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KINETIC EQUATIONS FOR LONGITUDINAL STOCHASTIC COOLING

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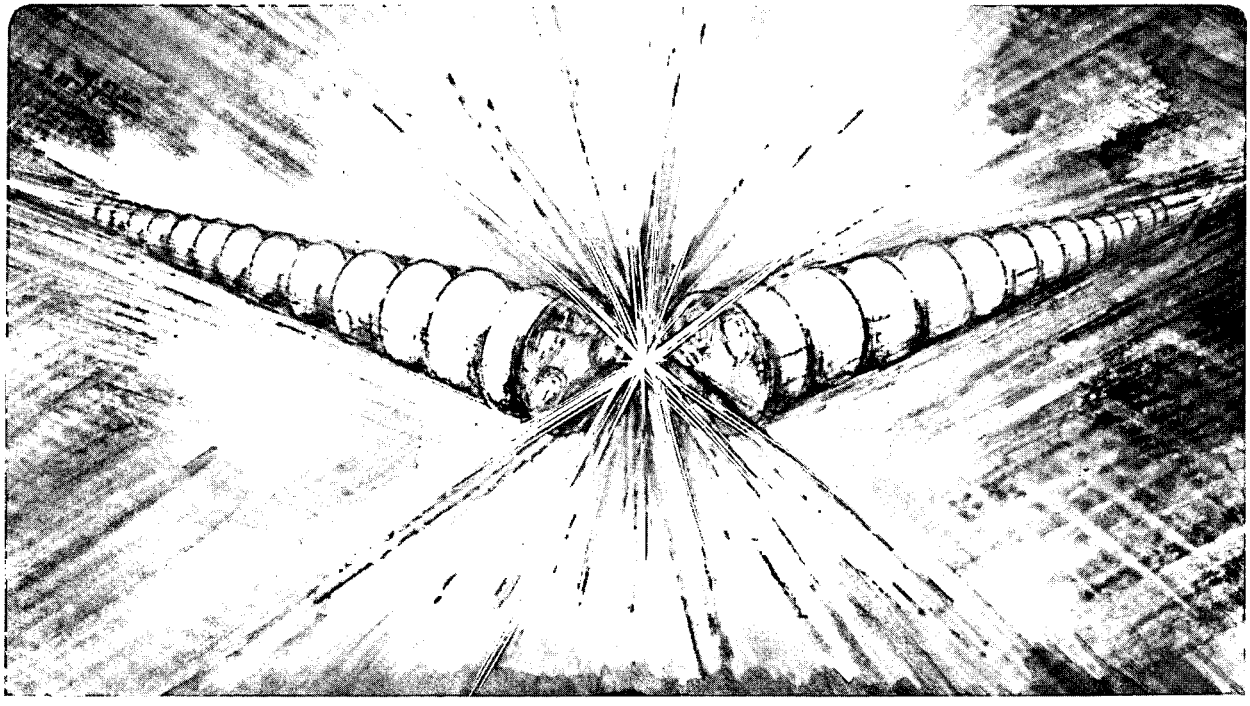

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KINETIC EQUATIONS FOR LONGITUDINAL STOCHASTIC COOLING

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Introduction

In this note a kinetic equation approach to longitudinal stochastic cooling is presented. Equations for one and two particle distribution functions are derived from the principle of the conservation of the number of ensemble systems. The violation of Liouville's theorem is expressed by certain self-interaction terms. In addition, the two particle distribution describes correlation or feedback effects and can be analyzed by techniques of the Lenard-Balescu equation for plasmas. The resulting equation is similar to the Fokker-Planck equation that has been used to describe experimental results at ICE.[1)]

I. Kinetic Equations for a Non-Liouvillian System

Consider the N-dimensional phase space distribution function $D(q_1,p_1,\ldots q_n,p_n)$, normalized so that

$$\int dq_1 \ldots dp_n \; D(q_1,\ldots,p_n) = 1 \qquad (1.1)$$

Conservation of the number of ensemble systems is expressed by

$$\frac{\partial D}{\partial t} + \vec{\nabla}\cdot(\vec{u}\,D) = 0 \qquad (1.2)$$

where $u = (\dot{q}_1,\dot{p}_1,\ldots,\dot{q}_n,\dot{p}_n)$. If the system dynamics are described by Hamiltonian equations, then (1.2) reduces to the condition of incompressible fluid flow[2)],

$$\frac{dD}{dt} = \frac{\partial D}{\partial t} + \vec{u}\cdot\vec{\nabla} D = 0 \qquad (1.3)$$

that is, Liouville's Theorem.

A Hamiltonian description is not applicable to longitudinal cooling. Instead, the dynamics are of the form

$$\dot{p}_i = \sum_j G(q_i,q_j,p_j) \qquad \dot{q}_i = Q(p_i) \qquad (1.4)$$

Equation (1.2) may still be integrated over 2(N-1) or 2(N-2) variables to yield equations for one and two particle distribution functions.

Define the one particle distribution by

$$f_1(q_1,p_1,t) = \int dq_2 \ldots dp_n \; D(q_1,p_1,\ldots,q_n,p_n,t) \qquad (1.5)$$

and similarly

$$f_2(q_1,p_1,q_2,p_2,t) = \int dq_3 \ldots dp_n \; D(q_1,p_1,\ldots,q_n,p_n,t) \qquad (1.6)$$

From equation (1.2) we have

$$\frac{\partial f_1}{\partial t} = -\int dq_2 \cdots dp_n \left(\frac{\partial}{\partial q_1} (\dot{q}_1 D) + \frac{\partial}{\partial p_1} (\dot{p}_1 D) \right) \qquad (1.7)$$

With dynamics of the form (1.4), this reduces under the usual symmetry assumption for D to

$$\frac{\partial f_1}{\partial t} + \dot{q}_1 \frac{\partial f_1}{\partial q_1} + (N-1) \frac{\partial}{\partial p_1} \int dq_2 dp_2 \; G(q_1, q_2, p_2) f_2 (q_1, p_1, q_2, p_2, t)$$
$$+ \left[\frac{\partial}{\partial p_1} \left(G(q_1, q_1, p_1) f_1 (q_1, p_1, t) \right) \right] = 0 \qquad (1.8)$$

The bracketed term is the addition to the usual kinetic equation without self-interaction and expresses the violation of Liouville's Theorem. It is of the form of the coherent part of a Fokker-Planck equation and induces compression of phase space. The integral in (1.8) describes interaction with other beam particles and includes the usual Vlasov average field effects and also correlation effects that tend to increase the phase space volume and suppress cooling through beam feedback. The corresponding equation for f_2 is

$$\frac{\partial f_2}{\partial t} + \dot{q}_1 \frac{\partial f_2}{\partial q_1} + \dot{q}_2 \frac{\partial f_2}{\partial q_2} + (N-2) \frac{\partial}{\partial p_1} \int dq_3 dp_3 \; G(q_1, q_3 p_3) f_3$$
$$+ (N-2) \frac{\partial}{\partial p_2} \int dq_3 dp_3 \; G(q_2, q_3, p_3) f_3$$
$$+ \frac{\partial}{\partial p_1} G(q_1, q_2, p_2) f_2 + \frac{\partial}{\partial p_2} G(q_2, q_1, p_1) f_2$$
$$+ \left[\frac{\partial}{\partial p_1} G(q_1, q_1, p_1) f_2 + \frac{\partial}{\partial p_2} G(q_2, q_2, p_2) f_2 \right] = 0 \qquad (1.9)$$

where again it is the bracketed term that is added.

As is the case with the usual kinetic equation of plasma physics, an infinite hierarchy of relations between the f_n is developing. We write

$$f_2 = f_1(q_1,p_1,t)\, f_1(q_2,p_2,t) + g(q_1,p_1,q_2,p_2,t) \quad (1.10a)$$

$$f_3 = f_1(q_1,p_1,t)\, f_1(q_2,p_2,t)\, f_1(q_3,p_3,t) + f_1(q_1,p_1,t)\, g(q_2,p_2,q_3,p_3,t)$$

$$+ f_1(q_3,p_3,t)\, g(q_1,p_1,q_2,p_2,t) + f_1(q_2,p_2,t)\, g(q_3,p_3,q_1,p_1,t) + h(q_1 \cdots p_3,t) \quad (1.10b)$$

and now assume that correlation affects are small, and in particular $h \approx 0$. With this assumption (1.8) and (1.9) yield (with the distinction N, N-1, N-2, dropped)

$$\frac{\partial f_1}{\partial t} + \dot{q}_1 \frac{\partial f_1}{\partial q_1} + N \frac{\partial f_1}{\partial p_1} \int dq_2\, dp_2\, G(q_1,q_2,p_2)\, f_1(q_2,p_2,t)$$

$$= -N \frac{\partial}{\partial p_1} \int dq_2\, dp_2\, G(q_1,q_2,p_2)\, g(q_1,p_1,q_2,p_2,t)$$

$$-\left[\frac{\partial}{\partial p_1} \left(G(q_1,q_1,p_1)\, f_1(q_1,p_1,t)\right]\right. \quad (1.11)$$

and

$$\frac{\partial g}{\partial t} + \dot{q}_1 \frac{\partial g}{\partial q_1} + \dot{q}_2 \frac{\partial g}{\partial q_2} + N \frac{\partial g}{\partial p_1} \int dq_3 dp_3 \; G(q_1, q_3, p_3) f_1(q_3, p_3, t)$$

$$+ N \frac{\partial g}{\partial p_2} \int dq_3 \, dp_3 \; G(q_2, q_3, p_3) f_1(q_3, p_3, t) =$$

$$- N \frac{\partial f_1}{\partial p_1} \int dq_3 \, dp_3 \; G(q_1, q_3, p_3) \, g(q_2, p_2, q_3, p_3, t)$$

$$- N \frac{\partial f_2}{\partial p_2} \int dq_3 \, dp_3 \; G(q_2, q_3, p_3) \, g(q_3, p_3, q_1, p_1, t)$$

$$\left.\begin{aligned} &- \frac{\partial}{\partial p_1} G(q_1, q_2, p_2) f_1(q_1, p_1, t) f_1(q_2, p_2, t) \\ &- \frac{\partial}{\partial p_2} G(q_2, q_1, p_1) f_1(q_1, p_1, t) f_1(q_2, p_2, t) \end{aligned}\right\} (1)$$

$$- \left\{ \frac{\partial}{\partial p_1} G(q_1, q_2, p_2) \, g(q_1, \ldots, p_2, t) + \frac{\partial}{\partial p_2} G(q_2, q_1, p_1) \, g(q_1, \ldots, p_2, t) \right\} (2)$$

$$- \left[\frac{\partial}{\partial p_1} G(q_1, q_1, p_1) \, g(q_1, \ldots, p_2, t) + \frac{\partial}{\partial p_2} G(q_2, q_2, p_2) \, g(q_1, \ldots, p_2, t) \right] (3)$$

(1.12)

The terms labeled (3) are the addition from the violation of Liouville's Theorem and are of the same order as terms (2). At this level of approximation, terms (2) are normally dropped as second order relative to (1)[3], and we likewise drop (3) from the analysis. The integrals in this equation are multiplied by the particle number N and in general cannot be considered negligible. With these approximations (1.12) is formally identical to the usual Liouvillian equations for two point correlations. The cooling of phase space appears only in the last term of the one-particle distribution equation (1.11). Equation (1.12), except for the explicit form of interaction, is identical with that of the Lenard-Balescu analysis of plasma physics[3].

This approach leads to a collision term which includes shielding effects (or for longitudinal cooling, feedback effects) and leads naturally to a Fokker-Planck equation of the curious form that has been successful in describing stochastic cooling. In its usual context, the Lenard-Balescu analysis leads to growth of the phase space volume. For stochastic cooling, instead, it leads to suppression of the perfect cooling rate described by the last term of equation (1.11).

2. Momentum cooling

We take as our variables the azimuthal angle θ, and $\epsilon = (E-E_0)$, where we assume small energy changes relative to E_0. For simplicity, consider the situation of nonoverlapping Schottky bands. Then the $\dot{p}$ equation (1.4) is of the form

$$\dot{\epsilon}_i = \sum_j \sum_n G_n(\epsilon_j)\, e^{in(\theta_i-\theta_j)} \tag{2.1}$$

where G is the system transfer function and $G_0(\epsilon)=0$. We also assume that f_1 is independent of θ and g is a function of $\theta_1-\theta_2$ only, that is

$$g(\theta_1,\theta_2,\epsilon_1,\epsilon_2,t) = \sum_\ell g_\ell(\epsilon_1,\epsilon_2,t)\, e^{i\ell(\theta_1-\theta_2)} \tag{2.2}$$

With these simplifying assumptions, (1.11) reduces to

$$\frac{\partial f_1(\epsilon,t)}{\partial t} = -\frac{\partial}{\partial \epsilon}\left(\sum_n G_n(\epsilon)\, f_1(\epsilon,t)\right)$$

$$- N\frac{\partial}{\partial \epsilon}\int d\epsilon' \sum_n G_n(\epsilon')\, g_{-n}(\epsilon,\epsilon',t) \tag{2.3}$$

and (1.12) reduces to

$$\frac{\partial g_\ell}{\partial t} + i\ell(\omega_1-\omega_2)\, g_\ell(\epsilon_1,\epsilon_2,t) =$$

$$-\frac{\partial}{\partial\epsilon_1}\left[G_\ell(\epsilon_2)\, f_1(\epsilon_1,t)\, f_1(\epsilon_2,t)\right] - \frac{\partial}{\partial\epsilon_2}\left[G_{-\ell}(\epsilon_1)\, f_1(\epsilon_1,t) f_1(\epsilon_2,t)\right]$$

$$- N\frac{\partial f_1}{\partial\epsilon_1}\int d\epsilon_3\; G_\ell(\epsilon_3)\; g_{-\ell}(\epsilon_2,\epsilon_3)$$

$$- N\frac{\partial f_1}{\partial\epsilon_2}\int d\epsilon_3\; G_{-\ell}(\epsilon_3)\; g_\ell(\epsilon_1,\epsilon_3) \qquad (2.4)$$

This equation may be solved by standard Laplace transform techniques developed in the Lenard-Balescu analysis under the assumption that the relaxation time for g is fast on the scale of variations of $f(\epsilon,t)$. That is, we solve (2.4) with $f(\epsilon,t)$ assumed constant in time. This leads to the expression [4,5]

$$N\int d\epsilon_2\; G_{\pm|\ell|}(\epsilon_2)\, g_{\mp|\ell|}(\epsilon_1,\epsilon_2) = \left[\frac{G_{\pm|\ell|}(\epsilon_1)\, f_1(\epsilon_1)}{\varepsilon_{\mp|\ell|}(\omega_1)} - G_{\pm|\ell|}(\epsilon_1)\, f_1(\epsilon)\right]$$

$$- \frac{N}{|\ell|}\int_{\eta\to 0_+} d\omega' \left|\frac{d\epsilon'}{d\omega'}\right| \left|\frac{G_{\pm|\ell|}(\epsilon')}{\varepsilon_{\mp|\ell|}(\omega')}\right|^2 \frac{\partial f_1}{\partial\epsilon'}\, f_1(\epsilon')\, \frac{1}{\eta \pm i(\omega_1-\omega')} \qquad (2.5)$$

with

$$\varepsilon_{\pm|\ell|}(\omega) = 1 + \frac{N}{|\ell|}\int_{\eta\to 0_+} d\epsilon'\, \frac{\frac{\partial f_1}{\partial\epsilon'}\, G_{\mp|\ell|}(\epsilon')}{\eta \pm i(\omega-\omega')} \qquad (2.6)$$

$$\omega = \omega(\epsilon)$$

If this relation is inserted into (2.3), after some cancellations between $\pm \ell$, we have

$$\frac{\partial f_1(\epsilon,t)}{\partial t} = -\sum_{\ell} \left\{ \frac{\partial}{\partial \epsilon} \left[\frac{G_\ell(\epsilon)\, f_1(\epsilon,t)}{\mathcal{E}_{-\ell}(\omega)} \right] - \frac{\partial}{\partial \epsilon} \left[\frac{N\pi}{|\ell|} \left|\frac{\partial \epsilon}{\partial \omega}\right| \left|\frac{G_\ell(\epsilon)}{\mathcal{E}_{-\ell}(\omega)}\right|^2 \frac{\partial f_1}{\partial \epsilon}\, f_1 \right] \right\} \qquad (2.7)$$

This result is of the form of a Fokker-Planck equation with one derivative acting on f alone[6]. The first term on the right hand side is the coherent cooling of a particle's energy error; the $\mathcal{E}_\ell$ factor describes the feedback of the coherent signal through the beam. The second term contains the effects of the beam signal, including again feedback. The form of interaction (2.1) is directly applicable to the Palmer method of momentum cooling, where the weighting function $G_n(\epsilon)$ derives from position measurements in a transverse pickup and the electronic gain is essentially constant over a Schottky band. For the filter method, in which energy information is obtained through variation of the electronic gain with frequency, the $\mathcal{E}_\ell$ factors are modified, with the corresponding $G_n(\epsilon)$ outside the integration in (2.6). If amplifier noise is included there will be an additional term on the right hand side:

$$\sum_{\ell} \frac{\partial}{\partial \epsilon} \left[\pi \left(\frac{e\omega}{2\pi}\right)^2 \frac{P(\ell\omega)}{|\mathcal{E}_{-\ell}(\omega)|^2} \frac{\partial f_1}{\partial \epsilon} \right] \qquad (2.8)$$

where P is the noise voltage power density, with normalization

$$\int_{-\infty}^{\infty} d\omega\, P(\omega) = \langle V^2 \rangle \qquad (2.9)$$

The condition $\varepsilon_\ell = 0$ corresponds to the onset of coherent motion and is the analog of the dispersion relations for space charge and wall impedence instabilities. In this kinetic equation approach, these additional forces can be included in equations (2.3) and (2.4) and will lead, for example, to modification of the ε_ℓ factors and cooling rates. This effect may be of importance in the latter stages of momentum cooling.

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References

1. G. Carron, et. al. Experiments in Stochastic Cooling in ICE. IEE Transactions in Nucl. Sci., NS-26, 3, 3456 (1979).

2. R. Liboff, Introduction to the Theory of Kinetic Equations. John Wiley and Sons, (1969), p. 52.

3. S. Ichimaru, Basic Principles of Plasma Physics. W. A. Benjamin (1973).

4. Yu. Klimontovich, Statistical Theory of Non-equilibrium Processes in a Plasma, MIT Press (1967), p. 122.

5. J. Bisognano, internal report, BECON 9.

6. F. Sacherer, Stochastic Cooling Theory, CERN-ISR-TH/78-11 (1978).

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