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TRANSVERSE TWO-STREAM INSTABILITY IN THE  
PRESENCE OF STRONG SPECIES-SPECIES  
AND IMAGE FORCES

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TRANSVERSE TWO-STREAM INSTABILITY IN THE PRESENCE  
OF STRONG SPECIES-SPECIES AND IMAGE FORCES

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ABSTRACT

The theory of coherent transverse oscillations of two particle species is extended to include strong species-species and image forces. It is shown that in general the species-species force can considerably alter the instability threshold. Conversely, it is shown that the limit on the performance of an electron ring accelerator imposed by the requirement of stable ion electron oscillations, is not significantly improved by the inclusion of images.

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## 1. INTRODUCTION

The transverse coupling instability of relativistic stabilized beams has long been a subject of intensive study (e.g. ref. 1-5). Recently, it has been emphasized,<sup>5)</sup> that this two-stream instability can impose a severe limit to the acceleration rate attainable in an electron ring accelerator (ERA).

A similar type of instability can also occur in synchrotrons or storage rings when particles of the opposite charge are trapped in the main beam.<sup>6),7),8)</sup>

In the present note we extend the theory to include -- in an approximate way -- the influence of space-charge forces acting between particles of the same beam ("species-species forces"), as well as image forces due to the presence of walls. For simplicity, and because they are the most unstable modes, we shall concentrate on dipole oscillations.

We find that species-species forces and images can considerably -- and in many cases adversely -- affect the instability threshold. In fact, to explain the instability in the Bevatron it seems vital to include electron-electron forces in the theory.

In an electron ring accelerator acceleration column, where axial focussing is provided only by ion-electron forces and electron images, we hoped that the inclusion of images would relax the ion-electron instability threshold. We shall show this is not the case.

## 2. OUTLINE OF THE SOLUTION

We start with the equation of motion for a test particle of each species. We include three types of forces, a "single particle force", a "coherent force" and a "coupling force". The single particle force is proportional to the displacement of the test particle, the coherent force is proportional to the displacement of the entire same beam of particles similar to the test particle, and the coupling force is proportional to the displacement of the other beam. Each of these force coefficients is modified by images and/or species-species forces.

We assume harmonic oscillation of the beam centers and average the single particle response over all beam particles. The averaging process takes frequency spread into account. The eigenvalues and eigenvectors, of the coupled system which describes the motion of the two beam centers,

determine the mode frequency (and hence thresholds and growth rates) and the relative amplitudes of the two beams.

### 3. EQUATIONS OF MOTION

To be specific, and clearly without loss of generality, we take the beam species to be electrons (the replacements for proton beams is made in Section 7). We normalize all frequencies to the average electron revolution frequency ( $\Omega_0$ ) and denote the beam (electron) frequencies by lower case  $q$ 's and the stationary species (proton) frequencies by capital  $Q$ 's. The equations of motion of the two test particles are

$$\frac{1}{\Omega_0^2} \left( \frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \theta} \right)^2 x + q^2 x + q_u^2 \bar{x} - q_c^2 \bar{y} = 0, \quad (1)$$

$$\frac{1}{\Omega_0^2} \frac{\partial^2 y}{\partial t^2} + Q^2 y + Q_u^2 \bar{y} - Q_c^2 \bar{x} = 0,$$

where  $x$  and  $y$  are the transverse coordinates (in the same direction) of the test electron and test ion.

The quantities  $q^2$ ,  $Q^2$ ,  $q_u^2$ , etc. will be discussed in more detail in the examples given below. We remark, here, that in the absence of species-species forces and of images:

$$q^2 = q_c^2 + q_o^2 \frac{\Omega^2}{\Omega_0^2} \quad ( = Q_e^2 = Q_1^2 + \lambda^2 ),$$

$$q_u^2 = 0,$$

$$(q_c^2 = Q_1^2),$$

$$Q^2 = Q_c^2 = \frac{m\gamma}{M} \text{ f } q_c^2 \quad ( = Q_i^2 ),$$

$$Q_u^2 = 0,$$

where we give in parentheses the notation of Koshkarev and Zenkevitch<sup>5)</sup>. The external focussing is characterized by  $q_c^2$ ,  $\frac{m\gamma}{M}$  is the (relativistic) mass ratio between electrons and ions, and  $f = \frac{N_i}{N_e}$  the fractional ion loading. The quantities  $q_c$  and  $Q_c$  are -- in this approximation -- the electron and ion bounce frequencies in the potential well of the other beam. The quantities  $q_u^2$  and  $Q_u^2$  are in general closely related to the coefficient  $(U + V + iV)$  of Ref. 9) which determines single beam stability (resistive wall effect, etc.). For the electrons we have, e.g.,

$$q_u^2 \approx -2q_c^{-1}(U + V + iV). \quad (3)$$

#### 4. SOLUTION

We solve (1) by assuming that the beam centers oscillate harmonically in time and space:

$$\begin{aligned} \bar{x} &= \bar{\xi} \exp[i(n\theta - v\Omega_0 t)], \\ \bar{y} &= \bar{\zeta} \exp[i(n\theta - v\Omega_0 t)], \end{aligned} \quad (4)$$

and regarding the  $\bar{x}$ - and  $\bar{y}$ - terms in (1) as driving forces. In finding the response of the test particle we, as is usual in Landau damping calculations, ignore transients and take  $\text{Im}(v) \geq +0$  -- hence concentrating on the unstable range.

In the case where nonlinearity in the oscillation direction is negligible the single particle response  $\xi$  and  $\zeta$  is simply the steady state solution of a driven harmonic oscillator. In the case of important nonlinearity in the oscillation we use the results of Ref. 10) to obtain approximate expressions for  $\xi$  and  $\zeta$  valid for small amplitude and small nonlinearity.

We introduce normalized distribution functions  $f(p)$ ,  $g(a^2)$ ,  $h(b^2)$  for the electrons, and  $F(p)$ ,  $G(a^2)$ ,  $H(b^2)$  for the ions, that describe the momentum distribution, and the distribution of the incoherent betatron amplitudes of the particles, with  $b$  referring to the direction of the oscillation. We assume that the distributions are uncorrelated so that the beam center is determined by:

$$\bar{\xi} = \int \xi f(p)g(a^2)h(b^2) dp da^2 db^2, \quad (5)$$

$$\zeta = \int \zeta F(p)G(a^2)H(b^2) dp da^2 db^2. \quad (5)$$

Thus we obtain:

$$\bar{\xi} (1 + i_u) - \zeta i_c = 0, \quad (6)$$

$$\zeta (1 + I_u) - \bar{\xi} I_c = 0,$$

where

$$i_u = \int_{q_u}^2 \frac{f(p)[-b^2 h'(b^2)]g(a^2)}{q^2 - (v-n \frac{\Omega}{\Omega_0})^2} dp da^2 db^2, \quad (7)$$

$$i_c = \int_{q_c}^2 \frac{f(p)[-b^2 h'(b^2)]g(a^2)}{q^2 - (v-n \frac{\Omega}{\Omega_0})^2} dp da^2 db^2,$$

and  $I_u$  and  $I_c$  are similar dispersion integrals for the ions (with  $\Omega = 0$ ).

## 5. APPROXIMATIONS

We know that the values of dispersion integrals, such as (7), are primarily determined by the width of the distribution functions.<sup>9),11)</sup> Hence we approximate (7) by neglecting the variation of the  $q^2$ -coefficients in the numerator and keeping only the first-order variation of the coefficients in the denominator. Furthermore, we circumvent questions of self-consistency and assume that the coefficients and the distribution functions can be independently selected. Thus, we write the characteristic equation, associated with (6), in the form

$$(\Delta q^2 + q_u^2)(\Delta Q^2 + Q_u^2) - q_c^2 Q_c^2 = 0 \quad (8)$$

where:

$$1/\Delta q^2 = \int \frac{f(p)g(a^2)[-b^2 h'(b^2)]}{q^2 - (v-n)^2} dp da^2 db^2, \quad (9)$$

$$1/\Delta Q^2 = \int \frac{F(p)G(a^2)[-b^2 H'(b^2)]}{Q^2 - v^2} dp da^2 db^2.$$

The combined effect of three spreads can be treated only with difficulty. Double dispersion integrals have in fact been treated in Ref. 11). The result is that the spread effective for damping is not the sum of the spreads, but rather the Landau damping is mainly determined by the larger of the two spreads. Hence we shall consider only the effect of a single spread; namely the largest.

Finally, one may make a further approximation which we call the "slow wave approximation"; namely we expand the denominators of (9) in partial fractions and keep the term which is largest when  $\nu \approx (n-q)$ , and when  $\nu \approx Q$ . In this approximation -- and by expanding  $q \cong q_0 + s(\partial q/\partial s)_0$ , etc. -- (8) takes the form

$$\left( \Delta q - \frac{q_u^2}{2q_0} \right) \left( \Delta Q + \frac{Q_u^2}{2Q_0} \right) + \frac{q_c^2 Q_c^2}{4q_0 Q_0} = 0 \quad (10)$$

with

$$\Delta q = \left[ \int \frac{f(s) ds}{(n-q_0) - \nu + \Delta'_e s} \right]^{-1},$$

$$\Delta Q = \left[ \int \frac{F(s) ds}{Q_0 - \nu + \Delta'_i s} \right]^{-1}, \quad (11)$$

and

$$\Delta'_e = \left[ \frac{\partial}{\partial s} \left( n \frac{\Omega(s)}{\Omega_0} - q(s) \right) \right] .$$

$$\Delta'_i = \left( \frac{\partial Q}{\partial s} \right)_0 . \quad (12)$$

The quantity  $s$  is one of the spreading parameters  $p$ ,  $a^2$  or  $b^2$ , and

$$f(s) ds = \begin{cases} f(p) dp & \text{or} \\ g(a^2) da^2 & \text{or} \\ -b^2 \frac{dh(b^2)}{db^2} db^2 & \text{[see ref. 8],} \end{cases} \quad (13)$$

$$\int f(s) ds = 1.$$

Alternatively, we employ the term "improved slow wave approximation" for the approximation in which we retain (8) but approximate the factors that

arise from the fast-wave terms by the  $\nu$ -value  $(n-q)$  and  $Q$ .

## 6. STABILITY CONDITIONS

### A. Analytic Results

Stability conditions can be obtained from (8) or (10), by finding the boundary (on which  $\nu$  is real) of the unstable zone (in which the imaginary part of  $\nu$  is positive). To this end, the integrals of (9), or (11), need to be evaluated; and in Table I we summarize results, for the essential component of (9) and (11), resulting from two different choices of distribution functions. The Lorentzian distribution is studied, despite its unphysically long tails, because the analysis is simple and because it can be employed to establish an interesting general result (see Sect. 8). For accurate results, a truncated distribution is required.

In the case of  $\delta$ -functions for  $f(s)$  and  $F(s)$  (no frequency spreads) the eigenfrequencies are determined from

$$[(\nu-n)^2 - q^2 - q_u^2][\nu^2 - Q^2 - Q_c^2] - q_c^2 Q_c^2 = 0. \quad (14)$$

For a Lorentzian line, [ $f(s) \propto (4s^2 + \Delta_e^2)^{-1}$ ,  $F(s) \propto (4s^2 + \Delta_i^2)^{-1}$ ], with equal slow wave and fast wave frequency spread and with  $\Delta_e$  and  $\Delta_i$  the full widths at half maximum, equation (14) is valid with  $\nu \rightarrow \nu + i \Delta_e/2$  in the first factor and  $\nu \rightarrow \nu + i \Delta_i/2$  in the second factor. If, in addition,  $\Delta_e \approx \Delta_i = \Delta$ , the condition for stability is

$$\Delta \geq 2(1/\tau_0), \quad (15)$$

where  $\tau_0^{-1}$  is the growth rate in the absence of dispersion.

In the neighborhood of a resonance we may use the improved slow-wave approximation. In the absence of frequency spread, (14) yields

$$\nu = \tilde{Q} + d \pm i \sqrt{\frac{q_c^2 Q_c^2}{4\tilde{Q}\tilde{q}} - d^2}, \quad (16)$$

where

$$d = \frac{1}{2}(n - \tilde{q} - \tilde{Q}), \quad (17)$$

$$\tilde{Q}^2 = Q^2 + Q_u^2,$$

$$\tilde{q}^2 = q^2 + q_u^2, \quad (17)$$

and resonance occurs when  $d \approx 0$ . For the Lorentzian line, and in improved slow wave approximation,

$$\nu = \tilde{Q} + d - i \left( \frac{\Delta_e + \Delta_i}{4} \right) \pm i \sqrt{\frac{q_c^2 Q_c^2}{4\tilde{q}\tilde{Q}} - \left[ d - i \left( \frac{\Delta_e - \Delta_i}{4} \right) \right]^2}, \quad (18)$$

with  $\Delta_p$  and  $\Delta_e$  the full widths at half maximum in the frequencies  $Q$  and  $\left| n \frac{\Omega}{\Omega_0} - q \right|$ . Stability of the solution (18) requires spreads such that

$$\Delta_e \Delta_i \geq \frac{q_c^2 Q_c^2}{\tilde{q}\tilde{Q}} \left[ 1 + \left( \frac{4d}{\Delta_e + \Delta_i} \right)^2 \right]^{-1}. \quad (19)$$

To suppress an instability that occurs within a narrow resonant frequency band (where  $d$  will be close to zero), (19) provides the convenient sufficient condition

$$\Delta_e \Delta_i \geq \frac{q_c^2 Q_c^2}{\tilde{q}\tilde{Q}} \quad (20)$$

For values of  $q_0$  etc. that are considered to be essentially known (e.g., from Table II). It is of interest to note, from (19) or (20), that both  $\Delta_e$  and  $\Delta_i$  must be non-zero to suppress the instability.<sup>5)</sup>

Finally we turn to the case of the semi-circular distribution (see Table I). For this distribution the damping is very different for the fast and the slow waves and hence it is not reasonable to assume  $\Delta_+ = \Delta_-$ . Rather, we employ the slow wave approximation and completely disregard the non-resonant fast wave to obtain:

$$\nu = Q + \frac{Q_u^2}{Q} + d_1 - \frac{i}{2} (\bar{\Delta}_e + \bar{\Delta}_i) \pm i \sqrt{\frac{q_c^2 Q_c^2}{qQ} - \left[ d_1 - \frac{i}{2} (\bar{\Delta}_e - \bar{\Delta}_i) \right]^2}, \quad (21)$$

with

$$d_1 = \frac{1}{2} \left[ \left( n - q - \frac{q_u^2}{q} \right) - \left( Q + \frac{Q_u^2}{Q} \right) \right], \quad (22)$$

$$\bar{\Delta}_e = \sqrt{\Delta_e^2 - (n-q-v)^2} ,$$

$$\bar{\Delta}_i = \sqrt{\Delta_i^2 - (Q-v)^2} .$$
(22)

After considerable algebraic manipulation it can be seen that stability requires:

$$\bar{\Delta}_e \bar{\Delta}_i \geq \frac{q_c^2 Q_c^2}{qQ} \left[ 1 + \left( \frac{2d_1}{\bar{\Delta}_e + \bar{\Delta}_i} \right)^2 \right]^{-1}$$
(23)

Again within a narrow band of instability, associated with the resonance  $d_1 \cong 0$  (where  $n - q - v \cong q_u^2/q$  and  $|Q-v| \cong Q_u^2/Q$ ), we may write

$$\bar{\Delta}_e \bar{\Delta}_i \geq \frac{q_c^2 Q_c^2}{qQ}$$
(24)

or

$$\left[ \Delta_e^2 - \left( \frac{q_u^2}{q} \right)^2 \right]^{1/2} \left[ \Delta_i^2 - \left( \frac{Q_u^2}{Q} \right)^2 \right]^{1/2} \geq \frac{q_c^2 Q_c^2}{qQ}$$

as a sufficient condition for suppression of the instability. The second of the forms (24) clearly implies that we must require

$$\Delta_e > \left| \frac{q_u^2}{q} \right|$$

and

$$\Delta_i > \left| \frac{Q_u^2}{Q} \right| .$$
(25)

The condition (23) is similar to the condition (19) found for the Lorentzian distribution -- or (24) is similar to (20) -- but with the width parameters modified to correct for the anomolous results arising from the extensive tails of the Lorentz distribution [e.g., in the manner suggested by (29) of Sect. 6B below]. It is evident that for wave frequencies removed from the central beam frequency there is reduced Landau damping. With the abruptly terminated semi-circular distributions that led to (23) et seq., this limitation is explicitly indicated by the conditions (25). Again we note that both  $\bar{\Delta}_e$  and  $\bar{\Delta}_i$  must be non-zero to insure stability.

## B. Numerical Formulation

For numerical work we proceed directly from (8) and (9) and again employ Lorentzian distributions in  $s$ , with  $\delta_{\pm}$  denoting full widths at half maximum in the quantities  $\left| n \frac{\Omega}{\Omega_0} \pm q \right|$  for the fast and slow waves of the electron component and  $\Delta_{\pm}$  correspondingly for the ions. If we then let

$$x_{\mp} = v - (n \mp q_0), \quad X_{\mp} = v \pm Q_0, \quad (26)$$

and write

$$g_{\mp} = x_{\mp} + i \delta_{\mp}, \quad G_{\mp} = X_{\mp} + i \Delta_{\mp}, \quad (27)$$

$$h = g_+ - g_- - 2q_0, \quad H = G_+ - G_- - 2Q_0,$$

we then find

$$\begin{aligned} & \left[ q_0(x_- + g_-)(x_+ + g_+) + hq_u^2 \right] \left[ Q_0(X_- + G_-)(X_+ + G_+) + Hq_u^2 \right] \\ & - hHq_c^2 Q_c^2 = 0. \end{aligned} \quad (28)$$

The imaginary parts of the expressions written above for  $g_{\mp}$ ,  $G_{\mp}$  are seen to imply a damping that is independent of the distance by which the actual frequency is displaced from the peak of the distribution. This results from the unphysically extensive tails of the Lorentz distributions that were assumed for evaluation of  $\Delta q^2$  and  $\Delta Q^2$ . For this reason we have elected to replace, in the numerical work, these expressions by

$$g_{\mp} = x_{\mp} + i \frac{\delta_{\mp}^2}{\left| x_{\mp} \right|^2 + \delta_{\mp}^2} \left| \delta_{\mp} \right|, \quad (29)$$

$$G_{\mp} = X_{\mp} + i \frac{\Delta_{\mp}^2}{\left| X_{\mp} \right|^2 + \Delta_{\mp}^2} \left| \Delta_{\mp} \right|$$

in the expectation that a more realistic type of distribution will be described in this way. With this replacement we obtain an equation for

which roots have been sought computationally.<sup>12)</sup>

From computational tests that employed parameters similar to those introduced in the example of the following Section, it was found (i) that the values of the fast-wave dispersion parameters  $\delta_+$ ,  $\Delta_-$  for the two species had little effect on the stability threshold (although it may be necessary that they be, for example, some 3% of the respective slow-wave quantities  $\delta_-$ ,  $\Delta_+$ ), and (ii) that (19) [or (20)] can be safely taken as a stability criterion to be applied to the slow-wave dispersion parameters after modification in the manner indicated by (29). It was confirmed, moreover, that, as expected,<sup>5)</sup> stabilization could not be obtained by introducing dispersion into just one of the two species.

## 7. PROTON SYNCHROTRONS AND STORAGE RING

We assume that electrons created by scattering with the background gas remain trapped in the circulation beam. Further, we assume the electrons to be uniformly distributed around the circumference, and we neglect the influence of the background gas ions. We take the proton and electron minor radii as equal.

The proton and electron frequencies relevant to this case are given in Table II. In many situations of interest one can use, to a good approximation, simplified relations obtained by taking  $q \approx v_{z0}$  and neglecting images.

In this approximation the stability conditions (24) and (25) are conveniently expressed in terms of the "space charge q-shift",  $q_1$ , so that one requires

$$\begin{aligned} \bar{\Delta}_e \bar{\Delta}_p &> f \frac{q_1^3}{v_{z0}} \sqrt{\frac{M\gamma}{m}} , \\ \Delta_e &> f q_1 \sqrt{\frac{M\gamma}{m}} , \\ \Delta_p &> \frac{q_1^2}{\gamma^2 v_{z0}} , \end{aligned} \quad (30)$$

where

$$q_1^2 = (2/\pi) \frac{N_p r_p R}{\gamma b(a+b)} . \quad (31)$$

Let us, as an example, discuss the case of the Bevatron, where an instability of the debunched beam at 6 GeV has been observed, and has been cured by the provision of clearing fields.<sup>7)</sup>

For typical operating conditions at 6 GeV we find  $q_1^2 = 4 \times 10^{-3}$  and  $v_{z_0} \approx 0.9$ . (More details may be found in Ref. 13.) Hence, for stability we require

$$\sqrt{\Delta_p^2 - (1.2 \times 10^{-4})^2} \sqrt{\Delta_e^2 - (6.3f)^2} \geq 2.5 \times 10^{-2}f. \quad (32)$$

If we assume the relatively large spreads  $\Delta_e = 1.5$ ,  $\Delta_p = 0.04$ , we find a threshold neutralization  $f = 0.24$ . Neglecting electron-electron forces, the threshold  $f$  would be a tolerable  $f > 1$ . Hence, in the case of the Bevatron, species-species forces appear to play a dominant role in the determination of the threshold. This situation is generally the case in a proton ring if the proton frequency spread is large and/or  $\gamma$  is small.

## 8. AXIAL STABILITY IN THE ERA

In the acceleration column of an ERA, or of any similar system in which translational invariance of the configuration can be legitimately assumed, it follows from equation (1) that

$$\begin{aligned} q^2 + q_u^2 - q_c^2 &= 0; \\ Q^2 + Q_u^2 - Q_c^2 &= 0. \end{aligned} \quad (33)$$

From (17), we may write

$$\begin{aligned} \tilde{q}^2 &= q_c^2, \\ \tilde{Q}^2 &= Q_c^2. \end{aligned} \quad (34)$$

As we shall see below, the invariance conditions imply that images hardly effect the axial stability conditions in an ERA.

Frequency parameters for an ERA have been derived in Ref. (14). They are presented in Table III. These formulas can be simplified by assuming

$$1 \gg f \gg \frac{1}{\gamma^2} \quad (35)$$

$$\frac{R^2}{b(a+b)} \gg P/8, \quad \frac{\epsilon_e}{(S_e-1)^2},$$

in which case image contributions only appear in  $q_u^2$  and  $q_o^2$ .

For the Lorentzian distribution, equation (14) is valid with the replacement discussed just following equation (14). In view of (33), this equation is independent of images. Thresholds are as has been discussed in the literature,<sup>5)</sup> and above threshold we have the stability condition (20), which takes the image-independent form:

$$\Delta_e \Delta_i \geq q_c Q_c. \quad (36)$$

We note that the condition (36) will normally not be satisfied except for working points with very small values of  $q_c$  and/or  $Q_c$ . Such working points, however, are unattractive because both  $q_c$  and  $Q_c$  are "figures of merit" of an ERA device -- since  $Q_c^2$  is a measure of the holding power of the ring and  $q_c^2$  determines the fractional ion loading.

For the semi-circular distribution, or the modified Lorentz distribution, the thresholds and damping conditions depend slightly upon the image terms. We have undertaken numerical studies in order to ascertain the effect, on the instability, of images and dispersion. We concentrate on the  $n = 1$  (dipole) instability and we refer to Table III and postulate parameters such that  $\gamma = 40$ ,  $C_1 \equiv 4\left(\frac{\mu}{N_e}\right) \frac{R^2}{b(a+b)} = 5.0 \times 10^{-13}$ ,

$$C_2 \equiv 4\left(\frac{\mu}{N_e}\right) \frac{P}{8} = \frac{1}{60} \times 10^{-13}, \quad \text{and} \quad C_3 \equiv 4\left(\frac{\mu}{N_e}\right) \frac{\epsilon_e}{(S_e-1)^2} = 0.05 \times 10^{-13}.$$

(Such coefficients might result, approximately, from  $R = 3.5$  cm,  $a = 0.30/\sqrt{2}$  cm,  $b = 0.15/\sqrt{2}$  cm, and  $|S_e-1| = 0.625/3.5$ ). Then, with  $M/m = 1836$ , we write

$$q_u^2 = (C_1/1600 + C_2 - \mathcal{K}C_3) \cdot N_e$$

$$q_c^2 = (C_1 + C_2 - \mathcal{K}C_3) \cdot N_i$$

$$q^2 = q_c^2 - q_u^2$$

$$Q_u^2 = (40/1836)q_c^2$$

$$Q_c^2 = (40/1836)(C_1 + C_2 - \mathcal{K}C_3) \cdot N_e$$

and

$$Q^2 = Q_c^2 - Q_u^2,$$

where  $\mathcal{K}$  is a "flag" that, if set equal to unity, introduces the effect of a strong electrostatic focussing. The dispersion may be controlled by means of a parameter  $\eta$  such that  $\delta_- = \delta_+ = \eta q$  and  $\Delta_- = \Delta_+ = \eta Q$ .

With these substitutions introduced into (28), as modified by (29), one may solve for the roots computationally, along a trajectory on which (for example)  $f = N_i/N_e$  is held constant, and so examine the variation of the threshold vs. the damping coefficient  $\eta$ . With the ratio  $N_i:N_e$  equal to one and one-half percent, and with images absent ( $\mathcal{K} = 0$ ), one finds in this way virtually no change of the threshold until  $\eta > 0.4$ , and even with  $\eta$  as large as unity the particle abundances are permitted to increase by only 43 percent. Under similar circumstances dispersion is found to be somewhat more effective when image focussing is present ( $\mathcal{K} = 1$ ), but the gains are trivial until  $\eta > 0.4$  and  $\eta$  should exceed 0.93 to achieve a doubling of the permissible particle numbers.

In examining an alternative trajectory on which the ratio  $N_i:N_e$  is taken to be one-half of one percent, it appears desirable to have image focussing present ( $\mathcal{K} = 1$ ) since the ion focussing can be expected to be weak. Under these conditions the effect of the dispersion coefficient  $\eta$  has been found to be somewhat greater than was the case for the trajectory mentioned earlier, although the effect remains small until  $\eta$  exceeds 1/2. Somewhat more striking effects do develop at the larger values of  $\eta$  -- thus, with  $N_i/N_e = 0.005$ , dispersion characterized by  $\eta = 0.88$  permits a doubling of the particle numbers and, at  $\eta = 1$  and  $N_e = 5 \times 10^{13}$ , stability is obtained for  $N_i \leq 4.67 \times 10^{11}$ , i.e. for  $f \leq 0.0093$  (cf. the Figure on p. 5 of ERAN-177,<sup>12</sup>) which suggests the ability of strong dispersion to open up a narrow stable corridor through a region of small  $N_i$ ).

In summary, the numerical studies have shown that with physically achievable damping terms the stability threshold is only slightly changed from that which obtains in the absence of damping; a result in accord with (36) and with the conclusions of Zenkevich and Koshkarev.<sup>5</sup>) We conclude that neither Landau damping nor image effects and species-species forces are capable of any considerable extension of the stable working range in an ERA-column.

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Table I - The Dispersion Integral I

a.) Definitions

$$I = \int \frac{f(s) ds}{q^2 - (v - n \frac{\Omega}{\Omega_0})^2} \cong \frac{1}{2q} \left[ \int \frac{f(s) ds}{v - n \frac{\Omega}{\Omega_0} + q} - \int \frac{f(s) ds}{v - n \frac{\Omega}{\Omega_0} - q} \right]$$

$$q = q_0 + \frac{\partial q}{\partial s} s$$

$$\Omega = \Omega_0 + \frac{\partial \Omega}{\partial s} s$$

$$\Delta_{\pm} = s_{1/2} \left| \frac{\partial q}{\partial s} \pm \frac{1}{\Omega_0} \frac{\partial \Omega}{\partial s} \right|$$

$s_{1/2}$  is the half width of  $f(s)$  (full width at half maximum or half width at bottom).

b.) Lorentzian Distribution

$$f(s) = \frac{2}{\pi} \frac{s_{1/2}}{(s_{1/2}^2 + 4s^2)}$$

$$I = \frac{1}{2q_0} \left[ \frac{1}{v - (n - q_0) + \frac{i\Delta_-}{2}} - \frac{1}{v - (n + q_0) + \frac{i\Delta_+}{2}} \right]$$

If  $\Delta_+ = \Delta_-$ :  $I = \frac{1}{q^2 - (v + \frac{i\Delta}{2} - n)^2}$

c.) Semi-circle Distribution

$$f(s) = \begin{cases} \frac{2}{\pi s_{1/2}} \sqrt{s_{1/2}^2 - s^2} & |s| \leq s_{1/2} \\ 0 & |s| \geq s_{1/2} \end{cases}$$

Table I (cont.)

$$I = \frac{1}{q_0} \left[ \frac{1}{v-(n-q_0) + i\bar{\Delta}_-} - \frac{1}{v-(n+q_0) + i\bar{\Delta}_+} \right]$$

$$\bar{\Delta}_+ = \sqrt{\Delta_+^2 - [v-(n+q_0)]^2}$$

$$\bar{\Delta}_- = \sqrt{\Delta_-^2 - [v-(n-q_0)]^2}$$

Table II - Frequencies of Vertical Oscillation of a Coasting Proton Beam Partially Neutralized by Electrons\*

1. Proton frequencies:†

$$q^2 = v_{z_0}^2 + 4\mu \left[ \frac{1}{b(a+b)} \left( f - \frac{1}{\gamma^2} \right) + \frac{\epsilon_1}{h^2} (f-1) - \frac{\epsilon_2 \beta^2}{g^2} \right] R^2$$

$$q_u^2 = \frac{4\mu}{\gamma^2} \left[ \frac{1}{b(a+b)} + \frac{\epsilon_1 - \xi_1}{h^2} \right] R^2$$

$$q_c^2 = 4\mu f \left[ \frac{1}{b(a+b)} + \frac{\epsilon_1 - \xi_1}{h^2} \right] R^2 = q_u^2 f \gamma^2$$

2. Electron frequencies ( $\beta_e = 0$ ):

$$Q^2 = 4\mu \frac{M\gamma}{m} \left[ \frac{1}{b(a+b)} (1-f) + \frac{\epsilon_1}{h^2} (1-f) \right] R^2$$

$$Q_u^2 = 4\mu \frac{M\gamma}{m} f \left[ \frac{1}{b(a+b)} + \frac{\epsilon_1 - \xi_1}{h^2} \right] R^2 = q_c^2 \frac{M\gamma}{m}$$

$$Q_c^2 = 4\mu \frac{M\gamma}{m} \left[ \frac{1}{b(a+b)} + \frac{\epsilon_1 - \xi_1}{h^2} \right] R^2 = Q_u^2 / f$$

where:

$$\mu = \frac{N_p r_p}{2\pi R \gamma} \quad f = \frac{N_e}{N_p},$$

\* Curvature effects are ignored, and the beam is assumed to be centered in the vacuum chamber.

† In writing the proton frequencies, we have set  $\beta = 1$ , save in the last (magnetostatic) term of the equation for  $q_o^2$ .

$\epsilon_1, \epsilon_2, \xi_1$ : Image coefficients

h: Half height of vacuum chamber

g: Half height of magnet gap

$r_p$ : Classical proton radius

Table III - Electron and Ion Frequencies for Axial Motion in an ERA-Column \* ††

1. Electron frequencies:†

$$q_o^2 = 4\mu \left[ \frac{R^2}{b(a+b)} \left( f - \frac{1}{\gamma^2} \right) - \frac{P}{8} + \frac{\epsilon_e(1-f)}{(S_e-1)^2} - \beta^2 \frac{\epsilon_m}{(S_m-1)^2} \right]$$

$$q_u^2 = 4\mu \left[ \frac{R^2}{b(a+b)\gamma^2} + \frac{P}{8} - \frac{\epsilon_e}{(S_e-1)^2} + \beta^2 \frac{\epsilon_m}{(S_m-1)^2} \right]$$

$$q_c^2 = 4\mu f \left[ \frac{R^2}{b(a+b)} - \frac{\epsilon_e}{(S_e-1)^2} \right]$$

2. Ion frequencies:

$$Q^2 = 4\mu \left( \frac{m\gamma}{M} \right) (1-f) \left[ \frac{R^2}{b(a+b)} - \frac{\epsilon_2}{(S_e-1)^2} \right]$$

$$Q_u^2 = q_c^2 \frac{m\gamma}{M}$$

$$Q_c^2 = q_c^2 \frac{m\gamma}{Mf}$$

where:

$$\mu = \frac{N_e r_e}{2\pi R \gamma}, \quad f = \frac{N_i}{N_e}$$

\* Uniform external guide field assumed.

†  $\beta_e \approx 0$

$$P = 2 \ln[16R/(a+b)]$$

$S_e$  = Radius of Electric image cylinder/R

$S_m$  = Radius of magnetic image cylinder/R

$\epsilon_e \approx \epsilon_m \approx 0.125$  image coefficients

$r_e$ : Classical electron radius

†† We are indebted to Prof. M. Reiser for a recent communication concerning his analysis of toroidal field gradients [Max-Planck-Institute for Plasma Physics Report IPP O/14 (Munich-Garching, July 1972)] that called to our attention the appropriate form of certain terms indicated in Table III.