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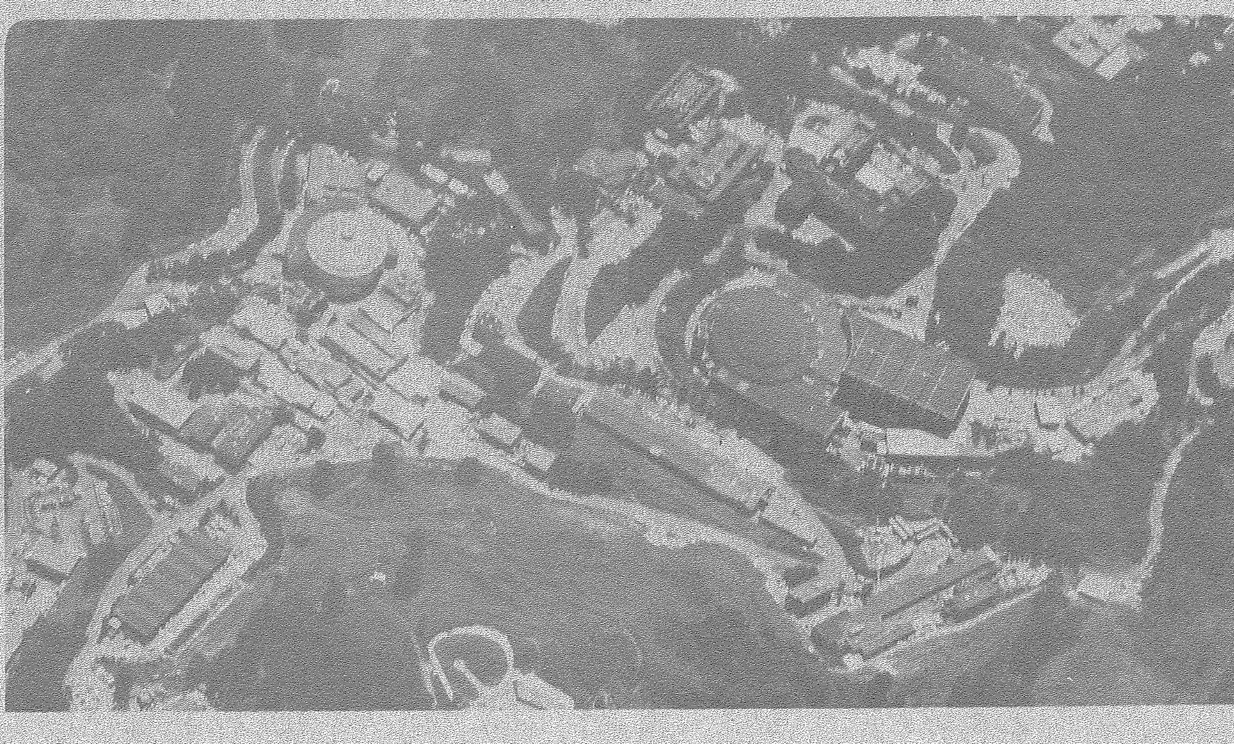

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TARGET MASS CORRECTIONS IN THE QCD PARTON MODEL

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ABSTRACT

We show how target mass corrections can be incorporated to all orders of the QCD parton model, for leading and non-leading logarithms. This algorithm reporduces the ξ scaling analysis of totally inclusive leptoproduction. Target mass corrections to semi-inclusive leptoproduction are computed. We show that the simplest final state variable to use is $\omega_H = -\frac{2P'\cdot q}{Q^2}$, where P' is the observed hadron's momentum. We define double moments for this process for which scaling violations and factorization breaking are target mass independent.

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INTRODUCTION

The usual treatment of the QCD parton model neglects all contributions which are suppressed by powers of $\frac{1}{Q^2}$, where Q is the (large) momentum scale of the process. Included in this category are kinematic corrections associated with the mass m of an initial state hadron, which are of order m^2/Q^2. In the case of a target nucleon, these corrections may be significant even when Q^2 is large enough so that perturbation theory in $\alpha_s(Q^2)$ is reliable. The other neglected effects (e.g. the amount by which the partons in the target are off shell, coherence effects etc.) should all be characterized by the fundamental scale of the strong interactions M_O (e.g. the inverse size of the proton or Λ) and are therefore of order M_O^2/Q^2. Since $M_O^2/m^2 \approx .2$ [f1] for nucleon targets, it makes sense to neglect the latter effects while keeping the target amss corrections.

Target mass corrections are anomalously important when they compete with higher order QCD corrections. Since $\frac{\alpha_s(Q^2)}{\pi} = \frac{m^2_{proton}}{Q^2}$ at $Q^2 \approx 6.5\ \text{GeV}^2$, any attempt to see higher order corrections at a moderate value of Q^2 must take proper account of the target mass. For instance in electroproduction at $Q^2 = 5\text{GeV}^2$, ξ scaling (target mass) corrections account for about 35% of the total scaling violation in the fourth moment (QCD scaling violations are an order α_s effect).

The paper is organized as follows: In section I, we present the algorithm for including target mass corrections in the QCD parton model. In section II, we use this algorithm to rederive the ξ scaling analysis for totally inclusive electroproduction. In section III, we analyze semi-inclusive leptoproduction. We show that the use of the variable ω_H instead of the usual z_H leads to the simplest result. Scaling violations and factorization breaking are studied using double moments which are constructed to give target mass independent results.

I. A. THE QCD PARTON MODEL

The standard parton model predictions for cross-sections involving an initial hadron of momentum P are of the form

$$d\sigma_H(P) = \sum_k \int_0^1 d\eta \; d\sigma_k(\eta P) f_k(\eta) \tag{1.1}$$

where $d\sigma_{H(k)}$ is the hadronic (parton of type k) cross-section, and $f_k(\eta)$ (the parton distribution function) is the probability of finding within the hadron a parton of type k with momentum p where

$$p = \eta P \tag{1.2}$$

The QCD perturbation theory expansion for $d\sigma_k$ is plagued with infrared (IR) singularities. In order to preserve the usefulness of eq. (1.1), we must factor these singularities out of $d\sigma_k$ and absorb them into a redefinition of f_k. To regulate these singularities while keeping the quarks massless, we will take $p^2 < 0$. The following factorization theorem has been proven to all orders of QCD and for all logs[1,2]

$$d\sigma_k(p) = \sum_j \int_0^1 d\beta \; d\tilde{\sigma}_j(\beta p, \frac{M^2}{Q^2}) \; T_{jk}(\beta, \frac{p^2}{M^2}) + O(p^2) \tag{1.3}$$

The "renormalized" cross section $d\tilde{\sigma}_j$ is to be evaluated at $p^2 = 0$ (it is IR finite). All of the p^2 dependence (and thus all of the IR sensitivity) resides in Γ. M is an arbitrary scale which is introduced to allow factorization of the logs.

Equations (1.1) and (1.3) can be combined to give

$$d\sigma_H(P) = \sum_j \int_0^1 d\eta \; d\tilde{\sigma}_j \left(\eta P, \frac{M^2}{Q^2}\right) \tilde{f}_j(\eta, M^2) \qquad (1.4)$$

where the Γ factor has been absorbed into a scale dependent "renormalization" of f

$$\tilde{f}_j(\eta, M^2) = \sum_k \int_\eta^1 \frac{d\beta}{\beta} \Gamma_{jk}\left(\beta, \frac{p^2}{M^2}\right) f_k\left(\frac{\eta}{\beta}\right) \qquad (1.5)$$

Equation (1.4) gives the hadronic cross-sections in terms of renormalized partonic cross-sections (which can be calculated in perturbation theory) and process independent[f2], renormalized distribution functions. The latter must be inferred from experiment; however their M^2 dependence is calculable in perturbation theory from the following equation:

$$M \frac{d}{dM} \tilde{f}_j^{(n)}(M^2) = -\sum_k \gamma_{jk}^{(n)}(\bar{g}(M^2)) \tilde{f}_k^{(n)}(M^2) \qquad (1.6)$$

where $\gamma^{(n)}$ is defined by

$$\left(M \frac{\partial}{\partial M} + \beta(\bar{g}(M)) \frac{\partial}{\partial \bar{g}}\right)\Gamma^{(n)}_{jk}\left(\frac{p^2}{M^2}, \bar{g}(M^2)\right)$$
$$= - \sum_{\ell} \gamma^{(n)}_{j\ell}(\bar{g}(M^2))\Gamma^{(n)}_{\ell k}\left(\frac{p^2}{M^2}, \bar{g}(M^2)\right) \tag{1.7}$$

We have implicitly chosen M to be the renormalization point of the theory. The index n denotes the n^{th} moment of a function, i.e.

$$h^{(n)} \equiv \int_0^1 \alpha^{n-1} h(\alpha)\, d\alpha \tag{1.8}$$

B. THE COVARIANT QCD PARTON MODEL

We now seek to extend this analysis to include target mass [f3] effects. Equation (1.2) must be modified, since it makes sense only if the initial hadron is massless. Further, it is assumed in eq.(1.1) that the partonic and hadronic fluxes are the same. This is true only if both the parton and hadron are massless.

The latter difficulty is easily handled by considering squared amplitudes W instead of cross sections. The most general statement consistent with incoherence is then

$$W_H(P) = \sum_k \int d^4p\, \delta(p^2)\, \theta(p^o) W_k(p) G_k(p,P) \frac{2}{\pi m^2} \tag{1.9}$$

where $W_{H(k)}$ is the squared matrix element for an initial state hadron (type k parton), G_k describes the decomposition of the hadron into an on-shell, massless type k parton plus anything else, and m is the hadron's mass.

G_k must be a dimensionless Lorentz invariant quantity. The only possibility for spin averaged initial states[f4] is

$$G_k(p,P) = G_k(u) \tag{1.10a}$$

$$u \equiv \frac{2\, p \cdot P}{m^2} \tag{1.10b}$$

Let $\vec{P}$ be in the 3 direction. Defining light cone coordinates

$$v_\pm \equiv v^0 \pm v^3 \tag{1.11}$$

we then define η via

$$\eta = p^+/P^+ \tag{1.12}$$

which is analagous to eq. (1.2). We then find

$$\frac{2}{\pi m^2}\, d^4p \quad \delta(p^2) \rightarrow d\eta\, dv\, \frac{d\phi}{2\pi} \tag{1.13}$$

where ϕ is the azimuthal angle of $\vec{p}_T$ (the component of $\vec{p}$ orthogonal to the 3 direction).

Kinematic limits can be placed on u and η by requiring that the momentum of the hadron's fragments $(P - p)$ correspond to a positive energy, positive mass state. This gives

$$0 < \eta < 1 \ ; \quad \eta < u = \eta + \frac{p_T^2}{\eta m^2} < 1 \tag{1.14}$$

Equation (1.14) thus gives rise to kinematically generated partonic transverse momenta bounded by

$$p_{T\,max}^2 = \frac{1}{4}\frac{m^2}{Q^2} \tag{1.15}$$

At this point, it is easy to see why we cannot (as of yet) include corrections involving the mass of a produced hadron in a semi-inclusive process. This would involve a covariant description of the decay of an on-shell massless parton into the massive hadron plus a positive energy, positive mass state. This is kinematically impossible. Thus, to include these corrections, we would have to take the parton off shell and extend the factorization theorem to include the $O(p^2)$ terms.

With u, η and θ as integration variables, eq. (1.9) becomes

$$W_H(P) = \sum_k \int_0^1 du \int_0^u d\eta \ \frac{d\phi}{2\pi} W_k(p) G_k(u) \tag{1.16}$$

This equation (with W_k replaced by its Born approximation) is the usual covariant parton model [5]. We seek to extend this to the QCD covariant parton model by allowing W_k to be calculated to all orders of QCD, factorizing out the IR sensitivity, and absorbing it into a renormalization of G_k, in analogy to the massless case. The essential point in this argument is that the factorization theorem of eq. (1.3) has nothing to do with target masses; it is simply a property of QCD perturbation theory.

It is easy to rewrite eq. (1.3) in terms of W's rather than cross-sections since these differ only by trivial flux factors. Under scalings of p

$$W_k(p) \sim p d\sigma_k(p) \tag{1.17}$$

the extra $\frac{1}{p}$ being induced by the flux factors. Thus we must replace Γ in eq. (1.3) by $\frac{\Gamma}{\beta}$, i.e.

$$W_k(p) = \sum_j \int_o^1 \frac{d\beta}{\beta} \tilde{W}_j\left(\beta p, \frac{M^2}{Q^2}\right) \Gamma_{jk}\left(\beta, \frac{p^2}{M^2}\right) + O(p^2) \tag{1.18}$$

where $\tilde{W}_j$ is to be evaluated with $p^2 = 0$.

We now show that Γ can be absorbed into G. From eqs. (1.16) and (1.18)

$$W_H(P) = \sum_{j,k} \int_o^1 \frac{d\beta}{\beta} \int_o^1 du \int_o^u d\eta \int \frac{d\phi}{2\pi} \tilde{W}_j\left(\beta p, \frac{M^2}{Q^2}\right) \Gamma_{jk}\left(\beta, \frac{p^2}{M^2}\right) G_k(u) \tag{1.19}$$

Let $\tilde{p} = \beta p$, $\tilde{u} = \beta u$, $\tilde{\eta} = \beta\eta$, and $\tilde{\phi} = \phi$, then eq. (1.19) becomes

$$W_H(P) = \sum_{j,k} \int_o^1 d\tilde{u} \int_o^{\tilde{u}} d\tilde{\eta} \int \frac{d\tilde{\phi}}{2\pi} \tilde{W}_j\left(\tilde{p}, \frac{M^2}{Q^2}\right) \int_{\tilde{u}}^1 \frac{d\beta}{\beta^3} \Gamma_{jk}\left(\beta, \frac{p^2}{M^2}\right) G_k\left(\frac{\tilde{u}}{\beta}\right) \tag{1.20}$$

Equivalently

$$W_H(P) = \sum_j \int_o^1 du \int_o^u d\eta \int \frac{d\phi}{2\pi} \tilde{W}_j(p, \frac{M^2}{Q^2}) \tilde{G}_j(u, M^2) \tag{1.21}$$

where

$$\tilde{G}_j(u, M^2) \equiv \sum_k \int_u \frac{d\beta}{\beta^3} \Gamma_{jk}(\beta, \frac{p^2}{M^2}) G_k(\frac{u}{\beta}) \tag{1.22}$$

As in the massless case, all of the IR sensitivity has been absorbed into the definiton of scale dependent distribution functions[f5]. The scale dependence dictated by eq. (1.22) is given by

$$M \frac{d}{dM} \tilde{G}_j^{(n)}(M^2) = - \sum_k \gamma_{jk}^{(n-2)}(\bar{g}(M^2)) \tilde{G}_k^{(n)}(M^2) \tag{1.23}$$

which is the analogue of eq. (1.6).

Equations (1.21) and (1.23) constitute the covariant QCD parton model. As in the massless case, hadronic quantities (W_H) can be computed in terms of partonic quantities ($\tilde{W}_j$) which have IR finite perturbation expansions, and scale dependent, process independent functions of a single variable ($\tilde{G}_k$), one for each parton type. The only complications are kinematic.

The extension of this analysis to semi-inclusive processes (neglecting the mass of the observed hadron) is trivial. Equation (1.16) becomes

$$\frac{dW_H(P,P')}{d^3\vec{P}'/P'^0} = \sum_{j,k} \int_o^1 du \int_o^1 d\eta \int \frac{d\phi}{2\pi} \int_o^1 \eta'^2 d\eta' D_k(\eta') \frac{dW_{kj}(p,p')}{d^3\vec{p}'/p'^0} G_j^{(u)} \tag{1.24}$$

where $\dfrac{dW_H}{d^3\vec{P}'/P'^0}$ $\left(\dfrac{dW_{kj}}{d^3\vec{p}'/p'^0}\right)$ is the squared hadronic (partonic) matrix element involving a final state of a hadron (type k parton) of momentum $P'(p')$ plus anything else. The decay function $D_k(\eta')$ is the probability of the decay of the final state parton into the final state hadron where their momenta are related by

$$P' = \eta' p' \tag{1.25}$$

The relevant factorization theorem is[1,2]

$$\frac{dW_{kj}}{d^3\vec{p}'/p'^0}(p,\ p') = \sum_{\ell,m} \int_0^1 d\beta \int_0^1 \frac{d\beta'}{\beta'^2}\, \Gamma'_{k\ell}(\beta',\ \frac{p'^2}{M^2})$$

$$\frac{d\tilde{W}_{\ell m}}{d^3\vec{p}'/p'^0}(\beta p,\ \beta' p',\ \frac{M^2}{Q^2})\Gamma_{mj}(\beta,\ \frac{p^2}{M^2}) + O(p^2,\ p'^2) \tag{1.26}$$

Manipulations analagous to those leading to eq. (1.20) give

$$\frac{dW_H(P,\ p')}{d^3\vec{P}/P^0} = \sum_{j,k} \int_0^1 du \int_0^u d\eta \int \frac{d\phi}{2\pi} \int_0^1 \eta'^2 d\eta'\, \tilde{D}_k(\eta',M^2) \frac{d\tilde{W}_{kj}(p,p',\frac{M^2}{Q^2})}{d^3\vec{p}'/p'^0}\, \tilde{G}_j(u,M^2) \tag{1.27}$$

where

$$\tilde{D}_k(\eta',\ M^2) \equiv \sum_\ell \int_{\eta'}^1 \frac{d\beta'}{\beta'}\, D_\ell(\ \frac{\eta'}{\beta'}\ )\ \Gamma'_{\ell k}(\beta',\ \frac{p'^2}{M^2}\ ) \tag{1.28}$$

Thus the factorization of IR singularities associated with $p^2 \to 0$ and their reabsorption into G is unchanged. The singularities associated with $p'^2 \to 0$ are absorbed into the decay functions exactly as in the massless case.

Taking M to be the renormalization point, the scale dependence of the moments of $\tilde{D}$ is given by

$$M \frac{d}{dM} \tilde{D}_j^{(n)}(M^2) = - \sum_k \tilde{D}_k^{(n)}(M^2)\gamma'^{(n)}_{kj}(\bar{g}(M^2)) \qquad (1.29)$$

where $\gamma'^{(n)}$ is defined via

$$\left(M\frac{\partial}{\partial M} + \beta(\bar{g}(M))\frac{\partial}{\partial g}\right)\Gamma'^{(n)}_{jk}\left(\frac{p^2}{M^2}, \bar{g}(M^2)\right)$$

$$= \sum_\ell \Gamma'^{(n)}_{j\ell}\left(\frac{p^2}{M^2}, \bar{g}(M^2)\right)\gamma'^{(n)}_{\ell k}(\bar{g}(M^2)) \qquad (1.30)$$

The extension of eq. (1.27) to the case of several incoming and or outgoing hadrons is obvious.

II. INCLUSIVE LEPTOPRODUCTION

It has been shown elsewhere[5] that the QCD parton model and the twist-two operator product expansion (OPE) descriptions of inclusive leptoproduction are equivalent in the absense of target mass corrections. These corrections are easily identified in the twist-two OPE. They arise in the nucleon matrix elements of the twist-two operators as follows

$$< P|O^{u_1 \cdot\cdot u_n}|P > = A_k^{(n)} (P^{u_1} \cdots P^{u_n} - \text{traces}) \qquad (2.1)$$

The traces in eq. (2.1) are uniquely determined by the tracelessness of the operators; they involve powers of $P^2 = m^2$.

Taking account of the traces gives rise to the ξ scaling formulae of refs. 3 and 4. It is our purpose to derive these results from the QCD covariant parton model[f6].

The squared matrix elements relevant to leptoproduction are

$$W_H^{uv} \equiv \frac{1}{4\pi}\int d^4x\ e^{iq\cdot x} <P|J^{u^\dagger}(x)J^v(0)|P> \qquad (2.2)$$

where the J's are electromagnetic or weak currents. Equation (1.20) becomes

$$W_H^{uv} = \sum_j \int_0^1 du \int_0^u d\eta \int \frac{d\phi}{2\pi} \tilde{W}_j^{uv}(p)\tilde{G}_j(u, Q^2) \qquad (2.3)$$

where W_j^{uv} is defined by analogy to W_H^{uv} with the hadron replaced by a type j parton; and we have chosen M, Q, and the renormalization point to be identical (thus $\tilde{W}$ has an implicit Q^2 dependence through $\bar{g}(Q^2)$).

We now introduce the usual Bjorken scaling variables

$$x_H \equiv \frac{Q^2}{2P\cdot q} \quad x \equiv \frac{Q^2}{2p\cdot q} \qquad (2.4)$$

and trade the integration variable η for x[f7]. Then eq. (2.3) becomes

$$W_H^{\mu\nu}(P,q) = R \sum_j \int_\xi^1 \frac{dx}{x^2} \int_{\xi/x}^1 du \int \frac{d\phi}{2\pi} \tilde{W}_j^{\mu\nu}(p,q)\, \tilde{G}_j(u,Q^2) \qquad (2.5)$$

where ξ is the usual ξ variable

$$\xi = \frac{2x_H}{1 + \sqrt{1 + \frac{4m^2}{Q^2} x_H^2}} \qquad (2.6)$$

the determinential factor R is

$$R = \frac{x_H}{\sqrt{1 + \frac{4m^2}{Q^2} x_H^2}} = \frac{\xi}{1 + \frac{m^2}{Q^2} \xi^2} \qquad (2.7)$$

and we have made use of the kinematic constraint $0 \leq x \leq 1$.

We define the usual structure functions as follows

$$W_H^{\mu\nu} = \frac{Q^2}{(P \cdot q)^2} \left(P^\mu + q^\mu \frac{P \cdot q}{Q^2}\right) \left(P^\nu + q^\nu \frac{P \cdot q}{Q^2}\right) \frac{\nu W_{2H}}{2x_H} (x_H, Q^2)$$

$$+ \left(-g^{\mu\nu} - \frac{q^\mu q^\nu}{Q^2}\right) W_{1H}(x_H, Q^2) + i\varepsilon^{\mu\nu\lambda\rho} \frac{P_\lambda q_\rho}{P \cdot q} \nu W_{3H}(x_H, Q^2) \qquad (2.8)$$

with analagous definitions for $\tilde{W}_{r,k}(x)$, $r = 1, 2, 3$.

We now form two projections of $W_H^{\mu\nu}$ which determine the non-parity-violating structure functions W_{1H} and W_{2H}:

$$W^{\mu}_{H\mu} = (1 + \frac{4m^2}{Q^2} x_H^2) \frac{\nu W_{2H}}{2x_H} - 3W_{1H} = W_{LH} - 2W_{TH}$$

(2.9a)

$$= R \sum_j \int_\xi^1 \frac{dx}{x^2} \int_{\xi/x}^1 du \, \tilde{W}^{\mu}_{j\mu}(x) \tilde{G}_j(u, Q^2)$$

$$4R^2 \frac{P_\mu W_H^{\mu\nu} P_\nu}{Q^2} = [1 + 4 \frac{m^2}{Q^2} x_H^2] \frac{\nu W_{2H}}{2x_H} - W_{1H} = W_{LH}$$

$$= R \sum_j \int_\xi^1 \frac{dx}{x^2} \int_{\xi/x}^1 du (4R^2 \frac{P_\mu \tilde{W}_j^{\mu\nu}(x) P_\nu}{Q^2}) \tilde{G}_j(u, Q^2)$$

(2.9b)

The partonic contractions are given by

$$\tilde{W}^{\mu}_{j\mu} = \frac{\nu \tilde{W}_{2j}}{2x} - 3\tilde{W}_{1k} \qquad (2.10a)$$

$$4R^2 \frac{P_\mu \tilde{W}_j^{\mu\nu} P_\nu}{Q^2} = \frac{\nu \tilde{W}_{2j}}{2x} - \tilde{W}_{1j} + 4R \frac{m^2}{Q^2} [(ux - \xi) + R \frac{m^2}{Q^2} (ux - \xi)^2] \frac{\nu \tilde{W}_{2j}}{2x}$$

(2.10b)

Since $\frac{\nu W_{2j}}{2x}$ and $\tilde{W}_{ij}$ depend only on x (and of course $g(Q^2)$), these contractions are ϕ independent, thus the ϕ integral has been dropped in eq. (2.9). Note also that the u dependence of the contractions is explicit and trivial.

Equation (2.9) involves integrals of the form

$$I_n(\xi) = \int_\xi^1 \frac{dx}{x^2} \int_{\xi/x}^1 du\, \tilde{W}(x)\tilde{G}(u)(ux-\xi)^n \,. \qquad (2.11)$$

These can be simplified by introducing functions $\tilde{g}_j$ defined by

$$\tilde{g}_j(\alpha,Q^2) \equiv \int_\alpha^1 du\, \tilde{G}_j(u,Q^2) \qquad (2.12)$$

Note that, in terms of moments (defined by eq.(1.8))

$$\tilde{g}_j^{(n)}(Q^2) = \frac{1}{n}\tilde{G}_j^{(n+1)}(Q^2) \qquad (2.13)$$

thus the scaling law of eq. (1.23) becomes

$$Q\frac{d}{dQ}\tilde{g}_j^{(n)}(Q^2) = -\sum_k \gamma_{jk}^{(n-1)}(\bar{g}(Q^2))\tilde{g}_k^{(n)}(Q^2) \qquad (2.14)$$

We now prove the following theorem

$$I_n(\xi) = n!\int_\xi^1 d\alpha_1 \int_{\alpha_1}^1 d\alpha_2 \cdots \int_{\alpha_n}^1 \frac{d\alpha_{n+1}}{\alpha_{n+1}} \left(\frac{w(\alpha_{n+1})}{\alpha_{n+1}}\right)\tilde{g}\left(\frac{\alpha_n}{\alpha_{n+1}}\right) \qquad (2.15)$$

This is easy to show for $n = 0$. The result for larger n can follows by induction since both sides of eq. (2.11) obey

$$\frac{dI_n(\xi)}{d\xi} = -n\, I_{n-1} \qquad (2.16)$$

and the boundary condition

$$I_n(1) = 0 \qquad (2.17)$$

Equation (2.15) shows that the distribution functions $\tilde{g}$ always appear in convolutions with partonic structure functions. This leads us to the following definitions

$$\tilde{h}_1(\alpha,Q^2) \equiv 4 \sum_j \int_\alpha^1 \frac{d\beta}{\beta^2} \tilde{W}_{1,j}(\beta,\bar{g}(Q^2))\tilde{g}_j(\frac{\alpha}{\beta}, Q^2) \qquad (2.18a)$$

$$\tilde{h}_2(\alpha,Q^2) \equiv 4 \sum_j \int_\alpha^1 \frac{d\beta}{\beta^2} \frac{\nu\tilde{W}_{2,j}(\beta,\bar{g}(Q^2))}{2\beta} \tilde{g}_j(\frac{\alpha}{\beta}, Q^2) \qquad (2.18b)$$

or, in terms of moments

$$\tilde{h}_1^{(n)}(Q^2) = 4 \sum_j \tilde{W}_{1,j}^{(n-1)}(\bar{g}(Q^2))\tilde{g}_j^{(n)}(Q^2) \qquad (2.19a)$$

$$\tilde{h}_2^{(n)}(Q^2) = 4 \sum_j \left(\frac{\nu\tilde{W}_{2,j}}{2x}\right)^{(n-1)}(\bar{g}(Q^2))\tilde{g}_j^{(n)}(Q^2) \qquad (2.19b)$$

Then, from eqs. (2.9)→(2.11), (2.15), and (2.18) we obtain

$$W_{LH} - 2W_{TH} = \frac{1}{4} R\, [\tilde{h}_2(\xi,Q^2) - 3h_1(\xi,Q^2)] \qquad (2.20a)$$

$$W_{LH} = \frac{1}{4} R [\tilde{h}_2(\xi,Q^2) - \tilde{h}_1(\xi,Q^2) + 4R\frac{m^2}{Q^2} \int_\xi^1 \tilde{h}_2(\xi',Q^2)d\xi' + 8R^2 \frac{m^4}{Q^4} \int_\xi^1 d\xi' \int_{\xi'}^1 d\xi'' \tilde{h}_2(\xi'',Q^2)] . \qquad (2.20b)$$

Equation (2.20) has exactly the same form as the ξ scaling equations of ref. (3)[f8], with the replacements

$$\tilde{h}_{1,2} \to F_{1,2} \tag{2.21}$$

The F's of ref. 3 are defined in terms of their moments:

$$F^{(n)}_{(1,2)}(Q^2) = \sum_j C^{(n-1)}_{(1,2)j}(\bar{g}(Q^2))A^{(n-1)}_j(Q^2)$$

where the $C^{(n)}$ are the coefficient functions of the twist two operators of spin n, and the $A^{(n)}$ are the corresponding reduced matrix elements defined in eq. (2.1). The index j completes the identification of the operator; there is one j for each relevant parton type.

The Q^2 dependence of the $A^{(n)}$ is governed by the anomalous dimension matrix $\bar{\gamma}\,(n)$ of the associated operators

$$Q \frac{d}{dQ} A^{(n)}_j (Q^2) = - \sum_k \bar{\gamma}^{(n)}_{jk}(\bar{g}(Q^2))A^{(n)}_k(Q^2) \ . \tag{2.23}$$

The coefficent functions and anomalous dimensions of the OPE can be related to the $\tilde{W}$ and γ of the parton model by computing $W^{\mu\nu}_k$ from the OPE and comparing the results to eq. (1.18). Using a particular definition of the operators[7] we see that the following is a possible realization of eq. (1.17)[f9]

$$\left(\frac{\nu\tilde{W}_{2k}}{2x}\right)^{(n)} = \frac{1}{4}\, C^{(n)}_{2k}; \quad (\tilde{W}_{1k})^{(n)} = \frac{1}{4}\, C^{(n)}_{1k} \tag{2.24a}$$

$$\bar{\gamma}^{(n)} = \gamma^{(n)} \qquad (2.24b)$$

Then from eq. (2.19), we see that the results are equivalent if we choose

$$A_k^{(n-1)}(Q^2) = \tilde{g}_k^{(n)}(Q^2) \qquad (2.25)$$

The Q^2 dependence of these quantities is governed by the same anomalous dimension matrix $\bar{\gamma}^{(n-1)} = \gamma^{(n-1)}$; so the equivalence is preserved at all Q^2.

The analysis of the W_3 structure function is entirely analagous. The result is

$$\nu W_3^H = \frac{1}{4}\frac{R^2}{x_H}[\tilde{h}_3(\xi,W^2) + \frac{2m^2}{Q^2} R\int_\xi^1 d\xi'\tilde{h}_3(\xi',Q^2)d\xi'] \qquad (2.26)$$

where

$$\tilde{h}_3(\alpha,Q^2) = 4 \sum_j \int_\alpha^1 \frac{d\beta}{\beta}\frac{\nu\tilde{W}_{3j}(\beta,\bar{g}(Q^2))}{\beta}\tilde{g}_j(\frac{\alpha}{\beta}, Q^2) \qquad (2.27a)$$

or

$$\tilde{h}_3^{(n)}(\alpha,Q^2) = 4 \sum_j (\nu\tilde{W}_{3j})^{(n-1)}(\bar{g}(Q^2))\,\tilde{g}_j^{(n)}(Q^2) \quad . \qquad (2.27b)$$

The Nachtmann moments [9] of the structure functions extract out the contributions of the operators of a given spin. In the parton model language, they extract out the contribution of a given moment of $\tilde{h}$. The Nachtman moments are

$$M_a^{(n)}(Q^2) \equiv \int_0^1 \xi^{n-1} d\xi [W_{LH} - 2W_{TH}](1 + \frac{m^2}{Q^2}\xi^2)$$

$$= \frac{1}{4}\tilde{h}_2^{(n+1)}(Q^2) - \frac{3}{4}\tilde{h}_1^{(n+u)}(Q^2)$$

$$= \sum_j [(\frac{\nu\tilde{W}_{2,j}}{2x})^{(n)}(\bar{g}(Q^2)) - 3\tilde{W}_{1,j}^{(n)}(\bar{g}(Q^2))]\tilde{g}_j^{(n+1)}(Q^2)$$

(2.28a)

$$M_b^{(n)}(Q^2) \equiv \int_0^1 \xi^{n-1} d\xi [W_{LH} + W_{TH}](1 + \frac{m^2}{Q^2}\xi^2)\Big[1 - \frac{6}{(n+2)} \frac{\frac{m^2}{Q^2}\xi^2}{(1 + \frac{m^2}{Q^2}\xi 2)^2}$$

$$- \frac{6(n+1)}{(n+2)(n+3)} \frac{\frac{m^2}{Q^4}\xi^4}{(1 + \frac{m^2}{Q^2}\xi^2)^2}\Big]$$

$$= \frac{1}{4}\tilde{h}_2^{(n+1)}(Q^2) = \sum_j (\frac{\nu W_{2j}}{2x})^{(n)}(\bar{g}(Q^2))g_j^{(n+1)}(Q^2) \qquad (2.28b)$$

$$M_c^{(n)}(Q^2) \equiv \int_0^1 \xi^{n-1} d\xi \, \nu W_{3H} \frac{(1 + \frac{m^2}{Q^2} 2)^2}{(1 - \frac{m^2}{Q^2}\xi^2)} \left[1 - \frac{2\frac{m^2}{Q^2}\xi^2}{(n + 2)(1 + \frac{m^2}{Q^2}\xi^2)}\right]$$

$$= \frac{1}{4}\tilde{h}_3^{(n+1)}(Q^2) = \sum_j (\nu\tilde{W}_{3j})^{(n)}(\bar{g}(Q^2))\tilde{g}_j^{(n+1)}(Q^2) \; . \qquad (2.28c)$$

The Nachtmann moment have the following important virtues: firstly, they give target mass independent results, so that all scaling violations are logarithmic. Secondly, each Nachtmann moment depends only on a single moment of the decomposition functions. These have the simple scale dependence dictated by eq. (2.14).

III. SEMI-INCLUSIVE LEPTOPRODUCTION:

A. DOUBLE MOMENT ANALYSIS

Consider the process

$$B^{\nu}(q) + N(P) \to M(P') + X \tag{3.1}$$

where B is a spacelike gauge boson (photon, $W^{\dagger}$ or W^{-}), N is a nucleon and M is a light meson[f10]. Suppose we charaterize the meson momentum P' by a single invariant ω_H The relevant squared matrix element is then

$$\frac{dW_H^{\mu\nu}}{d\omega_H} \equiv \frac{1}{4\pi} \sum_X d^4x e^{iq\cdot x} \int \frac{d^3\vec{P}'}{(2\pi)^3 2P'^0} \delta(\omega_H - \omega_H(P',P,q))$$
$$< P|J^{\mu\dagger}(x)|P',X><P',X|J^{\nu}(0)|P>$$

where J is the appropriate weak or electromagnetic current.

Equation (1.26) gives the covariant QCD parton model prediction for this matrix element. Schematically

$$\frac{dW_H^{\nu\mu}}{d\omega_H}(P,q,\omega_H) = \sum_{j,k} \int dp\,dp'\,dP'\,\delta(\omega_H - \omega_H(P',P,q))\tilde{D}_k(P',p')\,\frac{d\tilde{W}_{kj}^{\mu\nu}}{dp'}(p,p',q)\tilde{G}_j(p,P) \tag{3.3}$$

Equation 3.3 describes the hadronic process in 3 steps: a) The decomposition of the hadron (momentum P) into a j parton of momentum p plus anything, described by $\tilde{G}_j(p,P)$

b) Scattering of this parton to produce a k type parton of momentum p' and anthing else, described by $\frac{d\tilde{W}_{kj}}{dp'}(p,p',q)$.

c) Decay of the final parton into a meson of momentum P' and anything else, described by $\tilde{D}_k(P',p')$.

The final state momentum P' enters only in step c (the decay function), the initial state momentum P enters only in step a (the decomposition function).

If ω_H depends on P, the δ function ties the argument of the decay function to P. This obscures the 3 step nature of the process and therefore leads to unnecessarily complicated results.

Thus we choose ω_H to P independent. The only possibility which scales with P' (this makes for simple moment statements) is (up to trivial scalings)

$$\omega_H \equiv \frac{-2P'\cdot q}{Q^2} \tag{3.4}$$

The partonic analogue of this variable is

$$\omega = \frac{-2p'\cdot q}{Q^2} = \eta'\omega_H; \quad \eta' = \frac{P'}{p'} \tag{3.5}$$

where η' is the argument of the decay function in eq. (1.27).

Kinematic limits can be placed on ω_H by requiring that the X of eq. (3.1) be a positive energy state with mass $\geq m$ (baryon number conservation). This, and analagous considerations for the partonic process, give

$$-\frac{1-x_H}{x_H+\frac{x_H}{\xi}-1} \le \omega_H \le \frac{1-x_H}{\frac{x_H}{\xi}-x_H} \qquad (3.6a)$$

$$-\frac{1-x}{x} \le \omega \le 1 \qquad (3.6b)$$

The factorization theorem breaks down in the initial parton fragmentation region, i.e. whenever

$$p' = \alpha p; \quad \alpha > 0 \qquad (3.7)$$

This implies

$$\omega = -\frac{2\alpha p \cdot q}{Q^2} = -\frac{\alpha}{x} < 0 \qquad (3.8)$$

We must therefore study only those values of ω_H which cannot arise from outgoing partons with $\omega < 0$. From eq. (3.5), and from the restriction $\eta' > 0$; this requirement becomes[f11]

$$\omega_H > 0 \,. \qquad (3.9)$$

With this restriction, and with eq. (3.5), eq. (3.3) becomes

$$\frac{dW_H^{\mu\nu}}{d\omega_H} = R \sum_{j,k} \int_\xi^1 \frac{dx}{x^2} \int_{\xi/x}^1 du \int \frac{d\phi}{2\pi} \left\{ \int_{\omega_H}^1 \frac{d\omega}{\omega} \tilde{D}_k\left(\frac{\omega_H}{\omega}, Q^2\right) \frac{d\tilde{W}_{kj}^{\mu\nu}}{d\omega}(p,q,\omega) \right\} \tilde{G}_j(u,Q^2) \qquad (3.10)$$

$$(\omega_H > 0)$$

where we have parameterized p by u,x, and ϕ; and p' by ω. Note that all of the dependence on ω_H resides in the convolution in curly brackets. Thus, taking moments with respect to ω_H gives

$$\left(\frac{dW_H^{\mu\nu}}{d\omega_H}\right)^{(m)}(P\ ,q) \equiv \int_0^1 d\omega_H \omega_H^{m-1} \frac{dW_H^{\mu\nu}}{d\omega_H}(P,q,\omega_H)$$

$$= R \sum_{j,k} \int_\xi^1 \frac{dx}{x^2} \int_{\xi/x}^1 du \int \frac{d\phi}{2\pi} \left\{\tilde{D}_k^{(m)}(Q^2)\ \frac{d\tilde{W}_{kj}^{\mu\nu}}{d\omega}^{(m)}(p,q)\right\} \tilde{G}_j(u,Q^2) \tag{3.11}$$

This equation has exactly the same form as eq. (2.5) with the replacements

$$W_H^{\mu\nu}(P,q) \rightarrow \left(\frac{dW_H^{\mu\nu}}{d\omega_H}\right)^{(m)}(P,q) \tag{3.12a}$$

$$\tilde{W}_j^{\mu\nu}(p,q) \rightarrow \sum_k \tilde{D}_k^{(m)}(Q^2) \left(\frac{d\tilde{W}_{kj}^{\mu\nu}}{d\omega}\right)^{(m)} (p,q) \quad .$$

Thus if we take Nachtman moments of the structure functions of $\frac{dW_H^{\mu\nu}}{d\omega_H}$, we get simple results. By analogy to eq. (2.28a), we get

$$M_a^{(m,n)}(Q^2) \equiv \int_0^1 \xi^{n-1} d\xi (1+\frac{m^2}{Q^2}\xi^2) \int_0^1 \omega_H^{m-1} d\omega_H \left[\frac{dW_{LH}}{d\omega_H} - 2\ \frac{dW_{TH}}{d\omega_H}\right]$$

$$= \sum_{j,k} \tilde{D}_k^{(m)}(Q^2) \left\{ \left(\frac{d\left(\frac{\nu \tilde{W}_{2kj}}{2x}\right)}{d\omega}\right)^{(m,n)} (\bar{g}\ (Q^2)) - 3 \left(\frac{d\tilde{W}_{1kj}}{d\omega}\right)^{(m,n)} (\bar{g}(Q^2))\right\} \bar{g}_j^{(n+1)}(Q^2) \tag{3.13}$$

Analagous equation are trivially derived from eqs. (2.28b) and (2.28c).

The structure functions of $\frac{dW_H^{\mu\nu}}{d\omega_H}$ and $\frac{dW_{kj}^{\mu\nu}}{d\omega}$ are defined by analogy to eq. (2.8). The superscripts (m,n) of the partonic structure functions in eq. (3.13) denote their double moments with respect to x (n^{th} moment) and ω (m^{th} moment) e.g.

$$\left(\frac{dW_{1kj}}{d\omega}\right)^{(m,n)}(\bar{g}(Q^2)) \equiv \int_0^1 dx\, x^{n-1} \int_0^1 d\omega\, \omega^{m-1} \frac{dW_{1kj}}{d\omega}(x,\omega,\bar{g}(Q^2)) \quad (3.14)$$

The double moments $M_r^{(m,n)}$ $(r = a,b,c)$ have all of the virtues of the Nachtmann moments in the inclusive case. They give rise to target mass independent results involving only one moment of the distribution and decay functions. The latter quantities have scaling violations governed by simple algebraic equations (eqs. (2.14) and (1.29)).

In appendix I, we will discuss the kinematic complications which occur when the variable $Z_H \equiv \frac{P'\cdot P}{P\cdot q}$ is used instead of ω_H, and x_H moments are used instead of Nachtmann moments.

B. Factorization Breaking

To zeroth order in α_s, the kinematics of the Feynman diagram of fig. 1 gives

$$\frac{d\tilde{W}_{kj}}{d\omega} = A_{kj}\,\delta(1-x)\,\delta(1-\omega) + O(\alpha_s) \qquad (3.15)$$

where $\frac{d\tilde{W}_{kj}}{d\omega}$ stands for any of the three partonic structure functions $\frac{d\tilde{W}_{1kj}}{d\omega}$, $\frac{d(\frac{\nu\tilde{W}_{2kj}}{2x})}{d\omega}$, and $\frac{d(\nu\tilde{W}_{3kj})}{d\omega}$. Thus the double moments of this quantity are independent of n and m to zeroth order, i.e.

$$\left(\frac{d\tilde{W}_{kj}}{d\omega}\right)^{(m,n)} = A_{jk} + O(\alpha_s) \qquad (3.16)$$

If we take appropriate non-singlet differences (which are desribed in appendix II), only one linear combination of distribution and decay functions will contribute, so the sum over parton types in eq. (3.13) collapses to one term. The result is then [f12]

$$M^{(m,n)} = A\,\tilde{D}^{(m)}(Q^2)\tilde{g}^{(n+1)}(Q^2) + O(\alpha_s(Q^2)) \qquad (3.17)$$

where A is a pure number (independent of m,n and Q^2), and $M^{(m,n)}$ is any of the three types of double moments.

The lowest order result factorizes into a function of m and a function of n[f13]. This factorization does not persist in higher orders, since the $x = 1$, $\omega = 1$ kinematic constraint no longer applies.

Sakai[10] has proposed a double moment ratio whose deviation from unity measures the breaking of factorization. It is

$$R^{m,n;k,\ell} \equiv \frac{M^{(m,n)}M^{(k,\ell)}}{M^{(k,n)}M^{(m,\ell)}} . \qquad (3.18)$$

From eq. (3.13), the moments of $\tilde{D}$ and $\tilde{g}$ drop out in this ratio, giving

$$R^{m,n;k,\ell} = \frac{\left(\frac{d\tilde{W}}{d\omega}\right)^{(m,n)} \left(\frac{d\tilde{W}}{d}\right)^{(k,\ell)}}{\left(\frac{d\tilde{W}}{d\omega}\right)^{(k,n)} \left(\frac{d\tilde{W}}{d\omega}\right)^{(m,\ell)}} = 1 + \frac{\alpha_s(Q^2)}{4\pi}\, \beta^{m,n;k,\ell} + O\,(\alpha_s^2). \quad (3.19)$$

Where $\frac{d\tilde{W}}{d\omega}$ is the (linear combination of) structure function(s) appropriate to the Nachtmann moment being taken.

The ratio R has the following useful properties

1. It measures the QCD induced factorization breaking in a target mass independent way.
2. It involves no decay or distribution functions, thus Λ is the only necessary phenomenological input.
3. The $\beta^{n,m;k,\ell}$ correspond to the non-leading logarithm in $\frac{dW}{d\omega}$ (the unrenormalized structure function). Thus R measures a non-leading log effect.

Sakai[10] has calculated the relevant parts of $\frac{d\tilde{W}}{d\omega}$, which arise from the graphs of figure 2[F 14]. The results for the three types of Nachtmann moments are

$$\beta_a^{(m,n;k,\ell)} = \frac{8}{3}\left(\sum_{j=1}^{m-1} \frac{1}{j} \sum_{i=1}^{n-1} \frac{1}{i} + \sum_{j-1}^{m+1} \frac{1}{j} \sum_{1=1}^{n+1} \frac{1}{i} \right)$$

$$+[(m,n) \to (k,\ell)] - [(m,n) \to (k,n)] - [(m,n) \to (m,\ell)] \quad (3.20a)$$

$$\beta_b^{(m,n;k,\ell)} = \frac{8}{3}\left(\sum_{j=1}^{m-1}\frac{1}{j}\sum_{i=1}^{n-1}\frac{1}{i} + \sum_{j=1}^{m+1}\frac{1}{j}\sum_{i=1}^{n+1}\frac{1}{i} + \frac{6}{m(n+2)} - \frac{6}{m(m+1)(n+3)}\right)$$

$$+ [(m,n) \rightarrow (k,\ell)] - [(m,n) \rightarrow (k,n)] - [(m,n) \rightarrow (m,\ell)] \qquad (3.20b)$$

$$\beta_c^{(m,n;k,\ell)} = \frac{8}{3}\left(\sum_{j=1}^{n-1}\frac{1}{j}\sum_{i=1}^{m-1}\frac{1}{i} + \sum_{j=1}^{n+1}\frac{1}{j}\sum_{i=1}^{m+1}\frac{1}{i} + \frac{2}{(m+1)(n+2)}\right)$$

$$+ [(m,n) \rightarrow (k,\ell)] - [(m,n) \rightarrow (k,n)] - [(m,n) \rightarrow (m,\ell)] \qquad (3.20c)$$

The subscripts a, b and c refer to the type of Nachtmann moment used.

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APPENDIX I: THE VARIABLE Z_H

Many treatments [1,11] of semi-inclusive leptoproduction use the variable Z_H:

$$Z_H \equiv \frac{P \cdot P'}{P \cdot q} \quad . \tag{I.1}$$

For massless targets, $P = \eta p$, so

$$Z_H \equiv \frac{p \cdot P'}{p \cdot q} = \eta' \frac{p \cdot p'}{p \cdot q} = \eta' Z \quad . \tag{I.2}$$

Thus, in this limit, Z_H is effectively independent of P, and the relationship between Z_H and Z is trivial. Equations for double moments with respect to x_H and Z_H can be derived which are analagous to eq. (3.13), and the results factorize to zeroth order.

We now show how target mass corrections alter the zero order x_H, Z_H double moments.

From eq. (1.24). we get

$$\frac{dW_H(x_H, Z_H, Q^2)}{dZ4} = R\int_{\xi}^{1} du \int_{\xi/u}^{1} \frac{dx}{x^2} \int_{o}^{1} d\eta' \int \frac{d^3p'}{2p'^0} \, \delta(Z_H - \eta' \frac{P}{P}\frac{p'}{q})$$

$$\cdot \, \tilde{D}(\eta') \frac{d\tilde{W}(p,p',q)}{d^3p' |2p'^0} \tilde{G}(u) \tag{I.3}$$

where we have taken a trace of the gauge boson indices and suppressed the parton type indices.

The relationship between Z_H and x_H, η', u, x, ω, and Z is very complex[f15].

However, if we work to zeroth order in α_s, the kinematics simplify since

$$p' = p + q \tag{I.4}$$

thus

$$Z_H = \eta' \frac{p' \cdot P}{P \cdot q} = \eta' \frac{2(p+q) \cdot P}{2P \cdot q} = \eta'(1 + ux_H \frac{m^2}{Q^2}) \tag{I.5}$$

Equation (I.3) becomes

$$\frac{dW_H}{dZ_H} = R\int_{\xi}^{1} du \int_{\xi/u}^{1} \frac{dx}{x^2} \int_{o}^{1} d\eta' \int_{o}^{1} dZ\, \tilde{D}(\eta') \frac{d\tilde{W}(x,Z)}{dZ} \tilde{G}(u) \quad \delta\left(Z_H - \eta'(1+ux_H\frac{m^2}{Q^2})\right) . \tag{I.6}$$

The zeroth order structure functions are kinematically constrained to be

$$\frac{d\tilde{W}}{dz} \propto \delta(1 - x)\delta(1 - Z) . \tag{I.7}$$

Thus

$$\frac{dW_H}{dZ_H} \propto R\int_{\xi}^{1} du\, \tilde{D}\left(\frac{Z_H}{1 + ux_H\frac{m^2}{Q^2}}\right)\left(\frac{1}{1 + ux_H \frac{m^2}{Q^2}}\right)\tilde{G}(u) + O(\alpha_s) . \tag{I.8}$$

Note that the argument of the decay function depends on the variable u, which describes the initial hadron's decomposition, so the three step nature of the process is obscured.

Taking double moments with respect to x_H and Z_H, and keeping only terms up to $O\,(\frac{m^2}{Q^2})$ gives

$$\frac{dW_H}{dZ_H}^{(m,n)} \equiv \int Z_H^{m-1}\, x_H^{n-1}\, dZ_H\, dx_H\, \frac{dW_H(x_H, Z_H, Q^2)}{dZ_H}$$

$$\propto \tilde{D}^{(m)}(Q^2)\left\{\tilde{g}^{(n+1)}(Q^2) + \frac{m^2}{Q^2}\,\tilde{g}^{(n+3)}(Q^2)\left[(n+1) + (m-1)\left(\frac{n+3}{n+2}\right)\right]\right\}$$

$$+ O\,(\frac{m^4}{Q^4}) + O(\alpha_s(Q^2)) \qquad (I.9)$$

The term proportional to $\frac{m^2}{Q^2}(n+1)$ comes from taking x_H moments instead of Nachtmann moments, and the term proportional to $(m-1)\,\frac{n+3}{n+2}\,\frac{m^2}{Q^2}$ comes from using Z_H instead of ω_H. Both of these terms give rise to kinematically generated scaling violations, the latter term also breaks factorization.

To get a quantitative estimate of these effects, we will take a distribution function of the form

$$\tilde{g}(\xi) \propto \frac{(1-\xi)^3}{\xi^2} \qquad (I.10)$$

so that

$$\frac{\tilde{g}^{(n+3)}}{\tilde{g}^{(n+1)}} = \frac{n(n-1)}{(n+3)(n+4)} \qquad (I.11)$$

The QCD induced scaling violations to lowest order are given by (ref.1)

$$Q \frac{d}{dQ} \tilde{D}^{(m)}(Q^2) = - \frac{2d^{(m)}}{\log(Q^2|\Lambda^2)} \tilde{D}^{(m)}(Q^2) \tag{I.11a}$$

$$Q \frac{d}{dQ} \tilde{g}^{(n+1)}(Q^2) = - \frac{2d^{(n)}}{\log(Q^2|\Lambda^2)} \tilde{g}^{(n+1)}(Q^2) \tag{I.11b}$$

$$d^{(n)} = \frac{4}{27}\left[4 \sum_{j=1}^{n} \frac{1}{j} - 3 - \frac{2}{n(n+1)}\right] \tag{I.12}$$

Thus, to lowest order in $\alpha_s(Q^2)$ and $\frac{m^2}{Q^2}$

$$\begin{aligned} Q \frac{d}{dQ} \log\left[\left(\frac{dW_H}{dZ_H}\right)^{(m,n)}\right] = &- \frac{2(d^{(n)} + d^{(m)})}{\log(Q^2|\Lambda^2)} \\ &- 2\frac{m^2}{Q^2} \frac{n(n-1)}{(n+3)(n+4)}\left[(n+1) + (m-1)\frac{n+3}{n+2}\right] \\ &+ O(\alpha_s^2(Q^2), \frac{m^4}{Q^4}, \alpha_s(Q^2)\frac{m^2}{Q^2}). \end{aligned} \tag{I.13}$$

For $m = 3$, $n = 4$, $Q^2 = 5\ \mathrm{GeV}^2$ and $\Lambda = .5$ GeV we get

$$\frac{2(d^{(n)} + d^{(m)})}{\log(Q^2|\Lambda^2)} = .93 \tag{I.14a}$$

$$2\frac{m^2}{Q^2} \frac{n(n-1)}{(n+3)(n+4)}\left[(n+1) + (m-1)\frac{n+3}{n+2}\right] = .55 \tag{I.14b}$$

Thus, even at a moderate value of Q^2, the target mass corrections are important. Furthermore, at large m and n, the target mass corrections go like m + n, whereas the logarithmic QCD scaling violations go like log(mn), so the target mass corrections become increasingly important.

We now proceed to compare target mass corrections with higher order QCD corrections as sources of factorization breaking. The treatment of the QCD corrections, neglecting the target mass proceeds by direct analogy to the derivation of eq. (3.19). The order α_s corrections to $(\frac{d\tilde{W}}{dZ})^{(m,n)}$ have been computed in ref. (11). The result of this analysis is

$$\left[\frac{\left(\frac{dW_H}{dZ_H}\right)^{(m,n)}\left(\frac{dW_H}{dZ_H}\right)^{(k,\ell)}}{\left(\frac{dW_H}{dZ_H}\right)^{(k,n)}\left(\frac{dW_H}{dZ_H}\right)^{(m,\ell)}}\right]_{QCD} - 1$$

$$= \frac{\alpha_s}{\pi}\ \frac{2}{3}\left[\sum_{j=1}^{m-1}\frac{1}{j}\sum_{i=1}^{n-1}\frac{1}{i} + \sum_{j=1}^{m+1}\frac{1}{j}\sum_{i=1}^{n+1}\frac{1}{i} + \frac{1}{(m+1)(n+1)} - \frac{1}{mn}\right]$$

$$+ \Big[(m,n)\to(k,\ell)\Big] - \Big[(m,n)\to(k,n)\Big] - \Big[(m,n)\to(m,\ell)\Big] \tag{I.15}$$

The target mass correction to this ratio, to zeroth order in α_s and first order in $\frac{m^2}{Q^2}$ is, from eqs. (I.9) and I.10)

$$\left[\frac{\left(\frac{dW_H}{dZ_H}\right)^{(m,n)}\left(\frac{dW_H}{dZ_H}\right)^{(k,\ell)}}{\left(\frac{dW_H}{dZ_H}\right)^{(k,n)}\left(\frac{dW_H}{dZ_H}\right)^{(m,\ell)}}\right]_{\text{target mass}} - 1$$

$$= \frac{m^2}{Q^2}\frac{n(n-1)}{(n+4)(n+2)}(m-1) + [(m,n)\to(k,\ell)] - [(m,n)\to(k,n)]$$

$$- [(m,n)\to(m,\ell)] \qquad (\text{I}.16)$$

Consider again $Q^2 = 5\ \text{GeV}^2$ and $\Lambda = \cdot 5$ GeV. A typical case involving low moments is $m = n = 4$, $k = \ell = 2$. Then the QCD correction of eq. (I.15) is $\cdot 084$, while the target mass correction of eq. (I.16) is $\cdot 058$. So once again, target mass corrections are important at a moderate value of Q^2.

APPENDIX II.

In this appendix, we discuss quantities which depend only on one linear combination of distribution and decay functions, so that the sum over parton types effectively collapses to one term.

The symmetries of the strong interactions which are relevant to arguments about distribution and decay functions are those which interchange parton types. The only possibilities are charge conjugation (C)

$$\begin{aligned} C: \quad & q \leftrightarrow \bar{q} \\ & G \rightarrow G \end{aligned} \tag{II.1}$$

(where q is any quark and G is a gluon) and an isospin rotation by π about the y axis (R)

$$R: \begin{array}{c} u \leftrightarrow d \\ \bar{u} \leftrightarrow \bar{d} \end{array} \quad \text{others unchanged.} \tag{II.2}$$

Consider the difference between the structure functions for π^+ and π^- final state hadrons. The π^+, π^- difference is odd under both R and C. This implies that only one linear combination of decay functions is relevant

$$\tilde{D}_k^{\pi_+} - \tilde{D}_k^{\pi_-} = \lambda_k \tilde{D} \qquad (I.3)$$

where

$$\lambda_k = \begin{array}{l} 1 \;\; k = u \\ -1 \;\; k = d \\ -1 \;\; k = \bar{u} \\ 1 \;\; k = \bar{d} \\ 0 \text{ otherwise} \end{array} \qquad . \qquad (I.4)$$

In order to study the symmetry properties of the initial state, we will consider seperately the cases of electromagnetic and weak currents.

A. Electromagnetic currents:

In schematic form, eq. (3.10) reads

$$\frac{dW_H}{d\omega_H} \sim \sum_{k,j} \tilde{D}_k \frac{d\tilde{W}_{kj}}{d\omega} \tilde{g}_j \qquad . \qquad (II.5)$$

Taking a π^+, π^- difference for the final state

$$\frac{dW_H}{d\omega_H}^{\pi+} - \frac{dW_H}{d\omega_H}^{\pi-} \sim \tilde{D} \sum_{k,j} \lambda_k \frac{d\tilde{W}_{kj}}{d\omega} \tilde{g}_j \qquad . \qquad (II.6)$$

The partonic structure functions $\frac{d\tilde{W}_{kj}}{d\omega_H}$ are even under charge conjugation since the electromagnetic current is C even. Since λ_k is C odd, only the C odd part of $\tilde{g}_j$ contributes, thus initial state gluons are irrelevant. Thus the relevant graphs to order α_s are those of fig. (2) and the interference between the graphs of figs (1) and (3). But all of these graphs have the following parton-type structure

$$\frac{d\tilde{W}_{kj}}{d\omega} = \frac{d\tilde{W}}{d\omega}\,\delta_{kj}Q_j^2 \quad + \quad O(\alpha_s^2) \qquad \text{(II.7)}$$

thus

$$\frac{dW_H^{\pi_+}}{d\omega_H} - \frac{dW_H^{\pi_-}}{d\omega_H} \sim \tilde{D}\,\frac{d\tilde{W}}{d\omega}\,\tilde{g} + O(\alpha_s^2) \qquad \text{(II.8)}$$

where

$$\tilde{g} \equiv \sum_j \; Q_j^2 \lambda_j \tilde{g}_j \qquad . \qquad \text{(II.9)}$$

Thus, up to order α_s^2 terms (which govern next-to-leading order scaling violations and factorization breaking and thus are virtually unobservable), the difference between the structure functions into $\pi^\dagger$ and π^- involves only the linear combinations $\tilde{g}$ and $\tilde{D}$.

B. <u>Charged Weak Currents:</u>

The situation here is complicated by the fact that the $\cos\theta_C$ and $\sin\theta_C$ currents (θ_C is the Cabibo angle) have different flavor structures. However the resulting $\sin^2\theta_C$ terms make a negligible contribution to the $\pi^\dagger - \pi^-$ difference. Consider first the case of an initial quark (or antiquark). The final partonic state, up to order α_s^2 is a quark (antiquark) with or without a gluon. The final state fermion must carry isospin $\frac{1}{2}$ in order to contribute to the $\pi^\dagger - \pi^-$ difference.

However, the $\sin\theta_C$ current carries isospin $\frac{1}{2}$, so the initial quark must be an isosinglet (sea) quark. Thus the contribution of quarks (antiquarks) struck by $\sin\theta_C$ currents is suppressed by $\sin^2\theta_C$ times sea quark distributions or $\sin^2\theta_C\alpha_s^2$. Initial gluons struck by $\sin\theta_C$ currents are suppressed by $\alpha_s\sin^2\theta_C$.

By similar reasoning, the charm-strange part of the $\cos\theta_C$ current does not contribute until order α_s^2.

Thus the relevant current for neutrino scattering is

$$J^\mu = \cos\theta_C(\bar{u}\gamma^\mu d - \bar{u}\gamma^\mu\gamma_5 d) = J_V^\mu - J_A^\mu . \quad \text{(II.10)}$$

The vector current (J_V) and the axial-vector current (J_A) ahve simple transformation properties under RC

$$\begin{aligned} RC:&\quad J_V \to J_V \\ RC:&\quad J \to -J_A \end{aligned} \quad \text{(II.11)}$$

The parity conserving terms in $\frac{d\tilde{W}_{kj}}{d\omega}$ (i.e. the 1 and 2 structure functions) come from V - V and A - A terms, thus

$$RC: \left(\frac{d\tilde{W}_{kj}}{d\omega}\right)_{p.C.} \to \left(\frac{d\tilde{W}_{kj}}{d\omega}\right)_{p.C.} . \quad \text{(II.12)}$$

The parity violating terms (i.e. $\frac{d\tilde{W}_{3kj}}{d\omega}$) come from V - A interference, thus

$$\text{RC:}\quad \frac{d\tilde{W}_{3kj}}{d\omega} \rightarrow -\frac{d\tilde{W}_{3kj}}{d\omega} \quad . \tag{II.13}$$

The π^+, π^- difference is given by eq. (II.6). Consider the parity violating case. By definition, λ_k is even under RC. But since $\frac{d\tilde{W}_{3kj}}{d\omega}$ is odd, only the part of $\tilde{g}_j$ which is RC odd contributes, so initial state gluons are irrelevant.

Thus the relevant graphs to order α_s are those of figs. (1), (2), and (3) which have the parton type structure

$$\left(\frac{d\tilde{W}_{3kj}}{d\omega}\right)^{\nu} = \frac{d\tilde{W}_3}{d\omega}\,\eta^{\nu}{}_{kj} + O(\alpha_s^2) \tag{II.14}$$

where

$$\eta^{\nu}_{kj} = \left\{\begin{array}{l} 1 \;\; j = d \;\; k = u \\ -1 \;\; j = \bar{u} \;\; k = d \\ 0 \;\; \text{otherwise} \end{array}\right\} \quad . \tag{II.15}$$

Thus eq. (II.6) becomes

$$\left(\frac{dW^{\pi_+}_{3H}}{d\omega_H} - \frac{dW^{\pi_-}_{3H}}{d\omega_H}\right)^{\nu} \sim \tilde{D}\,\frac{d\tilde{W}_3}{d\omega}\,\tilde{g}_3^{\nu} + O(\alpha_s^2) \tag{II.16}$$

where

$$\tilde{g}_3^{\nu} \equiv \sum_{j,k} \eta^{\nu}_{kj}\lambda_k\tilde{g}_j = \tilde{g}_d - \tilde{g}_{\bar{u}} \quad . \tag{II.17}$$

For antineutrino scattering, we need only interchange u with d in eq. (II.15), so the relevant distribution functions is

$$\tilde{g}_3^{\bar{\nu}} = -\tilde{g}_u + \tilde{g}_{\bar{d}} \quad . \tag{II.18}$$

The analysis of the parity conserving structure functions for the π^+, π^- difference is complicated by the fact that both initial state gluons and quarks contribute. The gluon contribution can be eliminated by also taking the difference between proton and neutron initial states. Then, by steps analagous to those leading to eq. (II.6), the combination $\tilde{g}_d + \tilde{g}_{\bar{u}}$ contributes to neutrino scattering, and $-\tilde{g}_u - \tilde{g}_{\bar{d}}$ contributes to antineutrino scattering. A neutrino-antineutrino sum may be used instead of the proton-neutron difference. The resulting $d\tilde{W}_{kj}/d\omega$ is even under R (which exchanges ν and $\bar{\nu}$), but λ_k is odd under R, so only the R odd part of $\tilde{g}_j$ contributes, and initial gluons are irrelevant. The relevant distribution function is $\tilde{g}_d - \tilde{g}_u + \tilde{g}_{\bar{u}} - \tilde{g}_{\bar{d}}$.

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FOOTNOTES:

f1. This is based on the phenomenological analysis of scaling violations in electroproduction in ref. 3. Clearly the scale M_o will be process dependent, so the situation in other processes may be better or worse.

f2. The proof of the factorization theorem shows that the Γ's can be chosen to process independent. Thus eq. (1.5) preserves the process independence of the $\tilde{f}$ for all M^2.

f3. By "target" mass, we mean the mass of any initial state hadron.

f4. It is easy to incorporate the spin of the initial hadron. For instance, if the hadron has spin $\frac{1}{2}$ (represented by the spin 4-vector s_H^μ), eq. (1.10a) becomes, by the parity invariance of the strong interactions.

$$G_{k,h} = G_k^{(o)}(u) + G_k^{(1)}(u)\,\frac{s_H \cdot p}{h\ um}$$

where the superscripts o and 1 represent respectively the hadron-spin-zero and the hadron-spin-one components of the distribution function, and h is the helicity of the parton. The spin sensitive variable $\frac{s_H \cdot p}{h\ um}$ is invariant under scaling of p, thus it has no effect on the factorization of the infrared sensitivity.

f5. This reabsorption of IR singularities is very similar to that similar to that of ref. 6.

f6. It is shown in ref. 5 that the covariant parton model (i.e. the QCD parton model to zeroth order) gives rise to the ξ-scaling formulae if all logarithmic scaling violations are ignored. These scaling violations do not appear in the parton model until the second order.

f7. The variable x is independent of ϕ in any frame in which $\vec{P}$ and $\vec{q}$ are collinear.

f8. We must take account of the fact that the definition of $W_H^{\mu\nu}$ adopted in ref. 3 is twice our definition.

f9. The predictions of the OPE are invariant under changes in the definition of the operators. Similarly, the predictions of the QCD parton model are invariant under changes in the choice of Γ. Thus we need only prove that the predictions of the two methods coincide for one set of choices. These matters are discussed in ref. 7.

f10. We choose a light meson because we do not know how to include final state mass corrections.

f11. This kinematic cut was used by Sakai (ref.10). Altarelli et. al. (ref. 11) object to it on the basis that the point $\omega_H = 0$ is "arbitrary". Our argument shows that is is not; any $\omega_H < 0$ includes contributions from the fragmentation region; $\omega_H > 0$ does not.

f12. The lowest order scaling violations of the non-singlet moments of eq. (3.17) are easy to unfold: the $\tilde{g}^{(n+1)}$ scale with anomalous dimension $\gamma^{(n)}$ and the $\tilde{D}^{(m)}$ scale with $\gamma^{(m)}$ [there is no destinction between uncoming and out going anomalous demensions to lowest order]. Thus, using the lowest order quark anomalous dimension and the lowest order β function gives

$$M^{(m,n)}(Q^2) = \left(\frac{\ln Q^2 | \Lambda^2}{\ln Q_0^2 | \Lambda^2}\right)^{-D_n - D_m} [1 + O(\alpha_s)]\; M^{(m,n)}(Q_0^2)$$

where

$$D_n = \frac{4}{(33-2n_f)} \left[4 \sum_{j=1}^{n} \frac{1}{j} - 3 - \frac{2}{n(n+1)}\right]$$

where n_f is the number of flavors.

f13. This type of factorization is not to be confused with the factorization theorem.

f14. Initial and final state gluons are irrelevant because we are taking non-singlet differences. Further, the vertex correction graph of fig. 3 does not contribute to order α_s, because it has the same kinematic structure as the zeroth order graph of fig. 1.

f15. The actual relationship is

$$Z_H = \eta'\{Z+x_H[(2Z-\omega)ux\frac{m^2}{Q^2} - \frac{m}{Q}\sqrt{Z(Z-\omega)}\sqrt{\frac{m^2}{Q^2}(ux-\xi)^2+\frac{1}{R}(ux-\xi)}\quad\cos\phi]$$

where ϕ of the azimuthal angle between the components of $\vec{p}\,'$ and $\vec{P}$ orthogonal to the $\vec{p}$, $\vec{q}$ axis in a frame in which $\vec{p}$ and $\vec{q}$ are collinear.

FIGURE CAPTIONS

Fig. 1. Zeroth order amplitude for the process $q(\bar{q}) + B^{\nu} \to q(\bar{q})$.

Fig. 2. First order amplitude for the process $q(\bar{q}) + B^{\nu} \to q(\bar{q}) + G$.

Fig. 3. Second order amplitude for the process $q(\bar{q}) + B^{\nu} \to q(\bar{q})$.

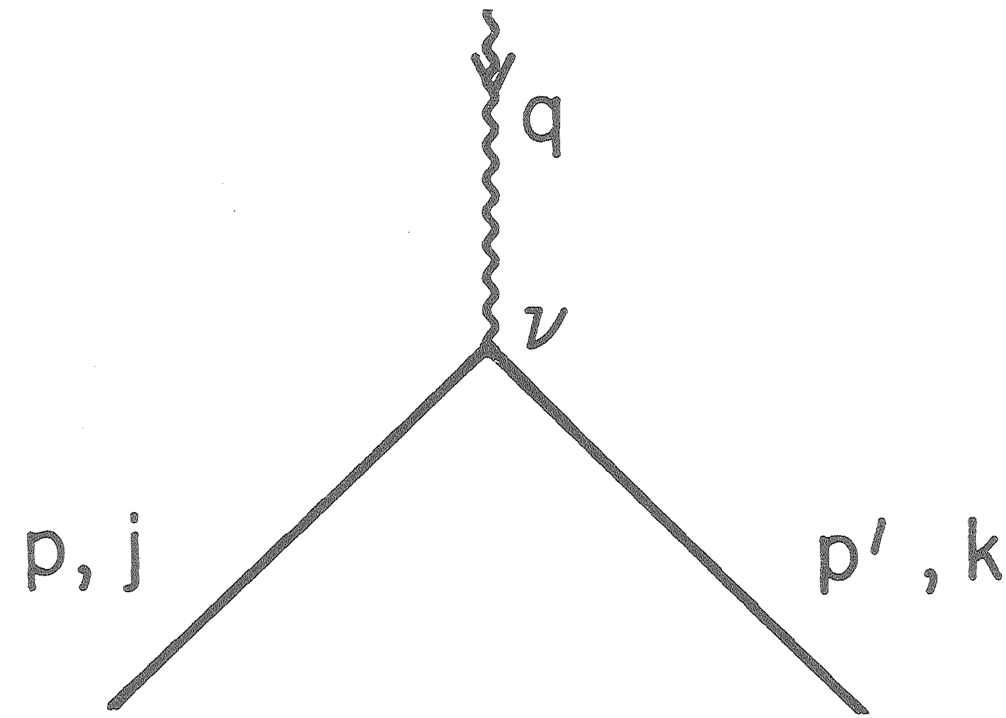

Figure 1

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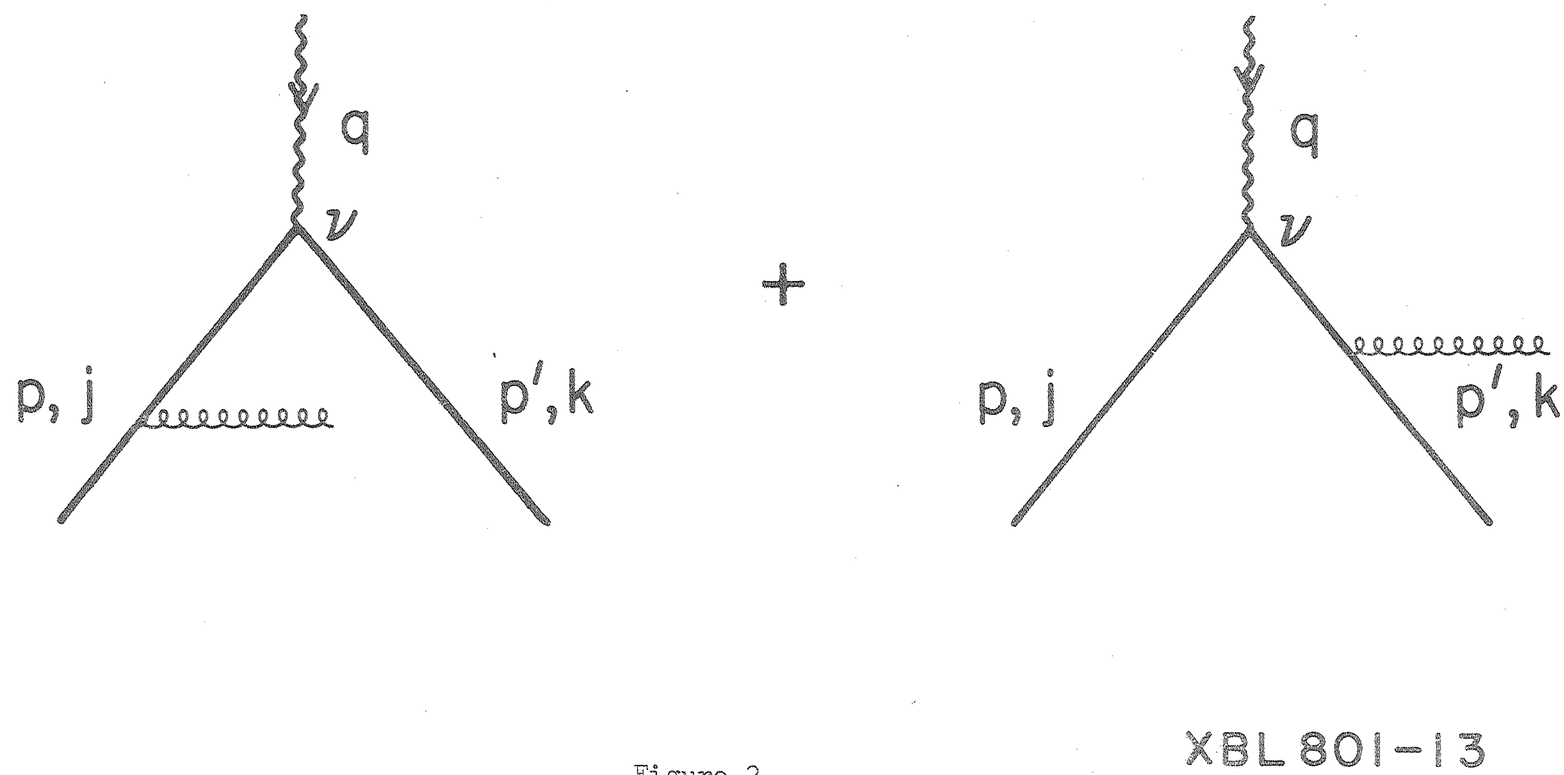

Figure 2

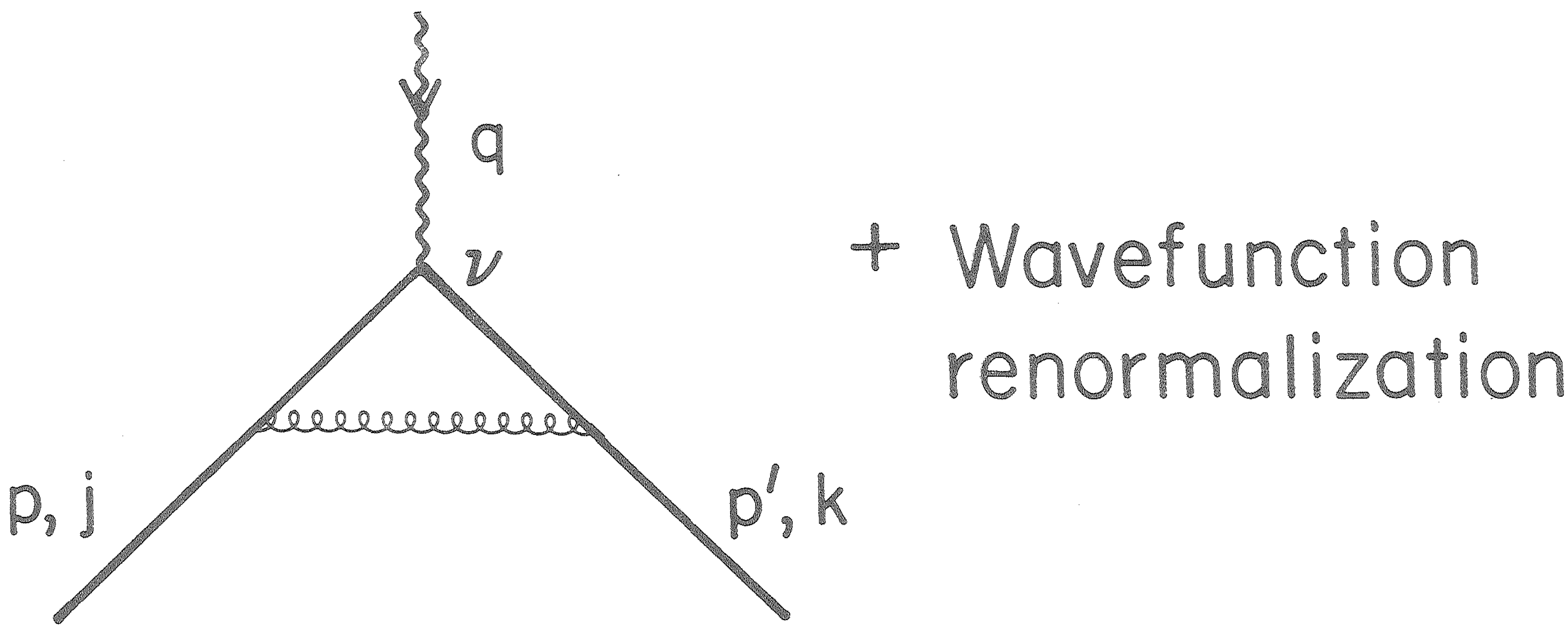

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Figure 3