory, but for this we need a more detailed understanding of the relation between brane geomety and the appearance of superconformal fixed points in field theory, as discussed in the previous section.

Finally, we consider the points where two mutually local dyons become massless. We take $m_1 = m_2 = 0$ for simplicity. There are two distinct possibilities, either

$$C_3(v)^2 - \Lambda_{N=2}^4 v^2 = v^4(v^2 \pm 2\Lambda^2) \tag{9.16}$$

with

$$u_2 = \mp 2\Lambda^2 \tag{9.17}$$

or

$$C_3(v)^2 - \Lambda_{N=2}^4 v^2 = (v-a)^2(v-\omega a)^2(v+2a)(v+2\omega a) \tag{9.18}$$

with

$$\omega^3 = 1, \quad a = \frac{2}{9}\Lambda^2(1-\omega), \quad u_2 = 2(3a^2 - \Lambda^2), \quad u_3 = -6a^3. \tag{9.19}$$

These two cases correspond to the non-baryonic branch roots with $r = 1, 0$ respectively. We take the first case as our example, and take $H(v) = h_0 v + h_1$. Then $N(v) = h_0 v^2 + h_1 v \pm h_0 \Lambda_{N=2}^2$, and we see that $3\mu_3 = h_0$ and $2\mu_2 = h_1$. The equation for w reads

$$w^2 - 2N(v)w - 2h_0 h_1 \Lambda_{N=2}^2 v + h_0^2 \Lambda_{N=2}^4 \mp 2h_1^2 \Lambda_{N=2}^2 = 0. \tag{9.20}$$

If we now make the substitution

$$v = q \mp \frac{1}{2}\Lambda_{N=2}^2 q^{-1} \tag{9.21}$$

then the equation for w factorizes,

$$(w - 2h_0 q^2 - 2h_1 q)(w \pm h_1 \Lambda_{N=2}^2 q^{-1} - \frac{1}{2} h_0 \Lambda_{N=2}^4 q^{-2}) = 0 \tag{9.22}$$

and so does the equation for t,

$$\frac{1}{4q^3}(t - 2q^3 \pm \Lambda^2 q)(\pm\Lambda^6 - 2\Lambda^4 q^2 + 4q^3 t) = 0. \tag{9.23}$$

The equation for the deformed brane is obtained by taking both for w and t the first factor in the equations. In particular, $w = 2h_0 q^2 + 2h_1 q$, in perfect agreement with (6.3) and (6.1). The parameters q_+ and q_- are given by $q_+ = \sqrt{\pm\frac{1}{2}}\Lambda_{N=2}$, $q_- = -q_+$. The meson vevs are simply given by substituting $q_\pm$ in the expression for w. The results agree with (6.4), which tells us that $w_\pm = 2h_1 q_\pm - 2h_0(q_\pm^2 + 2q_\pm q_\mp)$, upon using $q_- = -q_+$.

Finally, we compare these meson vevs with the ones obtained from (5.27). With two massless dyons, (5.27) reduces to

$$w_j = \frac{f_j(p_1) - f_j(p_2)}{(p_2 - p_1)}\Lambda_{N=2}^2, \tag{9.24}$$

where

$$f_j(p) = \frac{H(p)\det(m+p)^{1/2}}{p + m_j}. \tag{9.25}$$

As we take $p_1 \to p_2$, w_j approaches $-f_j'(p_1)$, but this does not have a well-defined limit as we send $p \to 0$ and $m_i \to 0$. This problem can be traced back to the derivation of

(5.27), where we assumed there that $m_j + p_i$ is not equal to zero. If $p_i = m_{2i-1} = m_{2i}$ for $i = 1, \ldots, r$, then $C_{N_c}(v)$ must also be divisible by $\prod_{i=1}^{r}(v - p_i)$, and (5.21) becomes

$$\tilde{C}_{N_c-r}(v)^2 - \Lambda_{N=2}^{2N_c-N_f} \prod_{i=2r+1}^{N_f} (v + m_i) = \prod_{i=r+1}^{l} (v - p_i)^2 T(v) \tag{9.26}$$

The derivation of (5.27) remains the same, except that in the final answer only the masses m_j with $j > 2r$ and the p_i with $i > r$ appear. Equation (5.27) therefore only provides us with the meson vevs of the mesons M_j with $j > 2r$. The other meson vevs M_j with $j \leq 2r$ cannot be written in a simple way. This also explains why when we consider a theory at a NB branch root, we only get at most one meson vev from (5.27).

Coming back to the example we are considering, after taking out a factor of v^2 from (5.21), we are left with no masses at all, and in this case (5.27) does unfortunately not yield any information about the meson vevs. This happens always at the NB branch root with $2r = N_f$.

10 Discussion

The present work points at several interesting directions to pursue further. We have shown, using the results obtained from the fivebrane, that while for vacua where the classical enhanced gauge group is $SU(2)$ the effective superpotential obtained by the "integrating in" method is exact, this is no longer true when the classical enhanced gauge group is $SU(r), r > 2$. It would be interesting to find W_Δ in this case, and to see which singular submanifolds of the $N = 2$ Coulomb branch the superpotential parametrizes. Finding W_Δ can presumably be done by computing the superpotential using the fivebrane configuration as suggested by [7].

We have derived several results from the fivebrane which should be understood from the field theory viewpoint. Among these phenomena are the splitting of the baryonic branch in section 5 and the simple expression for the meson vev (6.4). It is also curious to note that at the roots of the non-baryonic branch with a maximal number of massless dyons, there exists an expression for the gauge invariant order parameters u_k of the form

$$u_k = (-1)^k (k-1)(2N_c - N_f) q_+ q_- \sum_{l=0}^{k-2} q_+^l q_-^{k-2-l} \begin{pmatrix} k-1 \\ l \end{pmatrix} \begin{pmatrix} k-2 \\ l \end{pmatrix} \frac{1}{l+1} , \tag{10.1}$$

and the finite values of w (6.4) are proportional to the derivatives of u_k with respect to $q_\pm$. The field theory explanation for this should presumably rely on identifying an appropriate set of operators to describe the modified non-baryonic branches in view of

the perturbation ΔW (3.2). These are all parts of a bigger picture describing the complete structure of the Higgs branches and their intersection with the Coulomb branch when we perturb the $N = 2$ gauge theory by ΔW (3.2). We have clearly seen that this structure is much richer than in the case where the perturbation includes only a mass term for the adjoint chiral multiplet and it deserves further study.

Related to the above mentioned problem of identifying the appropriate set of operators in each situation are the various dualities the field theory can possess. The brane gives a uniform geometrical description, whereas the field theoretical descriptions depend on the relevant weakly coupled degrees of freedom. The field theory has has for example a dual magnetic description [60, 61], and as in the case of pure $N = 1$ gauge theory with matter this may be related to interchanging the role of v and w [37]. On the other hand, (4.19) looks like a curve for a $SU(N_c - l)$ gauge theory with $N_c - l - 1$ flavors, so by interchanging the role of t and w we may end up with such a completely novel dual description.

Other $N = 2$ theories like the product gauge theories in [4] can also be perturbed by superpotentials in the brane framework. If there are only mass terms for the adjoints, the relevant configuration is presumably the one given in [35]. Perturbing these theories by higher order superpotentials can lead to new families of $N = 1$ fixed points.

For fixed μ_k there are generally only a finite number of points on the Coulomb branch that are not lifted. Thus, there can be domain walls as in [7] and it would be interesting to know their behavior as a function of μ_k.

We have not completed the study of the $N = 1$ gauge theories with Landau-Ginzburg type superpotential in the fivebrane framework. This is an interesting direction by itself which should be pursued in order to learn about field theory from the fivebrane.

The study of non trivial IR fixed points using the M theory fivebrane is also not completed. In particular it is still not clear whether the fivebrane can geometrically distinguish between trivial and not trivial fixed points. We have not completed the calculation of the superpotential at the fixed points. A related issue is to study the case with vanishing superpotential [60, 61, 50] on which we commented in section 7.

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