INTERIM REPORT ON AN EXACT ANALYSIS OF A LIMITED PLANE PLASMA IN A MAGNETIC FIELD

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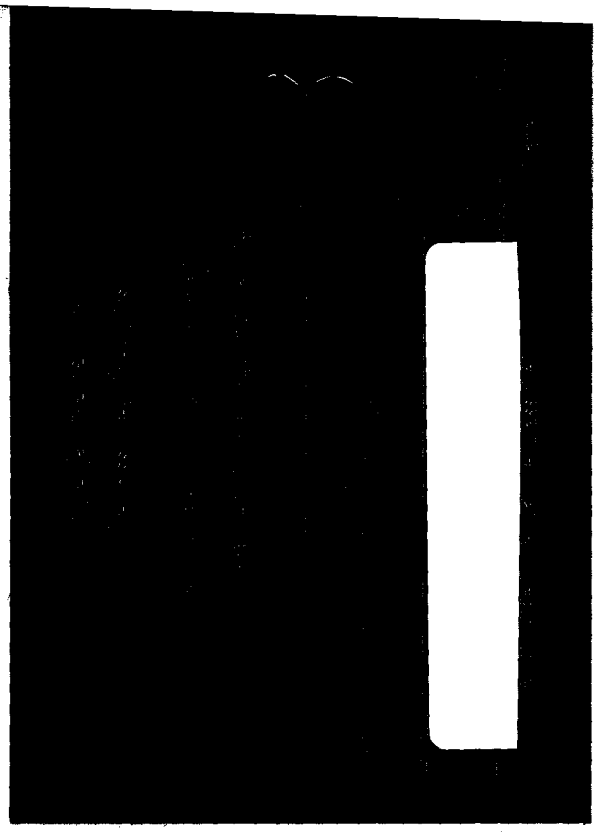


INTERIM REPORT

on

AN EXACT ANALYSIS OF A LIMITED PLANE PLASMA IN A MAGNETIC FIELD

Lewi Tonks

I. INTRODUCTION

Prior analyses of a plasma in a magnetic field have been limited, as far as the writer is aware, to cases in which the relative change in field over the orbital distance and the relative change in ion concentration are both small, or in which the relations have been viewed in a purely hydrodynamical way. The first approach excludes cases which can be of considerable interest, for it fails for a plasma edge. The second loses all sight of the structure imposed by the orbital motions.

II. FORMULATION OF PROBLEM

The present analysis attempts to overcome these limitations, although it may impose others. The problem attacked is that of describing exactly the relations which exist between a limited plane plasma and a vacuum. It is assumed that there are no collisions, and, for the present, that positive and negative particles are present in equal numbers, have the same mass m and are mono-energetic with speed v_0 in the plane perpendicular to the field.

It does not seem possible to start with an assumed density distribution because the distribution chosen might well be inconsistent with the existence of orbits, and the problem of distribution in velocity space remains open; certainly at the plasma boundary there are no particles moving perpendicular to it. Nor is it possible to begin with an assumed field distribution. Seemingly, the only approach is to describe the plasma as if it were being built up from its face inward essentially in the way chosen here.

Referring to Fig. 1, the vacuum magnetic field B_0 is uniform and in the z-direction. The plasma is uniform in the x- and z-directions and extends to the right from $y = 0$. Consider only positive particles which, in a field B have an angular velocity $\omega = eB/mc$. We now define a Class η particle as one which, (a) crosses an element of area $d\eta\, d\zeta$ at ξ, η, ζ, (b) has its velocity lying within an element of angle $d\theta$ pointing in the negative x-direction, and (c) crosses within an element of time dt at $t = 0$. The particle distribution in the plasma is completely defined when the spatial distribution of Class η particles is fixed. Let this be $\sigma(\eta)$. Then there are

$$\sigma(\eta)\, d\theta\, d\eta\, d\zeta\, dt \tag{1}$$

particles in the angle-space-time element. The angle $d\theta$ embraces a bundle of trajectories whose points of minimum y ($\dot{y} = 0$, $\ddot{y} > 0$) which will be called their apogees, lie very close to $d\eta\, d\zeta$ and extend over a range $d\xi$ of ξ given by

$$d\xi = (v_0/\omega(\eta))\, d\theta$$

These trajectories diverge until at some other locale, x, y, they cover a range of dy given by

$$dy = \left.\frac{\partial y}{\partial \xi}\right|_{t,\eta} d\xi$$

where, of course, $\partial y/\partial \xi$ refers to the family of trajectories

$$x = x(\xi, \eta, t), \qquad y = y(\xi, \eta, t) \tag{1.5 A,B}$$

At x, y the Class η particles will occupy an x-extent of

$$dx = -\dot{x}\, dt$$

By direct substitution we now find the number of Class η particles at x, y to be expressible as

$$\frac{(\sigma\, \omega(\eta)\, v_0)}{-\dot{x}\ \partial y/\partial \xi}\, dx\, dy\, dz\, d\eta$$

This can be simplified by noting that the uniformity with respect to x assures that when t is eliminated between Eq (1.5 A) and (1.5 B) y will be a function of $x - \xi$ rather than of either coordinate individually. Accordingly

$$-\dot{x}\ \partial y/\partial \xi = \dot{x}\ \partial y/\partial x = \dot{y}\ (\eta, y)$$

We note further that each particle makes a double contribution to the density, once on the outward leg ($\dot{y} > 0$) and again on the inward leg ($\dot{y} < 0$) of its journey. It follows that the density of Class η particles at x, y is

$$n(y) = \frac{2\ \sigma(\eta)\ \omega(\eta)}{v_o\ |\dot{y}\ (\eta, y)|}$$

and the density of all of the positive particles is

$$n(y) = \frac{2}{v_o} \int_{\eta(y)}^{y} \frac{\sigma(\eta)\ \omega(\eta)}{|\dot{y}\ (\eta, y)|}\ d\eta \qquad (2)$$

where, when y is small, $\eta(y) = 0$ because every trajectory encountered between the plasma edge and y reaches to y; but when y is so great that some trajectories do not penetrate to it, then $\eta(y)$ is the coordinate of the apogee of the trajectory whose perigee lies at y. A more general point of view is that trajectories are present throughout the vacuum as well as the plasma but that for $\eta < 0$, $\sigma(\eta) = 0$. With this concept no juggling of $\eta(y)$ at the lower limit is necessary. Such juggling can, however, be convenient.

The current density of all positive particles is evidently

$$j_x(y) = \frac{2e}{v_o} \int_{\eta(y)}^{y} \frac{\sigma(\eta)\ \omega(\eta)\ \dot{x}\ (\eta, y)}{|\dot{y}\ (\eta, y)|}\ d\eta \qquad (3)$$

A further integration is required to calculate the magnetic field change.

$$B(y) = B_o + \frac{8\pi e}{c\ v_o} \int_0^{y} dy' \int_{\eta(y')}^{y'} \frac{\sigma(\eta)\ \omega(\eta)\ \dot{x}\ (\eta, y')}{|\dot{y}\ (\eta, y')|}\ d\eta \qquad (4)$$

The equations of the particle trajectories are the usual

$$\ddot{y} = -\frac{eB(y)}{mc}\dot{x} ; \quad \ddot{x} = \frac{eB(y)}{mc}\dot{y} , \qquad (5), (6)$$

and because of symmetry we can proceed by letting σ be the sum of both positives and negatives.

This set of equations can be started toward dimensionless form by the substitution of the dimensionless capitals for the lower-case variables as follows:

$$\omega = \Omega \frac{eB_o}{mc} = \Omega\, \omega_o \qquad (7)$$

$$t = T/\omega_o; \quad (x, \eta, y, y') = \frac{v_o}{\omega_o} (X,H,Y,Y') \qquad (8),(9)$$

The new equations are

$$\Omega (Y) = 1 + \frac{8\pi e^2 v_o}{mc^2 \omega_o^2} \int_o^Y dY' \int_{H(Y')}^{Y'} \frac{\sigma (v_o H/ \omega_o)\Omega (H)\, dX/dT}{|dY/dT|} dH$$

$$\frac{d^2Y}{dT^2} + \Omega (Y) \frac{dX}{dT} = 0 ; \quad \frac{d^2X}{dT^2} - \Omega (Y) \frac{dY}{dT} = 0. \qquad (10),(11)$$

It only remains to put

$$S(H) = \frac{8\pi e^2 v_o}{mc^2 \omega_o^2} \sigma (v_o H/\omega_o \qquad (12)$$

to make the field equation dimensionless also:

$$\Omega (Y) = 1 + \int_o^Y dY' \int_{H(Y')}^{Y'} \frac{S(H)\Omega (H)\, dX/dT}{dY'/dT} dH \qquad (13)$$

If, now, we apply the same conversion to Eq (2) we find

$$n(y) = \frac{B_o^2}{4\pi m v_o^2} \int_{H(Y)}^{Y} \frac{S(H)\, \Omega (H)\, dH}{dY/dT} \qquad (14)$$

In the absence of collisions there is no natural distribution of particles to govern the choice of the build-up function $S(H)$. The simplest choice is $S(H)\,\Omega(H) = S$, constant, and in strong plasmas it may be that $S(\bar{H}) = S_0$ is more useful.

III. SOLUTION FOR WEAK PLASMAS

The solution of the equations is a machine job and is being undertaken on the Univac. Meanwhile, the only order of approximation which it is worth while to try to solve analytically is that in which the particle density is so low that the depression of B is so small that the deviation of the trajectories from circular is negligible. Then the orbit radius is unity, and for the trajectory

$$Y = H + 1 - \cos T \tag{15}$$

$$dY/dT = \sin T \tag{16}$$

$$d^2Y/dT^2 = \cos T = - dX/dT \tag{17}$$

and for fixed Y

$$dH = - \sin T \, dT \tag{18}$$

The first integration in Eq (13) for $S(H) = S$, constant, is conveniently carried out in the two regions

(A) $0 \leq Y' \leq 2$

$$S \int_0^{Y'} \frac{(Y' - H - 1)}{dY/dT} \, dH = S \int_0^{Y'} \frac{H \cos T \sin T \, dT}{\sin T} = -S \sin\left[\cos^{-1}(1-Y')\right]$$

$$= -S \left[1 - (1-Y')^2\right]^{1/2}$$

(B) $2 \leq Y'$

$$S \int_{Y'-2}^{Y'} \frac{(Y' - H - 1)}{dY/dT} \, dH = 0$$

Using

$$\psi = \cos^{-1}(1-Y); \quad 0 \leq \psi \leq \pi \tag{19}$$

the second integration leads to

$$\Omega(Y) - 1 = \begin{cases} -(S/2)\,(\psi - \sin 2\psi/2); & 0 \leq Y \leq 2 \\ -S\,\pi/2; & 2 \leq Y \end{cases} \tag{20}$$

which represents the depression of the magnetic field in dimensionless form. The quantity $(2/\pi)\,(\Omega-1)/S = (2/\pi)\Delta H/(H_o S)$ is plotted against Y as Curve A of Fig. 2.

Turning to the particle concentration given by Eq. (14):

$$n(y) = \frac{B_o^2\, S}{4\pi m\, v_o^2} \times \begin{cases} \psi; & 0 \leq Y \leq 2 \\ \pi; & 2 \leq Y \end{cases} \tag{21}$$

In Fig. 2, $n(Y)$ is shown as Curve B.

Now we can compare the depression in the field to the particle concentration and it will suffice to do this for $0 \leq Y \leq 2$ because for $Y > 2$ relations remain constant.

$$\frac{\Omega(Y) - 1}{n(y)} = \frac{B - B_o}{B_o} = \frac{2\pi m\, v_o^2}{B_o^2}\left(1 - \frac{\sin 2\psi}{2\psi}\right)$$

The best rough approximation for the introduction of temperature T is to put

$$m\, v_o^2/2 \simeq kT \tag{22}$$

and putting

$$B - B_o = \Delta B$$

$$B_o\,(B - B_o) = \Delta(B^2)/2$$

we have

$$\Delta(B^2) + 8\pi nkT\left[1 - \sin 2\psi/(2\psi)\right] = 0 \tag{23}$$

For $Y > 2$ the bracketed expression should be replaced by its $Y = 2$ value, namely, unity, and thus the generally accepted relation is confirmed. It is not strange that in the boundary layer, $Y < 2$, the accepted relation should be departed from because there the plasma is not isotropic. This departure is shown by Curve C of Fig. 2, which is a plot of the negative of the bracket in Eq (23).

Using an analysis based on the Boltzmann Equation Dr. L. Henrich has recently shown that in general

$$(\Delta B)^2 + 8\pi m n\overline{\dot{y}^2} = 0 \qquad (24)$$

where the bar denotes an average. This is easily confirmed for all Y in the present case of weak plasma and constant $S(H)$.

IV. PRESSURE INTEGRAL OF THE EQUATIONS WITH A SPEED DISTRIBUTION AND ANY PLASMA STRENGTH

For simplicity we shall make the substitution

$$W(H) \equiv S(H)\Omega(H) \qquad (25)$$

and shall use

$$\mathring{Y} = dY/dT, \quad \mathring{\mathring{Y}} = d^2Y/dT^2 \qquad (26)$$

In the integrals of Eqs (13) and (14)

$$\mathring{Y} = \mathring{Y}(H,Y)$$

but, also

$$\mathring{Y} = \mathring{Y}(H, Y(H,T))$$

Accordingly

$$\mathring{\mathring{Y}} = \frac{\partial \mathring{Y}}{\partial Y}\mathring{Y} \qquad (27)$$

Eq (14) applies to particles of a particular speed, say v, when there is a distribution of speeds, and this is indicated by re-writing it:

$$n_v\ (y) \ = \ \frac{B_o^2}{4\pi m v^2} \int_{H(v,Y(v))}^{Y(v)} \frac{W_v\ (H,v)}{\overset{\circ}{Y}\ (H(v),\ Y(v))}\ dH(v) \tag{28}$$

the dependence of Y and H on v arising through Eq (9) and the subscript v indicating that the number or quantity of the variable attributable to particles in the speed range dv is meant. Then, using the lead furnished by L. R. Henrich, we write:

$$\frac{d\ n_v \overline{\dot{y}^2}}{dy} = \frac{v}{\omega_o}\ \frac{d\ n_v\ \overline{\dot{y}^2}}{dY} = \frac{B_o^2\ v}{4\pi m \omega}\ \frac{d}{dY}\int_{H(Y)}^{Y} W_v(H)\ \overset{\circ}{Y}(H,Y)\ dH$$

$$= \frac{B_o^2\ v}{4\pi m\ \omega_o} \left\{ W_v\ (Y)\ \overset{\circ}{Y}\ (Y,Y) - W_v\ (H)\ \overset{\circ}{Y}\ (H(Y),Y) \right.$$

$$\left. + \int_{H(Y)}^{Y} W_v\ (H)\ (\partial \overset{\circ}{Y}/\partial Y)\ dH \right\}$$

The first and second terms within the braces vanish because $\overset{\circ}{Y}$ at a trajectory apogee, $H = Y$, and at a perigee, $H = H(Y)$, is zero. To the remaining term we apply Eq (27) so that

$$\frac{d\ n_v\ \overline{\dot{y}^2}}{dy} = \frac{B_o^2\ v}{4\pi m\ \omega_o} \int_{H(Y)}^{Y} \frac{W_v\ (H)\ \overset{\circ\circ}{Y}}{\overset{\circ}{Y}}\ dH \tag{29}$$

Similarly if we denote that part of $d\Omega/dy$ which is associated with v-speed particles as Ω'_v we have from Eq (13)

$$\Omega'_v\ (Y) = \frac{v}{\omega_o} \int_{H(Y)}^{Y} \frac{W_v(H)\ \overset{\circ}{X}}{\overset{\circ}{Y}}\ dH$$

Using Eq (10) and noting that $\Omega\ (Y)$ can be taken outside the integral

$$\Omega(Y)\,\Omega'_v(Y) = -\frac{v}{\omega_0}\int_{H(Y)}^{Y} \frac{W_v(H)\,\ddot{Y}}{\dot{Y}}\,dH \tag{30}$$

The integrals in Eqs (29) and (30) are identical so that it becomes apparent that

$$\Omega(Y)\,\Omega'_v(Y) + \frac{4\pi m}{B_0^2}\,\frac{d\,n_v\,\overline{\dot{y}^2}}{dy} = 0$$

and multiplying by dv, integrating over all speeds,

$$\int \Omega'_v(Y)\,dv = \Omega'(Y), \quad \int n_v\,\overline{\dot{y}^2}\,dv = n\,\overline{\dot{y}^2}$$

gives

$$\frac{d}{dy}\left(B^2 + 8\pi\,m\,n\,\overline{\dot{y}^2}\right) = 0 \tag{31}$$

which is valid for any velocity distribution. For the Maxwellian case it becomes, as expected,

$$\frac{d}{dy}\left(B^2 + 8\pi\,nkT\right) = 0.$$

V. PROBABLE NATURE OF RESULTS FOR STRONG PLASMAS

As a result of the calculations with finite S(H) we shall establish a set of curves for magnetic field v.s. distance which I conceive will look like A and B of Fig. 3 with increasing S(H). For a still larger value, B/B_0 will fall asymptotically to zero. For a further increased S(H) we shall find B/B_0 cutting zero with a finite slope and a symmetrical solution shown by Curve D becomes possible in which a plasma of high enough concentration is held between two regions of oppositely oriented magnetic field of equal magnitude. What may be the physical reality of such a situation is uncertain, and I am very dubious. Mathematically, the region shown could be matched by a second plasma layer to the right which would restore the vacuum field beyond it to the original B_0 value. But nowhere would the plasma thus "constructed" approach uniformity and it will not be possible to establish a self-consistent plasma with a distribution of velocities.

VI. ELECTRIC CURRENTS AND MASS MOTIONS IN THE BOUNDARY LAYER.

At a distance y_0 from the plasma edge where the plasma is assumed to be essentially Maxwellian and uniform imagine a plane P as shown in Fig. 4. To the right of P trajectories are circular so that there is no mass drift. At any point all directions of motion are equally likely for all velocity classes so that there is no net current density.

We shall be interested in the total current per unit z-depth of plasma, and we have just seen that all of it lies to the left of P. There we analyze the electric current not by volume element but by particles in their trajectories. Some trajectories cross P, many do not. The current, in terms of $\Delta\Omega$, arising from the former will be denoted by J_1, from the latter by J_2, and precise meaning is given these quantities by writing

$$\Omega - 1 = J_1 + J_2 \tag{32}$$

The cut trajectories are responsible for a current because, as with trajectory ζ_1 of Fig. 4 the particle makes its contribution to J_1 by its transport from A via B to C; its transport from C to A is balanced, as we have seen, among all the other particles to the right of P. Particles in circular orbits like ζ_2 which lie entirely within the region contribute nothing to J_2, but for particles in trajectories like ζ_3 their progress from D to E in each cycle does contribute to J_2.

Now J_1 and J_2 can be separated out of the double integral in Eq (13). The area of integration is shown as OABCDEO in Fig. 5. Since Y lies in a uniform region the orbit there is circular, constant in size, and actually of radius (dimensionless) $1/\Omega(Y)$; and $H = H(Y)$ is parallel to $H = Y$. A little reflection will show that reversing the order of integration is physically equivalent (1) to selecting one trajectory (H = constant) and summing its current contributions and then (2), summing over all the trajectories involved. It then becomes evident that the cut trajectories are those included in the area CDEC.

For this the double integral for a single velocity class is

$$J_{1v} = \int_{Y_c}^{Y_c + 2/\Omega} dY \int_{Y_c}^{Y} \frac{S_v\,(H)\,\Omega\,(H)\,\mathring{X}\,dH}{\mathring{Y}} = -\pi S_v/(2\Omega) \tag{33}$$

since $S_v\,(H)\,\Omega\,(H)$ is necessarily constant and the equations of motion are

$$Y = H + \Omega^{-1}\,(1 - \cos\omega T), \quad \mathring{X} = -\Omega^{-1}\,\ddot{Y} \tag{34}$$

We must now determine S_v to be consistent with a Maxwell distribution Applied to particles of speed v Eq (21) becomes

$$\left.\begin{aligned} n_v = \frac{B_o^2\,S_v}{4\,mv^2} &= 2\,nhm\,e^{-hmv^2}\,v \\ h &= 1/(2kT) \end{aligned}\right\} \tag{35}$$

Therefore

$$S = \int_0^{\infty} S_v\,dv = 8\,nkT/B_o^2 = (B_o^2 - B^2)/(\pi\,B_o^2) \tag{36}$$

After summing Eq (33) over all v we then have

$$J_1 = -\frac{1 - \Omega^2}{2\,\Omega} = -\frac{(1 - \Omega)\,(1 + \Omega)}{2\,\Omega} \tag{37}$$

and from Eq (32)

$$J_2 = \frac{(1 - \Omega)^2}{2\,\Omega} \tag{38}$$

These relations lead to the following comments:

(1) J_2 is a paramagnetic current which is more than overbalanced by the diamagnetic J_1.

(2) In weak enough plasmas where Ω is only slightly less than unity (very slight depression of B) J_2 is negligible compared to J_1 which is a confirmation of the earlier treatment of the weak plasma case.

(3) As B is severely depressed, Ω approaching zero, both currents grow without limit. For a particle of average velocity the orbital radius in the uniform plasma is $a_u = \bar{v}/(\omega_o\,\Omega)$ and the actual current corresponding to J_1

will be of the order of the product of this and $-ne\bar{v}$, or $-a_u ne\bar{v}$. The magnitude of J_2 will be of the order of $b\,\bar{n}_{gc}\,\bar{v}_D e$ where b is the thickness of the layer through which the paramagnetic current extends, $\bar{n}_{gc}$ is the average density of guiding centers, and $\bar{v}_D$ is the average paramagnetic drift velocity. Equating $|J_1|$ to J_2 for the very strong plasma case leads to

$$b \simeq \left(\frac{n_o}{\bar{n}_{gc}}\right)\left(\frac{\bar{v}}{\bar{v}_D}\right) a_u \tag{39}$$

Now the average particle density is less in the boundary layer and hence the guiding center density is less:

$$n_o > \bar{n}_{gc}$$

Also $\bar{v}$ is an upper limit which $\bar{v}_D$ can never nearly reach so that

$$\bar{v} > \bar{v}_D$$

probably by several fold. Accordingly

$$b > a_u \tag{40}$$

and the boundary layer is several times as thick as the radius of an average orbit in the uniform plasma. For a weak plasma we have already seen that the factor is not less than 2. This casts considerable doubt on the applicability of electrical skin effect ideas to strong plasmas.

(4) The current J_2 is associated with a mass motion of the charged particles which will transmit disturbances in one region of the boundary layer to adjacent regions, and suggests that here may lie a cause of instability.

FIG. 1

TRAJECTORIES IN PLASMA

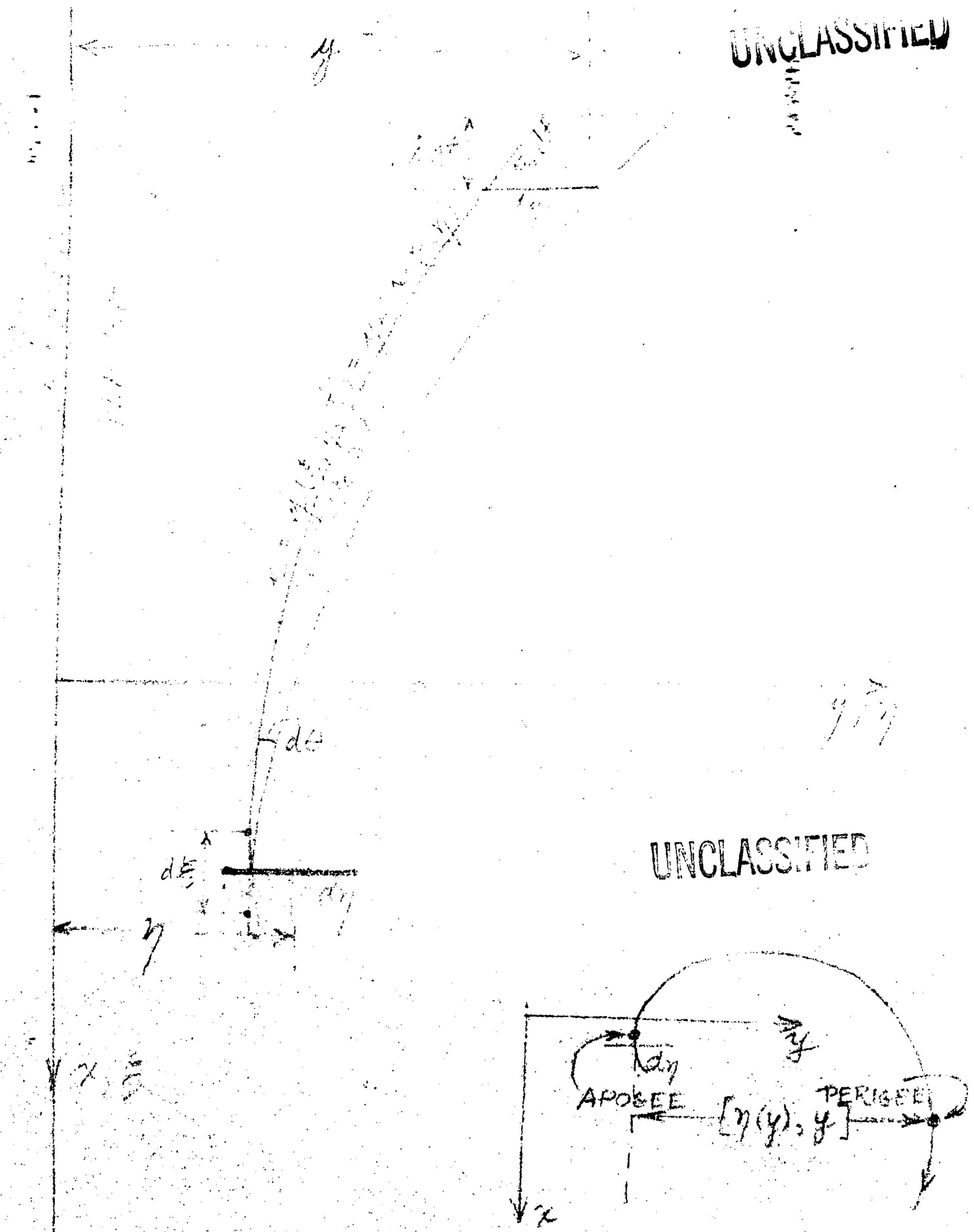
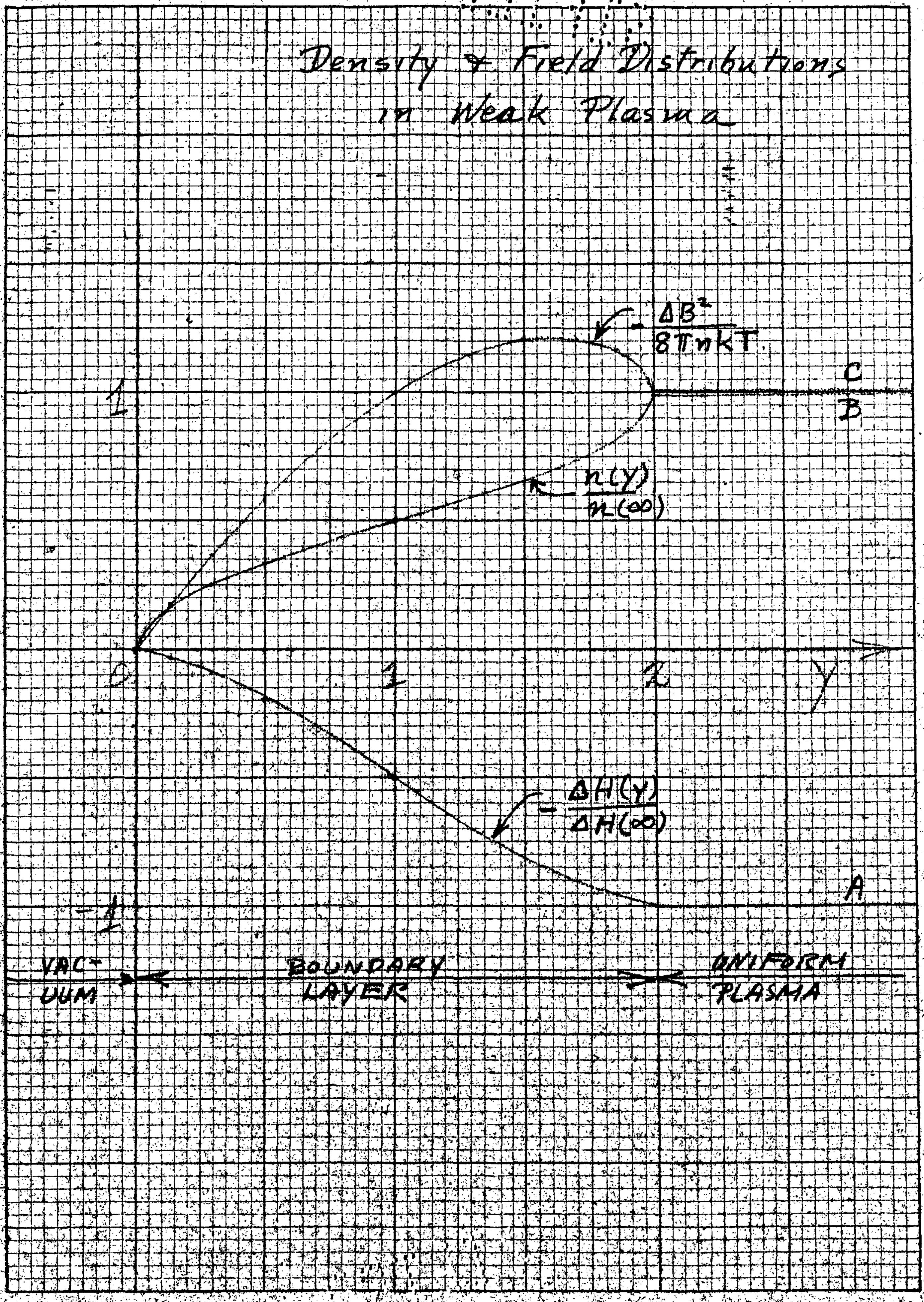


FIG. 2



Density & Field Distributions
in Weak Plasma
$-\frac{\Delta B^2}{8\pi nkT}$
C
B
1
$\frac{n(y)}{n(\infty)}$
0
1
2
Y
$-\frac{\Delta H(y)}{\Delta H(\infty)}$
A
-1
VAC-
UUM
BOUNDARY
LAYER
UNIFORM
PLASMA

FIG

FIELD DISTRIBUTIONS
IN FINITE-STRENGTH
PLASMAS
(Schematic)

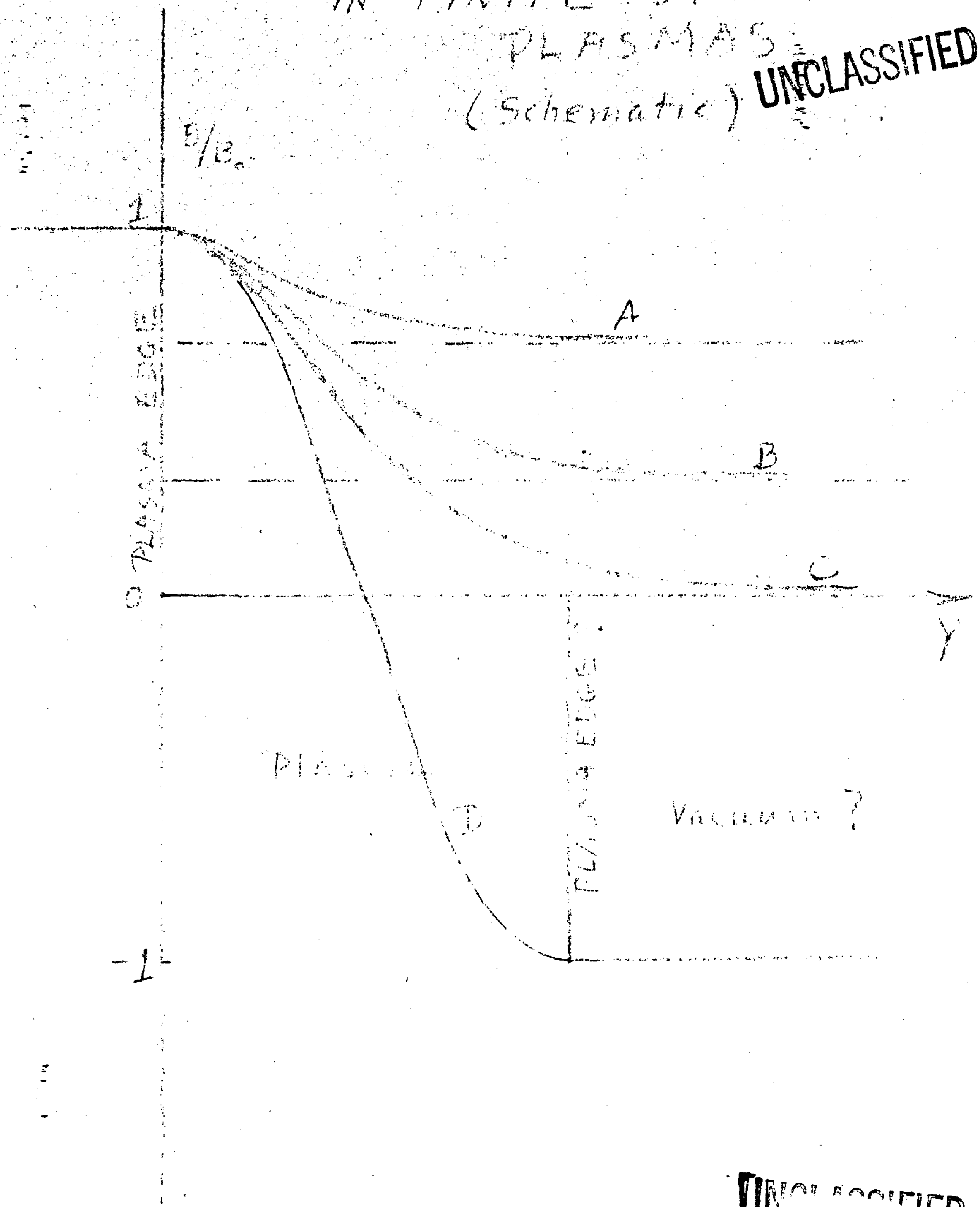


FIG. 4

ANALYSIS OF CURRENTS

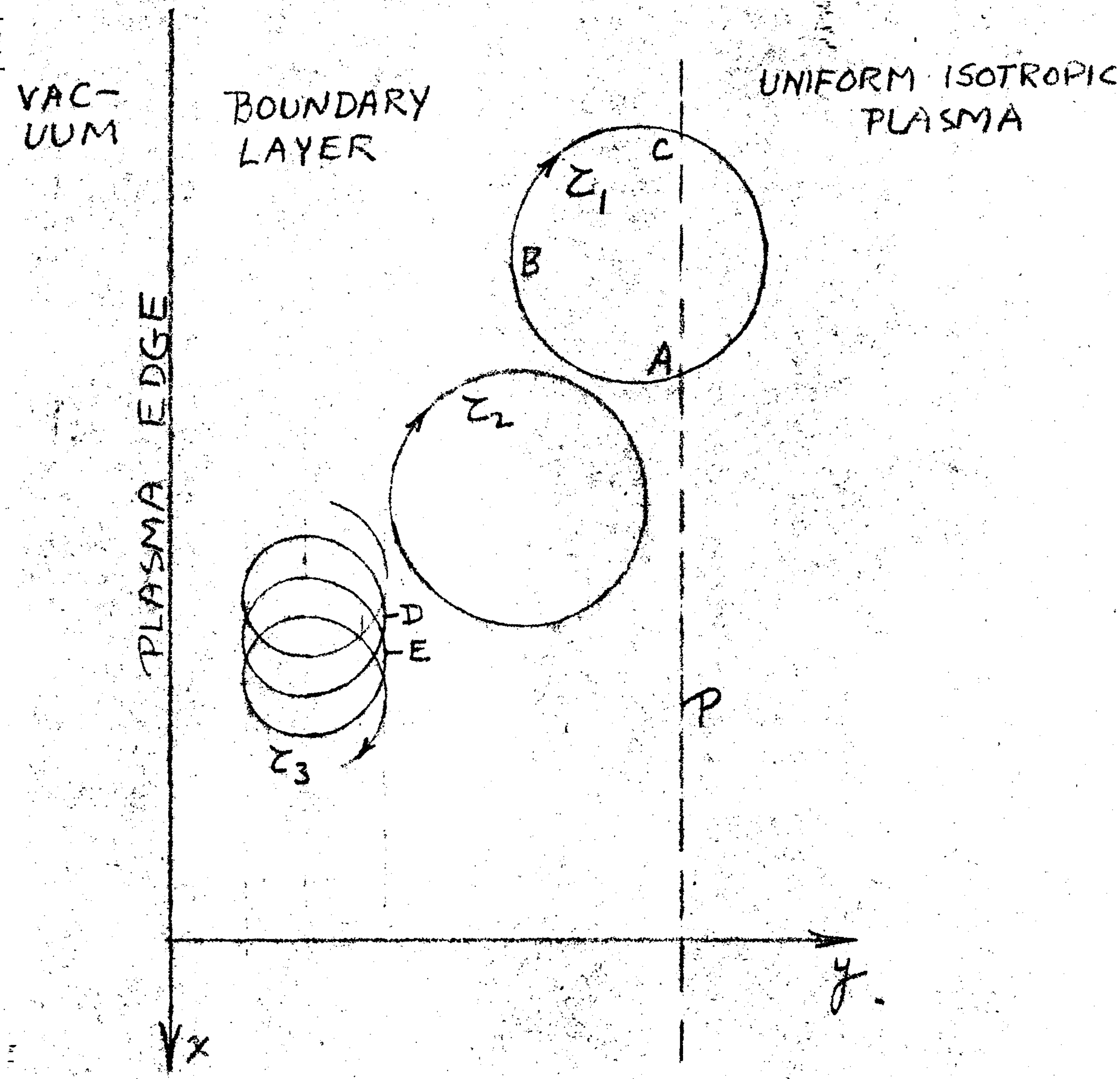


FIG. 5

INTEGRATION REGION.