Non-uniqueness of Beltrami-Schaefer Stress Functions

by

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Abstract

Beltrami and Schaefer derived solutions for the equilibrium equations of an elastic body free of body force, and Gurtin proved that the solutions are complete, i.e., proved that Beltrami–Schaefer stress functions are general solutions of the equilibrium equations. In this paper we show that the Beltrami – Schaefer stress functions are not unique, and we determine their degree of non-uniqueness. Finally, we present two applications of the non-uniqueness as remarks.

Keywords Beltrami-Schaefer stress functions· Non-uniqueness· General solution· Elasticity

Mathematics Subject Classification 2.080

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1 Introduction

Since Cauchy, Navier, and Poisson presented the basic governing equations for linear elasticity in the 19th century, scholars have devoted substantial effort toward seeking analytic solutions in elasticity. The Saint-Venant solutions are the most significant contributions towards solving twist and bending problems. As a result of the difficulties in establishing analytic solutions for 2D and 3D problems, scientists have extended their work to general solutions methods. The method of general solutions has been proven over decades to be a powerful tool in solving 3D problems in elastic theory.

It is well known that stress functions play an important role in the analysis of general solutions. Rostamian [7] studied the completeness of Maxwell's stress functions. These stress-function methods have been applied to a number of research fields, such as anisotropic elasticity [10], gradient elasticity [1], coupled stresses [6], and hyper-elasticity [9]. Fosdick and Royer [11] expanded Stokes' theorem for vector fields to Stokes' theorem for second-order tensor fields and implemented it in the Beltrami's stress functions.

Stress functions are the general solutions of the following set of equilibrium equations describing an elastic object free of body force:

$$\operatorname{div} \boldsymbol{S} = \boldsymbol{0} \tag{1.1}$$

where $\boldsymbol{S}$ is the stress tensor, which is a second-order symmetric tensor. Based on the work by Airy, Maxwell, Morera, and others, Beltrami [2] presented the following expression of stress functions:

$$\boldsymbol{S} = \operatorname{curl}\operatorname{curl} \boldsymbol{A}, \tag{1.2}$$

where $\boldsymbol{A}$ is an arbitrary second-order symmetric tensor. Schaefer [8] presented a more general form of what are called the Beltrami–Schaefer stress functions:

$$\boldsymbol{S} = \operatorname{curl}\operatorname{curl} \boldsymbol{A} + 2\tilde{\nabla}\boldsymbol{h} - \boldsymbol{I}\operatorname{div}\boldsymbol{h}, \tag{1.3}$$

where $\boldsymbol{h}$ is a harmonic vector, $\boldsymbol{I}$ is the unit tensor, $2\tilde{\nabla}\boldsymbol{h} = \nabla\boldsymbol{h} + \nabla\boldsymbol{h}^{\mathrm{T}}$, in which the superscript T denotes transpose. Gurtin [4, 5] proved the completeness of the Beltrami-Schaefer stress functions (1.3). That is, for any solution $\boldsymbol{S}$ which satisfies the equilibrium Equation (1.1), a symmetric tensor $\boldsymbol{A}$ and a harmonic vector $\boldsymbol{h}$ exist such that Equation (1.3) holds. Therefore, the Beltrami-Schaefer stress functions (1.3) provide a general solution of the equilibrium equation set (1.1).

The main objective of this paper is to investigate and prove that $\boldsymbol{A}$ and $\boldsymbol{h}$ in (1.3) are not unique. In Section 2 we present two theorems for proving and determining the degree of non-uniqueness, and we also provide some formulas for verification. In Section 3 we present the proofs of the two theorems. Finally, two applications of the two theorems are set forth in Section 4 as remarks.

2 Two Theorems and Formulas

To investigate and prove the non-uniqueness of the Beltrami-Schaefer stress functions (1.2) and their degree of non-uniqueness, we first introduce the following two theorems.

Theorem 2.1: *Assume that* $\boldsymbol{S}$ *, which is one solution of the equilibrium equation* (1.1)*, has been expressed in the form of equation* (1.3) *, as well as*

$$\tilde{\boldsymbol{A}} = \boldsymbol{A} + \boldsymbol{\Sigma} - \boldsymbol{I}\ J\left(\boldsymbol{\Sigma}\right) + 2\tilde{\nabla}\boldsymbol{p} \tag{2.1}$$

and

$$\tilde{\boldsymbol{h}} = \boldsymbol{h} + \operatorname{div}\boldsymbol{\Sigma}, \tag{2.2}$$

where $\boldsymbol{A}$ *and* $\boldsymbol{h}$ *are associated with* (1.3)*,* $\boldsymbol{p}$ *is an arbitrary vector,* $J\left(*\right)$ *is the trace of a tensor*, and $\boldsymbol{\Sigma}$ *is a second-order harmonic tensor, i.e.,*

$$\Delta\,\boldsymbol{\Sigma} = \boldsymbol{0}. \tag{2.3}$$

Then if we replace the symmetric tensor $\boldsymbol{A}$ *and the harmonic vector* $\boldsymbol{h}$ *in equation* (1.3) *by* $\tilde{\boldsymbol{A}}$ *in equation* (2.1) *and* $\tilde{\boldsymbol{h}}$ *in equation* (2.2) *, equation* (1.3) *still holds.*

Theorem 2.2: *If there are two sets of Beltrami-Schaefer stress functions for the equilibrium equation* (1.1) as

$$\boldsymbol{S} = \mathit{curl\ curl}\ \boldsymbol{A}_i + 2\tilde{\nabla}\boldsymbol{h}_i - \boldsymbol{I}\operatorname{div}\boldsymbol{h}_i \qquad \left(i = 1, 2\right), \tag{2.4}$$

then there exists a second-order symmetric harmonic tensor $\boldsymbol{\Sigma}$ *and a vector* $\boldsymbol{p}$ *such that*

$$\boldsymbol{A}_1 - \boldsymbol{A}_2 = \boldsymbol{\Sigma} - \boldsymbol{I}\ J\left(\boldsymbol{\Sigma}\right) + 2\tilde{\nabla}\boldsymbol{p} \tag{2.5}$$

and

$$\boldsymbol{h}_1 - \boldsymbol{h}_2 = \operatorname{div}\boldsymbol{\Sigma}. \tag{2.6}$$

In order to prove the above two theorems, we also need the following formulas

$$\operatorname{curl}\operatorname{curl}\tilde{\nabla}\boldsymbol{b} = \boldsymbol{0}, \tag{2.7}$$

$$\operatorname{div}\left(\boldsymbol{I}\,\varphi\right) = \nabla\varphi, \tag{2.8}$$

$$J\left(\nabla\boldsymbol{b} + \nabla\boldsymbol{b}^{\mathrm{T}} - \boldsymbol{I}\operatorname{div}\boldsymbol{b}\right) = -\operatorname{div}\boldsymbol{b}, \tag{2.9}$$

$$\operatorname{curl}\operatorname{curl}\left[\boldsymbol{I}\ J\left(\boldsymbol{\Pi}\right) - \boldsymbol{\Pi}\right] = 2\tilde{\nabla}\left(\operatorname{div}\boldsymbol{\Pi}\right) - \boldsymbol{I}\left(\operatorname{div}\operatorname{div}\boldsymbol{\Pi}\right) - \Delta\,\boldsymbol{\Pi} \tag{2.10}$$

and

$$\boldsymbol{I}\ J\left(\operatorname{curl}\operatorname{curl}\boldsymbol{\Pi}\right) - \operatorname{curl}\operatorname{curl}\boldsymbol{\Pi} = 2\tilde{\nabla}\left(\operatorname{div}\boldsymbol{\Pi}\right) - \nabla\nabla J\left(\boldsymbol{\Pi}\right) - \Delta\,\boldsymbol{\Pi}. \tag{2.11}$$

where $\boldsymbol{b}$ is a vector, φ is a scalar, $\boldsymbol{\Pi}$ is a symmetric tensor. Formulas (2.7) - (2.11) can be verified directly, or be found in Reference [5].

3 Proofs of Theorem 2.1 and Theorem 2.2

***The Proof of Theorem* 2.1** At first, we know that $\tilde{\boldsymbol{A}}$ is a symmetric tensor because $\boldsymbol{\Sigma}$ is a symmetric tensor and $\tilde{\boldsymbol{h}}$ is harmonic. Next by substituting (2.1) and (2.2) into the right side of equation (1.3), and considering equations (1.3), (2.7), (2.10) and (2.3), we can obtain

$$\begin{aligned}
&\operatorname{curl}\operatorname{curl}\tilde{\boldsymbol{A}} + 2\tilde{\boldsymbol{\nabla}}\tilde{\boldsymbol{h}} + \boldsymbol{\nabla}\tilde{\boldsymbol{h}}^{\mathrm{T}} - \boldsymbol{I}\operatorname{div}\tilde{\boldsymbol{h}} \\
&\quad = \boldsymbol{S} + \operatorname{curl}\operatorname{curl}\left[\boldsymbol{\Sigma} - \boldsymbol{I}\,J\left(\boldsymbol{\Sigma}\right) + 2\tilde{\boldsymbol{\nabla}}\boldsymbol{p}\right] + 2\tilde{\boldsymbol{\nabla}}\left(\operatorname{div}\boldsymbol{\Sigma}\right) - \boldsymbol{I}\operatorname{div}\operatorname{div}\boldsymbol{\Sigma} \\
&\quad = \boldsymbol{S} + \operatorname{curl}\operatorname{curl}\left[\boldsymbol{\Sigma} - \boldsymbol{I}\,J\left(\boldsymbol{\Sigma}\right)\right] + 2\tilde{\boldsymbol{\nabla}}\left(\operatorname{div}\boldsymbol{\Sigma}\right) - \boldsymbol{I}\operatorname{div}\operatorname{div}\boldsymbol{\Sigma} \\
&\quad = \boldsymbol{S} + \Delta\boldsymbol{\Sigma} = \boldsymbol{S}
\end{aligned} \tag{3.1}$$

Equation (3.1) indicates that for $\tilde{\boldsymbol{A}}$ and $\tilde{\boldsymbol{h}}$, equation (1.3) also holds.

***The Proof of Theorem* 2.2** Let

$$\boldsymbol{\Psi} = \mathcal{F}\left(\boldsymbol{A}_1 - \boldsymbol{A}_2\right), \quad \boldsymbol{q} = \mathcal{F}\left(\boldsymbol{h}_1 - \boldsymbol{h}_2\right), \tag{3.2}$$

where $\mathcal{F}\left(*\right)$ is Newton's potential , i.e.,

$$\Delta\boldsymbol{\Psi} = \boldsymbol{A}_1 - \boldsymbol{A}_2, \quad \Delta\boldsymbol{q} = \boldsymbol{h}_1 - \boldsymbol{h}_2. \tag{3.3}$$

Let

$$\boldsymbol{\Sigma} = \operatorname{curl}\operatorname{curl}\boldsymbol{\Psi} + 2\tilde{\boldsymbol{\nabla}}\boldsymbol{q} - \boldsymbol{I}\operatorname{div}\boldsymbol{q} \tag{3.4}$$

and

$$\boldsymbol{p} = \operatorname{div}\left[\boldsymbol{\Psi} - \frac{1}{2}\boldsymbol{I}\,J\left(\boldsymbol{\Psi}\right)\right] - \boldsymbol{q}. \tag{3.5}$$

Then, $\boldsymbol{\Sigma}$ and $\boldsymbol{p}$ defined in equations (3.4) and (3.5) satisfy equations (2.5) and (2.6). (We will verify this later.)

In view of equation (3.4), we observe that $\boldsymbol{\Sigma}$ is a symmetric second-order tensor. To prove that $\boldsymbol{\Sigma}$ is harmonic, we subtract the two formulas in (2.4) to obtain

$$\operatorname{curl}\operatorname{curl}\left(\boldsymbol{A}_1 - \boldsymbol{A}_2\right) + 2\tilde{\boldsymbol{\nabla}}\left(\boldsymbol{h}_1 - \boldsymbol{h}_2\right) - \boldsymbol{I}\operatorname{div}\left(\boldsymbol{h}_1 - \boldsymbol{h}_2\right) = \boldsymbol{0}. \tag{3.6}$$

Applying the Laplace operator to equation (3.4) , we derive

$$\boldsymbol{\Delta}\,\boldsymbol{\Sigma} = \operatorname{curl}\operatorname{curl}\left(\boldsymbol{\Delta}\boldsymbol{\Psi}\right) + 2\,\tilde{\boldsymbol{\nabla}}\left(\boldsymbol{\Delta}\,\boldsymbol{q}\right) - \boldsymbol{I}\,\operatorname{div}\left(\boldsymbol{\Delta}\,\boldsymbol{q}\right). \tag{3.7}$$

From equation (3.3) and (3.6) , we know that $\boldsymbol{\Sigma}$ obtained from (3.7) is a harmonic tensor.

Now, let us verify the correctness of equation (2.5). Upon substituting (3.4) and (3.5) into the right side of formula (2.5) , and considering (2.9) and (2.8) , the following expression is derived

$$\begin{aligned}
&\boldsymbol{\Sigma} - \boldsymbol{I}\;J\left(\boldsymbol{\Sigma}\right) + 2\,\tilde{\boldsymbol{\nabla}}\,\boldsymbol{p} \\
&\quad = \operatorname{curl}\operatorname{curl}\boldsymbol{\Psi} - \boldsymbol{I}\;J\left(\operatorname{curl}\operatorname{curl}\boldsymbol{\Psi}\right) + 2\,\tilde{\boldsymbol{\nabla}}\boldsymbol{q} - \boldsymbol{I}\;\operatorname{div}\boldsymbol{q} \\
&\qquad - \boldsymbol{I}J\left(2\,\tilde{\boldsymbol{\nabla}}\boldsymbol{q} - \boldsymbol{I}\;\operatorname{div}\boldsymbol{q}\right) + 2\,\tilde{\boldsymbol{\nabla}}\left\{\operatorname{div}\left[\boldsymbol{\Psi} - \frac{1}{2}\boldsymbol{I}\,J\left(\boldsymbol{\Psi}\right)\right] - \boldsymbol{q}\right\} \\
&\quad = \operatorname{curl}\operatorname{curl}\boldsymbol{\Psi} - \boldsymbol{I}\;J\left(\operatorname{curl}\operatorname{curl}\boldsymbol{\Psi}\right) + 2\,\boldsymbol{\nabla}\left(\operatorname{div}\boldsymbol{\Psi}\right) - \boldsymbol{I}\boldsymbol{\nabla}\boldsymbol{\nabla}\,J\left(\boldsymbol{\Psi}\right)
\end{aligned} \tag{3.8}$$

In terms of equation (2.11) and the first formula in equation (3.3) , equation (2.5) can be derived from (3.8) .

To prove the correctness of equation (2.6) , we substitute (3.4) into the right side of (2.6) , and by using equation (2.8) as well as the second formula in (3.3) , we obtain

$$\operatorname{div}\boldsymbol{\Sigma} = \operatorname{div}\left(\operatorname{curl}\operatorname{curl}\boldsymbol{\Psi} + 2\,\boldsymbol{\nabla}\,\boldsymbol{q} - \boldsymbol{I}\;\operatorname{div}\boldsymbol{q}\right) = \boldsymbol{\Delta}\,\boldsymbol{q} = \boldsymbol{h}_1 - \boldsymbol{h}_2\,. \tag{3.9}$$

Equation (3.9) is consistent with equation (2.6) .

4 Remarks

Two applications of the theorems are given in the following remarks.

Remark 4.1: A stress field is self-balancing if both the resultant force and resultant moment applied on the every closed surface inside the elastic domain are equal to zero. For the case of a self-balancing field, based on $\boldsymbol{A}$ and $\mathbf{h}$ as given by Gurtin [4, 5] in the proof of completeness, Carlson [3] found a symmetric harmonic tensor $\boldsymbol{\Sigma}$ that makes the vector $\tilde{\boldsymbol{h}}$ equal to zero. Therefore, *for a self-balancing field* , *the Beltrami's stress functions* (1.2) *are the complete solutions of equilibrium equation set* (1.1) . It has also been shown by Fosdick and Royer [11] in

view of Stokes' theorem for second-order tensor fields.

Remark 4.2: Consider the equation

$$\operatorname{curl}\operatorname{curl}\boldsymbol{A}+2\tilde{\nabla}\boldsymbol{h}-\boldsymbol{I}\operatorname{div}\boldsymbol{h}=\boldsymbol{0}. \tag{4.1}$$

where $\boldsymbol{A}$ is a symmetric second-order tensor, $\boldsymbol{h}$ is a harmonic vector.

In the view of Theorem 2.1 in section 2, we know that equation (4.1) has the following solutions $\boldsymbol{A}$ and $\boldsymbol{h}$:

$$\boldsymbol{A}=\boldsymbol{\Sigma}-\boldsymbol{I}\,J\left(\boldsymbol{\Sigma}\right)+2\tilde{\nabla}\boldsymbol{p} \tag{4.2}$$

and

$$\boldsymbol{h}=\operatorname{div}\boldsymbol{\Sigma}. \tag{4.3}$$

where $\boldsymbol{\Sigma}$ is a second-order symmetric harmonic tensor and $\boldsymbol{p}$ is an arbitrary vector.

On the other hand, for any solutions $\boldsymbol{A}$ and $\boldsymbol{h}$ of (4.1) , Theorem 2.2 indicates that there exists a harmonic tensor $\boldsymbol{\Sigma}$ and a vector $\boldsymbol{p}$ such that (4.2) and (4.3) hold, i.e., (4.2) and (4.3) give the complete solutions of equation (4.1) .

Now, we arrive at the following theorem:

Theorem 4.3: *The general solutions of equation* (4.1) *are given by* (4.2) *and* (4.3) .

For the special case $\boldsymbol{h}=\boldsymbol{0}$, we rewrite tensor $\boldsymbol{A}$ into the strain tensor $\boldsymbol{\Gamma}$, so equation (4.1) becomes

$$\operatorname{curl}\operatorname{curl}\boldsymbol{\Gamma}=\boldsymbol{0}. \tag{4.4}$$

Therefore, Theorem 4.3 leads to the following deduction:

Deduction: *The general solutions of the Saint-Venant strain compatibility equation* (4.4) *are*

$$\boldsymbol{\Gamma}=\boldsymbol{\Sigma}-\boldsymbol{I}\,J\left(\boldsymbol{\Sigma}\right)+2\tilde{\nabla}\boldsymbol{p}, \tag{4.5}$$

where $\boldsymbol{\Sigma}$ *is a symmetric tensor,* $\Delta\boldsymbol{\Sigma}=0,\ \operatorname{div}\boldsymbol{\Sigma}=0$, *and* $\boldsymbol{p}$ *is a vector.*

However, we know that the geometric equation in the theory of elasticity is described as in [5]:

$$\boldsymbol{\Gamma} = \tilde{\boldsymbol{\nabla}}\boldsymbol{u} = \frac{1}{2}\left(\boldsymbol{\nabla}\boldsymbol{u} + \boldsymbol{\nabla}\boldsymbol{u}^{\mathrm{T}}\right). \tag{4.6}$$

Generally speaking, for a multiple-connected domain, we cannot obtain a single-valued displacement field $\boldsymbol{u}$ from (4.6) for a given $\boldsymbol{\Gamma}$ if there are no additional conditions on $\boldsymbol{\Gamma}$, but we can always obtain a single-valued $\boldsymbol{\Sigma}$ and $\boldsymbol{p}$ in (4.5) by equation (4.4).

5. Conclusions

We have proved that the Beltrami-Schaefer stress functions are non-unique, and we have determined the degree of their non-uniqueness. In two applications of non-uniqueness, we deduced that Beltrami's stress functions are complete for self-balancing fields. Moreover, we derived a general solution for the Saint-Venant strain compatibility equation in a new form, which holds for both single-connected and multiple-connected domains.

Acknowledgments

The paper was supported by the Program for New Century Excellent Talents in University (NCET-07-0253), The National Natural Science Foundation (No. 51179060) and The Education Ministry Foundation（No. 20110094130002）of China, and in part, supported by the U.S. Dept. of Energy under contract No. DE-AC02-05CH11231.

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