

On the Geometry of the Quantum Poincaré Group

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Abstract

We review the construction of the multiparametric inhomogeneous orthogonal quantum group $ISO_{q,r}(N)$ as a projection from $SO_{q,r}(N+2)$, and recall the conjugation that for $N=4$ leads to the quantum Poincaré group. We study the properties of the universal enveloping algebra $U_{q,r}(iso(N))$, and give an R -matrix formulation. A quantum Lie algebra and a bicovariant differential calculus on twisted $ISO(N)$ are found.

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1 Introduction

By quantum group, or noncommutative deformation of a Lie group G , we understand a deformation of the algebra $Fun(G)$ of functions from G to the complex numbers \mathbf{C} . $Fun(G)$ besides being a commutative algebra is a Hopf algebra, i.e. it has three linear maps called coproduct $\Delta : Fun(G) \rightarrow Fun(G) \otimes Fun(G)$, counit $\varepsilon : Fun(G) \rightarrow \mathbf{C}$ and antipode (or coinverse) $\kappa : Fun(G) \rightarrow Fun(G)$, defined by $\Delta f(g \otimes g') = f(gg')$, $\varepsilon(f) = f(1_G)$ and $(\kappa f)(g) = f(g^{-1})$. These maps carry at the $Fun(G)$ level the information about the product in G , the existence of the neutral element 1_G and the existence of the inverse g^{-1} of any $g \in G$. The information contained in the Hopf algebra $Fun(G)$ is the same as that contained in G , and we can work with $Fun(G)$ instead of G .

A quantum group is a deformation $Fun_q(G)$ of the algebra $Fun(G)$ depending on one or more complex parameters q . $Fun_q(G)$ has the same rich structure of $Fun(G)$ but it is no more commutative; the noncommutativity is given by the parameters q , when $q \rightarrow 1$, $Fun_q(G) \rightarrow Fun(G)$.

Some motivations for studying q -groups in field theory have already been addressed in this conference by L. Castellani [1]; our main concern here is that a noncommutative space-time, with a deformed Poincaré symmetry group, is an interesting geometric background for the study of q -Minkowski space-time physics and, in particular, of Einstein-Cartan gravity theories based on the differential calculus on a q -Poincaré group [2], [3]. It is in this perspective that here, mainly reviewing [11] and [12], we investigate inhomogeneous orthogonal quantum groups, their quantum universal enveloping algebra, their quantum Lie algebras and more generally their differential structure.

Various deformations of the Poincaré group are known in the literature; here we will consider the quantum inhomogeneous orthogonal group $ISO_{q,r}(N)$ [and in particular $ISO_{q,r}(3, 1; \mathbf{R})$] found as a projection from $SO_{q,r}(N+2)$. This projection procedure is then exploited to obtain $U_{q,r}(iso(N))$, the universal enveloping algebra of $ISO_{q,r}(N)$, as a particular Hopf subalgebra of $U_{q,r}(so(N+2))$; an easy R -matrix formulation of $U_{q,r}(iso(N))$ is also given. These results are accomplished through a detailed study of the L^\pm generators of $U_{q,r}(so(N+2))$, that are the basic elements needed to construct q -Lie algebras and differential calculi. Using these techniques in Section 6 we find a q -Lie algebra and a differential calculus on twisted $ISO(N)$, i.e. the minimal deformation $ISO_{q,r=1}(N)$, obtained with the parametric restriction $r = 1$.

To conclude we mention that the differential geometry on $ISO_{q,r}(N)$ naturally induces a differential structure on the N -dimensional orthogonal quantum plane, that can be seen as the q -coset space $ISO_{q,r}(N)/SO_{q,r}(N)$. We thus have a canonical and straightforward way to obtain the differential calculus on the q -Minkowski plane.

2 $SO_{q,r}(N)$ multiparametric quantum group

The $SO_{q,r}(N)$ multiparametric quantum group is freely generated by the noncommuting matrix elements T^a_b (fundamental representation $a, b = 1, \dots, N$) and the unit element I , modulo the relation $\det_{q,r} T = I$ and the quadratic RTT and CTT (orthogonality) relations discussed below. The noncommutativity is controlled by the R matrix:

$$R^{ab}_{ef} T^e_c T^f_d = T^b_f T^a_e R^{ef}_{cd} \quad (2.1)$$

which satisfies the quantum Yang-Baxter equation

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12} . \quad (2.2)$$

The R -matrix components R^{ab}_{cd} depend continuously on a (in general complex) set of parameters q_{ab}, r . For $q_{ab} = r$ we recover the uniparametric orthogonal group $SO_r(N)$ of ref. [4]. Then $q_{ab} \rightarrow 1, r \rightarrow 1$ is the classical limit for which $R^{ab}_{cd} \rightarrow \delta^a_c \delta^b_d$: the matrix entries T^a_b commute and become the usual entries of the fundamental representation. The R matrix is upper triangular (i.e. $R^{ab}_{cd} = 0$ if $[a = c \text{ and } b < d]$ or $a < c$), and the parameters q_{ab} appear only in the diagonal components of R : $R^{ab}_{ab} = r/q_{ab}, a \neq b, a' \neq b$, where prime indices are defined as $a' \equiv N + 1 - a$. We also define $q_{aa} = q_{a'a'} = r$. The following relations reduce the number of independent q_{ab} parameters [5]: $q_{ba} = r^2/q_{ab}, q_{ab} = r^2/q_{a'b'} = r^2/q_{a'b} = q_{a'b'}$; therefore the q_{ab} with $a < b \leq \frac{N}{2}$ give all the q 's.

Orthogonality conditions are imposed on the elements T^a_b , consistently with the RTT relations (2.1):

$$C^{bc} T^a_b T^d_c = C^{ad} I, C_{ac} T^a_b T^c_d = C_{bd} I, \quad (2.3)$$

where the matrix C_{ab} and its inverse C^{ab} , that satisfies $C^{ab} C_{bc} = \delta^a_c = C_{cb} C^{ba}$, are the metric and its inverse. These matrices are antidiagonal; they are equal, and for example for $N = 4$ they read $C_{14} = r^{-1}, C_{23} = 1, C_{32} = 1$ and $C_{41} = r$. The explicit expression of R and C is given in [5], (see [6] for our notational conventions).

The coproduct Δ , the counit ε and the coinverse or antipode κ are given by

$$\Delta(T^a_b) = T^a_b \otimes T^b_c, \quad (2.4)$$

$$\varepsilon(T^a_b) = \delta^a_b, \quad (2.5)$$

$$\kappa(T^a_b) = C^{ac} T^d_c C_{db}. \quad (2.6)$$

One can also define a q -antisymmetric epsilon tensor and a quantum determinant $\det_{q,r} T$ [7, 12]; to obtain the special orthogonal quantum group $SO_{q,r}(N)$ we impose also the relation $\det_{q,r} T = I$.

The conjugation that from the complex $SO_{q,r}(N)$ leads to the real form $SO_{q,r}(n+1, n-1; \mathbf{R})$ and that is in fact needed to obtain the quantum Poincaré group, is

defined by $(T^a_b)^* = \mathcal{D}^a_c T^c_d \mathcal{D}^d_b$, \mathcal{D} being the matrix that exchanges the index n with the index $n+1$ [8]. This conjugation requires the following restrictions on the deformation parameters: $|q_{ab}| = |r| = 1$ for a and b both different from n or $n+1$; $q_{ab}/r \in \mathbf{R}$ when at least one of the indices a, b is equal to n or $n+1$.

3 $ISO_{q,r}(N)$ as a projection from $SO_{q,r}(N+2)$

A fruitful way to introduce the $ISO_{q,r}(N)$ quantum group is to express it as a quotient of the Hopf algebra $SO_{q,r}(N+2)$. Let T^A_B be the $SO_{q,r}(N+2)$ generators, and split the index A of $SO_{q,r}(N+2)$ as $A=(\circ, a, \bullet)$, with $a = 1, \dots, N$. With this notation:

$$ISO_{q,r}(N) = \frac{SO_{q,r}(N+2)}{H}, \quad (3.7)$$

where H is the left and right ideal in $SO_{q,r}(N+2)$ generated by the relations:

$$T^a_\circ = T^\bullet_b = T^\circ_\bullet = 0. \quad (3.8)$$

Following [6] the projection $P : SO_{q,r}(N+2) \rightarrow SO_{q,r}(N+2)/H$ is an epimorphism between Hopf algebras, and defining the projected matrix elements $t^A_B = P(T^A_B)$, we can give an R -matrix formulation of $ISO_{q,r}(N)$. We set $u \equiv P(T^\circ_\circ)$, $y_b \equiv P(T^\circ_b)$, $z \equiv P(T^\bullet_\bullet)$, $x^a \equiv P(T^a_\bullet)$ and (with abuse of notation) $T^a_b \equiv P(T^a_b)$, then we have

Theorem 3.1 The quantum group $ISO_{q,r}(N)$ is generated by the matrix entries

$$t \equiv \begin{pmatrix} u & y_b & z \\ T^a_b & x^a & \\ & & v \end{pmatrix} \quad \text{and the unity } I \quad (3.9)$$

modulo the Rtt and Ctt relations

$$R^{AB}{}_{EF} t^E_C t^F_D = t^B_F t^A_E R^{EF}{}_{CD}, \quad (3.10)$$

$$C^{BC} t^A_B t^D_C = C^{AD}, \quad C_{AC} t^A_B t^C_D = C_{BD}, \quad (3.11)$$

where R and C are the multiparametric R -matrix and metric of $SO_{q,r}(N+2)$, respectively.

The co-structures are given by :

$$\begin{aligned} \Delta(t^A_B) &= t^A_C \otimes t^C_B; \\ \varepsilon(t^A_B) &= \delta^A_B; \\ \kappa(t^A_B) &= C^{AC} t^D_C C_{DB}. \end{aligned}$$

□□□

Using the explicit expression of the R matrix one can check that relations (3.10), (3.11) contain in particular the $SO_{q,r}(N)$ relations (2.1), (2.3) and the quantum orthogonal plane commutation relations:

$$P^{ab}{}_{cd} x^c x^d = 0 \quad (3.12)$$

where the q -antisymmetrizer P_A is given by

$$P_A = \frac{1}{r + r^{-1}}[-\hat{R} + rI - (r - r^{1-N})P_0] \quad (3.13)$$

and $P_0^{ab} \equiv (C_{ef}C^{ef})^{-1}C^{ab}C_{cd}$. Moreover, due to the Ctt relations, the y and z elements are polynomials in u, x and T , and we also have $uv = vu = I$. A set of independent generators of ISO is then

$$T^a{}_b, x^a, u, v \equiv u^{-1} \text{ and the identity } I. \quad (3.14)$$

Their commutations are (2.1), (3.12) and

$$T^b{}_d x^a = \frac{r}{q_{d\bullet}} R^{ab}{}_{ef} x^e T^f{}_d, \quad (3.15)$$

$$u T^b{}_d = \frac{q_{b\bullet}}{q_{d\bullet}} T^b{}_d u, \quad (3.16)$$

$$u x^b = q_{b\bullet} x^b u. \quad (3.17)$$

The deformation parameters of $ISO_{q,r}(N)$ are the same as those of $SO_{q,r}(N+2)$; they are r and q_{AB} i.e. r, q_{ab} and $q_{a\bullet}$ ($q_{a\bullet} = r^2/q_{a0} = q_{\bullet a'} = q_{0a}$). In the limit $q_{a\bullet} \rightarrow 1 \forall a$, which implies $r \rightarrow 1$, the dilatation u commutes with x and T , and can be set equal to the identity I ; then, when also $q_{ab} \rightarrow 1$ we recover the classical algebra $Fun(ISO(N))$.

The $SO_{q,r}(N+2)$ real form mentioned in the previous section is inherited by $ISO_{q,r}(N)$. In particular the q -Poincaré group $ISO_{q,r}(3, 1; \mathbf{R})$ is obtained by setting $|q_{1\bullet}| = |r| = 1, q_{2\bullet}/r \in \mathbf{R}, q_{12} \in \mathbf{R}$. A dilatation-free q -Poincaré group is found after the further restriction $q_{1\bullet} = q_{2\bullet} = r = 1$. The only free parameter remaining is then $q_{12} \in \mathbf{R}$.

Note 1. The u and x^a elements generate a subalgebra of $ISO_{q,r}(N)$ because their commutation relations do not involve the $T^a{}_b$ elements. Moreover these elements can be ordered using (3.12) and (3.17), and the Poincaré series of this subalgebra is the same as that of the commutative algebra in $N+1$ indeterminates [4]. A linear basis of this subalgebra is therefore given by the ordered monomials: $\zeta^i = u^{i_0} (x^1)^{i_1} \dots (x^N)^{i_N}$. Then, using (3.15) and (3.16), a generic element of $ISO_{q,r}(N)$ can be written as $\zeta^i a_i$ where $a_i \in SO_{q,r}(N)$ and we conclude that $ISO_{q,r}(N)$ is a right $SO_{q,r}(N)$ -module generated by the ordered monomials ζ^i .

One can show that as in the classical case the expressions $\zeta^i a_i$ are unique: $\zeta^i a_i = 0 \Rightarrow a_i = 0 \forall i$, i.e. that $ISO_{q,r}(N)$ is a free right $SO_{q,r}(N)$ -module. [To prove this assertion one can show that (when $q_{a\bullet} = r \forall a$) $ISO_{q,r}(N)$ is a bicovariant bimodule over $SO_{q,r}(N)$. Since any bicovariant bimodule is free [9] one then deduce that, as a right module, $ISO_{q,r}(N)$ is freely generated by the ζ^i . Similarly, writing $a_i \zeta^i$ instead of $\zeta^i a_i$, we have that $ISO_{q,r}(N)$ is the free left $SO_{q,r}(N)$ -module generated by the ζ^i .

4 Universal enveloping algebra $U_{q,r}(so(N+2))$

Classically the universal enveloping algebra $U(g)$ of a Lie group G is the associative algebra generated by the Lie algebra g of G . Given a basis $\{\chi_i\}$ of g , any element of $U(g)$ is a finite formal power series in the elements χ_i . Because of the Lie algebra relations, it is always possible to order any product of χ_i 's; moreover (Poincaré-Birkhoff-Witt theorem) a linear basis of $U(g)$ is given by ordered products of these elements (for example $\chi_{i_1}\chi_{i_2}\dots\chi_{i_s}$, $i_1 \leq i_2 \leq \dots \leq i_s$, $s \geq 1$). Any element χ_i can be seen as a tangent vector to the origin of the group, and therefore it is a functional that associates the complex number $\chi_i(f)$ to any function f on the group. In the noncommutative case, for semi-simple g , there is a unique $U_q(g)$, while there is not a unique (categorical) definition of q -Lie algebra. In this section and the next we study the universal enveloping algebras $U_{q,r}(so(N+2))$ and $U_{q,r}(iso(N))$, while in Section 6 we will briefly consider a subspace (q -Lie algebra) of $U_{q,r}(iso(N))$ that defines a bicovariant differential calculus on $ISO_{q,r=1}(N)$.

$U_{q,r}(so(N+2))$ is the algebra over \mathcal{C} generated by the counit ε and by the functionals L^\pm defined by their value on the matrix elements T^A_B :

$$L^{\pm A}_B(T^C_D) = (R^\pm)^{AC}_{BD}, L^{\pm A}_B(I) = \delta^A_B \quad (4.1)$$

with

$$(R^+)^{AC}_{BD} \equiv R^{CA}_{DB}, (R^-)^{AC}_{BD} \equiv (R^{-1})^{AC}_{BD}. \quad (4.2)$$

To extend the definition (4.1) to the whole algebra $SO_{q,r}(N+2)$ we set $\forall a, b$:

$$L^{\pm A}_B(ab) = L^{\pm A}_C(a)L^{\pm C}_B(b). \quad (4.3)$$

From (4.1) using the upper and lower triangularity of R^+ and R^- we see that L^+ is upper triangular and L^- is lower triangular.

The commutations between $L^{\pm A}_B$ and $L^{\pm C}_D$ are induced by (2.2):

$$R_{12}L_2^\pm L_1^\pm = L_1^\pm L_2^\pm R_{12}, \quad (4.4)$$

$$R_{12}L_2^+ L_1^- = L_1^- L_2^+ R_{12}, \quad (4.5)$$

where the product $L_2^\pm L_1^\pm$ is the convolution product $L_2^\pm L_1^\pm \equiv (L_2^\pm \otimes L_1^\pm)\Delta$.

The $L^{\pm A}_B$ elements satisfy orthogonality conditions analogous to (2.3):

$$C^{AB}L^{\pm C}_B L^{\pm D}_A = C^{DC}\varepsilon \quad (4.6)$$

$$C_{AB}L^{\pm B}_C L^{\pm A}_D = C_{DC}\varepsilon \quad (4.7)$$

The co-structures of the algebra generated by the functionals L^\pm and ε are defined by duality [see (4.3)]: $\Delta'(L^{\pm A}_B)(a \otimes b) = L^{\pm A}_B(ab)$, $\varepsilon'(L^{\pm A}_B) = L^{\pm A}_B(I)$ and $\kappa'(L^{\pm A}_B)(a) = L^{\pm A}_B(\kappa(a))$ so that

$$\Delta'(L^{\pm A}_B) = L^{\pm A}_G \otimes L^{\pm G}_B \quad (4.8)$$

$$\varepsilon'(L^{\pm A}_B) = \delta^A_B \quad (4.9)$$

$$\kappa'(L^{\pm A}_B) = [(L^\pm)^{-1}]^A_B = C^{DA}L^{\pm C}_D C_{BC} \quad (4.10)$$

From (4.10) we have that κ' is an inner operation in the algebra generated by the functionals $L^{\pm A}_B$ and ε , it is then easy to see that these elements generate a Hopf algebra, the universal enveloping algebra $U_{q,r}(so(N+2))$ of $SO_{q,r}(N+2)$.

Note 2 : From the *CLL* relations we have

$$L^{\pm A}_A L^{\pm A'}_{A'} = L^{\pm A'}_{A'} L^{\pm A}_A = \varepsilon . \quad (4.11)$$

From the *RLL* relations we have that the subalgebra U^0 generated by the elements $L^{\pm A}_A$ and ε is commutative (use upper triangularity of R).

Note 3 : $U_{q,r}(so(N+2))$ is completely characterized by the relations (4.4), (4.5), (4.6), (4.7), and $L^{-A}_A = F(L^+_{A'})$. The functional dependence F of the L^{-A}_A in terms of the $L^+_{A'}$ is studied in [12]; in the uniparametric case ($q_{AB} \rightarrow r$) it becomes the simple expression:

$$L^+_{A'} L^{-A}_A = \varepsilon , \text{ i.e. } L^{-A}_A = L^+_{A'} \quad (4.12)$$

[indeed $L^+_{A'} L^{-A}_A(a) = \varepsilon(a)$ as can be easily seen when $a = T^A_B$ and generalized to any $a \in SO_{q,r}(N+2)$ using (4.3)]. In both cases the commutative subalgebra U^0 can be generated by the following elements: $L^{+\circ}_\circ, L^{+1}_1 \dots L^{+n}_n$ where $N = 2n$ for N even and $N = 2n+1$ for N odd.

The elements L^\pm [or $\frac{1}{r-r^{-1}}(L^{\pm A}_B - \delta^A_B \varepsilon)$] may be seen as the quantum analogue of the tangent vectors; then the *RLL* relations are the quantum analogue of the Lie algebra relations, and we can use the orthogonal *CLL* conditions to reduce the number of the L^\pm generators to $(N+2)(N+1)/2$, i.e. the dimension of the classical group manifold.

We have already given a reduced set of generators in the case of the diagonal $L^{\pm A}_A$. For the off-diagonal elements, for example from relation (4.7) choosing the free indices $(C, D) = (\bullet, d)$ and using $L^{+\circ}_\circ L^{+\bullet}_\bullet = \varepsilon$, we have:

$$L^{+\circ}_d = -(C_\bullet\bullet)^{-1} L^{+\circ}_\circ C_{ab} L^{+b}_\bullet L^{+a}_d . \quad (4.13)$$

Similar results hold for $L^{+\bullet}_\bullet, L^{-\bullet}_\bullet$ and $L^{-\circ}_\circ$. Iterating this procedure, from $C_{ab} L^{+b}_c L^{+a}_d = C_{ac} \varepsilon$ we find that L^{+i}_i (with $i = 2, \dots, N-1$) and L^{+1}_N are functionally dependent on L^{+i}_N and L^{+1}_1 . Similarly for L^{-N}_i and L^{-N}_1 . The final result is that the elements L^{+a}_J with $J < a < J'$ and L^{-a}_J with $J < a < J'$, whose number in both \pm cases is $\frac{1}{4}N(N+2)$ for N even and $\frac{1}{4}(N+1)^2$ for N odd, and the elements $L^{+\circ}_\circ, L^{+1}_1, \dots, L^{+n}_n$ generate all $U_{q,r}(so(N+2))$. The total number of generators is therefore $(N+2)(N+1)/2$.

It is also possible to show that, as in the classical case, these $(N+2)(N+1)/2$ generators can always be ordered. The proof [12] is based on the observation that the full set of L^\pm can be ordered [this is shown by induction on N , using that $U_{q,r}(so(N))$ is a subalgebra of $U_{q,r}(so(N+2))$] and that

$$P^{ab}_{A \ cd} L^{+d}_\bullet L^{+c}_\bullet = 0 , P^{ab}_{A \ cd} L^{-d}_\circ L^{-c}_\circ = 0 \quad (4.14)$$

[compare with (3.12)]. The conclusion is that using the set of $(N+2)(N+1)/2$ generators

$$L^{+a}_J \text{ with } J < a < J, \quad L^{-a}_J \text{ with } J < a < J' \quad (4.15)$$

and $L^{+\circ}, L^{+1}_1, \dots, L^{+n}_n$, a generic element of $U_{q,r}(so(N+2))$ can always be written as a linear combination of ordered monomials of the kind

$$\Pi(L^{+\alpha}_J)(L^{+\circ})^{i_0}(L^{+1}_1)^{i_1} \dots (L^{+n}_n)^{i_n} \Pi(L^{-\alpha}_J) \quad (4.16)$$

where $\Pi(L^{+\alpha}_J), [\Pi(L^{-\alpha}_J)]$ is a monomial in the off-diagonal elements $L^{+\alpha}_J$ with $J < \alpha < J$ [$L^{+\alpha}_J$ with $J < \alpha < J'$] where an ordering has been chosen.

We conjecture that monomials of the type (4.16) are linearly independent and therefore that the set of all these monomials is a basis of $U_{q,r}(so(N+2))$.

5 Universal enveloping algebra $U_{q,r}(iso(N))$

Consider a generic functional $f \in U_{q,r}(so(N+2))$. It is well defined on the quotient $ISO_{q,r}(N) = SO_{q,r}(N+2)/H$ if and only if $f(H) = 0$. It is easy to see that the set H^\perp of all these functionals is a subalgebra of $U_{q,r}(so(N+2))$: if $f(H) = 0$ and $g(H) = 0$, then $fg(H) = 0$ because $\Delta(H) \subseteq H \otimes S_{q,r}(N+2) + S_{q,r}(N+2) \otimes H$. Moreover H^\perp is a Hopf subalgebra of $U_{q,r}(so(N+2))$ since H is a Hopf ideal [16]. In agreement with these observations we will find the Hopf algebra $U_{q,r}(iso(N))$ [dually paired to $ISO_{q,r}(N)$] as a subalgebra of $U_{q,r}(so(N+2))$ vanishing on the ideal H .

Let

$$IU \equiv [L^{-A}_B, L^{+a}_b, L^{+\circ}, L^{+\bullet}, \varepsilon] \quad (5.1)$$

be the subalgebra of $U_{q,r}(so(N+2))$ generated by $L^{-A}_B, L^{+a}_b, L^{+\circ}, L^{+\bullet}, \varepsilon$.

Note 4: These are all and only the functionals annihilating the generators of H : T^a_\circ, T^\bullet_b and T^\bullet_\circ . The remaining $U_{q,r}(so(N+2))$ generators $L^{+\circ}_b, L^{+a}_\bullet, L^{+\circ}$ do not annihilate the generators of H and are not included in (5.1).

Since $\Delta(IU) \subseteq IU \otimes IU$ and $\kappa'(IU) \subseteq IU$ (as can be immediately seen at the generators level) we have that IU is a Hopf subalgebra of $U_{q,r}(so(N+2))$. Moreover one can also give the following R -matrix formulation [12]:

Theorem 5.1 The Hopf algebra IU is generated by the unit ε and the matrix entries:

$$L^- = (L^{-A}_B), \quad L^+ = \begin{pmatrix} L^{+\circ} & & \\ & L^{+a}_b & \\ & & L^{+\bullet} \end{pmatrix}$$

these functionals satisfy the q -commutation relations:

$$R_{12} \mathcal{L}^+_2 \mathcal{L}^+_1 = \mathcal{L}^+_1 \mathcal{L}^+_2 R_{12} \quad \text{or equivalently}$$

$$\mathcal{R}_{12}\mathcal{L}^+_2\mathcal{L}^+_1 = \mathcal{L}^+_1\mathcal{L}^+_2\mathcal{R}_{12} \quad (5.2)$$

$$R_{12}L_2^-L_1^- = L_1^-L_2^-R_{12} , \quad (5.3)$$

$$\mathcal{R}_{12}\mathcal{L}^+_2L_1^- = L_1^-\mathcal{L}^+_2\mathcal{R}_{12} , \quad (5.4)$$

where $\mathcal{R}_{12} \equiv \mathcal{L}^+_2(t_1)$ that is $\mathcal{R}_{cd}^{ab} = R_{cd}^{ab}$, $\mathcal{R}_{AB}^{AB} = R_{AB}^{AB}$ and otherwise $\mathcal{R}_{CD}^{AB} = 0$.

The orthogonality conditions and the costructures are the same as in (4.6)–(4.10), with the L^+ matrix replaced by the \mathcal{L}^+ matrix. □□□

Note 5 : From (5.2) applied to t we obtain the quantum Yang-Baxter equation for the matrix \mathcal{R} .

Note 6 : Following *Note 1* and using (4.14), a generic element of IU can be written as $\eta^i a_i$ ($a_i \eta^i$) where $a_i \in U_{q,r}(so(N+2))$ and η^i are ordered monomials in the L^- and L^- elements: $\eta^i = (L^-)^{i_1} (L^-)^{i_2} \dots (L^-)^{i_N}$. Moreover IU is a free right (left) $U_{q,r}(so(N))$ -module generated by the ordered monomials η^i .

Duality $U_{q,r}(iso(N)) \leftrightarrow ISO_{q,r}(N)$

We now show that IU is dually paired to $SO_{q,r}(N+2)$. This is the fundamental step allowing to interpret IU as the universal enveloping algebra of $ISO_{q,r}(N)$.

Theorem 5.2 IU annihilates H , that is $IU \subseteq H^\perp$.

Proof : Let \mathcal{L} and \mathcal{T} be generic generators of IU and H respectively. As discussed in *Note 4*, $\mathcal{L}(\mathcal{T}) = 0$. A generic element of the ideal is given by $a\mathcal{T}b$ where sum of polynomials is understood; we have (using Sweedler's notation for the coproduct):

$$\mathcal{L}(a\mathcal{T}b) = \mathcal{L}_{(1)}(a)\mathcal{L}_{(2)}(\mathcal{T})\mathcal{L}_{(3)}(b) = 0 , \quad (5.5)$$

since $\mathcal{L}_{(2)}(\mathcal{T}) = 0$ because $\mathcal{L}_{(2)}$ is still a generator of IU , indeed IU is a sub-coalgebra of $U_{q,r}(so(N+2))$. Thus $\mathcal{L}(H) = 0$. Recalling that a product of functionals annihilating H still annihilates the co-ideal H , we also have $IU(H) = 0$. □□□

In virtue of *Theorem 5.2* the following bracket is well defined:

$$\begin{aligned} \text{Definition. } \langle , \rangle : IU \otimes ISO_{q,r}(N) &\longrightarrow \mathcal{C} \\ \langle a', P(a) \rangle &\equiv a'(a) \\ \forall a' \in IU , \forall a \in SO_{q,r}(N+2) \end{aligned} \quad (5.6)$$

where $P : SO_{q,r}(N+2) \rightarrow SO_{q,r}(N+2)/H \equiv ISO_{q,r}(N)$ is the canonical projection, which is surjective. The bracket is well defined because two generic counter-images of $P(a)$ differ by an addend belonging to H .

Since IU is a Hopf subalgebra of $U_{q,r}(so(N+2))$ and P is compatible with the structures and costructures of $SO_{q,r}(N+2)$ and $ISO_{q,r}(N)$, the following theorem is then easily shown,

Theorem 5.3 The bracket (5.6) defines a pairing between IU and $ISO_{q,r}(N)$:
 $\forall a', b' \in IU, \forall P(a), P(b) \in ISO_{q,r}(N)$

$$\begin{aligned}\langle a'b', P(a) \rangle &= \langle a' \otimes b', \Delta(P(a)) \rangle \\ \langle a', P(a)P(b) \rangle &= \langle \Delta'(a'), P(a) \otimes P(b) \rangle \\ \langle \kappa'(a'), P(a) \rangle &= \langle a', \kappa(P(a)) \rangle \\ \langle I, P(a) \rangle &= \varepsilon(a) ; \quad \langle a', P(I) \rangle = \varepsilon'(a')\end{aligned}$$

□□□

We now recall that IU and $ISO_{q,r}(N)$, besides being dually paired, are free right modules respectively on $U_{q,r}(so(N))$ and on $SO_{q,r}(N)$. They are freely generated by the two isomorphic sets of the ordered monomials in the q -plane plus dilatation coordinates L^-_o, L^+_o and u, x^a respectively. We then conclude that IU is the universal enveloping algebra of $ISO_{q,r}(N)$: $U_{q,r}(iso(N)) \equiv IU$.

6 Projected differential calculus

In the previous sections we have found the inhomogeneous quantum group $ISO_{q,r}(N)$ by means of a projection from $SO_{q,r}(N+2)$. Dually, its universal enveloping algebra is a given Hopf subalgebra of $U_{q,r}(so(N+2))$. Using the same techniques a differential calculus on $ISO_{q,r}(N)$ can be found.

Classically the differential calculus on a group is uniquely determined by the Lie algebra g of the tangent vectors to the group. Similarly a differential calculus on a q -group A , with universal enveloping algebra \mathcal{U} ($\mathcal{U} \equiv U_q(g)$), is determined by a q -Lie algebra T : the q -deformation of g . It is natural to look for a linear space T , $T \subset \ker \varepsilon \subset \mathcal{U}$ satisfying the following three conditions:

$$T \text{ generates } \mathcal{U} \tag{6.1}$$

$$\Delta'(T) \subset T \otimes \varepsilon + \mathcal{U} \otimes T, \tag{6.2}$$

$$[T, T] \subset T \tag{6.3}$$

where the bracket is the adjoint action defined by

$$\forall \chi, \psi \in T \quad [\chi, \psi] = \chi_1 \psi \kappa'(\chi_2). \tag{6.4}$$

Condition (6.2) states that the elements of T are generalized tangent vectors, and in fact, if $\{\chi_i\}$ is a basis of the linear space T , we have $\Delta(\chi_i) = \chi_i \otimes \varepsilon + f_i^j \otimes \chi_j$ that is equivalent to

$$\chi_i(ab) = \chi_i(a) \varepsilon(b) + f_i^j(a) \chi_j(b) \tag{6.5}$$

where $f_i^j \in \mathcal{U}$ and $\varepsilon(f_i^j) = \delta_i^j$. [Hint: apply $(\varepsilon \otimes id)$ to (6.2)]. In the commutative limit we expect $f_i^j \rightarrow 1$. One can also consider the generalized left-invariant vector

fields χ_i^* defined by $\chi_i^* a \equiv a_1 \chi_i(a_2)$, then (6.2) states that the χ_i^* are generalized derivations:

$$\chi_i^* (ab) = (\chi_i^* a) b + (f_i^j * a) \chi_j^* b. \quad (6.6)$$

Condition (6.3) is the closure of T under the adjoint action, in the classical case, if χ is a tangent vector: $\Delta(\chi) = \chi \otimes \varepsilon + \varepsilon \otimes \chi$, $\kappa'(\chi) = -\chi$ and the adjoint action of χ on ψ is given by the commutator $\chi\psi - \psi\chi$.

Note 7: Woronowicz conditions for a bicovariant differential calculus are slightly weaker than (6.1)-(6.3). Relation (6.2) can also be written $\Delta(T \oplus \{\varepsilon\}) \subset \mathcal{U} \otimes (T \oplus \{\varepsilon\})$, where $T \oplus \{\varepsilon\}$ is the vector space spanned by χ_i and ε ; therefore $\kappa'(T \oplus \{\varepsilon\})$ is a right *co-ideal*, it is the space orthogonal to the Woronowicz [9] right *ideal* R : $[\kappa'(T \oplus \{\varepsilon\})](R) = 0$. Relations (6.1) and (6.3) imply that $\forall \varphi \in \mathcal{U}, \forall \chi \in T$, $\varphi_1 \chi \kappa'(\varphi_2) \in T$ (for example if $\varphi = \chi'' \chi'$ then $\varphi_1 \chi \kappa'(\varphi_2) = [\chi'', [\chi', \chi]]$); this last condition is then equivalent to the *ad* invariance of R : $ad(R) \subset R \otimes A$, where $ad(a) \equiv a_2 \otimes \kappa(a_1) a_3$. For further details see [13].

From (6.1)-(6.3), the construction of the differential calculus associated to the tangent space T is quite straightforward (see for example [14]): the main ingredients are

- i) the coordinates x^j , that are uniquely defined by $x^j \in \ker \varepsilon - \kappa(R)$, $\chi_i(x^j) = \delta_i^j$;
- ii) the adjoint representation $M_i^j \equiv \chi_i(x_2^j) x_3^j \kappa^{-1}(x_1^j)$, that satisfies $\Delta(M_i^j) = M_i^k \otimes M_k^j$ and $\varepsilon(M_i^j) = \delta_i^j$; note that if $y^j \in \ker \varepsilon$ and $\chi_i(y^j) = \delta_i^j$ then $M_i^j = \chi_i(y_2^j) y_3^j \kappa^{-1}(y_1^j)$.
- iii) the space of left invariant one-forms, defined as the space dual to that of the tangent vectors: $\omega^i : \langle \chi_i, \omega^j \rangle = \delta_i^j$. A generic one-form is then given by $\rho = \omega^i a_i$. [The space of one-forms is the bicovariant bimodule freely generated by the ω^i with $a \omega^i = \omega^j f_j^i * a \equiv \omega^j (id \otimes f_j^i) \Delta(a)$, $\Delta_L \omega^i \equiv I \otimes \omega^i$, $\Delta_R \omega^i \equiv \omega^j \otimes M_j^i$].
- iv) The differential, defined by $da = \omega^i \chi_i^* a$; it satisfies the undeformed Leibniz rule.

In our case $A = SO_{q,r}(N+2)$, and a differential calculus satisfying (6.1)-(6.3) is given by the χ functionals [see [15] for (6.1)]:

$$\chi_B^A = \frac{-1}{r-r^{-1}} [f_{BC}^A - \delta_B^A \varepsilon] ; \quad (6.7)$$

the f functionals and the adjoint representation read

$$f_{A_2 B_1}^{A_1 B_2} \equiv \kappa'^{-1} (L^{-B_2}_{A_2}) L^{+A_1}_{B_1}, \quad (6.8)$$

$$M_{BC}^A \equiv T^A_C \kappa(T^D_B); \quad (6.9)$$

see [10] and references therein (see also [11], and for notations [14], to obtain the χ_i of [11] act on our χ_i with $-\kappa'$).

We now consider the differential calculus on $ISO_{q,r}(N)$. From (6.2) and (6.3) it is immediate to see that $T' \equiv T \cap U_{q,r}(iso(N))$ satisfies $\Delta(T') \subset T' \otimes \varepsilon +$

$U_{q,r}(iso(N)) \otimes T'$ and $[T', T'] \subseteq T \cap U_{q,r}(iso(N)) = T'$; indeed $U_{q,r}(iso(N))$ is a Hopf subalgebra of $U_{q,r}(so(N+2))$. However, condition (6.1) is not fulfilled since the χ_b^a do not belong to $U_{q,r}(iso(N))$ and therefore to T' unless $r = 1$. [The χ_b^a contain the addend $\frac{-1}{r-r-1} f_{b\bullet}^a \notin U_{q,r}(iso(N))$, that vanish for $r \rightarrow 1$]. We therefore obtain an $ISO_{q,r=1}(N)$ bicovariant differential calculus. A basis of T' is given by

$$\chi_b^a = \lim_{r \rightarrow 1} \frac{1}{\lambda} [f_{bc}^a - \delta_b^a \varepsilon], \text{ with } a + b > N + 1; \quad (6.10)$$

$$\chi_b^\bullet = \lim_{r \rightarrow 1} \frac{1}{\lambda} f_{b\bullet}^\bullet; \quad \chi^\bullet = \lim_{r \rightarrow 1} \frac{1}{\lambda} [f_{\bullet\bullet}^\bullet - \varepsilon], \quad (6.11)$$

here $\lambda = r^{-1} - r$; notice that all the other χ_b^a in the $r \rightarrow 1$ limit are linearly dependent from the χ_B^A in (6.10) and (6.11) [11]. The q -Lie algebra is

$$\begin{aligned} \chi_{b_2}^{b_1} \chi_{c_2}^{c_1} - (q_{b_1 c_2} q_{c_1 b_1} q_{b_2 c_1} q_{c_2 b_2}) \chi_{c_2}^{c_1} \chi_{b_2}^{b_1} = \\ + (q_{b_1 c_2} q_{c_2 b_2} q_{b_2 b_1} \delta_{b_2}^{c_1}) \chi_{c_2}^{b_1} - (q_{c_1 b_1} q_{b_2 b_1} C_{b_2 c_2}) \chi_{c_1}^{b_1} \\ - (q_{c_2 b_2} q_{b_1 c_2} C^{c_1 b_1}) \chi_{c_2}^{b_2} + (q_{b_2 c_1} \delta_{c_2}^{b_1}) \chi_{c_1}^{b_2}, \end{aligned}$$

$$\begin{aligned} \chi_{b_2} \chi_{c_2}^{c_1} - \left(\frac{q_{c_1 \bullet}}{q_{c_2 \bullet}} q_{b_2 c_1} q_{c_2 b_2} \right) \chi_{c_2}^{c_1} \chi_{b_2} \\ = - \frac{q_{c_1 \bullet}}{q_{c_2 \bullet}} [C_{b_2 c_2} \chi_{c_1} - \delta_{b_2}^{c_1} q_{c_2 c_1} \chi_{c_2}], \end{aligned}$$

$$\chi_{b_2} \chi_{c_2} - \left(\frac{q_{b_2 \bullet}}{q_{c_2 \bullet}} q_{c_2 b_2} \right) \chi_{c_2} \chi_{b_2} = 0,$$

$$\chi^\bullet \chi_{c_2}^{c_1} - \chi_{c_2}^{c_1} \chi^\bullet = 0, \quad \chi^\bullet \chi_{c_2} - \chi_{c_2} \chi^\bullet = \chi_{c_2}$$

where we have defined $\chi_a \equiv \chi_a^\bullet$. The exterior differential reads

$$da = \omega_a^b (\chi_b^a * a) + \omega_\bullet^b (\chi_b^\bullet * a) + \omega_\bullet^\bullet (\chi^\bullet * a); \quad (6.12)$$

where ω_a^b , ω_\bullet^b , and ω_\bullet^\bullet are the one-forms dual to the tangent vectors (6.10) and (6.11).

In the limit $r \rightarrow 1$, $q_{a\bullet} \rightarrow 1$ we have seen from (3.16) and (3.17) that the element u can be set equal to the identity I , i.e. we obtain an $ISO_{q,r}(N)$ q -group without dilatations. Then the tangent vectors χ_b^a and χ_b^\bullet alone generate $U_{q,r}(iso(N))$, they satisfy (6.2) and (6.3) and therefore we have an inhomogeneous q -Lie algebra and a bicovariant differential calculus without dilatation generator χ^\bullet . In particular, for $N = 4$ we have a one deformation parameter ($q_{12} \in \mathbf{R}$) differential calculus on the q -Poincaré group discussed at the end of Section 3.

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